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AN INVESTIGATION OF SOME EXPERIMENTAL CONSEQUENCES OF ANALYTIC CONTINUATION IN ENERGY FOR STRONG-INTERACTION PARTIAL-WAVE AMPLITUDES

Berkeley, California

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Louis A. P. Balázs

(Ph. D. Thesis)

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ABSTRACT

A general method for setting up effective-range formulae is presented. Known nearby singularities can be taken into account exactly. The method is applied to the $^1S_0$ nucleon-nucleon state in the range 0 to 100 MeV and the $I = \frac{1}{2}$, $J = \frac{1}{2}$ state in πN scattering in the range 0 to 220 MeV. In the latter case, the two parameters can be calculated from the pion and nucleon masses and the πN coupling constant (f^2), on the assumption that the nucleon is a bound state. A method is also given for calculating effective-range parameters from partial-wave amplitudes in the crossed channel, with $\pi\pi$ scattering as an example. This, in turn, makes it possible to carry out dynamical calculations and to calculate Regge poles at low energies.

AN INVESTIGATION OF SOME EXPERIMENTAL CONSEQUENCES OF ANALYTIC CONTINUATION IN ENERGY FOR STRONG-INTERACTION PARTIAL-WAVE AMPLITUDES

Louis A. P. Balázs

June 25, 1962

INTRODUCTION

When partial-wave dispersion relations were first introduced, it was hoped that the relatively tractable nearby unphysical singularities (outer forces) would dominate the amplitude at low energies.[1] One could then systematically improve the accuracy of the calculation by bringing in the successively more complicated distant singularities (inner forces). Unfortunately, it was found in the pion-pion problem that such an approach is incapable of giving the physically observed P-wave resonance.[2,3] This is due to the strong distant unphysical singularities, which have to be represented by means of an effective-range formula.

The usual approach to the construction of effective-range formulae in dispersion theory consists of replacing unknown unphysical cuts by a small number of simple poles. Such a procedure is certainly justified for a physical energy range that is small compared with its distance from these cuts. This is often not the case in actual applications, however. The only way of justifying the pole approximation in such cases has been to solve the dispersion relations exactly for several simple assumed forms of the unphysical discontinuities and to compare these solutions with the pole formula. This also give a rough estimate of the possible error associated with the approximation.

The main defects of the above approach are that it may involve a large amount of computation in solving dispersion relations exactly and that one does not see in any natural way why the approximation works in the way it does. Indeed, it is comparable to the early work on low-energy nucleon-nucleon scattering, in which the problem was solved accurately for a number of standard potentials to establish the fact that the low-energy region could be characterized in terms of only two parameters per angular-momentum state. As in that problem, so here also, it would be desirable to find a shape-independent approach in which the approximate formula could be justified in a natural way, without the consideration of specific examples.

In the present method this is attempted by making certain approximations only for functions that are already definitely known from the general theory. These are made in such a way that we are left with an expression depending on only a small number of unknown constants. No approximations are made for unknown functions as in the pole approach. Thus our results are independent of the specific shape of the unphysical discontinuity. Moreover, the accuracy of our approximate formula in any given energy range, or, equivalently, its range of validity, can be easily estimated *a priori*, even before we have any knowledge of the orders of magnitude of our phenomenological parameters.

In Part II, the above method is applied to the $I = \frac{1}{2}$, $J = \frac{1}{2}$ state in pion-nucleon scattering. If we assume that the nucleon is a bound state, the two effective-range parameters can be fitted to the known position and residue of the nucleon pole. The formula then

depends only on the pion and nucleon masses and the πN coupling constant (f^2). No purely phenomenological parameters are involved. The formula gives roughly the correct δ_{11} phase shift in the range 0 to 220 MeV. This provides an experimental test of the bound-state hypothesis.

In Part III, a method is given for calculating effective-range parameters from the partial-wave amplitudes in the crossed channel. It is applied specifically to the pion-pion problem but can be used in other problems as well. When combined with the requirement of self-consistency, this makes it possible to calculate partial-wave amplitudes at low energies dynamically. A rough calculation is given in which only a narrow P-wave resonance is retained, and the only free parameter is the pion mass. The resulting resonance has a mass of 585 MeV and a half-width of 125 MeV. The method can also be extended to calculating Regge poles at low energies.[4,5]

PART I. THE EFFECTIVE-RANGE FORMALISM

A. Method for Setting Up Effective-Range Formulae

For simplicity, take the equal-mass case, and solve for a partial-wave amplitude $A(\nu)$, by the N/D method. This gives[6], in the elastic approximation,

$$A(\nu) = N(\nu)/D(\nu) \quad , \tag{I-1a}$$

where

$$N(\nu) = \frac{1}{\pi} \int_{-\infty}^{\nu_L} d\nu' \, \frac{\mathrm{Im}\, A(\nu')\, D(\nu')}{\nu' - \nu} \, , \tag{I-1b}$$

$$D(\nu) = 1 - \frac{\nu - \nu_0}{\pi} \int_0^{\infty} d\nu' \left(\frac{\nu'}{\nu' + m^2} \right)^{1/2} \frac{N(\nu')}{(\nu' - \nu_0)(\nu' - \nu)} \, , \tag{I-1c}$$

with $\nu = q^2 = (s/4) - m^2$, s = square of the total energy in the barycentric system, and m = mass of each particle. The constant ν_0 is some normalization point, and ν_L is the start of the left-hand cut of $A(\nu)$. The function $D(\nu)$ can be determined if $N(\nu)$ is known. Therefore we shall concentrate on $N(\nu)$.

If we set $y = (\nu_L/\nu')$, $E(y) = D(\nu')$, and $R(y) = -[\mathrm{Im}\, A(\nu')/\pi\, y]$, we have

$$N(q^2) = \int_0^1 dy \, H(q^2, y) \, R(y) \, E(y) \, , \tag{I-2}$$

where

$$H(q^2, y) = (1 + 4\, y\, q^2)^{-1} \, .$$

To obtain a shape-independent formula without making any specific assumptions about the unknown quantities that come into expression (I-2), we shall make approximations only for the kernel, which we already know. Suppose we have a finite set of functions $F_i(y)$ such that, to a good approximation in the energy range of interest, we may write

$$H(q^2,y) \simeq \sum_{i=1}^{N} B_i(q^2)\, F_i(y) \,, \tag{I-3}$$

where the $B_i(q^2)$ are functions such that, at each value of q^2 in our range, the approximation is as good as possible. The reason for making these approximations will be obvious presently. The simplest accurate way of making them would probably be to use some kind of polynomial interpolation through N carefully chosen points (whose positions may vary with q^2) at each value of q^2. The resulting approximations would then have the forms of Eq. (I-3), even if we used different polynomials in different regions rather than a single polynomial. In general, neither the $F_i(y)$ nor the $B_i(q^2)$ have to be continuous. However, one should choose the $F_i(y)$ in such a way that N is as small as possible for the desired degree of accuracy. Since $H(q^2,y)$ is known and is quite a smooth function of y, there should be no difficulty in doing this.

Substituting the approximations (I-3) into Eq. (I-2), we have

$$N(q^2) = \sum_{i=1}^{N} B_i(q^2)\alpha_i \quad ; \tag{I-4}$$

where

$$\alpha_i = \int_0^1 dy\, F_i(y)\, R(y)\, E(y) \; . \tag{I-5}$$

The reason for making the approximations (I-3) is now obvious, since we have a formula [namely, Eq. (I-4)] depending on only a small number N of unknown parameters α_i, which could be determined from experiment. A bound on the accuracy of this formula can be easily estimated because it is of the same order as the accuracy of the approximations (I-3), which we can obtain at once since we know what $H(q^2,y)$ actually is.[7] Equivalently, the range of validity of Eq. (I-4) is of the same order as the corresponding range of validity of the approximations (I-3). An additional advantage of this approach is that it is always much easier to make approximations of known functions than of unknown ones (as in the usual pole approach), since no guesswork is involved. Finally, we do not have to make any very special assumptions about $R(y)$, although a knowledge of some of its more detailed features could certainly be used to improve the accuracy of our results. For instance, if we know that $R(y)\,E(y)$ is comparatively large in some region, it would certainly be desirable to make the approximations (I-3) particularly accurate there.

In states with orbital angular momentum $\ell > 0$, ℓ of the α_i can be determined from the condition that the phase shift goes to zero as $q^{2\ell+1}$ for small q^2.

A particular way of obtaining the approximations (I-3) is to interpolate through N fixed points y_i at each value of q^2. Then, no matter what type of interpolation we use, we always obtain expressions having the forms

$$H(q^2,y) \simeq \sum_{i=1}^{N} H(q^2,y_i)\, F_i(y) \,. \tag{I-6}$$

This means that Eq. (I-4) becomes

$$N(q^2) = \sum_{i=1}^{N} H(q^2,y_i)\alpha_i \,, \tag{I-7}$$

which is identical with the result of applying the pole approximation. Of course, this time the positions of the poles are not free parameters.

Perhaps the most familiar interpolation formula is Lagrange's formula for polynomial interpolation. (One may use different polynomials in different regions.) In this case, one has (in any region)

$$F_i(y) = [\prod_{\substack{j \\ (j\neq i)}} (y - y_j)]/[\prod_{\substack{j \\ (j\neq i)}} (y_i - y_j)] \,. \tag{I-8}$$

B. Effect of Oscillations

For small y we can expect strong oscillations in the function $R(y)$,[8] and hence in the integrand of Eq. (I-2). In general, we may expect that, even if the oscillations grew rapidly for small y, the nearly exact cancellations would contribute little to the integrands compared with other contributions. This would be true

whether or not we made the approximation (I-3). Let us assume, however, that the oscillations are so large that even the small remainders after an almost exact cancellation are large compared with the other contributions. At first sight, it may appear that in such cases the errors introduced by the approximation (I-3) might be large compared with the contributions themselves. We shall show, however, that if both the value and the slope are well approximated by Eq. (I-3), the errors will be small compared with the contribution, no matter how violent the oscillations become.

To show this, we first note that the accuracy of $N(q^2)$ is determined by the errors in the integral compared with the values of that integral at all energies in the range of interest, and not just the value at one energy. For instance, the mere fact that the value of an integral varying with energy passes through zero at a slightly different energy than it would if it were given correctly is obviously not going to affect $N(q^2)$ by much, although it is technically true that at the point where the integral passes through zero the relative error is infinite.

