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ABSTRACT

Apparent ambiguities of definitions of masses and lifetimes of unstable particles that depend either on the introduction of unperturbed Hamiltonians and their eigenstates, or on an assumed correspondence between resonances and elementary fields, are noted. The S-matrix definition is unambiguous; the positions of poles in the first unphysical sheets are given by the zeros of the Fredholm denominator function, which is a function only of an appropriate center-of-mass energy. The mass and lifetime of a particle are consequently independent of the variables of the scattering process or of the particular process to which the particle contributes. The invariance of the Fredholm denominator under charge conjugation, which is a consequence of CPT invariance, ensures the equality of masses and lifetimes of relatively conjugate antiparticles.

Unstable particles are closely akin to stable ones; by the factorization of the residues of unstable-particle poles, unstable-particle scattering functions quite analogous to ordinary scattering functions can be unambiguously defined. Like ordinary scattering functions they are defined only on the mass shell, the fixed masses of the unstable particles being well-defined complex numbers. The needed factorizability of the residue is an immediate consequence of Fredholm's second theorem. The continuation, by means of unitarity, through the multiparticle physical cuts onto unphysical sheets is discussed.

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I. INTRODUCTION

Lüders and Zumino[1] have given a proof that the mass and lifetime of a particle are the same as those of its conjugate antiparticle. Their proof is based on an examination of the positions of poles in matrix elements of the formal resolvent of the exact Hamiltonian. The matrix elements used are those corresponding to a single-particle eigenstates of an unperturbed Hamiltonian. That such quantities exist, in a rigorous sense, is somewhat doubtful. In field theory exact and unperturbed Hamiltonians appear usually to act in mutually orthogonal subspaces, the action of one Hamiltonian on eigenstates of the other being undefined.[2-6] As a consequence, methods that avoid the use of unperturbed eigenstates are now generally employed in rigorous work. Whether these rigorous methods can provide answers to questions regarding lifetimes of unstable particles is not yet known.

Beyond this technical difficulty there is the practical question of whether the definition of the mass and lifetime introduced by these authors is unique. Does it depend, for instance, on the choice unperturbed Hamiltonians or on the choice of eigenstate? And how is the position of this pole related to experimental determinations of the mass?

A number of other definitions for the mass and lifetimes of unstable particles have been proposed and studied.[7-15] The most prominent general definition is the one of Peierls, who suggests that the mass and lifetime be defined by the position of a pole in the one-particle propagator, the two-point Green's function. This definition has the advantage of obviously not depending on an arbitrary separation into parts of the exact Hamiltonian. However, it introduces the very obscure question of the connection between fields and particles: which field, if any, is it that corresponds to a particular observed resonance? The work of Zimmerman[16] and Nishijima[17] has emphasized that a stable particle need not be associated with a fundamental field, and Schwinger[14] has stressed that a fundamental field need not possess a stable particle. As the situation is probably the same for unstable particles, the entire question of what relations, if any, exist between the basic fields and the observed stable and unstable particles (resonances) becomes an acute basic problem for all of field theory.[18] In the present context, since the Green's function depends on which field is used, the propagator definition apparently becomes ambiguous unless a unique correspondence between unstable particles and fields can be established, or the position of the pole can be shown to be the same in all propagators.

In this paper questions involving the masses and lifetimes of unstable particles are examined in the framework of S-matrix theory. An important virtue of this approach is the direct and unambiguous manner in which unstable particles are treated. That S-matrix theory should be

well adapted to the treatment of unstable particles might at first appear surprising, for in S-matrix theory, much more than in field theory, the stable particles have, at the outset, a manifestly preferred status. Yet in spite of this it develops that a natural framework for the discussion of unstable particles is provided, one in which stable and unstable particles are closely related and are treated on a quite similar footing.

In an S-matrix approach, the mass and lifetime of an unstable particle are defined by the position of a pole of the S matrix in a center-of-mass energy variable.[19,20] A first requirement on an acceptable definition of the mass and lifetime of an unstable particle is that it provide unique, unambiguous universal constants that do not depend on the particular process in which the particle occurs, or on other free variables.[21] Accordingly, our first task is to verify that the position in the center-of-mass energy plane of the pole corresponding to an unstable particle depends neither on the other variables of the S matrix, nor on the process in which it occurs. In doing this we shall obtain an equation defining the position of the pole. The use of CPT invariance allows this to be shown invariant under charge conjugation, which proves that the mass and lifetime of a particle are equal to those of its conjugate antiparticle. Further analysis shows a quite close relationship between stable and unstable particles. A general proof of the factorizability of residues of simple poles of the S matrix is also given.

