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Berkeley, California

Contract No. W-7405-eng-48

THE DECOMPOSITION OF THE S MATRIX AND THE CONNECTION BETWEEN SPIN AND STATISTICS

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June 18, 1962

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I. INTRODUCTION

In a previously given S-matrix proof of the normal connection between spin and statistics,[1] the observability of the magnitudes of linear combinations of particle and antiparticle amplitudes was assumed. This assumption is objectionable, as it has no physical basis and in fact conflicts with conjectured super selection rules. Also the assumption is ad hoc, being used in the proof of spin and statistics but not in the general development of the theory. The present paper gives a new proof in which this assumption is eliminated. Two new assumptions are used. The first requires that the S matrix contain disconnected parts that are essentially products of S matrices for separated subprocesses. The second requires that the unitarity relations give the discontinuities of the scattering function across physical cuts. Both these requirements are integral parts of S-matrix calculation procedures.

The decomposition law is stated and discussed in the next section. In Section III it is shown that this law, in conjunction with the general formalism developed earlier, implies that the scattering functions have the same symmetry under the interchange of adjacent conjugate variables as under the interchange of the corresponding like variables. This property is the analog of one proved for field theory by Dell' Antonio,[2] and it has the same physical basis.

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ABSTRACT

A previous S-matrix proof of normal connection between spin and statistics required an objectionable "ad hoc" assumption that magnitudes of linear combinations of particle and antiparticle amplitudes were observable. Also the decomposition requirement was not discussed in detail. This requirement specifies that the S matrix decompose into a sum of terms that are products of the S matrices for the various subgroups of particles that can interact independently, times a phase factor depending on the order of variables. The phase factor is subject to the consistency requirement that the nonprinciple parts of the unitarity relations be redundant products of principle part equation. The decomposition requirement is shown herein to imply that adjacent pairs of conjugate variables commute with other variables. This result, combined with the CPT identity and the requirement that the unitarity relations give the discontinuities of scattering functions across physical cuts, implies the normal connection between spin and statistics. The two new assumptions needed here, which replace the objectionable one, are required for S-matrix calculations, and have a physical basis.

In Section IV there is a discussion of the proposal to strengthen the analyticity requirement so that it implies that, as in potential scattering, the unitarity relation gives the discontinuity of the scattering function across physical cuts. It is shown that if this requirement is imposed, then the symmetry under the interchange of adjacent conjugate variables must be in accordance with the normal rule in the connection between spin and statistics, provided this symmetry is independent of the particular position of the variables among the arguments of the scattering function. In conjunction with the results of Section III, this gives the normal connection between spin and statistics.

In Appendix A it is shown that the specific formalism used in the proofs results from legitimate specifications of conventions, and its use does not prejudice the results. In Appendix B it is shown that the assumption that the symmetry rule for the interchange of adjacent conjugate variables be independent of the position of variables is not required for the proof.

II. DECOMPOSITION OF THE S MATRIX

The postulate of maximal analyticity asserts that the M functions are analytic functions regular in and on the boundary of their physical sheets, aside from singularities required by unitarity. The definition of the physical sheet is given in Appendix I of SI, where the physical sheet is asserted to be a collection of physical sheets lying in manifolds restricted by the mass constraints and various conservation laws. It is specified, in particular, that one is to allow the possibility that the M functions have terms restricted by conservation-law constraints involving subgroups of particles interacting only among themselves. This assumption may be expressed symbolically, in terms of S functions, by the equation

$$S\ (K) = \sum_{p} S^{p}\ (K) \quad . \tag{2.1}$$

Here p labels the partitions into disjoint subsets K^{ip} of the set of variables K, and S^{p} (K) is defined to be zero unless the sum of the momentum-energy vectors of each set K^{ip} vanishes. The physical sheet referred to in the analyticity postulate is, by definition, a collection of sheets, with one sheet in the manifold associated with each term S^{p} (K). The analyticity requirement makes unique the separation given by Eq. (2.1).

The possible existence of the various terms on the right of (2.1) was used in the construction of the singularities required by unitarity, and hence in the setting up of the general framework used in SI.