Suppose next that we have two successive oscillations that exactly cancel each other at some energy in the range of interest (see Fig. 1a). If this never happens, we can always divide our integrand into an oscillating part for which this does happen, and into a nonoscillating part, which does not cause any difficulties. Dividing the two oscillations at that energy into infinitesimal strips of equal area A, we see that at the second energy of Fig. 1b,

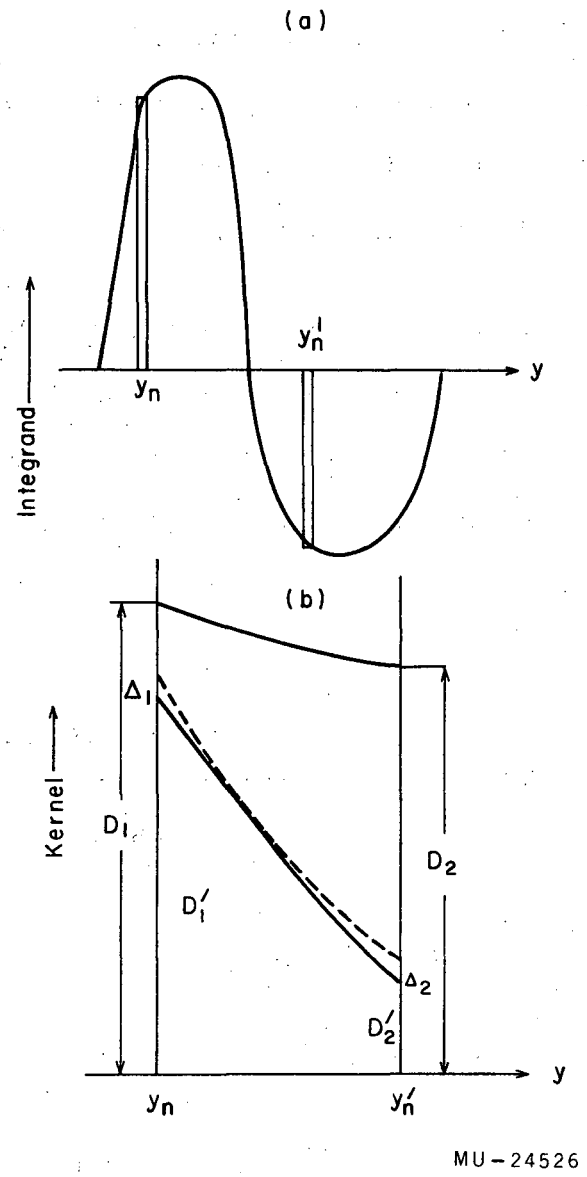

Fig. 1. (a) Two successive oscillations at the energy for which they cancel. The nth strips are shown in each case. (b) Values of the kernel (solid lines) and of the approximate kernel as given by Eq. (I-3) (dashed lines) between y_n' and y_n at two widely spaced energies in the range of interest. One of the energies is the same as in Fig. 1(a). There is no loss of generality in taking this to be the energy for the upper curve in this diagram.

the total contribution of the two nth strips of each oscillation is $A(D_1'/D_1 - D_2'/D_2)$. This is large compared with the error $A(\Delta_1/D_1 - \Delta_2/D_2)$ no matter how small $(y_n' - y_n)$ is, if both the slope and the value are well approximated by Eq. (I-3). Thus the error is small compared with the contribution. Since this is true of all such pairs of strips, it is also true of the oscillations.

C. Application to Moderately-High-Energy Nucleon-Nucleon Scattering

We now illustrate the above method in the simple case of S-wave nucleon-nucleon scattering. Here we use Eq. (I-7). If we substitute this expression into Eq. (I-1c), we have, in the nonrelativistic limit $(q^2/m^2) \to 0$,

$$q \cot \delta = \frac{\mathrm{Re}\, D}{N} = \frac{1 + \sum_{i=1}^{N} G(q^2, y_i)\alpha_i}{\sum_{i=1}^{N} H(q^2, y_i)\alpha_i} \qquad \text{(I-9)}$$

where δ is the phase shift and

$$G(q^2, y) = 2q^2 \sqrt{y} \; (1 + 4yq^2)^{-1} \; . \qquad \text{(I-10)}$$

This nonrelativistic approximation should be a reasonable one for energies up to 100 MeV.

Consider the $^1S_0$ state in the range 0 to 100 MeV. From Fig. 2 we see that, for $y > y_2$ (=0.35), we may approximate $H(q^2, y)$ by a straight line passing through its values at y_2 and y_1 (=0.85), and for $y < y_2$ by the sum of this same line and a parabola in such

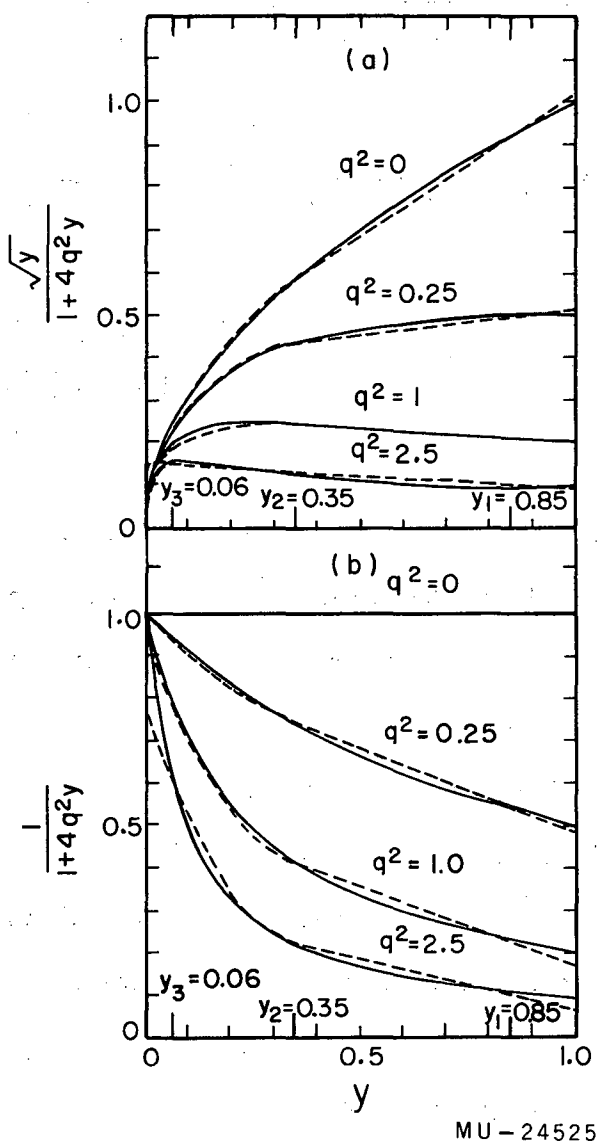

Fig. 2. Plots of $[1/(1 + 4q^2y)]$ (solid lines) compared with the corresponding approximations that have the forms of Eq. (I-11) (dashed lines).

a way that this sum passes through the value of $H(q^2,y)$ at y_3 (=0.06). In other words, we have

$$H(q^2,y) \simeq \left[\left(\frac{y - y_2}{y_1 - y_2}\right) H(q^2,y_1) + \left(\frac{y - y_1}{y_2 - y_1}\right) H(q^2,y_2)\right]$$
$$+ \theta(y_2 - y)\left(\frac{y - y_2}{y_3 - y_2}\right)^2 \left\{H(q^2,y_3) - \left[\left(\frac{y_3 - y_2}{y_1 - y_2}\right) H(q^2,y_1) + \left(\frac{y_3 - y_1}{y_2 - y_1}\right) H(q^2,y_2)\right]\right\}. \quad \text{(I-11)}$$

As can be seen from Fig. 2, the overall error associated with this approximation is of the order of only several percent. This accuracy beings to deteriorate after 100 MeV, and so we shall not go beyond this energy.

Now the approximation (I-11) is an interpolation through fixed points, and so we obtain just Eq. (I-9). Since we can neglect Coulomb effects only for energies $\gtrsim$ 40 MeV in p-p scattering, the α_i were determined by fitting this formula to the n-p scattering length and effective range, as well as the p-p phase shift at 68.3 MeV. The results are shown in Table I.

D. Exact Treatment of Known Outer Forces

We now give a method of treating known outer forces exactly by using a generalization of a method developed by Noyes. (Noyes,

Table I. The $^1S_0$ nucleon-nucleon phase shifts calculated from Eq. (I-9) with $\alpha_1 = -6.78$, $\alpha_2 = 23.10$, and $\alpha_3 = 0.43$ (which give a scattering length of 23.7×10^{-13} cm and an effective range of 2.66×10^{-13} cm).

Energy (MeV)	Phase shift (deg)	
	Calculated	Experimental[a]
39.4	41.9	44.0
68.3	34.6	34.5
95	30.4	27.0

[a] These values are taken from a recent phase-shift analysis made by H. P. Stapp, M. Moravcsik, and H. P. Noyes, Lawrence Radiation Laboratory (unpublished).

however, used the usual pole approach in describing the inner forces.)[9]

Let

$$R(y) = R_0(y) + R'(y) , \tag{I-12}$$

where $R_0(y)$ is the known part and gives the entire contribution to $R(y)$ for $y > a$. If we substitute Eq. (I-12) into expression (I-2) and make the approximation (I-3) only within the integral containing $R'(y)$ (which means that these approximations have to be made accurately only for $y < a$), we obtain

$$N(q^2) = \int_0^1 dy\, H(q^2,y)R_0(y)E(y) + \sum_{i=1}^{N} B_i(q^2)\alpha_i' \tag{I-13}$$

with

$$E(z) = 1 + \int_0^1 dy\, K(z,y)R_0(y)E(y) + \sum_{i=1}^{N} C_i(z)\alpha_i' , \tag{I-14}$$

where

$$K(z,y) = \frac{1}{\pi} \int_0^\infty d\nu' \left(\frac{\nu'}{\nu' + m^2}\right)^{1/2} \frac{H(\nu',y)(\nu_L - z\nu_0)}{(\nu' - \nu_0)(\nu_L - z\nu)} , \tag{I-15}$$

$$\alpha_i' = \int_0^a dy\, F_i(y)\, R'(y)E(y) , \tag{I-16}$$

and

$$C(z) = \frac{1}{\pi} \int_0^\infty d\nu' \left(\frac{\nu'}{\nu' + m^2}\right)^{1/2} \frac{B_i(\nu')(\nu_L - z\nu_0)}{(\nu' - \nu_0)(\nu_L - z\nu)} . \tag{I-17}$$

Now if we knew the α_i', we could solve Eq. (I-14) for $E(y)$, which would give us $N(q^2)$ through Eq. (I-13). However, since the α_i' must usually be determined by experiment, it would be desirable to have to solve integral equations that do not contain any unknown parameters. This may be accomplished by writing

$$E(y) = f(y) + \sum_{i=1}^{N} g_i(y)\alpha_i' , \tag{I-18}$$

where the $g_i(y)$ are defined such that

$$g_i(z) = C_i(z) + \int_0^1 dy\, K(z,y)R_0(y)g_i(y) . \tag{I-19}$$

Thus it follows from Eqs. (I-18) and (I-14) that $f(y)$ must obey the equation

$$f(z) = 1 + \int_0^1 dy\, K(z,y)R_0(y)f(y) . \tag{I-20}$$

Inserting expression (I-18) into Eq. (I-13), we obtain

$$N(q^2) = \int_0^1 dy\, H(q^2,y)R_0(y)f(y) + \sum_{i=1}^{N} \left[\int_0^1 dy\, H(q^2,y)R_0(y)g_i(y) + B_i(q^2) \right] \alpha_i' . \tag{I-21}$$

Thus, if we solve the integral equations (I-19) and (I-20) (which do not contain any α_i') we obtain $g_i(y)$ and $f(y)$ which, when inserted into the expression (I-21), give an explicit expression for the phase shift in terms of the α_i' .

PART II. THE $I = \frac{1}{2}$, $J = \frac{1}{2}$ STATE IN πN SCATTERING WITH THE NUCLEON AS A BOUND STATE

A. The Bound-State Hypothesis

In the S-matrix theory recently proposed by Chew[6] it is postulated that none of the strongly interacting particles is elementary. Thus, all stable particles must be bound states, and their masses and coupling constants can, in principle, be determined dynamically. Although a complete calculation would be difficult, it should be possible to test such a postulate by an experimental study of various scattering processes. In particular, one may investigate the consequences of the assumption that the nucleon is a bound state on the low-energy behavior of the P wave $I = \frac{1}{2}$, $J = \frac{1}{2}$ state in πN scattering, since the nucleon pole is present in this state.[10]

Therefore, with this assumption a two-parameter formula is first set up to approximate the effect of the interaction singularities. The parameters are then adjusted so as to give a bound-state pole with the position and residue of the nucleon pole. This removes all arbitrary constants and leaves a formula that depends only on the nucleon and pion masses and the πN coupling constant. Such a procedure is similar to the one followed in the ^3S state in n-p scattering, in which the scattering length and effective range can be calculated from the position and residue of the deuteron pole; i.e., from the binding energy and square of the asymptotic normalization coefficient for the bound-state wave function of the deuteron.[6] This procedure can also be compared with

that followed by Chew and Low,[11] who correlated the parameters of the interaction singularities with the position and width of the 3-3 resonance (which can be considered as an unstable bound state). The only important difference in the present case is that a part of the interaction singularities coincides with the nucleon pole--a difficulty that does not arise for the deuteron and the 3-3 resonance.