II. FUNDAMENTAL EQUATIONS

The basis of the analysis will be the S-matrix formalism developed in reference 22. A general process is described by a function $M(K', -\tilde{K}'')$, which is a covariant generalization of the scattering function $R = S - 1$. In this formalism the covariant unitarity relation takes the form

$$M(K', -\tilde{K}'') + M^*(K'', -\tilde{K}') = -\Sigma \int_K M(K', -\tilde{K})\, K\cdot\tilde{\sigma}\, M^*(K'', -\tilde{K}) . \tag{2.1}$$

Neglecting for the moment contributions associated with disconnected parts, which are discussed in Section V, we write the M function in the center-of-mass frame in terms of the matrix $M(E_+)$, defined by

$$M(K', -\tilde{K}'')\Big|_{c.m.} = \langle\, \Omega', \Lambda', \Gamma' \mid M(E_+) \mid \Omega'', \Lambda'', \Gamma''\rangle\, N(E) . \tag{2.2a}$$

where $N(E)$ is the normalization factor,[23]

$$N(E) = [\, (2\pi)^4\, \delta(E' - E)\, \delta(\underset{\sim}{P}')\, (2\pi)^4\, \delta(E'' - E)\, \delta(\underset{\sim}{P}'')\,]^{1/2} . \tag{2.2b}$$

Here E'' and E' are the total c.m. energies of the initial and final particles, and $\underset{\sim}{P}'$ and $\underset{\sim}{P}''$ are the corresponding total momenta. The symbols Λ, Ω, and Γ represent sets of variables describing spin states, angle variables, and the energies of various subsystems of particles, respectively. The plus on E_+ indicates we are considering $M(E_+)$ to be the limit from above the real energy axis of a function defined there. Type variables have been suppressed on the right of (2.2a).

In terms of the matrix $M(E)$ the unitarity relations take the form

$$M(E_+) + M^\dagger(E_+) = -M(E_+)\,\rho(E_+)\,M^\dagger(E_+) \quad , \tag{2.3}$$

where the dagger represents Hermitian conjugation and $\rho(E)$ is the density-of-states factor defined by

$$\prod_i (2\pi)^{-4}\, d^4k_i \;\; 2\pi\, \theta(k_i^{\,0})\; 2m\; \delta(k_i^{\,2} - m_i^{\,2})\; K\cdot\tilde{\sigma}$$

$$= d\Omega\, d\Gamma (2\pi)^{-4}\, dE\, d\underset{\sim}{P} \;\; \rho(E,\, \Omega,\, \Gamma) \quad , \tag{2.4}$$

$\rho(E,\, \Omega,\, \Gamma)$ being the diagonal elements of the matrix $\rho(E)$. Matrix multiplication is understood to mean integrations over $d\Omega$ and $d\Gamma$, together with sums over the spin states and the various particle combinations (configurations) that are energetically allowed at the specified energy.

The symbol $M(E_-)$ is defined to be the limit from below the real energy axis of the matrix defined there by[24]

$$M(E) = -M^\dagger(E^*) \qquad (\mathrm{Im}\, E < 0\ ) \ . \tag{2.5}$$

The unitarity relation, Eq. (2.3), can then be written in the alternative form

$$M(E_+) - M(E_-) = M(E_+)\,\rho(E_+)\,M(E_-) \ . \tag{2.3a}$$

The matrix $\rho(E)$ has a different form in each interval of the energy axis, a new subspace being added at each threshold energy. We shall represent by $\rho^N(E)$ the expression for $\rho(E)$ that is valid in the $\underline{N}$th interval. The continuation of $M(E_+)$ clockwise through the $\underline{N}$th

interval will be represented by $M_N(E)$, and $M_{N-1}(E)$ will represent the counterclockwise continuation of $M(E_-)$ through the Nth interval. In terms of these values on sheets adjacent to the physical sheet, Eqs. (2.3) and (2.3a) can be written, for E in the Nth interval, in the forms

$$M(E_+) + M_N^{\dagger}(E_-) = - M(E_+)\, \rho^N(E_+)\, M_N^{\dagger}(E_-) \tag{2.6}$$

and

$$M(E_+) - M_{N-1}(E_+) = M(E_+)\, \rho^N(E_+)\, M_{N-1}(E_+) \quad . \tag{2.6a}$$

By virtue of the postulated analyticity[25] of $M(E)$ on the boundary of the physical sheet, these relations, valid for real E, imply that the equations

$$M(E) + M_N^{\dagger}(E^*) = -M(E)\, \rho^N(E)\, M_N^{\dagger}(E^*) \tag{2.7}$$

and

$$M(E) - M_{N-1}(E) = M(E)\, \rho^N(E)\, M_{N-1}(E) \quad , \tag{2.7a}$$

considered as functions of the complex variable E, are valid in a neighborhood of the interior points of the Nth interval.

Multiplying Eqs. (2.7) and (2.7a) by $(\rho^N(E))^{1/2}$ on both the right and left, and solving, one obtains, formally,

$$(\rho^N(E))^{1/2}\, M_N^{\dagger}(E^*)\, (\rho^N(E))^{1/2} = \frac{-1}{S^N(E)}\, (\rho^N(E))^{1/2}\, M(E) (\rho^N(E))^{1/2} \tag{2.8}$$

and

$$(\rho^N(E))^{1/2}\, M_{N-1}(E)\, (\rho^N(E))^{1/2} = \frac{1}{S^N(E)}\, (\rho^N(E))^{1/2}\, M(E)\, (\rho^N(E))^{1/2} \, , \tag{2.8a}$$

where the quantity

$$S^N(E) \equiv I^N + (\rho^N(E))^{1/2} M(E) (\rho^N(E))^{1/2} \tag{2.9}$$

has been introduced. Here I^N is the unit matrix in the subspace where $\rho^N \not\equiv 0$. In terms of $S^N(E)$ alone, these equations become

$$[S_N{}^N(E^*)]^\dagger = \frac{1}{S^N(E)} \tag{2.10}$$

and

$$\epsilon^{1/2} [S_{N^{-1}}^{N}(E) - I^N] \epsilon^{1/2} = \frac{1}{S^N(E)} [S^N(E) - I^N] , \tag{2.10a}$$