To make practical calculations it appears necessary to make a more incisive assumption, the decomposition law. This law states that, aside from phases depending on the order of variables, each S^{p} (K) is just a product of the S functions associated with the separate sets of variables K^{ip} . Symbolically,

we shall write

$$S^p(K) = \alpha^p(K) \prod_i S_c(K^{ip}) . \tag{2.2}$$

Here $\alpha^p(K)$ is the phase factor that depends on the order of variables, and $S_c(K^{ip})$, called the connected part of the function $S(K^{ip})$, is a function only of its arguments K^{ip}, and not of the total K appearing on the left.

Equation (2.2) embodies the physical expectation that dynamical effects are associated with a transfer of momentum-energy. Mathematically, assumption (2.2), considered as an ansatz, tremendously simplifies the set of unitarity equations. Already by virtue of Eq. (2.1) these break into sets of equations defined over different manifolds. The part that is defined over the manifold constrained by only the single overall conservation law will be called the principal part of the unitarity relation; the other parts will be called the nonprincipal parts. When Eq. (2.2) is invoked, the nonprincipal parts become redundant equations automatically satisfied if the principal parts are satisfied, provided the $\alpha^p(K)$ are chosen appropriately. This greatly simplifies the system of equations, of course. It also brings the equations in accord with the physical expectation that, in a certain approximation, individual sets of particles interacting among themselves can be treated as an isolated system, without regard to the rest of the universe.

While it is possible to conceive theories in which the decomposition law given in Eq. (2.2) is not satisfied, the equation is an expression of current beliefs regarding the structure of physical laws. Accordingly, it is reasonable to propose that this decomposition law be added to the S-matrix postulates given in SI. The specific statement will be this: the S-matrix functions have the structure specified in Eqs. (2.1) and (2.2), with $\alpha^p(K)$ a phase factor that must be such as to ensure that the nonprincipal parts of the unitarity equations

become redundant products of the principal parts of the unitarity equations for the various function $S(K^{ip})$.

This condition on the $\alpha^p(K)$ is discussed in the next section, where it is shown that the redundancy requirement implies that, under interchange of adjacent conjugate variables, the M functions must experience the same sign change as under the interchange of the corresponding like variables. This is the S-matrix analog of a property proved for field theory by Dell' Antonio.[2]

III. THE PHASE FACTOR IN THE DECOMPOSITION LAW

Let us consider processes involving only particles of a single type, together with their conjugate antiparticles. We will also suppose that the particles carry an additive constant of the motion, which distinguishes them from their antiparticles. Then all the nonzero M functions for these processes will have an equal number of particle and antiparticle variables. The phase arbitrariness of the functions allows us to specify that the no-scattering part will be the unit operator for processes involving physical particles--no antiparticles--provided the variables are normal ordered (see Appendix A). This fixes the phase of only those functions with all their antiparticle variables on the right. The phases of the functions with variables in other orders will be determined by the symmetry rule for the interchange of adjacent conjugate variables.

To obtain conditions on the $\alpha^p(K)$, let us write, for the pth partition, the variables of K in the order

$$K = (K^{1p'}, K^{2p'}, \ldots, K^{np'}, -\tilde{K}^{np''}, \ldots, -\tilde{K}^{2p''}, -\tilde{K}^{1p''}), \tag{3.1}$$

where

$$K^{ip} \equiv (K^{ip'}, -\tilde{K}^{ip''}) \tag{3.2}$$

is the normal-ordered ith subset of the pth partition. Let this particular ordering be called the standard ordering for the pth partition. It depends, of course, on the numbering of the subsets of the partition.

The phase convention for the no-scattering parts determines $\alpha^p(K)$ to be unity for the case in which K has the standard ordering for the pth partition, provided $K^{ip'} = K^{ip''}$ contains only particle variables. For, the isolation in the unitarity equation for $S(K', -\tilde{K}')$ of the connected

contributions corresponding to the pth partition gives, in this case,

$$\alpha^p (K', -\tilde{K}') \prod_i M_c (K^{ip'}, -\tilde{K}^{ip'}) + \alpha^{p*} (K', -\tilde{K}') \prod_i M_c^* (K^{ip'}, -\tilde{K}^{ip'})$$

$$= - \prod_i \sum \int_{K^i} [\alpha^q (K', -\tilde{K}) \; M (K^{ip'}, -\tilde{K}^i)$$

$$\times \; K^i \cdot \tilde{\sigma} M^* (K^{ip'}, -\tilde{K}^i) \; \alpha^{r*} (K', -\tilde{K})]_c \quad , \tag{3.3}$$

where the c on M and outside the square bracket indicates the restriction to connected parts. The requirement on the α's is that this relation be identical to the one given by the product of the principle-part equations

$$M_c (K^{ip'}, -\tilde{K}^{ip'}) + M_c^* (K^{ip'}, -\tilde{K}^{ip'})$$

$$= -\sum \int_{K^i} [M (K^{ip'}, -\tilde{K}^i) \; K^i \cdot \tilde{\sigma} M^* (K^{ip'}, -\tilde{K}^i)]_c \quad . \tag{3.4}$$