B. The $I = \frac{1}{2}$, $J = \frac{1}{2}$ Partial-Wave Amplitude

Consider the partial-wave amplitude[12]

$$g(W) = \frac{W^2}{(W + M)^2 - 1} \frac{\eta_1(W) \exp 2i\delta_1(W) - 1}{2iq(W)} , \qquad \text{(II-1a)}$$

$$= \frac{-W^2}{(W + M)^2 - 1} \frac{\eta_{11}(-W) \exp 2i\delta_{11}(-W) - 1}{2iq(-W)} , \qquad \text{(II-1b)}$$

where we have used the MacDowell reflection property,[13] and where, taking the pion mass to be unity,

W = total energy in the barycentric system,

M = nucleon mass,

δ_1, η_1 = S-wave $I = \frac{1}{2}$, $J = \frac{1}{2}$ phase shift and inelastic parameter,

δ_{11}, η_{11} = P-wave $I = \frac{1}{2}$, $J = \frac{1}{2}$ phase shift and inelastic parameter, and

$$q(W) = \frac{1}{2} W^{-1} \left\{ [(W + M)^2 - 1][(W - M)^2 - 1] \right\}^{1/2} . \qquad \text{(II-2)}$$

Since we are concerned only with the amplitude on the real axis,

we replace the correct singularity structure[12] of $g(W)$, as shown in Fig. 3(a), by the equivalent structure of Fig. 3(b). That this is always possible to any order of approximation can be seen quite readily from the familiar Coulomb-law analogy.[6] Furthermore, we consider only the δ_{11} phase shift, because the corresponding physical cut--running from $-M-1$ to $-\infty$ --is quite close to the nucleon pole, which is at $W = -M$.

To solve the resulting problem, we use the N/D method of Chew and Mandelstam.[2] We write

$$g(W) = N(W)/D(W) \quad , \tag{II-3}$$

where we define $D(W)$ to be a function with a cut in the left-hand physical region that obeys the subtracted dispersion relation

$$D(W) = 1 + \frac{W + M + 1}{\pi} \int_{-\infty}^{-M-1} dW' \frac{\operatorname{Im} D(W')}{(W' - W)(W' + M + 1)} \quad , \tag{II-4}$$

with

$$\operatorname{Im} D(W) = -W^{-2} q(-W)[(W + M)^2 - 1] \operatorname{Re} N(W) \quad \text{for } W < -M-1 \quad , \tag{II-5}$$

where the function is approached from above the cut. This corresponds to $N(W)$ being real for $-M-1 > W > -W_I$, as can be seen from Eqs. (II-1b) and (II-3). Here W_I is the threshold of the inelastic region.

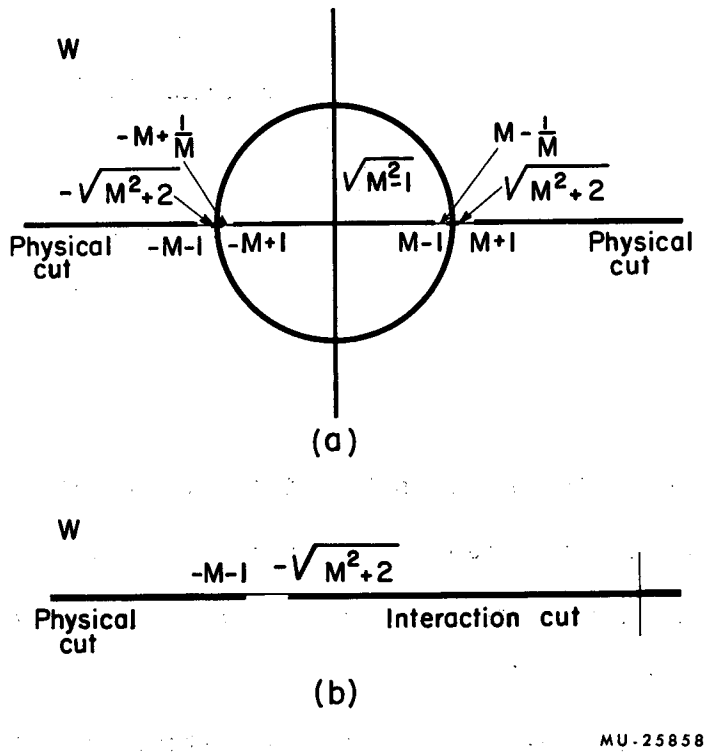

Fig. 3(a). The singularity structure of g(W) in the complex W plane.

Fig. 3(b). The singularity structure of g(W) with the unphysical singularities off the real axis replaced by an equivalent cut, that has its discontinuity adjusted so as to give the correct amplitude on the real axis.

We can find now the singularities of $N(W) = g(W)D(W)$. It is obvious that this function has no singularities in addition to those possessed by $g(W)$ and $D(W)$. Since, as we saw, $\mathrm{Im}\, N(W) = 0$ for $-M-1 > W > -W_I$, the dispersion relation for $N(W)$ becomes

$$N(W) = \frac{1}{\pi} \int_{-(M^2+2)^{1/2}}^{\infty} dW' \, \frac{\mathrm{Im}\, g(W')\, D(W')}{W' - W} + \frac{1}{\pi} \int_{-\infty}^{-W_I} dW' \, \frac{\mathrm{Im}[g(W')\, D(W')]}{W' - W} \, . \tag{II-6}$$

The second integral comes from inelastic effects. In practice the integrand is small until W' corresponds to 600 MeV, and so we may effectively take $W_I \approx M + 4$, even though the correct value is smaller.

Equations (II-4) and (II-6)--together with (II-1), (II-3), and (II-5)--can be solved if we know $\mathrm{Im}\, g(W)$ for $W > -M-1$ and $\eta_{11}(-W)$ for $W < -W_I$.[14] If--as we are assuming here--the nucleon is a bound state, we have $D(-M) = 0$;[15] i.e., there is a pole in $D^{-1}(W)$ at $W = -M$. Hence there is no pole in $N(W)$, since the corresponding δ-function in $\mathrm{Im}\, g(W)$ is canceled by $D(W)$, which is zero at the same point in Eq. (II-6). To obtain the residue of this pole, we expand

$$\frac{1}{g} = \frac{D(-M)}{N(-M)} + (W + M) \left[\frac{\partial}{\partial W} \left(\frac{D}{N} \right) \right]_{W=-M} + \cdots . \tag{II-7}$$

Since $D(-M) = 0$, this means that, near $W = -M$,

$$g(W) = \frac{-1}{-M - W} \left[\frac{\partial}{\partial W} \left(\frac{D}{N} \right) \right]^{-1}_{W=-M} = \frac{-[N(-M)/D'(-M)]}{-M - W} ; \tag{II-8}$$

i.e., we have a pole in $g(W)$ at $W = -M$, with residue

$$\gamma = -[N(-M)/D'(-M)] . \tag{II-9}$$

It is useful to note that we may always redefine $\mathrm{Im}\, D(W)$ by distorting $\mathrm{Re}\, N(W)$ in Eq. (II-5) into any other function $F(W)$ for $W < -W_I$ without changing the form of Eq. (II-6), provided only that the integrals (II-4) and (II-6) still exist. If $F(W)$ is such that on solving the resulting equations we are still able to obtain a bound state, then that bound state must be the nucleon, since there is no other in this problem.

C. The Approximation

We first neglect all unphysical singularities for $W < W_R \approx -M + 1$ except for the short cut at $W = -M$ arising from the nucleon pole in the crossed channel, which we shall call the S cut. This is equivalent to neglecting the nearby portions of the circular cut in Fig. 3(a), and should not be unreasonable, because this cut depends on $\pi\pi$ scattering and the $\pi\pi$ resonance has a large mass, which corresponds mostly to the more distant

portions of the cut. Nevertheless, this approximation can be expected to introduce some error into the function $N(W)$ at $W \approx -M$ in any quantitative calculation. For $W < -M-1$, however, this error is much smaller because, in addition to the fact that $D(W')$ is small in the neighborhood of $W' = -M$, $(W' - W)$ is much larger in Eq. (II-6).

Thus, putting $W' = (x^{-1} - M)$ and combining the two integrals (except for the S-cut contribution) in Eq. (II-6), we have

$$N(W) = \frac{1}{\pi} \int_{-(M^2+2)^{1/2}}^{-M+(1/M)} dW' \, \frac{\mathrm{Im}\, g(W')\, D(W')}{W' - W}$$

$$+ \frac{1}{\pi} \int_{x_I}^{x_R} dx \, \frac{\mathrm{Im}[g(x^{-1} - M)\, D(x^{-1} - M)]}{x[1 - (W_A + M)x]}$$

$$\times \frac{1 - (W_A + M)x}{1 - (W + M)x} , \qquad \text{(II-10)}$$

where $x_R = (W_R + M)^{-1} \approx 1$, $x_I = (M - W_I)^{-1} \approx -0.25$ (effectively), and W_A is some convenient value of W in the range of interest. We take $W_A = -1.5 - M$.

Now, within the first integral in Eq. (II-10)--since the interval is small, and since $D(-M) = 0$ --we may put

$$D(W) \approx D'(-M)(W + M) \quad . \qquad \text{(II-11)}$$

To treat the second integral in Eq. (II-10), we use the approach of I. From Fig. 4, we see that in the range $0 > (W + M) > -2.25$ we may make the straight-line approximation

$$\frac{1 - (W_A + M)x}{1 - (W + M)x} \approx \frac{1 - (W_A + M)x_1}{1 - (W + M)x_1}\left(\frac{x - x_2}{x_1 - x_2}\right) + \frac{1 - (W_A + M)x_2}{1 - (W + M)x_2}\left(\frac{x - x_1}{x_2 - x_1}\right), \tag{II-12}$$

where $x_1 = 0.081$ and $x_2 = 0.735$.

Substituting the approximations (II-11) and (II-12) into Eq. (II-10), we get

$$N(W) = \frac{D'(-M)}{\pi} \int_{-(M^2+2)^{1/2}}^{-M+(1/M)} dW' \frac{\mathrm{Im}\, g(W')(W' + M)}{(W' - W)} + \sum_{i=1}^{2} \frac{\alpha_i}{W_i - W}, \tag{II-13}$$

where

$$W_i = x_i^{-1} - M$$

and

$$\alpha_{1,2} = \frac{1 - (W_A + M)x_{1,2}}{\pi x_{1,2}} \int_{x_I}^{x_S} dx \frac{\mathrm{Im}[g(x^{-1} - M)D(x^{-1} - M)]}{x[1 - (W_A + M)x]} \left(\frac{x - x_{2,1}}{x_{1,2} - x_{2,1}}\right).$$

The first term in Eq. (II-13) can be neglected for $W < -M-1$, because the numerator in the integrand is small compared with the denominator. The second term has the same form as a two-pole formula. In this case, of course, the positions of these poles are not free parameters.

