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ABSTRACT

A continuation of the S matrix, for many-channel potential-scattering problems with arbitrary spin, away from physical values of the angular momentum is defined. It is shown that the scattering amplitude can be expressed as a sum over physical J values of a summand which is meromorphic in the entire finite J plane.

ON COMPLEX ANGULAR MOMENTUM IN MANY-CHANNEL POTENTIAL-SCATTERING PROBLEMS. I*

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1. INTRODUCTION

The purpose of this work is to extend the discussion of complex angular momentum given by Regge[1] and by Bottino et al.[2] for the single-channel, zero-spin, potential-scattering problem to the many-channel potential problem with arbitrary spin. We restrict our considerations at present to the case where all channels are two-body.

The intrinsic interest of the properties of partial wave amplitudes in the complex angular momentum plane in potential scattering problems is probably only academic, but, in so far as the results can be taken over to the relativistic problem, where at present many of the proofs are lacking, it is extremely useful (see, for example, Udgaonkar,[3] Chew et al.,[4] and Frautschi et al.[5]). For this purpose the essential result of Regge[1] is that the partial-wave amplitude, defined initially for physical ℓ values, can be continued to give a function of ℓ which is meromorphic in Re $\ell > -1/2$ and for which the Sommerfeld-Watson transform is possible. For potentials regular at $r = 0$, the meromorphy domain has been extended to include the whole ℓ plane except $\ell = \infty$.[6,7,8] In this paper we show that an analogous continuation can be made for the many-channel problem with spin, and yields an S matrix which is meromorphic in the finite J plane. The restrictions we make on the potential matrix are (i) that it be local in r in the sense defined in Eq. (3.4), (ii) that it be a superposition of Yukawa potentials with a finite maximum range, and (iii) that

it be regular at the origin. It should be noted that (i) does not exclude such interesting cases as, for example, the spin-orbit interaction. The restriction to a sum of Yukawa potentials is made because in the one-channel problem it ensures suitable behavior of the scattering amplitudes for large $|J|$. The regularity condition at $r = 0$ allows us to show meromorphy in the entire J plane rather than in a part of it. Note that these three restrictions are sufficient for our purpose but may well not be necessary. In section 2 we obtain a general expression for the scattering amplitudes in terms of solutions of the coupled Schrödinger equations, essentially following conventional treatments (see, for example, Newton[9]). The continuation to complex angular momentum, and the study of the analyticity properties of the scattering amplitudes are contained in section 3.

It should be noted that there are many continuations of the S matrix away from physical angular momenta.[8,10] Our particular choice is motivated by our belief that it is the only one that permits an analogue of the Sommerfeld-Watson transformation to be made (cf. the situation in the one-channel case[8]). We will discuss these and other related matters in a future paper on this subject.

2. GENERAL FORMULATION

The Coupled Schrödinger Equations

We are concerned with states of two particles, which in general are "compound", satisfying the Schrödinger equation

$$(E - H) \mid \Psi \rangle = 0, \tag{2.1}$$

where H contains the kinetic-energy operator of the relative motion and a general interaction operator V, and E is the energy of the system. We expand these states $|\Psi\rangle$ in terms of a complete set of states of total angular momentum, which we define as[11]

$$|JMc\ell s\rangle = \sum_{m_\ell m_s m_1 m_2} C(\ell s J;\, m_\ell m_s)\, C(s_1 s_2 s;\, m_1 m_2)\, |\ell m_\ell c_1 m_1 c_2 m_2\rangle. \tag{2.2}$$

Here J is the total angular momentum, ℓ is the orbital angular momentum of the relative motion, s_1 and s_2 are the spins of the two particles in intrinsic states c_1 and c_2, and M, m_ℓ, m_1, and m_2 are the z projections of J, ℓ, s_1, and s_2 respectively. The index c on the left is used for brevity instead of c_1 and c_2. Both $|JMc\ell s\rangle$ and $|\ell m_\ell c_1 m_1 c_2 m_2\rangle$ are normalized to unity.