where we have used the relation

$$[\rho_N{}^N(E^*)]^* = \rho^N(E) , \tag{2.11}$$

which follows from the reality of $\rho^N(E)$ in the Nth interval, and the quantity $\epsilon^{1/2}$ defined by

$$\epsilon^{1/2} [\rho_{N^{-1}}^{N}(E)]^{1/2} = [\rho^N(E)]^{1/2} . \tag{2.12}$$

Equation (2.10) is the simple expression of unitarity. The general relativistic kinematic factors for the multiparticle processes are displayed in (2.9). The quantity ϵ is a matrix that gives plus or minus unity when acting on states of odd or even numbers of particles respectively. If only even numbers of particles occur in the configurations associated with an interval N, Eq. (2.10a) simplifies to

$$S_{N^{-1}}^{N}(E) = \frac{1}{S^N(E)} , \tag{2.13}$$

a result which is well known.[26]

The sheets obtained by continuing clockwise from the physical sheet through some physical interval of the energy axis will be called first unphysical sheets. These are the unphysical sheets whose points are closest, on the Riemann surface, to the physical points, and whose poles, consequently, if close to the physical interval give the usual resonance effects. The value of $S(E)$ on these sheets is given directly by

$$[S_N{}^N(E^*)]^\dagger = S^N(E)^{-1} , \tag{2.14}$$

where $S^N(E)$ is evaluated on the physical sheet; S^N on the first unphysical sheet is the Hermitian adjoint of the inverse of S^N at the conjugate point on the physical sheet.

$S^N(E)$ will have various cuts on the physical sheet and these will evidently be reflected as cuts on the first unphysical sheets. The only other singularities on the first unphysical sheets will be at points E^* for which the inverse of $S^N(E)$ fails to exist.

If $S^N(E)$ were a matrix of finite order on discrete indices, the expression for the inverse would be

$$[S^N(E)]^{-1} = \frac{\text{adj } S^N(E)}{\det S^N(E)} . \tag{2.15}$$

Both the numerator and denominator, being finite combinations of matrix elements of $S^N(E)$, would be regular at regular points of $S^N(E)$. Hence the only possible singularities of the inverse in a domain of regularity of $S^N(E)$ would be points where the determinant in the

denominator vanishes. This determinant could vanish only at isolated points of any domain of regularity of $S^N(E)$ that included the physical points, since otherwise it would vanish identically, which would conflict with unitarity at the physical points. Thus the only singularities in the images of the interior of the physical sheet would be isolated poles. The positions of such a pole, being at a zero of $\det S(E)$, a function of E alone, would be independent of the other variables. It would consequently occur at the same point in any matrix element in which it occurred, not wander about as a function of the remaining variables of S.

Our matrix $S^N(E)$ is generally not of finite order in discrete indices. It is a function of the many continuous variables represented by the sets Ω and Γ. The direct generalization of the expression (2.15), used above, to the case of continuous variables is provided by Fredholm theory. So the immediate task is to verify that the conditions needed for the validity of Fredholm theory are satisfied, and to thereby establish rigorously the analogs of the properties described above.

III. APPLICABILITY OF FREDHOLM THEORY

The standard Fredholm equation is

$$\phi(x) = f(x) + \lambda \int_R K(x,y)\, \phi(y)dy \quad . \tag{3.1}$$

In matrix notation, the case $\lambda = -1$ becomes

$$(I + K)\phi \equiv S\,\phi = f \quad , \tag{3.2}$$

the formal solution of which is

$$\phi = (I + K)^{-1} f \equiv S^{-1} f \quad . \tag{3.3}$$

Fredholm theory gives an explicit expression for the inverse operator. This we write as

$$S^{-1} = \frac{\text{adj } S}{\det S} \quad , \tag{3.4}$$

where

$$\det S \equiv I + \text{Tr } K + (2!)^{-1}(\text{Tr})^2 K^{(2)} + (3!)^{-1}(\text{Tr})^3 K^{(3)} + \cdots, \tag{3.5}$$

and

$$\text{adj } S = I \det S - [\, K + \text{Tr } K^{(2)} + (2!)^{-1}(\text{Tr})^3 K^{(3)} + \cdots \,] \; . \tag{3.6}$$

Here $K^{(n)}$ is an operator in a space that is an n-fold tensor product of copies of the original space. It is defined in terms of the determinant of an n-by-n matrix of K's by the equation

$$\langle x_1, x_2, \cdots, x_n \,|K^{(n)}|\, y_1, y_2, \cdots, y_n \rangle$$

$$\equiv \begin{vmatrix} \langle x_1 |K| y_1 \rangle \langle x_1 |K| y_2 \rangle & \cdots & \langle x_1 |K| y_n \rangle \\ \cdots & \cdots & \cdots \\ \langle x_n |K| y_1 \rangle & \cdots & \langle x_n |K| y_n \rangle \end{vmatrix} \quad . \tag{3.7}$$

The symbol Tr stands for trace, the sum (integral) over equated corresponding initial and final variables.

To apply the formula to our case, we make the identification

(see Eq. 2.9)

$$K = [\rho^N(E)]^{1/2} M(E) [\rho^N(E)]^{1/2} \quad . \tag{3.8}$$

A sufficient condition[27] for the existence of the numerator and denominator in (3.4) is the boundedness of K over the region R, which will be chosen to be bounded.