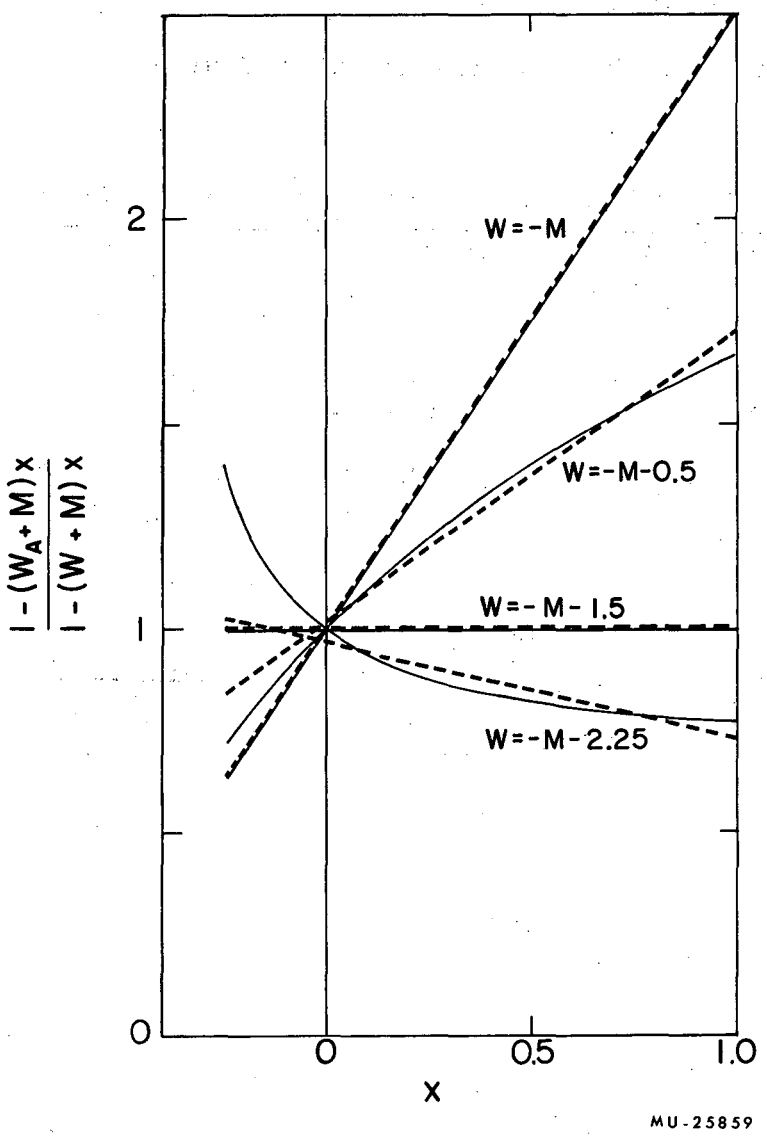

Fig. 4. Comparison of $[1 - (W_A + M)x]/[1 - (W + M)x]$ (solid lines) with the corresponding approximate forms given by Eq. (II-12) (dashed lines), as functions of x .

The accuracy of this term can be easily estimated, since it is the same order as the approximation (II-12), which, as can be seen from Fig. 4, is correct to within several percent in the range of interest. That possible oscillations, even growing ones, in $\mathrm{Im}\, g(W)$ should cause no difficulty was shown in I.B.

To obtain an expression for $D(W)$ valid in the same range, we substitute Eq. (II-13) into Eq. (II-5). Equation (II-4) then becomes

$$D(W) = 1 + (W + M + 1)[\alpha_1 H(W,W_1) + \alpha_2 H(W,W_2)] , \qquad \text{(II-14)}$$

where we have dropped the first term in Eq. (II-13), and where

$$H(W,W_i) = \frac{1}{\pi} \int_{-\infty}^{-M-1} dW' \frac{q(-W')[(W' + M)^2 - 1]}{W'^2(W' - W)(W' - W_i)(W' + M + 1)} . \qquad \text{(II-15)}$$

Since for $W < -M-1$, Eq. (II-2) becomes $q(-W) \approx \frac{1}{2} W^{-1}(W - M)[(W+M)^2-1]^{1/2}$, this latter term reduces to

$$H(W,W_i) = -\frac{M + 2\pi L(0)}{4\pi(M^2 - 1)} + \frac{C(W) - C(W_i)}{W - W_i} , \qquad \text{(II-16)}$$

where

$$C(W) = \frac{(W - M)(W + M - 1)}{W} \left\{ \frac{M + 2\pi L(0)}{4\pi(M^2 - 1)} + \frac{1 - 2\pi M L(0)}{2\pi W} + \frac{[(W + M)^2 - 1]\, L(W) - [M^2 - 1]L(0)}{W^2} \right\} ,$$

and

$$L(W) = \frac{\ln\{(W + M) - [(W + M)^2 - 1]^{1/2}\}}{2\pi[(W + M)^2 - 1]^{1/2}} \quad \text{for} \quad |(W + M)| > 1 ,$$

$$= - \frac{1}{2\pi[1 - (W + M)^2]^{1/2}} \left\{ \frac{\pi}{2} - \tan^{-1} \frac{W + M}{[1 - (W + M)^2]^{1/2}} \right\} ,$$

$$\text{for} \quad |(W + M)| < 1 .$$

Now it is true that we have a good approximation for $N(W)$ in only a limited range of W, whereas the range in the integral of Eq. (II-4) is from $-M-1$ to $-\infty$. That this should not give rise to any difficulties, however, was shown in the last paragraph of II.B.

The two parameters α_1 and α_2 can now be calculated from the two conditions necessary for the nucleon pole to be a bound-state pole, namely,

$$D(-M) = 0 , \tag{II-17}$$

and from Eqs. (II-9) and (II-13),

$$\gamma + \frac{1}{\pi} \int_{-[M^2+2]^{1/2}}^{-M+(1/M)} dW' \, \mathrm{Im}\, g(W') = \gamma_{eq} = \frac{-1}{D'(-M)} \sum_{i=1}^{2} \frac{\alpha_i}{W_i - W} , \tag{II-18}$$

with $D(W)$ given by Eq. (II-14) in each case. Of these two conditions, Eq. (II-17) is the more accurate, since it depends only on the value

of $N(W)$ for $W < -M-1$. Thus it could be used even if we did not know the singularities at $W \approx -M$. In Eq. (II-18), γ_{eq} is just the residue of the equivalent pole in $g(W)$ that would be obtained by replacing the S cut by a pole at $W = -M$ and adding to this the nucleon pole. This is equal to the corresponding pole in the static theory,[11] i.e., $\gamma_{eq} \approx \frac{8}{3} f^2 M^2$, with f^2 equal to the renormalized pseudovector coupling constant. Taking $f^2 = 0.08$, we obtain $\alpha_1 = 68.5$ and $\alpha_2 = 3.83$.

Using Eqs. (II-1b), (II-3), (II-13), and (II-14)--and remembering that we may drop the integral in Eq. (II-13) for $W < -M-1$ --we obtain, for δ_{11},

$$\mathrm{Re}\,\frac{D}{N} = \frac{(W+M)^2 - 1}{-W^2}\, q(-W)\, \cot \delta_{11}(-W)$$

$$= \frac{1 + (W + M + 1) \sum_{i=1}^{2} \alpha_i \,\mathrm{Re}\, H(W, W_i)}{\sum_{i=1}^{2} [\alpha_i/(W_i - W)]}, \qquad \text{(II-19)}$$

which may be compared with the experimental phase shifts in the range of interest.[16] Such a comparison is given in Table II and Fig. 5 and is not unreasonable in view of the crude approximations that were made. Moreover, the experimental δ_{11} phase shifts are not very reliably known at present and the values given should probably not be taken too seriously.

Table II. The δ_{11} phase shift calculated from Eq. (II-19) with $\alpha_1 = 68.5$ and $\alpha_2 = 3.83$ (which corresponds to $f^2 = 0.08$).

Lab energy (MeV)	Phase shift (deg) Calculated	Phase shift (deg) Experimental[a]
30	- 1.15	-0.11 ± 6.30
41.5	- 1.76	-0.34 ± 2.12
98	- 4.96	-1.20 ± 0.23
150	- 7.76	$-3^{+1.3}_{-1.5}$
170	- 8.90	$-3^{+2}_{-2.3}$
220	-11.39	-5.4^{+6}_{-4}
225	-11.59	-4.3 ± 2.5

[a] These values were taken from S. W. Barnes, H. Winick, K. Miyake, and K. Kinsey, Phys. Rev. 117, 238 (1960) (30 to 98 MeV); from Solution A of H. Y. Chiu and E. L. Lomon, Ann. Phys. 6, 50 (1959) (150 to 220 MeV); and from the Fermi (ii) solution of J. Deahl, J. Fetkovich, T. Fields, and G. B. Yodh, Phys. Rev. 124, 1987 (1961) (225 MeV).

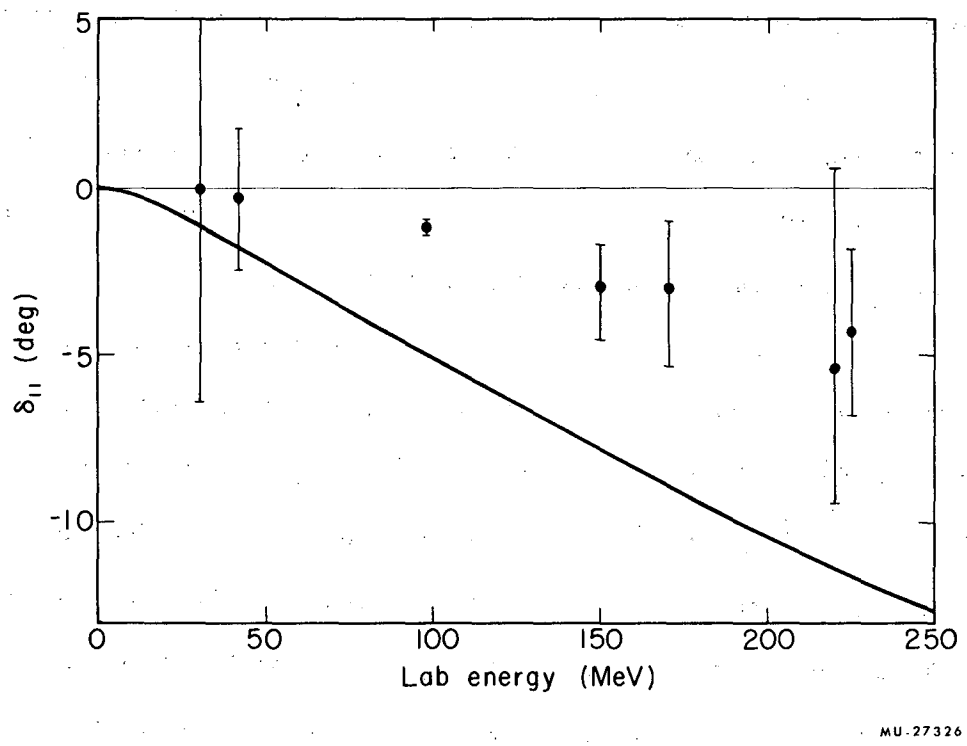

Fig. 5. Plot of the δ_{11} phase shift calculated with $f^2 = 0.08$ as a function of lab energy.

D. Discussion

We have shown that the energy dependence of the δ_{11} phase shift in the 0- to 220-MeV range can be roughly predicted by assuming that the nucleon is a bound state. We have not considered higher energies, since phase-shift analyses normally neglect inelastic effects at such energies, and it is not known how the results of such analyses would be modified by their inclusion. Even if they do not affect a particular solution substantially, it is not certain whether the best solution (statistically) without the consideration of inelastic effects remains the best when such effects are taken into account, and vice versa.

The fact that the nucleon is a bound state in the πN system means that its mass and the πN coupling constant could be calculated by solving Eqs. (II-4) and (II-6) together with (II-1), (II-3), and (II-5) to obtain $N(W)$ and $D(W)$, and then finding M and γ from $D(-M) = 0$ and Eq. (II-9). The nucleon pole would not have to be inserted at the beginning of the calculation, because, as we have seen, it is absent from $N(W)$. It is true that the pole parameters in the crossed channel would indeed contribute to these equations in the complete problem. But we could treat their values as variables, and then find those values for which the calculated parameters equal the assumed ones. Inelastic effects may be quite important in such a calculation, however. A similar procedure could be followed for any bound state.

PART III. LOW-ENERGY PION-PION DYNAMICS

A. The General Approach

There have been several attempts to calculate the low-energy pion-pion amplitude from the requirements of analyticity, elastic unitarity, crossing symmetry, and self-consistency.[2,3,17,18,19] Of these, only the method of Zachariasen[18] does not involve any arbitrary parameters in the physically interesting P-dominant case, at least if only the lowest-order term is retained. However, when this method is extended to higher orders, one has to introduce cutoffs,[20] and so one once again introduces such parameters. The main difficulty in all these calculations arises from the strong incalculable distant unphysical singularities within each partial wave.