The α^q and α^r come from the decomposition of the M and M* on the right. For the particular term in which $-\tilde{K}^i$ happens to be $-\tilde{K}^{ip'}$, and the M and M* on the right are connected, the α^q and α^r become equal. Hence for this particular contribution to the right side the factors α^q and α^{r*} multiply to give unity. Therefore, the $\alpha^p (K', -\tilde{K}')$ on the left must also be unity; otherwise (3.3) would not be identical to the product of the (3.4), and other relations would be implied by the simultaneous validity of both (3.3) and (3.4).

Knowing that α^p (K) is unity when the variables of K are in the standard order and $K^{ip'} = K^{ip''}$ contains only particle variables, one can determine the $\alpha^p(K)$ for any other ordering in terms of the symmetry rules for the

for the interchange of adjacent conjugate variables. The general value of $\alpha^p(K)$ in

$$S^p(K) = \alpha^p(K) \prod_i S_c(K^{ip}) \tag{3.5}$$

is obtained by letting $\alpha^p(K)$ be the phase change induced by taking the variables from this original order to the order specified in K. If the rule for the interchange of adjacent conjugate variables is independent of the other variables, then one can place the variables of K^{ip} in the same order that they appear in K and compensate by allowing variables within the sets K^{ip} to be considered to commute in the calculation of $\alpha^p(K)$.

A particularly simple result is obtained if, in the interchange of adjacent conjugate variables, the phase change is the same as for the interchange of the corresponding like variables. In this case a pair of conjugate variables commutes with a single variable. This means that the set $K^{np} = (K^{np'}, -\tilde{K}^{np'})$ can be moved through the other variables and placed at the end. Next the set $K^{(n-1)p}$ can be moved to the end, and so forth. Thus the rule for the phase α gives the simple result

$$S^p(K) = \prod_i S_c(K^{ip}), \tag{3.6}$$

provided the order of the variables of K is the same as the order of the variables on the right. And this is independent of the order in which the factors on the right are written. Alternatively, in (3.6), the variables of K can be written in any standard order; the factor $\alpha^p(K)$ will be unity independent of the order of the subsets or of the division of particle and antiparticle variables between $K^{ip'}$ and $-K^{ip''}$, provided the relative ordering of the variables of each K^{ip} are the same on both sides.

If pairs of conjugate variables do not commute with single variables

these simple results do not follow. One must then use the general rule (3.5).

In the formalism developed in SI the unitarity relations are expressed in the form

$$\sum \int_K S(K', -\tilde{K})\, S^*(K'', -\tilde{K}) = \delta(K', K''), \tag{3.7}$$

where K', K'', and K are normal-ordered sets of positive energy variables. The principle part of this equation is

$$\sum \int_K [S(K', -\tilde{K})\; S^*(K'', -\tilde{K})]_c = \delta^o(K', K''), \tag{3.8}$$

where $\delta^o(K', K'')$ is zero unless $K' = K''$ contains only a single variable. The nonprincipal part corresponding to the pth partition of $(K', -K'')$ is

$$\sum \int_K \alpha^q(K', -\tilde{K}) \prod_i [S(K^{ip'}, -\tilde{K}^i)\; S^*(K^{ip''}, -\tilde{K}^i)]_c\, \alpha^{r*}(K'', -\tilde{K})$$

$$= \prod_i \delta^o(K^{ip'}, K^{ip''}). \tag{3.9}$$

The factors $\alpha^q(K', -\tilde{K})$ and $\alpha^r(K'', -\tilde{K})$ differ in general from term to term; they depend on the partitions of $S(K', -\tilde{K})$ and $S(K'', -\tilde{K})$ that are contributing to the particular term. However, if adjacent pairs of conjugate variables commute with other variables, so that one can use (3.6) in place of the more general (3.5), then the various factors of (3.9) labeled by i become independent of each other, and one obtains the required redundancy.