The postulate of maximal analyticity states that $M(E)$ is a meromorphic function of all its variables at all values of these variables corresponding to points on the physical sheet, including its boundary points, except at certain boundary points called singularities required by unitarity. These are the generalized branch points terminating the cuts that bound the physical sheet, and except also at points at which the expressions $k_i(\Omega, \Gamma, E)$ for the individual momentum-energy four-vectors are not regular. The only poles allowed on the physical sheet are those associated with the stable particles. The points of the physical sheet are points in the space of reduced variables $(\Omega', \Gamma', E, \Omega'', \Gamma'')$ such that the set of points $\{ k_i(\Omega', \Gamma', E), k_j(\Omega'', \Gamma'', E)\}$ is a point of the physical sheet constructed in the manner given in reference 22.

Consider first values of E^* such that the points represented by $(\Omega', \Gamma', E, \Omega'', \Gamma'')$ are regular points of the physical sheet for all (real) physical values of the (Ω, Γ) in R. For these E^*, the $M(E)$ is analytic, hence continuous, over the closed region R. Thus $M(E)$ is bounded over R, and the numerator and denominator on the right of

$$[S_N{}^N(E^*)]^* = [S^N(E)]^{-1} = \frac{\text{adj } S^N(E)}{\text{def } S^N(E)} \tag{3.9}$$

are well-defined functions, provided the $(\rho^N(E))^{1/2}$ in

$$S^N(E) = I^N + [\rho^N(E)]^{1/2} M(E) [\rho^N(E)]^{1/2} \tag{3.10}$$

are bounded over R .

The matrix $\rho^N(E)$ is a tensor product extending over the density-of-states matrices $\rho_c(E)$ of the various configurations that are available in interval N . Specifically, $\rho^N(E)$ is given by

$$\rho^N(E) = \Pi \; \theta_c^{\,N} \rho_c(E) \equiv P^N \; \Pi \; \rho_c(E) \quad , \tag{3.11}$$

where $\rho_c(E)$ acts in the subspace corresponding to the $\underline{c}$th configuration, $\theta_c^{\,N}$ is unity if the $\underline{c}$th configuration is available (energetically allowed) in the $\underline{N}$th interval, otherwise zero, and P^N is the operator that projects onto the subspace of configurations available in the $\underline{N}$th interval. To show the existence of the numerator and denominator of (3.9), it is sufficient to show that the variables Ω and Γ can be chosen so that the physical region R is a bounded region and the corresponding functions $\rho(\Omega, \Gamma, E)$ bounded over R .

The density matrix $\rho(E)$ is easily computed by using the relativistic formulas (see Appendix A)

$$\prod_{i=0}^{N} \frac{d^3 \underset{\sim}{k}_i}{\omega_i} = \frac{d^3 \underset{\sim}{P}}{E} \prod_{n=1}^{N} \frac{d^3 \underset{\sim}{q}_n}{\omega_n^{\,r}} = \frac{d^3 P}{E} \prod_{n=1}^{N} q_n \, dE_n \, d\Omega_n \quad .$$

Here (k_i, ω_i) are the momentum-energy vectors of the $(N + 1)$ particles, $(\underset{\sim}{P}, E)$ is the total energy-momentum, and E_n is the center-of-mass energy of the system Σ_n consisting of particles zero through n . The vector

$\underset{\sim}{q}_n$ is the momentum of particle n measured in the rest frame of the system Σ_n, and Ω_n and q_n represent its angles and magnitude. The $\omega_n^{\,r}$ is the reduced energy

$$\omega_n^{\,r} = \left\{ [q_n^{\,2} + m_n^{\,2}]^{1/2} \; [q_n^{\,2} + (E_{n-1})^2]^{1/2} \right\} / E_n \quad ,$$

where

$$E_n = [q_n^{\,2} + m_n^{\,2}]^{1/2} + [q_n^{\,2} + (E_{n-1})^2]^{1/2}$$

Making the identifications

$$d\Omega \equiv \prod_{n=1}^{N} d\Omega_n \quad , \quad \text{and}$$

$$d\Gamma = \prod_{n=1}^{N-1} dE_n/2\pi \quad ,$$

we have, using the definition (2.4), the relativistic multiparticle phase-space factor

$$\rho = \frac{1}{E_N} \prod_{n=1}^{N} \frac{q_n}{(2\pi)^2} \prod_{i=0}^{N} m_i \; K\cdot\tilde{\sigma} \quad .$$

The boundedness of ρ is apparent; in fact, it vanishes like q_n on the boundary where the nth relative momentum vanishes, and like

$$(E - \Sigma\, m_i)^{[3(N+1)-5]/2} \qquad \text{at the energy threshold.}$$

Inspection of the Fredholm solution shows the uniqueness of the definition of the masses and lifetimes associated with the poles of the

type we have been discussing; the zeros of the $\det S^N(E)$ can occur only at isolated points of the domain of analyticity of $S^N(E)$, and the associated poles cannot wander about as functions of the remaining variables. It is, of course, not necessary that the pole be present in every matrix element of M, as there can be compensating zeros in the numerator. Indeed, if the nonvanishing matrix elements of M were, in some representation, to group into small submatrices along the diagonal--as would happen if there were conservation laws--then the vanishing of the determinant associated with a submatrix would produce poles only in this submatrix, at the image point in the unphysical sheet. This accords with the physical expectation that a resonance will appear in all reactions having the same quantum numbers, but not in others.