In the method presented here, which does not contain any arbitrary parameters, the nearby unphysical singularities are treated by the usual polynomial method of CM-I (reference 2). An effective-range formula is then set up to represent the remaining unphysical singularities. The parameters of this formula can be determined by requiring that the resulting partial-wave amplitude has the correct value and derivatives at some point between these singularities and the physical region. The amplitude at this point can, in turn, be calculated from the absorptive part in the crossed channel, through a fixed momentum-transfer dispersion relation. This absorptive part is always expandable in Legendre polynomials in the region of interest. Thus we have a self-consistency problem in which we must find partial-wave amplitudes such that the assumed forms equal the calculated ones.

Finally, the same type of procedure can be followed for complex values of the angular momentum, ℓ , once the physical problem has been solved at low energies. The main usefulness of such a calculation is that it gives the trajectories of the poles in the complex ℓ plane, as they move with energy. These Regge poles, in turn, dominate high-energy scattering in the crossed channel.[4,5]

B. The Effective-Range Formula

If the amplitude for a given isotopic spin state I is $A_I(\nu, \cos\theta)$, the partial-wave amplitude is

$$A_{(\ell)I}(\nu) = \frac{1}{2} \int_{-1}^{1} d(\cos\theta)\, P_\ell(\cos\theta)\, A_I(\nu, \cos\theta) , \qquad \text{(III-1)}$$

where $\nu = (s/4) - 1$, s is the square of the total energy with pion mass $\mu = 1$, and θ is the scattering angle in the barycentric system. When this is solved by the N/D method, we have, from CM-I,[2]

$$A_{(\ell)I}(\nu) = N_\ell^{\,I}(\nu)/D_\ell^{\,I}(\nu) , \qquad \text{(III-2)}$$

with

$$N_\ell^{\,I}(\nu) = A_{(\ell)I}(\nu_0) + \frac{\nu - \nu_0}{\pi} \int_{-\infty}^{-1} d\nu' \frac{\operatorname{Im} A_{(\ell)I}(\nu')\, D_\ell^{\,I}(\nu')}{(\nu' - \nu_0)(\nu' - \nu)} \qquad \text{(III-3)}$$

and

$$D_\ell^{\,I}(\nu) = 1 - \frac{\nu - \nu_0}{\pi} \int_0^{\infty} d\nu' \left(\frac{\nu'}{\nu' + 1}\right)^{1/2} \frac{R_\ell^{\,I}(\nu')\, N_\ell^{\,I}(\nu')}{(\nu' - \nu_0)(\nu' - \nu)} , \qquad \text{(III-4)}$$

where we have made subtractions at some point ν_0. The function $R_\ell^{\ I}(\nu)$ is the ratio of total to elastic partial-wave cross section. In the elastic approximation we have $R_\ell^{\ I}(\nu) = 1$, a relation which is exact for $0 < \nu < 3$.

Now for any range $\nu_L < \nu < -1$ with $\nu_L > -9$, the function $\mathrm{Im}\, A_{(\ell)I}(\nu)$ can be calculated from the partial-wave cross sections in the crossed channel through Eq. (IV.7) of CM-I,[2] according to which

$$\mathrm{Im}\, A_{(\ell)I}(\nu) = \frac{2}{\nu} \int_0^{-\nu-1} d\nu'\, P_\ell\left(1 + 2\,\frac{\nu' + 1}{\nu}\right) \mathrm{Im}\, \tilde{A}_I\left(\nu',\ 1 + 2\,\frac{\nu + 1}{\nu'}\right), \tag{III-5}$$

where

$$\mathrm{Im}\, \tilde{A}_I\left(\nu',\ 1 + 2\,\frac{\nu + 1}{\nu'}\right) = \sum_{\ell'=0}^{\infty} (2\ell' + 1) \sum_{I'=0}^{2} \beta_{II'}\, \mathrm{Im}\, A_{(\ell')I'}(\nu') P_{\ell'}\left(1 + 2\,\frac{\nu + 1}{\nu'}\right), \tag{III-6}$$

with

$$\beta_{II'} = \begin{pmatrix} 1/3 & 1 & 5/3 \\ 1/3 & 1/2 & -5/6 \\ 1/3 & -1/2 & 1/6 \end{pmatrix}.$$

To treat the region $-\infty < \nu < \nu_L$, we use the approach of I. When we put $\nu' = -x^{-1}$ for $\nu' < \nu_L$, Eq. (III-3) becomes

$$N_\ell^I(\nu) = A_{(\ell)I}(\nu_0) + \frac{\nu - \nu_0}{\pi} \int_{\nu_L}^{-1} d\nu' \frac{\mathrm{Im}\, A_{(\ell)I}(\nu')\, D_\ell^I(\nu')}{(\nu' - \nu_0)(\nu' - \nu)}$$

$$+ \frac{\nu - \nu_0}{\pi} \int_0^{x_L} \frac{dx}{x} \frac{\mathrm{Im}\, A_{(\ell)I}(-x^{-1})\, D_\ell^I(-x^{-1})}{(\nu_0 + x^{-1})(1 + x\nu)} , \tag{III-7}$$

where $x_L = -\nu_L^{-1}$. Now, we can approximate the kernel in the second integral in the same way as in I. If we use an interpolation formula as in Eq. (I-6), we have, for a given range of ν and for $0 < x < x_L$,

$$\frac{1}{1 + x\nu} \approx \sum_{i=1}^{n} \frac{G_i(x)}{1 + x_i\nu} , \tag{III-8}$$

which, inserted into Eq. (III-7), gives

$$N_\ell^I(\nu) = A_{(\ell)I}(\nu_0) + \frac{\nu - \nu_0}{\pi}$$

$$\times \int_{\nu_L}^{-1} d\nu' \frac{\mathrm{Im}\, A_{(\ell)I}(\nu')\, D_\ell^I(\nu')}{(\nu' - \nu_0)(\nu' - \nu)} + (\nu - \nu_0) \sum_{i=1}^{n} \frac{F_{(\ell)I}^{\;i}}{\omega_i + \nu} , \tag{III-9}$$

where $\omega_i = x_i^{-1}$ and the $F_{(\ell)I}^{\;i}$ are constant parameters. The last term has the same form as an n-pole formula. In this case, of course, the positions of these poles are not free parameters. The accuracy of this term can be easily estimated, since it is of the same order

as the approximation (8). That possible oscillations, even growing ones in $\mathrm{Im}\, A_{(\ell)I}(\nu)$, should cause no difficulty was shown in I.B.

In states with $\ell > 0$, ℓ of the $F^{i}_{(\ell)I}$ can be eliminated by the condition that $A_{(\ell)I}(\nu)$ go to zero as ν^{ℓ} for small ν . (This is automatically guaranteed if we consider $\nu^{-\ell} A_{(\ell)I}(\nu)$, as is done in CM-I,[2] in which case a subtraction in $N_{\ell}^{I}(\nu)$ would likewise be unnecessary. However, this method is somewhat more inconvenient in other respects.)

C. Evaluation of the Effective-Range Parameters

To evaluate the remaining parameters in Eq. (III-9), we use the fixed momentum-transfer dispersion relation (CM-I)[2]

$$A_I\left[\nu(s),\ 1 + \frac{t}{2\nu(s)}\right] = \frac{1}{\pi}\int_4^{\infty} dt'\ \mathrm{Im}\,\tilde{A}_I\left[\nu(t'),\ 1 + \frac{s}{2\nu(t')}\right] \times \left[\frac{1}{t' - t} + \frac{(-1)^I}{t' - (4 - s - t)}\right] , \tag{III-10}$$

where $t = -2\nu(s)(1 - \cos\theta)$, and $\nu(s) = (s/4) - 1$. From Fig. 6 we see that the integration is along some line s = constant. But, from the Regge pole analysis of Chew, Frautschi and Mandelstam,[4] we have along such a line

$$\mathrm{Im}\,\tilde{A}_I\left[\nu(t),\ 1 + \frac{s}{2\nu(t)}\right] \propto t^{\alpha_I(s)} , \quad \text{as} \quad t \to \infty, \tag{III-11}$$

where the $\alpha_I(s)$ are shown in Fig. 7.[5] From this diagram, and

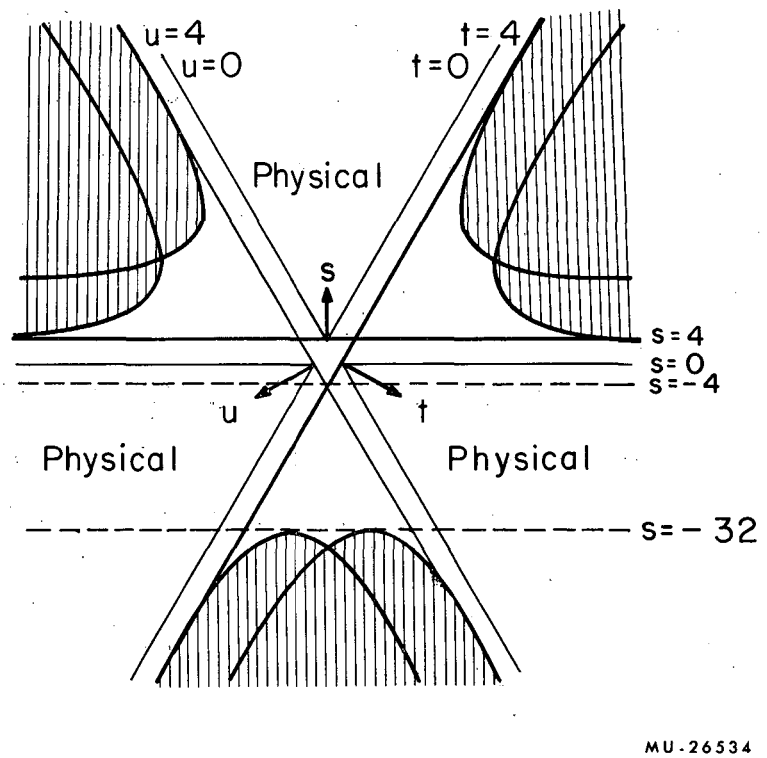

Fig. 6. The Mandelstam diagram for π-π scattering, with $u = 4-s-t$. The double-spectral functions are nonvanishing in the shaded regions. The expansion (III-6) converges between $s = 4$ and $s = -32$.

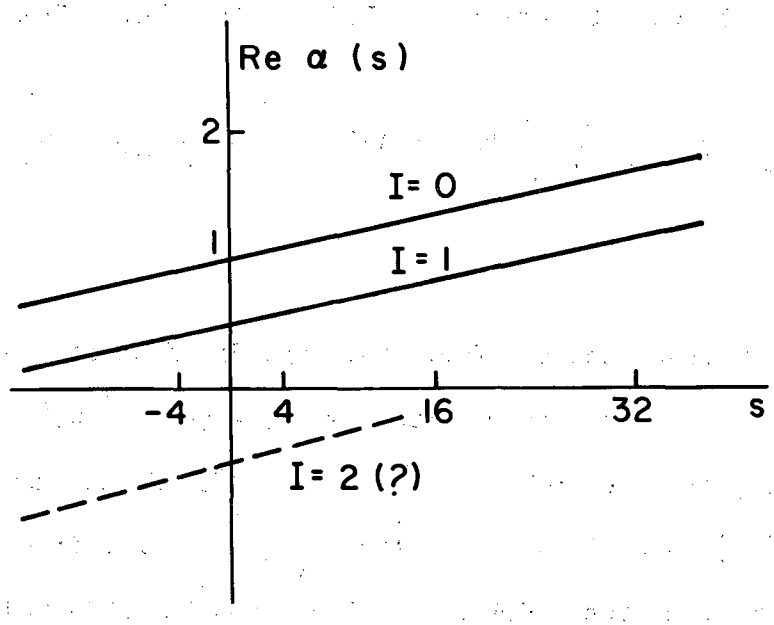

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Fig. 7. Schematic plots of the real parts of the complex angular momentum $\alpha_I(s)$ as functions of s for the topmost Regge poles. These functions are real for $s < 4$.

from Eq. (III-10), we see that in the region of convergence of the polynomial expansion (III-6), i.e., for $4 > s > -32$, the integral in Eq. (III-10) converges for $I = 1$ and 2, while an S-wave subtraction has to be made for $I = 0$.