If adjacent pairs of conjugate variables do not commute with other variables, then the nonprincipal parts will not be redundant in general. To see a contradiction, consider the case in which $K^{ip'} = K^{ip''}$ contains only particle variables for $i \neq n$. Assume that $K^{np''}$ also contains only particle variables, but $K^{np'}$ differs from $K^{np''}$ by the added presence of one pair of conjugate

variables. The antiparticle variable can be supposed to be on the far right in $K^{np'}$. For intermediate configurations described by $-\tilde{K}^i$ containing only antiparticle variables, the arguments of both functions have all the antiparticle variables on the far right. Thus the ordering of the variables is the original order specified in (3.5) and the α^q and α^r will be unity for contributions with variables in the standard order.

But consider next a contribution in which one of the $-\tilde{K}^i$, for $i \neq n$, has a single pair of conjugate variables. The particle variable can be supposed to be on the left in $-\tilde{K}^i$. Now the α^q $(K', -\tilde{K})$ and α^r $(K'', -\tilde{K})$ will generally be different from unity, since the variables are not in the original order specified in (3.5). These two factors differ only because the particle variable of $-\tilde{K}^i$ must be moved through the extra pair that is in K' but not in K''. For the case that the commutation rules are independent of the other variables occurring in a function, the necessary and sufficient condition that these be the same is that the particle variable commute with the adjacent pair of conjugate variables.

A consequence of the above result is the nonredundancy of the nonprincipal parts for the case that adjacent pairs of conjugate variables do not commute with other variables. For, the relative signs of various contributions associated with the ith subset of the partition are generally altered by the presence of the other subsets. Hence, the redundancy requirement in the decomposition law implies that pairs of conjugate variables must commute with other variables, provided the symmetry under interchange of adjacent conjugate variables is always the same, independent of the position of variables or of the particular function in which the variables occur. In Appendix B it is shown that this last proviso is not actually needed to prove the connection between spin and statistics.

IV. THE SYMMETRY RULE FOR THE INTERCHANGE OF CONJUGATE VARIABLES AND THE CONNECTION BETWEEN SPIN AND STATISTICS

In SI the no-scattering part of the S matrix is defined to be the unit operator, provided the variables are normal ordered. This gives a unitarity relation of the form

$$\begin{aligned} & M(K', -\tilde{K}'') + M^{*}(K'', -\tilde{K}') \\ &= -\Sigma \int_K M(K', -\tilde{K})\, K \cdot \tilde{\sigma}\, M^{*}(K'', -\tilde{K}) \\ &= -\Sigma \int_K M^{*}(K, -\tilde{K}')\, K \cdot \tilde{\sigma}\, M(K, -\tilde{K}'') \quad . \end{aligned} \tag{4.1}$$

The introduction of

$$M'(K', -\tilde{K}'') = -M^{*}(K''^{*}, -\tilde{K}'^{*}) ,$$

which is an analytic function of its arguments , if M is, gives

$$\begin{aligned} M(K', -\tilde{K}'') - M(K'^{*}, -\tilde{K}''^{*}) &= \Sigma \int_K M(K', -\tilde{K})\, K \cdot \tilde{\sigma}\, M'(K^{*}, -\tilde{K}''^{*}) \\ &= \Sigma \int_K M'(K'^{*}, -\tilde{K}^{*})\, K \cdot \tilde{\sigma}\, M(K, -\tilde{K}'') , \end{aligned} \tag{4.2}$$

a form that is suited to analytic continuation. The asterisks on the real variables K', K'', and K indicate that one is to take limits from the lower-half energy planes.

One obtains equations analogous to (4.2) in potential scattering. There they continue to be valid also below threshold, where the right side vanishes. Thus one obtains the important relation

$$M(K', -\tilde{K}'') = M'(K', -\tilde{K}'') = -M^{*}(K''^{*}, -\tilde{K}'^{*}) . \tag{4.3}$$

This anti-Hermitian analyticity property of M goes over to the real analyticity of iM, in certain representations, if time reversal invariance is maintained.

For the relativistic many-particle case, this property does not follow from the postulates given in SI. Yet for practical calculations in S-matrix theory the property is crucial; the basis of these calculations is that one has a single function with cuts having discontinuities specified by unitarity.