So far we have considered only those E^* such that for all (Ω, Γ) in R the points $(\Omega', \Gamma', E, \Omega'', \Gamma'')$ are on the physical sheet. The denominator, $\det S^N(E)$, is of course a function of E alone, so that a zero of this function persists when Ω and Γ are continued outside of R. (The integrations in Ω and Γ will continue to be over R, of course. It is the <u>free</u> variables in $\operatorname{adj} S$ that will be freed.) For the original values of E^*, the Fredholm formula will continue to define the function so long as the kernels remain regular. As we continue in the various variables, boundary points of the domain of validity of the Fredholm formula will be reached when singularities of the kernels reach the region of integration R. The usual situation will be that a single singularity comes to some interior point of R. But if this occurs, one can distort the contour of integration away from the singularity and thereby extend the domain of validity of the Fredholm theory. As remarked by Polkinghorne, the situation is very similar

to what he and others have been doing using Feynman-type formulas. One can continue to use the Fredholm formula unless a singularity of the kernel either comes to an end point of the integration region, or pinches the contour against another of its singularities, or retreats to infinity carrying the contour with it. One would expect that a detailed analysis would show that the limits of the domain of applicability would be just at the second sheet cuts given by the Landau equations, with both first and second type singularities included. But the unstable particle poles, and their associated branch points, would now come out automatically from the vanishing of the Fredholm denominator and from the pinching of the contours against these poles. The pursuit of these questions is outside the scope of this paper.

There are similar expressions defining the scattering function in the sheets obtained by counterclockwise continuation up from the bottom of the energy cuts [see (2.10a)]. Using the expressions obtained for the values on the various unphysical sheets, one can continue the right-hand side of (3.9) through cuts onto unphysical sheets and establish the existence of $S^N(E)$ on second-order unphysical sheets, and so on.

The methods discussed here allow continuation only through the energy cuts associated with physical processes, where the simple unitarity relations are valid; what types of singularities lurk behind other cuts we do not know. However, it is reasonable to apply the name "particle poles" to the simple poles of the type we have been discussing, which are associated with the vanishing of Fredholm denominator $\Delta(E)$. If this terminology is adopted, the masses and lifetimes of the unstable

particles will be fixed constants of nature, just like those of the stable particles. By reversing Eq. (3.9) and considering it an expression for S on the physical sheet in terms of S on the first unphysical sheet, the stable particle poles are seen to be a special case of the particle poles defined above; they correspond to the vanishing of the Fredholm determinant on the first unphysical sheet (see below).

In view of these similarities between stable and unstable particles, the idea of introducing unstable-particle scattering functions presents itself. These functions would, like ordinary scattering functions, be defined only on a manifold consistent with mass and conservation-law constraints, but the masses would now be fixed complex numbers, the fixed masses of the unstable particles.

These generalized scattering functions can be defined by factorization of residues of particle poles. That is, for ordinary M functions the unitarity relations imply that the residue of a stable-particle pole of an M function is a product of two other M functions, provided the class of M functions is extended to include also the three-particle functions usually called coupling constants. Accordingly, let us apply the name "generalized M function" to each of the two factors of the residue of a single-particle pole of any (connected) generalized M function, where a single-particle pole is defined to be the pole associated with a simple zero of a Fredholm denominator $\Delta(E)$.

That the residue of a pole associated with a simple zero of the Fredholm determinant is just a product of two factors is an immediate consequence of Fredholm's second theorem. This theorem says that if

$\det S(E_r) = 0$, then for any V' the quantity

$$V = \text{adj } S(E_r)\, V' \tag{3.12}$$

is a solution of the equation

$$S(E_r)V = 0 \quad . \tag{3.13}$$

Moreover, it says that aside from multiplicative constants, this solution is unique, provided $\text{adj } S(E_r)$ is not zero. The immediate consequence is factorizability:

$$\langle \alpha \,|\text{adj } S(E_r)|\, \beta \rangle = V_\alpha W_\beta^{\dagger} \quad . \tag{3.14}$$

If $\text{adj } S(E_r)$ were zero, then, by virtue of its analyticity in E , each element of $\text{adj } S(E)$ would contain at least one power of $(E - E_r)$. But then if $\det S(E)$ had a simple zero, the inverse

$$S^{-1}(E) = \frac{\text{adj } S(E)}{\det S(E)} \tag{3.15}$$

would be regular at $E = E_r$, and there would be no pole.

It will be observed that Fredholm theory not only gives a simple proof of factorizability, but also provides an explicit formula for the product of the two factors, and hence also, aside from an indeterminant scale factor, for the individual M functions: every M function will be

expressed explicitly in terms of others by means of Fredholm formulas. This gives, for example, explicit expressions for coupling constants in terms of the scattering functions at image points. The expressions also provide a basis for the analysis of the properties of the generalized M functions. For instance, from the anti-Hermitian analyticity property of M functions,

$$\langle \alpha \, |M_c(E)| \, \beta \rangle \equiv M_c(\alpha;\, E;\, \beta) = -M_c^{\dagger}(\alpha^*;\, E^*;\, \beta^*) \,, \tag{3.16}$$

and the relation

$$\lim_{E \to E_r} (E - E_r)\, M_c(\alpha;\, E;\, \beta) \equiv \operatorname{Res} M_c(\alpha;\, E_r;\, \beta) \tag{3.17}$$

$$= -iM_1(\alpha;\, E_r;\, \gamma)\, \rho\, M_1(\gamma;\, E_r;\, \beta) \,, \tag{3.18}$$

which defines the generalized M function M_1, one concludes that, with the appropriate choice of the indeterminant scale factor, M_1 is also anti-Hermitian analytic. In (3.18) the variable γ corresponds to one particle of mass E_r.