The r representative of a state $|\Psi\rangle$, where r is the separation of particles 1 and 2, can be written in the form

$$\langle r \mid \Psi \rangle = \sum_{JMc\ell s} r^{-1} \psi^{JM}_{c\ell s}(r) \mid JMc\ell s \rangle. \tag{2.3}$$

Inserting this into Eq. (2.1) and using the orthonormality of the states defined in Eq. (2.2), we obtain

$$\ddot{\psi}^{JM}_{c\ell s}(r) + \left(k_c^2 - \frac{\ell(\ell+1)}{r^2}\right) \psi^{JM}_{c\ell s}(r) - \sum_{c'\ell's'} \langle c\ell s|V^J(r)|c'\ell's'\rangle\, \psi^{JM}_{c'\ell's'}(r) = 0. \tag{2.4}$$

Here we have put $\hbar^2/2m = 1$, where m is the reduced mass of the two particles, and introduced

$$k_c^2 = E - E_c, \tag{2.5}$$

where E_c is the energy threshold for channel c, i.e., the channel in which the two particles are in intrinsic states c_1 and c_2.

In writing Eq. (2.4) we have used the locality restriction on the potential, and also its rotational invariance. Since we shall always use boundary conditions that are independent of M, it follows from Eq. (2.4) that the index M on the ψ's is irrelevant; we shall therefore omit it in future. It will also be convenient in much of the following to replace the suffixes $c\ell s$ by μ. We can then order the channels in some way so that μ takes on integral values between 1 and N, where N is the number of channels. The value of a parameter in the μth channel will then be denoted by giving it a suffix μ, e.g., ℓ_μ, k_μ.

Special Solutions

We first define regular solutions of Eq. (2.4) which, for physical values of the angular momenta, satisfy

$$\lim_{r \to 0} \phi_\mu^J(r) = 0. \tag{2.6}$$

We choose N linearly independent solutions satisfying this condition, where N is the number of channels, and denote them by $\phi_{\mu p}^J$, with $p = 1,2\ldots N$.

We further define 2N linearly independent solutions $\chi_{\mu p}^J(\pm)$, each of which satisfies a boundary condition for large r, viz.

$$\lim_{r\to\infty} e^{\pm ik_\mu r} \chi_{\mu p}^J(\pm) = (k_\mu)^{-1/2}(\pm i)^{\ell_\mu} \delta_{\mu p}. \tag{2.7}$$

Thus, at infinity the wave function $X^J_{\mu p}(\pm)$ will have an ingoing or outgoing wave in the pth channel, and nothing in the other channels. The possibility of being able to define wave functions by using Eq. (2.7) depends on the fact that the potential falls off faster than $1/r$ for large r, which follows from the second restriction on V given in section 1.

The $2N$ solutions of the N second-degree coupled equations (2.4) defined by Eq. (2.7) form a basis for all solutions of these equations. Therefore we can write, in matrix notation,

$$\underset{\sim}{\phi}^J = \frac{i}{2}\,[\underset{\sim}{f}^J(-)\,\underset{\sim}{X}^J(+) - \underset{\sim}{f}^J(+)\,\underset{\sim}{X}^J(-)], \tag{2.8}$$

where $\underset{\sim}{\phi}^J$, $\underset{\sim}{X}^J(\pm)$ are N-by-N matrices with components $\phi^J_{\mu p}$ and $X^J_{\mu p}(\pm)$, respectively. Here we have introduced the matrices $\underset{\sim}{f}^J(\pm)$, with components $f^J_{qp}(\pm)$, which are independent of r and are generalizations of the well-known Jost functions. To obtain explicit expressions for $\underset{\sim}{f}^J(\pm)$, we note first that

$$\underset{\sim}{W}\,[\underset{\sim}{X}^J(\pm),\ \underset{\sim}{X}^J(\pm)] = 0, \tag{2.9a}$$

and

$$\underset{\sim}{W}[\underset{\sim}{X}^J(+),\ \underset{\sim}{X}^J(-)] = 2i. \tag{2.9b}$$