It will be recalled that the origin of the analyticity postulate was the premise that the S matrix for the relativistic many-particle theory should exhibit the general analyticity properties found in potential theory. It is in complete accord with this idea, and apparently necessary for calculations, to strengthen the analyticity requirement by asserting that also in the relativistic case the unitarity relation should give the discontinuity of the scattering function across the physical cut. Specifically, for each physical cut the unitarity relation should give the difference of two functions that are, in fact, a single function by virtue of the anti-Hermitian analyticity requirement.

If the analyticity property is strengthened in this way, the only the normal symmetry under the interchange of adjacent conjugate variables is allowed. It was shown in Appendix I of SI that if other symmetries are allowed, then the left side of the unitarity relation will not always be simply $M(K', -\tilde{K}'') + M^*(K'', -\tilde{K}')$. In cross channels having pairs of particles with abnormal symmetries the two terms will sometimes have a relative phase difference. But then the added analyticity requirement gives both

$$M(K'^*, -\tilde{K}''^*) = -M^*(K'', -\tilde{K}')$$

and

$$M(K'^*, -\tilde{K}''^*) = -(\exp i\alpha)\, M^*(K'', -\tilde{K}'),$$

which together imply either $M \equiv 0$, in which case $\exp i\alpha$ can be redefined, or

$(\exp i\alpha) = 1$. The latter can, because of the CPT identity, be universally true only if the symmetry under the interchange of adjacent conjugate variables coincides with the normal rule. In conjunction with the results of Section III, this implies the normal connection between spin and statistics. It has been assumed, in the above, that the symmetry under interchange of adjacent conjugate variables is independent of the other variables of the function.

ACKNOWLEDGMENT

This paper is the outgrowth of a suggestion by Dr. Marcel Froissart that a detailed examination of the decomposition requirement might supply the needed element for a more satisfactory proof of the normal connection between spin and statistics.

APPENDIX A. DISCUSSION OF ARBITRARY CONVENTIONS

The arguments in this paper are based on the unitarity relation

$$\Sigma \int_K S\,(K',\ -\tilde{K})\ \ S^*\,(K'',\ -\tilde{K}) \ = \ \delta(K',\ K'')\ . \tag{A.1}$$

Here the variables are normal-ordered. It was argued in SI that, independently of what form the unitarity equation takes in some primitive formalism, one can define the S-functions with normal-ordered variables to be equal to the primitive transformation function; one can construct a formalism in which the unitarity relation takes the form (A.1), even though it might originally be expressed in terms of quite different functions, for which the normal ordering of variables has no significance. Thus the expression of unitarity in the particular form (A.1) does not prejudice the arguments, it just allows us to write a specific equation.

A second convention was to choose the no-scattering part to be the unit operator for the case of normal-ordered variables. In SI this was merely a convention, which did not affect the arguments. In this paper the two additional assumptions place limitations on the phase of the unit part. However, each different function, having a different set of variables for arguments, can be multiplied by an arbitrary phase. This affects neither the unitarity relations, the decomposition law, nor the symmetries. The phase change cancels out of the unitarity relations. It does not affect the decomposition law because this contains the factor $\alpha^p(K)$, which absorbs any phase change. Finally the symmetries are not affected, because all the different functions having the same set of variables in different orders are multiplied by the same phase. This leaves symmetries the same. Thus one does indeed have the freedom to assign the phase of the no-scattering part for the particular normal ordering with all antiparticle variables on the right.

APPENDIX B. DEPENDENCE OF SYMMETRY RULE ON OTHER VARIABLES

Here we consider the question of the dependence of the symmetry rule for the interchange of adjacent conjugate variables on the remaining variables of the function. If in the unitarity relation

$$M\,(K',\ -\tilde{K}'') + M^*\,(K'',\ -\tilde{K}') \;=\; -\,\Sigma \int_K M\,(K',\ -\tilde{K})\ K \cdot \tilde{\sigma}\, M^*\,(K'',\ -\tilde{K}) \tag{B.1}$$

one continues the two right-most variables of K' to a relatively non-normal order and then interchanges them, the result is

$$\sigma\,(K',\ -\tilde{K}'')\ [M\,(K_1',\ -\tilde{K}'') + M^*\,(K'',\ -\tilde{K}_1')]$$