If we substitute Eq. (III-10) into Eq. (III-1), we have

$$A_{(\ell)I}(\nu) = \frac{4}{\pi\nu} \int_0^\infty d\nu' \, \mathrm{Im}\, \tilde{A}_I(\nu', 1 + 2\frac{\nu + 1}{\nu'}) \, Q_\ell(1 + 2\frac{\nu' + 1}{\nu}) , \tag{III-12}$$

where we have used the fact that Legendre functions of the second kind $Q_\ell(z)$ obey the relation $Q_\ell(z) = (-1)^{\ell+1} Q_\ell(-z)$, and that in the pion-pion problem, $(-1)^\ell = (-1)^I$. We can now determine $A_{(\ell)I}(\nu_0)$ and $(n - \ell)$ of the $F^i_{(\ell)I}$ by requiring that $A_{(\ell)I}(\nu)$ --as given by Eqs. (III-2), (III-4), and (III-9)--and $(n - \ell)$ of its derivatives be equal to the corresponding quantities as given by Eq. (III-12) at $\nu = \nu_F$. The point ν_F may be taken anywhere in the region $\nu_L < \nu < 0$, except at the branch points -1 and -4, where the higher derivatives are infinite. The most appropriate value of ν_F is one that is not too close to ν_L, but at the same time avoids the strong peaking of the higher waves in the crossed channel that occurs in the neighborhood of $s = 0$. Therefore, a natural choice would be the lowest point of the central triangle in Fig. 6, i.e., at $s = -4$ or $\nu_F = -2$. It is also usually convenient to put $\nu_0 = \nu_F$, although it may be more appropriate to make ν_0 coincide with one of the $-\omega_i$ in an accurate calculation, since this reduces the number of parameters.

The above procedure is actually a generalization of a technique used by Ball and Wong in the $\pi\pi \to N\bar{N}$ problem.[21] These authors, however, chose the branch point -1 for ν_F, which enabled them to evaluate only the value and first derivative at $\nu = \nu_F$. An alternative generalization would be to take the value, and perhaps derivatives, at several points in the region $\nu_L < \nu < 0$.

In Eq. (III-12), $\mathrm{Im}\,\tilde{A}_I$ can always be evaluated from the partial-wave cross sections in the crossed channel through Eq. (III-6). This expansion always converges at $\nu = \nu_F$, as is obvious from Fig. 6. At large values of ν' it may be more convenient to approximate this function by retaining the contribution of only a small number of Regge poles.[4,5] This contribution is quite small, however.

The method of this section can be carried to any order of accuracy if $R_\ell^I(\nu)$ is known, or, equivalently, if inelastic effects are inserted. This is because the accuracy of the approximation (III-8) can be increased indefinitely as n increases, and any number of derivatives can be taken at $\nu = \nu_F$. It is, in fact, essentially an analytic continuation from $\nu = \nu_F$ into the physical region of the function

$$N_\ell^I(\nu) - A_{(\ell)I}(\nu_0) - \frac{\nu - \nu_0}{\pi} \int_{\nu_L}^{-1} d\nu' \, \frac{\mathrm{Im}\, A_{(\ell)I}(\nu')\, D_\ell^I(\nu')}{(\nu' - \nu_0)(\nu' - \nu)} ,$$

which is free of singularities in the region $\nu_L < \nu < \infty$. Such a function can be regarded in some sense as the partial-wave amplitude from which the physical and nearby unphysical singularities have been removed by unitarity and crossing respectively.

Since we have to make an S-wave subtraction in the $I = 0$ state, the above procedure cannot be applied to the $I = 0$, $\ell = 0$ state. But, once the other waves have been calculated, the constants $A_{(0)0}(\nu_0)$ and $F_{(0)0}^{\ i}$ can be computed by using the exact crossing conditions of CM-II (reference 3). As there is an infinite number of such conditions, this state can also be determined to any order of accuracy.

D. The P-Wave Approximation

We now illustrate the preceding techniques in the approximation of consistently neglecting everything except the P wave. This is a valid approximation because, although the S wave in the crossed channel distorts the P-wave amplitude at $\nu = \nu_F$, it can be shown that the amplitude at the resonance will not be affected much by a consistent neglect of the S wave. We shall also use a zero-width resonance approximation. From Eqs. (III-5) and (III-6) this means that if the resonance is at $\nu = \nu_R$, the left-hand cut starts at $\nu = -\nu_R - 1$. If we take $\nu_L = -\nu_R - 1$, this in turn means that the first integral in Eqs. (III-7) and (III-9) vanishes. In addition, we take $R_\ell^I(\nu) = 1$. This is a reasonable approximation for $0 < \nu \lesssim 10$ (CM-I).[2]

In the range $0 < \nu \lesssim 5$, which we would expect to be the dominant region, we can approximate the kernel $(1 + x\nu)^{-1}$ by a straight line for $0 < x \lesssim 0.21$, which is the appropriate interval if $\nu_R \gtrsim 3.5$. In other words, in Eq. (III-8) we set $n = 2$, and put

$$G_{1,2}(x) = (x - x_{2,1})/(x_{1,2} - x_{2,1}) \quad , \tag{III-13}$$

with $x_1 = 0.16$ and $x_2 = 0.02$, which corresponds to $\omega_1 = 6.25$ and $\omega_2 = 50$ in Eq. (III-9). Within this range of ν, the accuracy is of the order of several percent, as can be seen from Fig. 8. Of course the accuracy continues to be reasonable for much higher values of ν.

Now, the above approximations obviously distort $R_1^{\,1}(\nu)N_1^{\,1}(\nu)$ at large values of ν. This distortion arises from the inadequacy of both the elastic approximation and the effective-range formula at high energies. However, as was shown in II.B, Eqs. (III-3) and (III-4) are still valid provided that we add to the right side of Eq. (III-3) the additional term

$$\begin{aligned} \Delta N_1^{\,1}(\nu) &= \frac{\nu - \nu_0}{\pi} \int_{\nu_I}^{\infty} d\nu' \, \frac{\mathrm{Im}\,[A_{(1)1}(\nu')D_1^{\,1}(\nu')]}{(\nu' - \nu_0)(\nu' - \nu)} \\ &= \frac{\nu - \nu_0}{\pi} \int_{x_I}^{0} \frac{dx}{x} \, \frac{\mathrm{Im}[A_{(1)1}(-x^{-1})D_1^{\,1}(-x^{-1})]}{(\nu_0 + x^{-1})(1 + x\nu)} \quad , \end{aligned} \tag{III-14}$$

where $\nu_I = -x_I^{-1}$ is the lowest value of ν at which this distortion begins to become large. But the approximation given by Eq. (III-8) holds also for $x_I < x < 0$, as can be seen from Fig. 8. Thus the form of Eq. (III-9) is unchanged by the addition of $\Delta N_1^{\,1}(\nu)$ if we insert the approximation (III-8) into Eq. (III-14).

To set up the zero-width resonance approximation we use the fact that, for a dynamical resonance, since

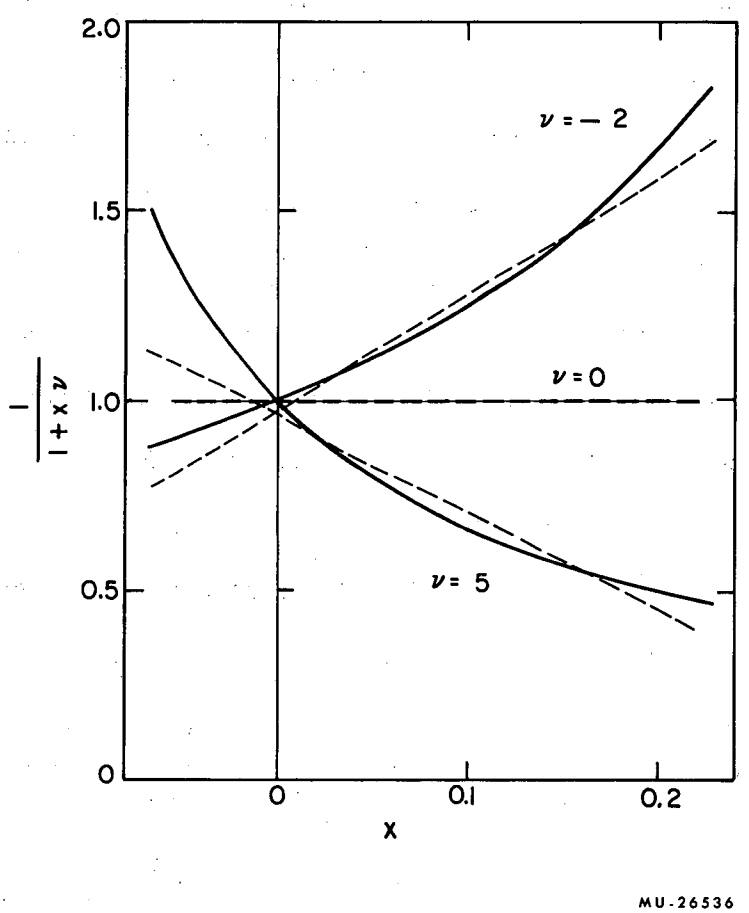

Fig. 8. Plots of $(1 + x\nu)^{-1}$ (solid lines) and the approximate form given by Eqs. (III-8) and (III-13) (dashed lines) for various values of ν .

$$\mathrm{Re}\, D_1^{\,1}(\nu_R) = 0 \quad , \tag{III-15}$$

we may to a good approximation[6] put $\mathrm{Re}\, D_1^{\,1}(\nu) \approx -(\nu - \nu_R)/(\nu_R - \nu_0)$, and $N_1^{\,1}(\nu) \approx (\nu/\nu_R)\, N_1^{\,1}(\nu_R)$, as can be seen a posteriori from Fig. 9. From Eq. (III-2), since $\mathrm{Im}\, D_1^{\,1}(\nu) = -[\nu/(\nu + 1)]^{1/2}\, N_1^{\,1}(\nu)$, this leads to the Breit-Wigner form for $\nu > 0$, namely

$$\mathrm{Im}\, A_{(1)1}(\nu) = \frac{\nu(\Gamma_1^{\,1})^2\, [\nu^3/(\nu + 1)]^{1/2}}{(\nu - \nu_R)^2 + (\Gamma_1^{\,1})^2[\nu^3/(\nu + 1)]} \quad , \tag{III-16}$$

where

$$\nu_R\, \Gamma_1^{\,1} = (\nu_R - \nu_0)\, N_1^{\,1}(\nu_R) \quad . \tag{III-17}$$

In the zero-width approximation, Eq. (III-16) becomes

$$\mathrm{Im}\, A_{(1)1}(\nu) = \pi\, \nu_R\, \Gamma_1^{\,1}\, \delta(\nu - \nu_R) \; . \tag{III-18}$$

Taking $\nu_0 = \nu_F = -2$, we can now calculate the constants $A_{(1)1}(\nu_0)$, $F_{(1)1}^{\,1}$, and $F_{(1)1}^{\,2}$ from the requirements that $A_{(1)1}(0) = 0$ and that Eqs. (III-9) and (III-4) give the same values of $A_{(1)1}(\nu_0)$ and $A'_{(1)1}(\nu_0)$ as would be obtained from Eqs. (III-12), (III-6), and (III-18). We can then calculate ν_R and $\Gamma_1^{\,1}$ through Eqs. (III-9), (III-4), (III-15) and (III-17). Thus we have to find ν_R and $\Gamma_1^{\,1}$ such that these calculated values equal the original assumed ones. This is a straightforward--if somewhat tedious--calculation that can be carried out by hand and does not entail any numerical integration. It leads to $\nu_R = 3.4$