Here $\underset{\sim}{W}[\underset{\sim}{\psi}^{(1)}, \underset{\sim}{\psi}^{(2)}]$ is the Wronskian of two solutions of Eq. (2.4), defined by

$$\underset{\sim}{W}[\underset{\sim}{\psi}^{(1)}, \underset{\sim}{\psi}^{(2)}] = \underset{\sim}{\psi}^{(1)T}\,\dot{\underset{\sim}{\psi}}^{(2)} - \dot{\underset{\sim}{\psi}}^{(1)T}\,\underset{\sim}{\psi}^{(2)}, \tag{2.10}$$

where $\underset{\sim}{\psi}^T$ is the transpose of $\underset{\sim}{\psi}$ and $\dot{\underset{\sim}{\psi}}$ is the derivative of $\underset{\sim}{\psi}$ with respect to r. Since the potential matrix in Eq. (2.4) is symmetrical, the Wronskian is independent of r. Then, from Eqs. (2.8) and (2.9) we have

$$\underset{\sim}{W}[\underset{\sim}{X}^J(\pm),\ \underset{\sim}{\phi}^J] = \underset{\sim}{f}^J(\pm). \tag{2.11}$$

The S Matrix

From Eqs. (2.7) and (2.8) the asymptotic form of $\underset{\sim}{\phi}^J$ is given by

$$\phi^J_{\mu p} \underset{r\to\infty}{\sim} \frac{i}{2}\, i^{\ell_\mu}(k_\mu)^{-1/2} \left[f^J_{\mu p}(-)\, e^{-ik_\mu r} - e^{-i\pi\ell_\mu} f^J_{\mu p}(+)\, e^{+ik_\mu r} \right]. \quad (2.12)$$

To obtain the S matrix, we define N linear combinations of the $\phi^J_{\mu p}$, each having zero ingoing flux in all but one channel, i.e. we introduce $\sum_p \phi^J_{\mu p}\, \lambda^J_{pq}$, with

$$\underset{\sim}{f}^J(-)\, \underset{\sim}{\lambda}^J = \underset{\sim}{1}. \quad (2.13)$$

Then we have

$$\sum_p \phi^J_{\mu p}\, \lambda^J_{pq} \underset{r\to\infty}{\sim} \frac{i}{2}\, i^{\ell_\mu}(k_\mu)^{-1/2} \left[\delta_{\mu p}\, e^{-ik_\mu r} - e^{-i\pi\ell_\mu} \sum_p f^J_{\mu p}(+)\, \lambda^J_{pq}\, e^{ik_\mu r} \right], \quad (2.14)$$

and, by definition, the S matrix is given by[9]

$$S^J_{q\mu} = f^J_{\mu p}(+)\, \lambda^J_{pq}. \quad (2.15)$$

From Eqs. (2.13) and (2.15) we obtain

$$\underset{\sim}{S}^J = [\underset{\sim}{f}^J(-)^T]^{-1}\, \underset{\sim}{f}^J(+)^T. \quad (2.16)$$

We can write Eq. (2.16) in a simpler form if we note that $\underset{\sim}{W}[\underset{\sim}{\phi}^J, \underset{\sim}{\phi}^J] = 0$, from the boundary condition at $r = 0$, and, from Eqs. (2.8) and (2.9):

$$\underset{\sim}{W}[\underset{\sim}{\phi}^J, \underset{\sim}{\phi}^J] = \frac{i}{2}\, [\underset{\sim}{f}^J(-)^T\, \underset{\sim}{f}^J(+) - \underset{\sim}{f}^J(+)^T\, \underset{\sim}{f}^J(-)] = 0.$$

Hence we have

$$\underset{\sim}{S}^J = \underset{\sim}{f}^J(+)\,[\underset{\sim}{f}^J(-)]^{-1}. \tag{2.17}$$

The symmetry of the $\underset{\sim}{S}$ matrix follows from comparison of Eqs. (2.16) and (2.17).