$$= -\,\Sigma \int \sigma\,(K',\ -\tilde{K})\ M\,(K_1',\ -\tilde{K})\ K \cdot \tilde{\sigma}\, M^*\,(K'',\ -\tilde{K}) \tag{B.2}$$

where $\sigma(K',\ -\tilde{K})$ is the phase change induced by interchanging the order of these two variables of K' in $M(K',\ -\tilde{K})$ and K_1' is the new normal-ordered set of variables. Taking $K'' = K_1'$ and comparing to the unitarity relation for $M\,(K_1',\ -\tilde{K}_1')$, one concludes, from the positive definiteness of the right side of the latter, that $\sigma\,(K',\ -\tilde{K})$ is independent of K; the phase change is the same for all these functions. The relation $M^*\,(K'',\ -\tilde{K}') = -M\,(K'^*,\ \tilde{K}''^*)$ has been used to equate the two σ's that originally appear on the left of (3.11).

Since the two variables interchanged were chosen to be the two right-most variables of K', the sign change has been shown to be independent of the variables occurring to the right of the two exchanged variables, provided there are at least two of these. A similar argument shows that the change is independent of the variables occurring to the left of the exchanged variables.

This is not quite sufficient to obtain the general nondependence upon the remaining variables of the function, because the class of functions differing only by the set $-\tilde{K}$, with always the same K', may be limited by conservation laws. However, the argument is sufficient to establish the nondependence for the particular cases needed for the argument of Section III. There the two functions differ only by the presence of an adjacent pair of conjugate variables, and the foregoing result is applicable. Thus from the arguments of that section, one can conclude that the adjacent pair of conjugate variables sitting at the boundary between the particle and antiparticle variables can be commuted through either variable adjacent to it.

The argument is easily extended to more general cases. One can take the variables of $K^{np'}$ and $K^{np''}$ to be any sets differing by the insertion of one adjacent pair of conjugate variables, and compare the contributions to the unitarity relation coming from two intermediate states that differ by a pair in $-\tilde{K}^i$, for $i \neq n$. In order to bring the variables to the original order, the extra particle variable in $-\tilde{K}^i$ must be brought to the right through $K^{np'}$ or alternatively, through $K^{np''}$. This means that the relative phase of the contributions of the two intermediate states to the unitarity relation will not be the same as in the corresponding principle-part equation, unless the extra pair in $K^{np'}$ commutes with the particle variable arriving from the right.

Since all the like variables have the same symmetry, the above argument shows that this same symmetry must obtain also for the interchange of adjacent conjugate variables, provided there are at least two variables lying to the outside of the pair being exchanged. This limitation arises from the requirement that there must be sets of variables $K^{ip'}$ and $K^{ip''}$ for $i \neq n$.

One would like to show that the same rule must hold also for the interchange of adjacent pairs of conjugate variables that are near the end positions.

However, for the proof of spin and statistics, only the interchange of adjacent conjugate variables near the center is important.

The quantity that enters into the determination of the statistics is the sign change under a complete reversal of the order variables of a function $M(K', -\tilde{K}')$. Suppose, for example, that K' contains only particle variables. One could then represent the set of variables $(K', -K')$ in the form

$$(K', -\tilde{K}') = (\phi_1, \phi_2, \cdots, \phi_n, \bar{\phi}_n, \cdots, \bar{\phi}_2, \bar{\phi}_1) . \tag{B.3}$$

To effect the complete reversal of order, one can first reverse the order within the sets $(\phi_1, \phi_2, \cdots, \phi_n)$ and $(\bar{\phi}_n, \cdots, \bar{\phi}_2, \bar{\phi}_1)$ individually. This leaves M unchanged, because the sign change for the interchange of like variables is the same as for the conjugate like variables. This follows either by examination of the unitarity relation for $M(K', -\tilde{K}')$, or from the relation

$$M(K', -\tilde{K}'') = - M^*(K''^*, -\tilde{K}'^*) ; \tag{B.4}$$

or directly from a continuation to new values of the arguments of $M(K', -\tilde{K}')$.

The variables are now in the order

$$(\phi_n, \cdots, \phi_2, \phi_1, \bar{\phi}_1, \bar{\phi}_2, \cdots, \bar{\phi}_n) . \tag{B.5}$$

Next one can interchange ϕ_1 and $\bar{\phi}_1$. According to the above arguments this gives the same sign change as the interchange of like variables. Then one can move $\bar{\phi}_1$ to the left and ϕ_1 to the right. According to the result shown at the beginning of this Appendix, the phase change for these interchanges does not depend on the order in which the changes are made; the phase changes for these interchanges are independent of the variables occurring to the right or left of the exchanged variables, provided there are at least two of these.