The usual spinor transformation properties are easily established for generalized M functions, which may play an important role in the development of S-matrix theory. We have mentioned them here to emphasize the close kinship of stable and unstable particles.

IV. EQUALITY OF THE MASSES AND LIFETIMES OF CONJUGATE ANTIPARTICLES

The mass and lifetime of a particle are determined by the position of the zero of a Fredholm denominator, $\Delta(E)$. That a particle has the same mass and lifetime as its conjugate antiparticle follows from the

invariance of Δ under charge conjugation. This, as will now be shown, is a consequence of invariance under CPT. Consider first the first term in the formula for $\det S^N(E)$. It may be written

$$\mathrm{Tr}\, K(E, T) = \sum \int_{K'K''} M^N(K', -\tilde{K}'')\, K'\cdot\tilde{\sigma}\ \delta(E, T) \quad , \tag{4.1}$$

where

$$\delta(E, T) = N(E)\delta_{T'T}\, \delta_{T''T}\, \delta(\Omega' - \Omega'')\, \delta(\Gamma' - \Gamma'') \quad . \tag{4.2}$$

Here T is a set of type variables and $N(E)$ is defined in (2.2). The CPT identity is, for $T' = T''$,[28]

$$M^N(K', -\tilde{K}'') = M^N(\tilde{K}''_T, -K'_T) \quad , \tag{4.3}$$

where the subscript T on K indicates a transposition of the order of the variables of K. Substitution of (4.3) into (4.1) gives

$$\mathrm{Tr}\, K(E, T) = \sum \int_{K'K''} M^N(\tilde{K}''_T, -K'_T)\, K'\cdot\tilde{\sigma}^{tr}\ \delta(E, T) \ , \tag{4.4}$$

where the superscript tr on $\tilde{\sigma}^{tr}$ indicates the change of order of spin indices needed to compensate for the reversed order of variables on the two sides of (4.3).

This term, $\mathrm{Tr}\, K(E, T)$, is to be compared to the charge-conjugate expression

$$\mathrm{Tr}\, K(E, T_c) = \sum \int_{K'K''} M^N(K', -\tilde{K}'')\, K'\cdot\tilde{\sigma}\ \delta(E, T_c) \ , \tag{4.5}$$

in which T_c is the set of type variables obtained by changing each type index of the set T to the index specifying the conjugate antiparticle.

An equivalent expression is

$$\mathrm{Tr}\, K(E,\, T_c) = \sum \int_{K'K''} M^N(K''_c,\, -\tilde{K}'_c)\;\; K''\cdot\tilde{\sigma}\;\; \delta(E,\, T)\;, \qquad (4.6)$$

where K'_c and K''_c have type variable T'_c and T''_c respectively. The fact that K' and K'' are dummy variables has been used to interchange the prime and double prime.

Recalling that $\tilde{K}$ is the set obtained from K by reversing the order of variables, changing all particle-type indices to those specifying the corresponding antiparticle, and dotting the spinor indices, we see that K''_c and $\tilde{K}''_T$ differ only in that their spinor indices are relatively dotted.

The rules for changing the spinor index types on M functions have been derived in SI and SII. The "metric tensors" that effect the changes are

$$\begin{aligned}
g^{\dot{\alpha}\beta} &= (k\cdot\tilde{\sigma}^{\dot{\alpha}\beta})m_i^{-1} &&,\\
g_{\beta\dot{\alpha}} &= (k\cdot\sigma_{\beta\dot{\alpha}})m_i^{-1} &&,\\
g^{\beta\alpha} &= (C^{-1})^{\beta\alpha} &&,\\
g_{\alpha\beta} &= C_{\alpha\beta} &&, \qquad (4.7)\\
g_{\dot{\alpha}\dot{\beta}} &= C_{\dot{\alpha}\dot{\beta}} &&,\\
g^{\dot{\beta}\dot{\alpha}} &= (C^{-1})^{\dot{\beta}\dot{\alpha}} &&.
\end{aligned}$$

When these are contracted with M functions, following the usual contraction

rule that upper indices contract with lower, one obtains the M functions with altered index type. The matrices C and C^{-1} are to multiply M from the left; transposed matrices should be used if they multiply from the right. Application of these rules gives

$$M^N(\tilde{K}''_T, -K'_T) = \underset{(K'')}{\mathcal{C}} K''\cdot\tilde{\sigma}\, M^N(K''_c, -\tilde{K}'_c)\, K'\cdot\tilde{\sigma} \underset{(K')}{\mathcal{C}}^{-1} . \quad (4.8)$$

Here it has been assumed, with no loss of generality, that the original indices of K' and K'' are all lower undotted. Using the relations

$$\mathcal{C}^{-1} K'\cdot\tilde{\sigma}^{tr} \mathcal{C} = K'\cdot\sigma \quad (4.9)$$

and

$$K'\cdot\tilde{\sigma}\; K'\cdot\sigma = I \quad , \quad (4.10)$$

one finds that

$$\mathrm{Tr}\, K(E, T) = \mathrm{Tr}\, K(E, T_c) \quad . \quad (4.11)$$

Thus the first term of the series for Δ is invariant under charge conjugation, as a consequence of CPT invariance.