The Schrödinger equation (2.4) is real for physical k and ℓ, so that we can choose the $\underset{\sim}{\phi}^J$ to be real. From Eq. (2.7), however, we have

$$\underset{\sim}{\chi}^J(\pm)^* = \underset{\sim}{\chi}^J(\mp). \tag{2.18}$$

Hence we obtain

$$\underset{\sim}{f}^J(\pm)^* = \underset{\sim}{f}^J(\mp), \tag{2.19}$$

from which follows the unitarity of $\underset{\sim}{S}^J$,

$$\underset{\sim}{S}^{J\dagger}\,\underset{\sim}{S}^J = \underset{\sim}{1}. \tag{2.20}$$

<u>The Scattering Amplitude</u>

The cross section for scattering from intrinsic states denoted by c to states denoted by c' is given in terms of the scattering amplitude $f(\Theta, \phi)$ by

$$\left.\frac{d\sigma(\Theta)}{d\Omega}\right|_{c'\lambda_1'\lambda_2',\,c\lambda_1\lambda_2} = \left|\langle c'\lambda_1'\lambda_2'\,|f(\Theta,\phi)|\,c\lambda_1\lambda_2\rangle\right|^2, \tag{2.21}$$

where the λ are the respective helicities and Θ and ϕ are respectively the center-of-mass polar and azimuthal angles of scattering. In terms of the S matrix we have[12]

$$\langle c'\lambda_1'\lambda_2'\,|f(\Theta,\phi)|\,c\lambda_1\lambda_2\rangle = \langle \underset{\sim}{k}_c, c'\lambda_1'\lambda_2'\,|S-1|\underset{\sim}{k}_c c\lambda_1\lambda_2\rangle/2i(k_c,k_c)^{1/2} \tag{2.22}$$

and

$$\langle c'\lambda_1'\lambda_2'|f(\theta,\phi)|c\lambda_1\lambda_2\rangle = (2i)^{-1}(k_{c'}k_c)^{-1/2} \sum_{J\ell\ell'ss'} (2\ell'+1)^{1/2}(2\ell+1)^{1/2}$$

$$\times\ C(\ell's'J;0,\lambda')\ C(s_1's_2's';\ \lambda_1',-\lambda_2')\ C(\ell sJ;\ 0,\lambda)\ C(s_1s_2s;\ \lambda_1,-\lambda_2)\ i^{\ell-\ell'}$$

$$\times\ \exp[i(\lambda-\lambda')\phi]\ d^J_{\lambda\lambda'}(\theta)\ \langle\ k_{c'}JMc'\ell's'|\ S-1\ |\ k_cJMc\ell s\rangle; \tag{2.23}$$

where

$$\lambda = \lambda_1 - \lambda_2 \tag{2.24a}$$

and

$$\lambda' = \lambda_1' - \lambda_2'. \tag{2.24b}$$

In the shorthand notation used previously, the S-matrix elements in Eq. (2.23) are simply $S^J_{\mu'\mu}$.

This completes the review of the formal analysis for physical values of the angular momentum, and we now turn to the problem of the continuation to general values.

3. ANALYTIC PROPERTIES IN J

The Potential

Before we can determine the analyticity properties of the solutions of Eq. (2.4), we must consider the potential matrix. Ignoring the index c, which is irrelevant here, we have

$$\langle r'JM\ell's|V|rJM\ell s\rangle = \frac{1}{4\pi} \sum_{m_1 m_2 m_1' m_2'} \left[\frac{(2\ell+1)(\ell-m)!(2\ell'+1)(\ell'-m')!}{(\ell+m)!\,(\ell'+m')!}\right]^{1/2}$$

$$\times\, C(\ell's'J;\, m', m_1' + m_2')\, C(s_1's_2's';\, m_1'm_2')\, C(\ell sJ;\, m,\, m_1 + m_2)$$

$$\times\, C(s_1s_2s;\, m_1m_2) \int d\Omega'\, d\Omega\, \exp\left[-i(m'\phi' - m\phi)\right] P_{\ell'}^{m'}(\cos\theta')\, P_{\ell}^{m}(\cos\theta)$$

$$\langle \underset{\sim}{r}'m_1'm_2' \mid V \mid \underset{\sim}{r}m_1m_2\rangle, \tag{3.1}$$

where

$$d\Omega = \sin\theta\, d\theta\, d\phi \tag{3.2}$$

and

$$m = M - m_1 - m_2, \tag{3.3}$$

and similarly for $d\Omega'$ and m'. Using now the first restriction on the potential, we can express it in the form

$$\langle \underset{\sim}{r}'m_1'm_2'|V|\underset{\sim}{r}m_1m_2\rangle = \sum_{\mathcal{L}\mathcal{M}} \langle m_1'm_2'|V_{\mathcal{L}\mathcal{M}}(r)|m_1m_2\rangle\, r^{-2}(2\pi)^3$$

$$\times\, \delta(\underset{\sim}{r}-\underset{\sim}{r}')\, Y_{\mathcal{L}\mathcal{M}}(\theta,\phi). \tag{3.4}$$