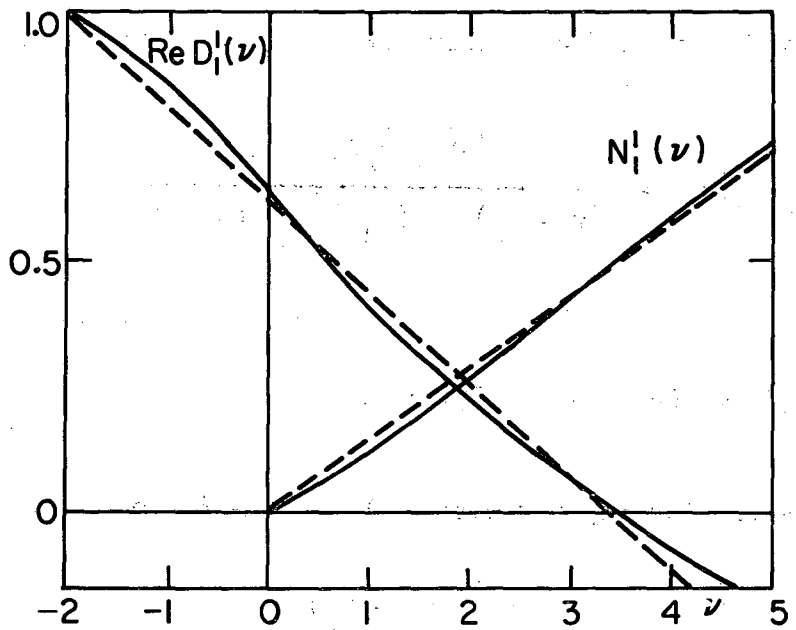

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Fig. 9. The functions $N_1^1(\nu)$ and Re $D_1^1(\nu)$ calculated from Eqs. (III-9) and (III-4) (solid lines) compared with the approximate forms $(\nu/\nu_R)N_1^1(\nu_R)$ and $-[(\nu - \nu_R)/(\nu_R - \nu_0)]$ (dashed lines), with the parameters of III.D.

and $\nu_R \Gamma_1^{\,1} = 2.6$. The corresponding values of $A_{(1)1}(-2)$, $F_{(1)1}^{\,1}$, and $F_{(1)1}^{\,2}$ are -0.0979, -1.326, and 13.05 respectively, which can be used to calculate the phase shift $\delta_1^{\,1}$ through Eq. (V.20) of CM-I,[2] according to which

$$[\nu/(\nu + 1)]^{1/2} \cot \delta_1^{\,1} = \mathrm{Re}\, D_1^{\,1}(\nu)/N_1^{\,1}(\nu) \quad . \qquad \text{(III-19)}$$

The corresponding cross section is plotted in Fig. 10. The half-width of the distribution is 125 MeV, whereas the positon of the resonance, at which $\delta_1^{\,1} = 90$ deg, is 585 MeV.

E. Calculation of Regge-Pole Trajectories

It has been proposed by Froissart and shown by Squires that Eq. (III-12) can be used to uniquely continue the function $A_{(\ell)I}(\nu)$ to complex values of ℓ .[22] Actually, in this case it is more appropriate to consider the function $B_I(\nu,\ell) = \nu^{-\ell} A_{(\ell)I}(\nu)$, since $A_{(\ell)I}(\nu)$ is singular for $-1 < \nu < 0$, whereas $B_I(\nu,\ell)$ is not.[23] Now for physical ℓ , the N/D solution in the elastic approximation,[2] except for possible subtractions, has the form

$$\nu^{-\ell} A_{(\ell)I}(\nu) = B_I(\nu,\ell) = N_I(\nu,\ell)/D_I(\nu,\ell) \quad , \qquad \text{(III-20)}$$

where

$$N_I(\nu,\ell) = \frac{1}{\pi} \int_{-\infty}^{-1} d\nu' \, \frac{\mathrm{Im}\, B_I(\nu',\ell) D_I(\nu',\ell)}{\nu' - \nu} \, , \qquad \text{(III-21)}$$

and

$$D_I(\nu,\ell) = 1 - \frac{\nu - \nu_0}{\pi} \int_0^{\infty} d\nu' \left(\frac{\nu'^{2\ell+1}}{\nu' + 1} \right)^{1/2} \frac{N_I(\nu',\ell)}{(\nu' - \nu_0)(\nu' - \nu)} \, . \qquad \text{(III-22)}$$

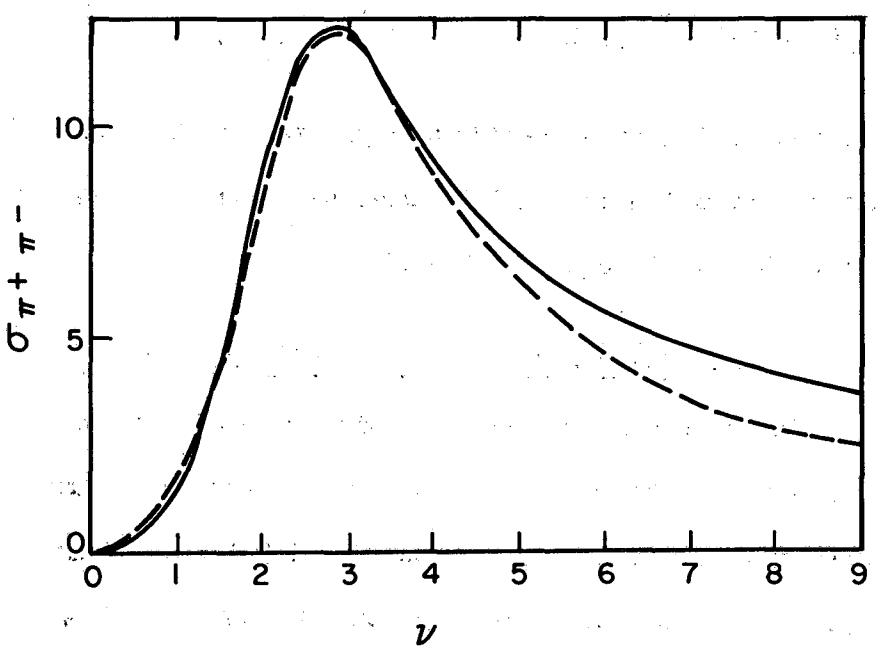

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Fig. 10. The $I = 1$, P-wave total cross section = $(12\pi/\nu)\sin^2\delta_1^1 = (12\pi/\nu)[\nu/(\nu+1)]^{1/2}\,\mathrm{Im}\,A_{(1)1}(\nu)$ calculated from Eqs. (III-9), (III-4), and (III-19) (solid lines), and from Eq. (III-16) (dashed line), with the parameters of III.D.

It has been proposed by Prosperi that equations of this type, together with Eq. (III-12), can now be continued to complex ℓ.[24] The above integral for $D_I(\nu,\ell)$ is not defined for large values of Re ℓ. However, the integral equation for $D_I(\nu,\ell)$ formed by substituting Eq. (III-21) into Eq. (III-22) can always be continued to such values of ℓ.

The treatment of the above equations can now be carried out exactly as in III.B and III.C. Inelastic effects can be taken into account by the method of the third paragraph of the preceding section for small ν. For complex ℓ, it is important not so much to solve for the partial-wave amplitude as to find the poles of $B_I(\nu,\ell)$, i.e., to find those values of $\ell = \alpha(\nu_P)$ for which, at some value $\nu = \nu_P$,

$$D_I\,[\nu_P,\,\alpha(\nu_P)] \;=\; 0 \;. \tag{III-23}$$

Equation (III-23) is thus the equation of motion of the topmost Regge trajectory for a given value of I, i.e., the relation between the position $\alpha(\nu_P)$ in the complex ℓ plane and the energy. The residue of this pole in $B_I(\nu,\ell)$ as a function of ν for a given ℓ is

$$\Gamma_I[\nu_P,\,\alpha(\nu_P)] \;=\; -N_I[\nu_P,\,\alpha(\nu_P)]\left\{\partial D_I[\nu,\,\alpha(\nu_P)]/\partial\nu\right\}^{-1}_{\nu=\nu_P} . \tag{III-24}$$

The residue of the corresponding pole in $A_{(\ell)I}(\nu)$ in the complex ℓ plane is then

$$\beta(\nu_P) = \nu_P^{\alpha(\nu_P)} \Gamma_I[\nu_P, \alpha(\nu_P)][d\alpha(\nu)/d\nu]_{\nu=\nu_P} . \qquad \text{(III-25)}$$

Now this method can be used only for $\mathrm{Re}\ \ell > \mathrm{Re}\ \alpha(\nu_F)$, since Eq. (III-12) diverges for smaller values of $\mathrm{Re}\ \ell$. To calculate high-energy cross sections,[4,5] however, we need $\alpha(\nu)$ and $\beta(\nu)$ for $\nu < 0$, which can only be obtained in a limited region by the above procedure. The values in the remaining region can be calculated from the dispersion relations

$$\alpha(\nu) = \frac{1}{\pi} \int_0^\infty d\nu' \ \frac{\mathrm{Im}\ \alpha(\nu')}{\nu' - \nu} \qquad \text{(III-26)}$$

and

$$\beta(\nu) = \frac{1}{\pi} \int_0^\infty d\nu' \ \frac{\mathrm{Im}\ \beta(\nu')}{\nu' - \nu} , \qquad \text{(III-27)}$$

where we have assumed that $\alpha(\nu)$ and $\beta(\nu)$ each has only the right-hand cut.[25] Suppose now that ν_B is the largest value of ν for which $\alpha(\nu)$ and $\beta(\nu)$ can be calculated. Then, should Eqs. (III-26) and (III-27) prove insufficient because of a large contribution from $\nu > \nu_B$, one may represent the singularities in that region by effective-range formulae as in III.B. The parameters of these formulae may then be determined by requiring that they give the correct $\alpha(\nu)$ and $\beta(\nu)$ and perhaps their derivatives at some point or points ν_A , where $\alpha(\nu_A) > \alpha(\nu_F)$, and $\nu_A < \nu_B$.

The above procedure gives only the topmost Regge pole for a particular isotopic spin state. To obtain the next pole, we must

first subtract the contribution of the topmost pole for $\mathrm{Im}\,\tilde{A}_I$ in Eq. (III-12), and then explicitly insert that pole into $B_I(\nu, \ell)$. This is to assure the necessary convergence of Eq. (III-12). We may now proceed as before in calculating this next pole. This procedure may be repeated any number of times.

F. Discussion

A general method for calculating the low-energy pion-pion amplitude has been given, both for physical and unphysical angular momenta. The method takes into account both long-range and short-range forces (nearby and distant unphysical singularities), the latter being calculated with the help of the fixed momentum-transfer dispersion relation (III-10). Incidentally, such a relation was also used by Chew and Mandelstam (CM-II).[3] Those authors, however, used it essentially to calculate the left-hand cut, a calculation leading to divergences in the more distant singularities. In this work, this relation is used to calculate the full amplitude at $\nu = \nu_F$, which does not lead to any divergences.

A rough calculation, in which only a sharp P-wave resonance is retained, gives a mass of 585 MeV and a half-width of 125 MeV. The mass is smaller and the width larger than the latest experimental values of 725 MeV and 75 Mev respectively.[26,27] This is not unreasonable, however, in view of the crude approximations made. Moreover, the experimental pion-pion cross sections are deduced from pion-production experiments with the help of simple models which may not be very reliable. The calculated width is also larger than that obtained by Serebryakov and Shirkov,[17] who found a

width smaller than the experimental value. On the other hand, it is smaller than the width obtained by Chew and Mandelstam (CM-II)[3] and Zachariasen.[18] The latter also calculated the mass of the resonance without arbitrary parameters and obtained 350 MeV, which is smaller than the value calculated here.

It is interesting to note that the calculation by Serebryakov and Shirkov depends quite sensitively on the assumed value of the pion-pion coupling constant, whereas all the other calculations do not. This appears to be a peculiarity of the particular approximation scheme used by these authors, rather than a feature of the general theory. Indeed, as was first pointed out by Chew,[28] a calculation of the $I = 2$ state, for which an S-wave subtraction is unnecessary, would automatically give this constant. Such a calculation is certainly possible with the present method.