The proof carries over with minor changes to the trace of any power of $K(E, T)$. Since every term in the absolutely convergent series for Δ is a combination of traces of products of powers of K , we obtain the desired result,

$$\Delta\,(E, T) = \Delta\,(E, T_c) \quad . \quad (4.12)$$

In the extension of the proof to traces of products of $K(E, T)$,

one problem regarding phase factors arises. The general CPT identity is

$$M(K', -\tilde{K}'') = (-1)^{N(K', -\tilde{K}'')} M(-K', \tilde{K}'')$$

$$= (-1)^{N(K', -\tilde{K}'')} \sigma(K', -\tilde{K}'') M(\tilde{K}''_T, -K'_T) \quad , \quad (4.13)$$

where $N(K)$ is the number of dotted indices of the set K, and $\sigma(K)$ is the phase factor associated with the reversal of order of variables of K. The normal connection between spin and statistics requires $\sigma(K', -\tilde{K}'')$ to be $(-1)^{N(K', -\tilde{K}'')}$ if $T' = T''$. This fact was used in Eq. (4.3). The unitarity relations, together with analyticity and the normal connection between spin and statistics, require that

$$\sigma(K', -\tilde{K}'') = \sigma^*(K'', -\tilde{K}') \quad (4.14)$$

and

$$\sigma(K', -\tilde{K}'') = \sigma(K', -\tilde{K}) (-1)^{N(K, -\tilde{K})} \sigma(K, -\tilde{K}'') \quad . \quad (4.15)$$

These conditions ensure that, in the trace of a product of K's, the factors $(-1)^N$ will just cancel the σ's, as they do for the first power of K.

V. CONCLUDING REMARKS

Complications due to disconnected parts have been ignored in the preceding sections, In the original version of this paper,[29] section V was devoted to the discussion of these complications, which occur for cuts involving three or more particles. It was found that certain pole terms prevented the direct application of Fredholm theory, for the case of continuation through three-particle cuts. Publication was accordingly

delayed until these problems were resolved. Olive[30] has now examined the question in detail and has shown that the pole terms can be absorbed into regular terms by a redefinition of certain contours and that Fredholm theory is indeed applicable. Although Olive's analysis was for a specific cut involving only three particles, it nonetheless now seems very likely that Fredholm theory will continue to be applicable in the general case, and that the discussion of unstable particles given here will be generally applicable.

If this is the case then it may be concluded that the mass and lifetime of an unstable particle can be defined directly by the position of a pole in the S-matrix; this definition provides universal constants that depend neither on other variables such as scattering angles or subenergies, nor on the process in which the pole occurs. The position of the pole is at a zero of a Fredholm denominator. Because this is expressed in terms of traces, the operation of charge conjugation is equivalent to that of CPT-inversion, and the equality of the masses and lifetimes of conjugate antiparticles follows from CPT-invariance. By virtue of Fredholm's second theorem the residue of a simple pole is the product of two factors one of which contains all the dependence on initial variables, the other of which contains all dependence on the final variables, where the separation between initial and final is made in the channel in which the pole occurs. This factorization allows the connected part of the scattering functions for unstable particles to be defined unambiguously, up to a phase factor. Because the factorization property

is valid only at the pole value, these functions are defined only at the physical mass value, which is, however, of course complex.

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APPENDIX A: RELATIVISTIC PHASE-SPACE FACTOR

The phase-space density matrix ρ has been defined by the equation

$$\prod_{i=0}^{N}\left(\frac{d^3\underset{\sim}{k}_i}{(2\pi)^3}\,\frac{m_i}{\omega_i}\right) K\cdot\tilde{\sigma} = \frac{d^3\underset{\sim}{P}}{(2\pi)^3}\,\frac{dE}{2\pi}\,d\Omega\,d\Gamma\,\rho(\Omega,\Gamma,E) \quad . \tag{A.1}$$

The coordinates Ω and Γ will be introduced in the following way: Let the subsystem composed of particles zero through n be called the nth subsystem. Let its rest frame and rest mass be denoted by Σ_n and M_n, respectively. Let k_{ij} and K_{ij} denote the momentum vectors of the ith particle and the ith subsystem, respectively, as measured in Σ_j. These quantities are related by the equations

$$\begin{aligned} M_n &= (k_{nn}{}^2 + m_n{}^2)^{1/2}\left[(K_{n-1,n})^2 + (M_{n-1})^2\right]^{1/2} \\ &= (k_{nn}{}^2 + m_n{}^2)^{1/2} + \left[k_{nn}{}^2 + (M_{n-1})^2\right]^{1/2} \end{aligned} \tag{A.2}$$

The total rest energy is $E = M_N$. For angle variables let us choose the angles describing the N vectors k_{nn} $(n = 1,2,\cdots,N)$, and for the Γ the $N - 1$ quantities M_n $(n = 1,\cdots,N-1)$. These, together with $E = M_N$ and $\underset{\sim}{P}$, give the required $3(N + 1)$.