The spherical symmetry of the potential imposes certain restrictions on the $\underset{\sim}{V}_{\mathcal{L}\mathcal{M}}$; we assume that these are satisfied but otherwise do not use them explicitly. Insertion of this into Eq. (3.1) yields

$$\langle r'JM\ell's'|V|rJM\ell s\rangle = \langle \ell's'|V^J(r)|\ell s\rangle r^{-2}(2\pi)^3\delta(r-r')$$

$$= \sum_{\mathcal{L}\mathcal{M}} \sum_{m_1 m_2 m_1' m_2'} r^{-2}(2\pi)^3\delta(r-r')$$

$$\times \langle m_1' m_2'|V_{\mathcal{L}\mathcal{M}}(r)|m_1 m_2\rangle \left[\frac{(2\ell+1)\ (2\mathcal{L}+1)}{4\pi(2\ell'+1)}\right]^{1/2}$$

$$\times\ C(s_1's_2's';\ m_1'm_2')\ \ C(s_1s_2s;\ m_1m_2)\ \ C(\ell\mathcal{L}\ell';\ 00)\ \ C(\ell\mathcal{L}\ell';m\mathcal{M})$$

$$\times\ C(\ell's'J;\ m',\ M-m')\ \ C(\ell sJ;\ m, M-m). \tag{3.5}$$

We introduce new variables t and t' by

$$t = \ell - J \tag{3.6a}$$

and

$$t' = \ell' - J, \tag{3.6b}$$

and substitute for ℓ and ℓ' in Eq. (3.5). The right-hand side of this equation is then used to define the potential matrix $V^J(r)$ for all J. It is a holomorphic function of J in the whole J plane, apart from isolated singularities. To study these, we use Wigner's expression[13] for the Clebsch-Gordan coefficients, from which it follows that

$$\langle t's'|\tilde{V}^J(r)|ts\rangle = \frac{N(Jts)}{N(Jt's')}\ \langle J+t', s'|V^J(r)|J+t, s\rangle \tag{3.7}$$

is holomorphic in J, where

$$N(Jts) = \left[\frac{(2J+t+s+1)!}{(2J+t-s)!\ (2J+2s+1)} \right]^{1/2} \tag{3.8}$$

Notice, however, that $\tilde{\underset{\sim}{V}}^J$ is generally not symmetrical.

Reformulation of the Scattering Amplitude

In this section we modify the formal theory to take account of the singularities of the factors $N(J,t,s)$ introduced in the last section. If the problem has only zero-spin particles, these modifications are unnecessary since these factors are then identically unity.

We introduce new wavefunctions $\tilde{\psi}_{cts}^{J}(r)$ defined by

$$\tilde{\psi}_{cts}^{J}(r) = \psi_{c\ell s}^{J}(r)\, N(Jts)^{-1}, \tag{3.9}$$

where $\ell = J + t$. These satisfy the equations

$$\ddot{\psi}_{cts}^{J}(r) + \left[k_c^{\,2} - \frac{(J+t)(J+t+1)}{r^2} \right] \tilde{\psi}_{cts}^{J}(r)$$

$$-\sum_{c't's'} \langle cts\, |\tilde{V}^J(r)|\, c't's' \rangle\, \tilde{\psi}_{c't's'}^{J}(r) = 0. \tag{3.10}$$

The "potential" in this equation is an analytic function of J, so that this form of the Schrödinger equation is more suitable for considering the analyticity properties of the wavefunctions than is Eq. (2.4). First we show how the S matrix can be expressed in terms of particular solutions of Eq. (3.10).

For physical J regular solutions $\tilde{\phi}_{\mu p}^{J}(r)$ satisfy

$$\lim_{r\to 0} \tilde{\phi}^{J}_{\mu p}(r) = 0, \tag{3.11}$$

where μ denotes c, t, and s, and p (= 1,2...N) again labels N linearly independent solutions.