From (B.4) one sees that the phase change induced by moving $\tilde{\phi}_1$ to the left must be just the conjugate of the phase change induced by moving ϕ_1 to the right. Thus the effect of these cancel.

Proceeding, one interchanges next the ϕ_2 and $\tilde{\phi}_2$, and moves these to the ends, and so forth. The only phase changes that do not cancel are the ones associated with the interchanges of ϕ_i and $\tilde{\phi}_i$ at the center. For these interchanges, the induced phase change is equivalent to the sign change induced by the interchange of the corresponding like variables.

The completion of the proof is as outlined in Appendix I of SI. The above result, combined with the CPT identity, gives

$$M\,(K',\ -\tilde{K}') = (-1)^{N(K')}\ M(\tilde{K}_T',\ -K_T')\ , \tag{B.6}$$

where $N(K')$ is the number of particles with abnormal symmetries in K'. The substitution of this equation into the unitarity relation

$$\begin{aligned} M(K',\ -\tilde{K}') - M\,(K'^*,\ -\tilde{K}'^*) &= \Sigma \int_K M\,(K',\ -\tilde{K})\ K\cdot\tilde{\sigma}\ M(K^*,\ -\tilde{K}'^*) \\ &= \Sigma \int_K M\,(K'^*,\ -\tilde{K}^*)\ K\cdot\tilde{\sigma}\ M\,(K,\ -\tilde{K}')\ , \end{aligned} \tag{B.7}$$

which is valid if K' contains only particle variables, gives

$$\begin{aligned} &(-1)^{N(K')}\ M\,(\tilde{K}_T',\ -K_T') - M\,(\tilde{K}_T'^*\ -K_T'^*)\ (-1)^{N(K')} \\ &= \Sigma \int_K M\,(\tilde{K}_T'^*\ ,\ -\tilde{K}^*)\ \ K\cdot\tilde{\sigma}\ \ M\,(K,\ -K_T') \\ &= \Sigma \int_K M\,(\tilde{K}_T'\ ,\ -\tilde{K})\ \ K\cdot\tilde{\sigma}\ \ M\,(K^*\ ,\ -K_T'^*)\ . \end{aligned} \tag{B.8}$$

Here the fact is used that the phases for the corresponding changes on the right side cancel. Equation (B.8) is not obtained by analytic continuation of (B.7), but by the substitution of equalities into it. However, it implies that the no-scattering part of $S(K', -\tilde{K}')$ must be taken to be $(-1)^{A(K')}$, where $A(K')$ is the number of antiparticles of K' having the abnormal statistics. But then the general unitarity relation is

$$M(K', -\tilde{K}'')(-1)^{A(K'')} + (-1)^{A(K')} M^*(K'', -\tilde{K}')$$

$$= -\Sigma \int_K M(K', -\tilde{K})\, K \cdot \tilde{\sigma}\, M^*(K'', -\tilde{K}'). \tag{B.9}$$

For $A(K'') = 0$ and $A(K') = 1$, this gives

$$M(K', -\tilde{K}'') - M^*(K'', -\tilde{K}') = -\Sigma \int_K M(K', -\tilde{K})\, K \cdot \sigma\, M^*(K'', -\tilde{K}'). \tag{B.10}$$

From the strengthened analyticity postulate, we conclude that

$$M^*(K'', -\tilde{K}') = M(K'^*, -\tilde{K}''^*), \tag{B.11}$$

which when compared to the corresponding equation from the direct channel,

$$M(K'^*, -\tilde{K}''^*) = -M^*(K'', -\tilde{K}'), \tag{B.12}$$

gives the contradiction we seek, since one can analytically continue M from the region where one equation is valid to one where the other is valid. The conclusion is that M vanishes if it contains like variables that do not obey the normal connection between spin and statistics; these particles do not interact.

FOOTNOTES AND REFERENCES

* This work was done under the auspices of the U. S. Atomic Energy Commission.

1. Henry P. Stapp, Phys. Rev. 125, 2139 (1962) (Referred to as SI).

2. G. F. Dell'Antonio, Annals of Physics 16, 153 (1961).

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