The above technique can, of course, be applied to other strong-interaction problems. In such problems one must also deal with energetically unavailable quasi-physical regions, for instance, the region $0 < \nu < (m^2 - 1)$ in the process $\pi\pi \rightarrow N\bar{N}$, where m = nucleon mass. While such regions are, strictly speaking, unphysical, the techniques used to handle them are essentially the same as for the physical regions.

G. Conclusion

A general approach for treating low-energy partial-wave amplitudes has been presented. This approach consists of first setting up an effective-range formula in an a priori manner. This

can then be used as a phenomenological formula. However, a method is also given for calculating the effective-range parameters from the partial-wave amplitudes in the crossed channel. Thus one can also calculate amplitudes dynamically. Alternatively, one may calculate only some of the parameters dynamically and leave the remaining ones to be determined experimentally. This would reduce the number of arbitrary constants but would still give a more accurate formula then could be obtained dynamically.

To increase the accuracy of the calculation without adding additional phenomenological constants would require some method for calculating inelastic effects at intermediate energies. Such a method does not yet exist. In practice, however, it may be possible to make approximate calculations by treating resonances as stable particles. For instance, the ω meson may emerge as a resonance in π-ρ scattering. Such a calculation may be necessary for explaining the hard core in the nucleon-nucleon problem--for instance --where the ω probably does play an important role.

Although inelastic effects would probably be difficult to include, they do appear to be small, at least in the crossed channel. In the direct channel they are approximately taken into account by the mechanism described in the third paragraph of III.D. (See also Appendix A.) This means, for instance, that it is possible to calculate states with small phase shifts, where such effects could play an important role. In particular, a calculation of the $I = \frac{1}{2}$, $J = \frac{1}{2}$ state in πN scattering should yield the nucleon mass and coupling constant, with the nucleon as a bound state.

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APPENDIX A. HIGH-ENERGY EFFECTS AT LOW ENERGIES

We now discuss the mechanism of the third paragraph of III.D. in greater detail. It arises essentially from the nonuniqueness of the N/D method. There is, in fact, an infinite number of N/D methods. Suppose that, instead of defining $D_\ell^I(\nu)$ by Eq. (III-4), we define it by

$$D_\ell^I(\nu) = 1 + \frac{\nu - \nu_0}{\pi} \int_0^\infty d\nu' \frac{\operatorname{Im} D_\ell^I(\nu')}{(\nu' - \nu_0)(\nu' - \nu)} , \qquad \text{(A-1)}$$

with

$$\operatorname{Im} D_\ell^I(\nu) = -\left(\frac{\nu}{\nu + 1}\right)^{1/2} \operatorname{Re} N_\ell^I(\nu) , \quad \text{for} \quad \nu < \nu_I , \qquad \text{(A-2)}$$

$$= H_\ell^I(\nu) , \qquad \text{for} \quad \nu > \nu_I , \qquad \text{(A-3)}$$

where $H_\ell^I(\nu)$ is any function for which the integral of (A-1) does not diverge, and ν_I any value of ν such that the relation $R_\ell^I(\nu) = 1$ is valid--at least approximately--for $\nu < \nu_I$.

We can now find the singularities of $N_\ell^I(\nu) = A_{(\ell)I}(\nu) D_\ell^I(\nu)$. It is obvious that this function cannot have any singularities in addition to those possessed by $A_{(\ell)I}(\nu)$ and $D_\ell^I(\nu)$. Thus it has, at most, a right-hand cut running from 0 to $+\infty$ and a left-hand cut running from -1 to $-\infty$. But, from Eqs. (III-2) and (A-2) combined with the elastic unitarity condition $\operatorname{Im}[A_{(\ell)I}^{-1}(\nu)] = -[\nu/(\nu + 1)]^{1/2}$, it follows that $N_\ell^I(\nu)$ is real for $0 < \nu < \nu_I$. In other words, the discontinuity across

the right-hand cut in the interval $0 < \nu < \nu_I$ is zero. This, in turn, means that $N_\ell^I(\nu)$ satisfies the dispersion relation

$$N_\ell^I(\nu) = A_{(\ell)I}(\nu_0) + \frac{\nu - \nu_0}{\pi} \int_{-\infty}^{-1} d\nu' \frac{\mathrm{Im}\, A_{(\ell)I}(\nu')\, D_\ell^I(\nu')}{(\nu' - \nu_0)(\nu' - \nu)}$$

$$+ \frac{\nu - \nu_0}{\pi} \int_{\nu_I}^{\infty} d\nu' \frac{\mathrm{Im}[A_{(\ell)I}(\nu')\, D_\ell^I(\nu')]}{(\nu' - \nu_0)(\nu' - \nu)} \tag{A-4}$$

since, according to Eq. (A-1), $D_\ell^I(\nu)$ is real for $\nu < 0$. Thus Eq. (III-7) is unchanged except for the additional term

$$\Delta N_\ell^I(\nu) = \frac{\nu - \nu_0}{\pi} \int_{x_I}^{0} \frac{dx}{x} \frac{\mathrm{Im}[A_{(\ell)I}(-x^{-1})\, D_\ell^I(-x^{-1})]}{(\nu_0 + x^{-1})(1 + x\nu)} . \tag{A-5}$$

Suppose now that the approximation (III-8) is such that it also holds for $x_I < x < 0$. Then, if we insert that approximation into Eq. (A-5), we find that the form of Eq. (III-9) is unchanged by the addition of $\Delta N_\ell^I(\nu)$, although the $F_{(\ell)I}^{\,i}$ will, of course, be different. Now, this is true no matter what $H_\ell^I(\nu)$ may be. In particular, $H_\ell^I(\nu)$ may be just the function obtained by using Eq. (A-2) for $\nu > \nu_I$, but with $N_\ell^I(\nu)$ given by Eq. (III-9) instead of Eq. (A-4). This justifies the use of Eqs. (III-4) and (III-9) for $\nu < \nu_I$, despite the failure of Eq. (III-9) and the elastic approximation at large values of ν' within the integral of Eq. (III-4).

APPENDIX B. S-WAVE EFFECTS IN THE $\pi\pi$ CROSSED CHANNEL

We now justify the neglect of the S waves in the calculation of III.D. In other words, although the S waves in the crossed channel may distort the amplitude in the neighborhood of ν_F, a consistent neglect of these waves has little effect on the P-wave amplitude in the physical region. To see this, we note that the partial-wave amplitude satisfies the dispersion relation[2]

$$A_{(1)1}(\nu) = \frac{\nu}{\pi} \int_{-\infty}^{-1} d\nu' \frac{\operatorname{Im} A_{(1)1}(\nu')}{\nu'(\nu' - \nu)} + \frac{\nu}{\pi} \int_{0}^{\infty} d\nu' \frac{\operatorname{Im} A_{(1)1}(\nu')}{\nu'(\nu' - \nu)} . \tag{B-1}$$

Let $\operatorname{Im} A_{(1)1}^{(s)}(\nu)$ be the part of $\operatorname{Im} A_{(1)1}(\nu)$ that arises from the S waves in the crossed channel. From Eqs. (III-5) and (III-6), this has the form

$$\operatorname{Im} A_{(1)1}^{(s)}(\nu) = \frac{2}{\nu} \int_{0}^{-\nu-1} d\nu' \left(1 + 2\,\frac{\nu' + 1}{\nu}\right)[\beta_{10} \operatorname{Im} A_{(0)0}(\nu') + \beta_{12} \operatorname{Im} A_{(0)2}(\nu')] . \tag{B-2}$$

If this is substituted into Eq. (B-1) the contribution to $A_{(1)1}(\nu)$ is just

$$\frac{4}{\pi\nu} \int_{0}^{\infty} d\nu'\, Q_1\left(1 + 2\,\frac{\nu' + 1}{\nu}\right)[\beta_{10} \operatorname{Im} A_{(0)0}(\nu') + \beta_{12} \operatorname{Im} A_{(0)2}(\nu')] .$$

But this is exactly the contribution that would be obtained from Eq. (III-12). Thus, if we neglect the S wave in Eq. (III-12) and, at the same time, the S-wave contribution to the left-hand cut, we will be calculating the function

$$A_{(1)1}(\nu) - \frac{\nu}{\pi} \int_{-\infty}^{-1} d\nu' \frac{\operatorname{Im} A_{(1)1}^{(s)}(\nu')}{\nu'(\nu' - \nu)} .$$

But this function is known to be a good approximation to $A_{(1)1}(\nu)$ in the physical region.[19] This fact at the same time justifies the application of physical unitarity (and, hence, of the N/D method) to this function and also the consistent neglect of S waves.

FOOTNOTES AND REFERENCES

1. Geoffrey F. Chew, Ann. Revs. Nuclear Sci. 9, 29 (1959).

2. G. F. Chew and Stanley Mandelstam, Phys. Rev. 119, 467 (1960); hereafter referred to as CM-I.

3. G. F. Chew and S. Mandelstam, Nuovo cimento 19, 752 (1961); hereafter referred to as CM-II.

4. G. F. Chew, Steven C. Frautschi, and S. Mandelstam, Regge Poles in π-π Scattering, Phys. Rev. (to be published).

5. G. F. Chew and S. C. Frautschi, Phys. Rev. Letters 7, 394 (1961); ibid. 8, 41 (1962), to which the reader is referred for additional references on Regge poles.

6. G. F. Chew, S-Matrix Theory of Strong Interactions (W. A. Benjamin, Inc., New York, 1961).

7. We are referring here to the percentage accuracy, since the product of an approximate (known) function by an exact (unknown) function always has the same percentage accuracy as the approximate function at any point.

8. G. F. Chew and S. C. Frautschi, Phys. Rev. 123, 1478 (1961).

9. H. Pierre Noyes, Phys. Rev. 119, 1736 (1960).

10. For a possible test at high energies, see reference 5.

11. G. F. Chew and Francis E. Low, Phys. Rev. 101, 1570 (1956).

12. S. C. Frautschi and J. D. Walecka, Phys. Rev. 120, 1486 (1960). See also William R. Frazer and Jose R. Fulco, Phys. Rev. 119, 1420 (1960).

13. S. W. MacDowell, Phys. Rev. 116, 774 (1960).

14. This is not the most convenient method in an actual calculation, but is more appropriate for setting up effective-range formulae. For an alternative method see; for instance, M. Froissart, Nuovo cimento 22, 191 (1961).

15. G. F. Chew and S. C. Frautschi, Phys. Rev. 124, 264 (1961).

16. It is important to note that this is true only in a limited energy range. For instance, although Levinson's theorem is satisfied in this case, any considerations based on this theorem are probably meaningless.

17. V. V. Serebryakov and D. V. Skirkov, Some Resonance Solutions of the Low-Energy Pion-Pion Scattering Equations (submitted to Zhur. Eksptl. Teoret. Fiz.).

18. Frederick Zachariasen, Phys. Rev. Letters 7, 112, 268 (E) (1961).

19. James S. Ball and David Y. Wong, Phys. Rev. Letters 7, 390 (1961), to which the reader is also referred for additional references on the π-π problem.

20. Frederick Zachariasen (California Institute of Technology) and Charles Zemach (University of California), private communication.

21. James S. Ball and David Y. Wong, Phys. Rev. Letters 6, 29 (1961).

22. Euan J. Squires, On the Continuation of Partial-Wave Amplitudes to Complex ℓ, Lawrence Radiation Laboratory Report UCRL-10033, January 25, 1962 (unpublished).

23. Azim O. Barut and Daniel Zwanziger (University of California, Berkeley), private communication.

24. Giovanni Prosperi (Lawrence Radiation Laboratory) private communication.

25. Philip G. Burke (Lawrence Radiation Laboratory), private communication.

26. Duane D. Carmony and Remy T. Van de Walle, Phys. Rev. Letters 8, 73 (1962).

27. Edwin Pickup, D. K. Robinson, and Edward O. Salant, Phys. Rev. Letters 7, 192 (1961). This paper contains an extensive bibliography on the experimental situation for π-π scattering.

28. Geoffrey F. Chew (Lawrence Radiation Laboratory), private communication.

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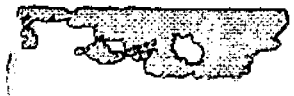
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