We also define solutions $\tilde{\chi}^{J}_{\mu p}(\pm)$ by the boundary condition

$$\lim_{r\to\infty} e^{\pm ik_\mu r} \tilde{\chi}^{J}_{\mu p}(\pm) = (k_\mu)^{-1/2} (\pm i)^{\ell_\mu} \delta_{\mu p}, \tag{3.12}$$

and, as before, write

$$\tilde{\underset{\sim}{\phi}}^{J} = \frac{i}{2} \left[\tilde{\underset{\sim}{f}}^{J}(-) \tilde{\underset{\sim}{\chi}}^{J}(+) - \tilde{\underset{\sim}{f}}^{J}(+) \tilde{\underset{\sim}{\chi}}^{J}(-) \right], \tag{3.13}$$

with

$$\tilde{\underset{\sim}{f}}^{J}(\pm) = \lim_{r\to\infty} \underset{\sim}{W} \left[\tilde{\underset{\sim}{\chi}}^{J}(\pm), \tilde{\underset{\sim}{\phi}}^{J} \right]. \tag{3.14}$$

Note that the Wronskian is not now independent of r, owing to the nonsymmetrical form of $\tilde{V}$. The limit in Eq. (3.14) exists since the potential tends exponentially to zero as r tends to infinity.

From Eqs. (3.12) and (3.13) we obtain

$$\tilde{\phi}^{J}_{\mu p} \underset{r\to\infty}{\sim} \frac{i}{2} i^{\ell_\mu} (k_\mu)^{-1/2} \left[\tilde{f}^{J}_{\mu p}(-) e^{-ik_\mu r} - e^{-i\pi\ell_\mu} \tilde{f}^{J}_{\mu p}(+) e^{+ik_\mu r} \right], \tag{3.15}$$

whence

$$\phi^{J}_{\mu p} \underset{r\to\infty}{\sim} \frac{i}{2} i^{\ell_\mu} (k_\mu)^{-1/2} \left[\tilde{f}^{J}_{\mu p}(-) N(J\mu) e^{-ik_\mu r} - e^{-i\pi\ell_\mu} \tilde{f}^{J}_{\mu p}(+) N(J\mu) e^{+ik_\mu r} \right], \tag{3.16}$$

where $N(J\mu) = N(Jt_\mu s_\mu)$. Therefore the S matrix is given by

$$S^{J}_{\nu\mu} = \frac{N(J\nu)}{N(J\mu)} \tilde{S}^{J}_{\nu\mu}, \tag{3.17}$$

where

$$\tilde{\underset{\sim}{S}}^{J} = \left[\tilde{\underset{\sim}{f}}^{J}(-)^{T}\right]^{-1} \tilde{\underset{\sim}{f}}^{J}(+)^{T}. \tag{3.18}$$

We shall define $\tilde{\underset{\sim}{S}}^{J}$ for all J by means of this equation, and show below that it is meromorphic.[14]

Before doing this we return to the scattering amplitude, given by Eq. (2.23), which we rewrite as

$$\langle c'\lambda_1'\lambda_2' \mid f(\theta, \phi) \mid c\lambda_1\lambda_2\rangle = (2i)^{-1}(k_c, k_c)^{-1/2} \exp\left[i(\lambda - \lambda')\phi\right]$$

$$\sum_J \left\{ \sum_{tt'ss'} (2J + 2t' + 1)^{1/2} (2J + 2t + 1)^{1/2} \times C(J+t', s'J;\ 0, \lambda') \right.$$

$$\times\ C(s_1's_2's';\ \lambda_1', -\lambda_2')C(J+t, sJ;\ 0, \lambda)C(s_1s_2s;\ \lambda_1, -\lambda_2)i^{t-t'}d^{J}_{\lambda\lambda'}(\theta)$$

$$\left. \times\ N(Jt's')N(Jts)^{-1}\langle k_c, JMc'\ J+t', s'|\tilde{S} - 1|\ k_c JMc\ J+t,\ s\rangle \right\}. \tag{3.19}$$