The Jacobian of the transformation is calculated in two steps. First we take the variables $\underset{\sim}{k}_i$ $(i = 0,\cdots,N)$ to the variables $\underset{\sim}{k}_{ii}$ $(i = 0,\cdots,N)$ with $\underset{\sim}{k}_{00}$ defined to be the total momentum $\underset{\sim}{P}$. The Jacobian of this transformation is

$$J_1 = \left[\prod_{i=0}^{N} m_i\right]\left[\prod_{n=1}^{N}(\omega_n{}^r)\right]^{-1} M_N{}^{-1} \quad , \tag{A.3}$$

where the reduced energy of the nth subsystem is

$$\omega_n^{\,r} = \frac{(k_{nn}^{\,2} + m_n^{\,2})^{1/2}\,(k_{nn}^{\,2} + M_{n-1}^{\,2})^{1/2}}{\left[(k_{nn}^{\,2} + m_n^{\,2})^{1/2} + (k_{nn}^{\,2} + M_{n-1}^{\,2})^{1/2}\right]}$$

$$= (k_{nn}^{\,2} + m_n^{\,2})^{1/2}\,(k_{nn}^{\,2} + M_{n-1}^{\,2})^{1/2} \Big/ M_n \quad . \qquad \text{(A.4)}$$

Transforming to these new variables, we have

$$\prod_{n=1}^{N} \left[(k_{nn}\, dk_{nn}^{\,2}) \,/\, 2(2\pi)^3\right] K\cdot\tilde{\sigma}\; J_1 = \frac{dE}{2\pi}\, d\Gamma\, \rho \equiv \prod_{n=1}^{N} \frac{dM_n}{2\pi}\, \rho \,, \qquad \text{(A.5)}$$

where $d\Gamma$ has been interpreted as $\prod_{n=1}^{N-1} (dM_n/2\pi)$. The Jacobian function $\partial(M_n)/\partial(k_{nn}^{\,2})$, which is the determinant of a triangular matrix, is

$$J_2 = \prod_{n=1}^{N} \partial M_n/\partial k_{nn}^{\,2}$$

$$= \prod_{n=1}^{N} (1/2) \left[(k_{nn}^{\,2} + m_n^{\,2})^{1/2} + (k_{nn}^{\,2} + M_{n-1}^{\,2})^{1/2}\right] \Big/ (k_{nn}^{\,2} + m_n^{\,2})^{1/2}\,(k_{nn}^{\,2}\; M_{n-1}^{\,2})^{1/2}$$

$$= \prod_{n=1}^{N} (1/2\, \omega_n^{\,red}) = J_1\, M_N \left(2^N \prod_{i=0}^{N} m_i\right)^{-1} \quad . \qquad \text{(A.6)}$$

Thus we have

$$\rho = \left\{(1/M_N) \prod_{i=0}^{N} m_i \prod_{n=1}^{N} \left[K_{nn}/(2\pi)^2\right]\right\} K\cdot\tilde{\sigma} \quad . \qquad \text{(A.7)}$$

To display the behavior near threshold, one may replace the M_n by

$$M'_n = M_n - M_n^0 = M_n - (M_{n-1} + m_n) \approx k_{nn}^2/2m_n^{red} \qquad (A.8)$$

and introduce

$$E' = \sum_{n=1}^{N} M'_n \approx \sum_{i=1}^{N} k_{nn}^2/2m_n^{red} \quad . \qquad (A.9)$$

In terms of the dimensionless quantities

$$X_n^2 = M'_n/E' \qquad (A.10)$$

we have

$$\rho \prod_{n=1}^{N} \frac{dM'_n}{2\pi} = \frac{1}{M_N} \prod_{i=0}^{N} m_i \prod_{n=1}^{N} \frac{k_{nn}\, dM'_n}{(2\pi)^3} K\cdot\tilde{\sigma}$$

$$= \rho' \frac{dE'}{2\pi} \prod_{n=1}^{N-1} dX_n^2 \, , \qquad (A.11)$$

with

$$\rho' \approx \frac{2\pi}{M_N} \prod_{i=0}^{N} m_i \prod_{n=1}^{N} \left[(2m_n^{red})^{1/2} X_n / (2\pi)^3 \right] (E')^{\left[3(N+1)-5\right]/2} K\cdot\tilde{\sigma} \, ,$$

$$\approx 2\pi \prod_{n=1}^{N} \left[(2m_n^{red})^{3/2} Xn \Big/ 2(2\pi)^3 \right] E^{\left[3(N+1)-5\right]/2} K\cdot\tilde{\sigma} \quad . \qquad (A.12)$$

The boundaries of the physical region are at $X_n = 0$, for any n , and at $E = 0$. The X_n are bounded by the condition that $-\Sigma X_n^2 = 1$.

FOOTNOTES AND REFERENCES

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23. It is the matrix $M(E)$, considered as a function of its arguments $(\Omega', \Gamma', E, \Omega'', \Gamma'')$ that is, if disconnected contributions are ignored, the finite function with postulated analyticity properties. See Ref. 22.

24. The discontinuity equation, (2.3a), is of extreme generality, being a consequence of the definition of S in terms of "in" and "out"

states and the superposition principle. It is, in particular, independent of the unitary equation, and of special symmetries such as time reversal or TP invariance, The discontinuity equation together with unitarity imply the Hermitian analyticity of M expressed in (2.5). Time reversal invariance implies that M is symmetric in certain representations. This, together with Hermitian analyticity, implies real analyticity.

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