The t sum, for example, is initially limited to values ranging by integer steps from $|J - s| - J$ through s. However, for physical J, we can replace this by the range $-s$ through s without altering the value of the sum, since the Clebsch-Gordan coefficients are zero at the additional values. The limits on the t and t' sums thus become independent of J. We now observe that the J summand in Eq. (3.19) is holomorphic in J apart from the singularities of $\tilde{\underset{\sim}{S}}^{J}$. To see this we use[13]

$$d_{\lambda\lambda'}{}^{J}(\theta) = \left[\frac{(J-\lambda')!\,(J+\lambda)!}{(J+\lambda')!\,(J-\lambda)!}\right]^{1/2} \frac{1}{(\lambda-\lambda')!}(\cos\theta/2)^{2J}(-\tan\theta/2)^{\lambda-\lambda'}$$

$$\times \quad F(\lambda - J,\ -\lambda' - J;\ \lambda - \lambda' + 1;\ -\tan^2\theta/2)$$

for $\lambda > \lambda'$, and a similar expression derived from

$$d_{\lambda\lambda'}{}^{J}(\theta) = (-1)^{\lambda-\lambda'}\ d_{\lambda'\lambda}{}^{J}(\theta)$$

for $\lambda' \geqslant \lambda$. Here the F is the hypergeometric function (${}_2F_1$ in Pochhammer's notation). It can be shown to be holomorphic in J for all θ for which $-\tan^2\theta/2$ is not a real number greater than or equal to unity. It may then be easily verified that the singularities in J of the $d_{\lambda\lambda'}{}^{J}$ and of the Clebsch-Gordan coefficients in Eq. (3.19) cancel with those of the factors $N(Jt's')\,N(Jts)^{-1}$.

Analyticity Properties

In this section we shall extend the definitions of the $\tilde{\chi}^{J}(\pm)$ and $\tilde{\phi}^{J}$ to nonphysical J, and show that they are holomorphic in J.

The regular solutions $\tilde{\phi}^{J}$ satisfy Eq. (3.10) for all J and Eq. (3.11) for Re J sufficiently large (for example, in the zero-spin case, Re $J > -1/2$). In this restricted region of the J plane, the holomorphic property of the $\tilde{\phi}^{J}$ can be obtained by consideration of the iterative solution of an integral equation with specified boundary conditions at $r = 0$. As observed by Newton,[9] some care is required in writing this integral equation when the ℓ values are different in different channels. To avoid this difficulty and to extend the holomorphy domain to the whole J plane, we introduce the third assumption on the potential, viz. that $rV(r)$ is regular in r at $r = 0$. This enables us to obtain a power-series solution[7,8] for $\tilde{\phi}^{J}$.

Explicitly, we assume that

$$(r\,\underset{\sim}{K}^2 - r\,\underset{\sim}{\tilde{V}}^J) = \sum_{n=0}^{\infty} \underset{\sim}{a}^J(n) r^n \tag{3.20}$$

converges for $r < R$. Here the matrix $\underset{\sim}{K}$ is given by

$$K_{\mu\nu} = \delta_{\mu\nu}\, k_\mu . \tag{3.21}$$

We make the Ansatz

$$\underset{\sim}{\tilde{\phi}}^J(r) = r^{\ell_0+1} \sum_{n=0}^{\infty} \sum_{q=0}^{q_{max}} \underset{\sim}{\alpha}^J(n,q) r^n \,(r \ln r)^q , \tag{3.22}$$

where

$$\ell_0 = J - s_{max} \tag{3.23}$$

and

$$q_{max} = 2\, s_{max} + 1 , \tag{3.24}$$

s_{max} being the maximum value of s.

Substituting Eq. (3.22) into Eq. (3.10) and equating coefficients of $r^n(r \ln r)^q$, we obtain the recurrence relation

$$[(\ell_0 + n + q + 1)(\ell_0 + n + q) - \underset{\sim}{L}(\underset{\sim}{L} + 1)]\underset{\sim}{\alpha}^J(n,q)$$

$$= \sum_m \underset{\sim}{a}^J(m)\, \underset{\sim}{\alpha}^J(n - m - 1, q) - (2\ell_0 + 2n + 2q + 1)(q + 1)\, \underset{\sim}{\alpha}^J(n-1, q+1)$$

$$- (q + 2)(q + 1)\, \underset{\sim}{\alpha}^J(n-2, q+2), \tag{3.25}$$

subject to the choice of $\underset{\sim}{\alpha}(n,q) = 0$ for $n < 0$, $q < 0$, or $q > q_{max}$. The matrix $\underset{\sim}{L}$ is defined by

$$L_{\mu\nu} = \delta_{\mu\nu}\, \ell_\mu . \tag{3.26}$$

This recurrence relation determines all the $\alpha^J_{\mu p}(n,q)$ in terms of N linearly independent sets of N arbitrary parameters. These are $\alpha_{\mu p}(0,0)$ for $\ell_\mu = \ell_0$, $\alpha_{\mu p}(1,0)$ for $\ell_\mu = \ell_0 + 1$, and in general $\alpha_{\mu p}(n,0)$ for $\ell_\mu = \ell_0 + n$. If we choose these parameters to be holomorphic functions of J, then Eq. (3.25) shows that the $\alpha^J_{\mu p}(n,q)$ are meromorphic in J. As in the single-channel case,[7,8] the poles of $\underset{\sim}{\alpha}^J$ give rise to fixed poles in $\underset{\sim}{\tilde{\phi}}^J$ which therefore cancel when we form the $\tilde{S}$ matrix. Alternatively they can be removed by a suitable choice of the arbitrary parameters. If we do this, Eq. (3.22) defines $\underset{\sim}{\tilde{\phi}}^J(r)$ to be a holomorphic function of J, at least for $r \leq R$.

To extend this result[17] to all r, we use the integral form of Eq. (3.10) with boundary conditions specified at some r_0, with $0 < r_0 < R$. In order to write down the integral equation, we must first consider the solutions of Eq. (3.10) when $V \equiv 0$. Denote those solutions that behave at $r = 0$ like $r^{\ell_\mu + 1}$ by $u^J_{\mu p}(r)$, and those that behave like $r^{-\ell_\mu}$ by $v^J_{\mu p}(r)$, where p again labels N linearly independent solutions of each kind. Then the integral equation is

$$\underset{\sim}{\tilde{\phi}}^J(r) = \underset{\sim}{\tilde{\phi}}^J(r) - \underset{\sim}{W}\,[\underset{\sim}{u}, \underset{\sim}{v}]^{-1} \int_{r_0}^{r} [\underset{\sim}{u}^T(r)\, \underset{\sim}{v}(r') - \underset{\sim}{u}^T(r')\, \underset{\sim}{v}(r)]\, \underset{\sim}{\tilde{V}}^J(r')\, \underset{\sim}{\tilde{\phi}}^J(r')\, dr' , \tag{3.27}$$

where $\underset{\sim}{\tilde{\phi}}_0{}^J(r)$ is the unperturbed solution satisfying

$$\underset{\sim}{\tilde{\phi}}_0{}^J(r_0) = \underset{\sim}{\tilde{\phi}}^J(r_0) \tag{3.28a}$$

and

$$\dot{\underset{\sim}{\tilde{\phi}}}_0{}^J(r_0) = \dot{\underset{\sim}{\tilde{\phi}}}^J(r_0) . \tag{3.28b}$$

Since the $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are in fact just spherical Bessel functions, the kernel of the integral equation (3.27) is bounded for $r \geqslant r_0 > 0$, and is holomorphic in J for all J.[18] Thus the iterative solution is convergent, and shows that $\underset{\sim}{\tilde{\phi}}^J(r)$ is holomorphic in J for all J and all real r.

For the solution $\underset{\sim}{\tilde{\chi}}^J(\pm)$, we can immediately use the integral equation with boundary conditions specified at $r = \infty$. This is discussed in detail for the one-channel case by Bottino et al.,[2] and a similar argument obtains in the present case.

Thus we have the result that the $\underset{\sim}{\tilde{\phi}}^J$ and $\underset{\sim}{\tilde{\chi}}^J(\pm)$ are holomorphic functions of J for all J. It follows from Eq. (3.14) that the same result holds for $\underset{\sim}{\tilde{f}}^J(\pm)$ and, hence, that $\underset{\sim}{\tilde{S}}^J$ is meromorphic.

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18. Compare the treatment of a similar integral equation in reference 2.

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