UCRL-10700

University of California

Ernest O. Lawrence Radiation Laboratory

TWO-WEEK LOAN COPY

This is a Library Circulating Copy which may be borrowed for two weeks. For a personal retention copy, call Tech. Info. Division, Ext. 5545

ANALYTICITY OF REGGE POLE TERMS

Berkeley, California

DISCLAIMER

This document was prepared as an account of work sponsored by the United States Government. While this document is believed to contain correct information, neither the United States Government nor any agency thereof, nor the Regents of the University of California, nor any of their employees, makes any warranty, express or implied, or assumes any legal responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by its trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof, or the Regents of the University of California. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof or the Regents of the University of California.

Submitted for pub. in Phys. Rev.

UCRL-10700

UNIVERSITY OF CALIFORNIA

Lawrence Radiation Laboratory
Berkeley, California

Contract No. W-7405-eng-48

ANALYTICITY OF REGGE POLE TERMS

C. Edward Jones

February 26, 1963

ANALYTICITY OF REGGE POLE TERMS

C. Edward Jones

Lawrence Radiation Laboratory
University of California
Berkeley, California

February 26, 1963

ABSTRACT

A formula for single Regge pole terms is proposed which produces the correct analyticity in momentum-transfer, while retaining the correct behavior in the asymptotic region of the crossed channel and near the pole.

ANALYTICITY OF REGGE POLE TERMS*

C. Edward Jones

Lawrence Radiation Laboratory
University of California
Berkeley, California

February 26, 1963

Regge pole terms in relativistic scattering are assumed to give a good representation for the scattering amplitude in two regions: (a) near a resonance or bound state and (b) in the asymptotic region for the crossed channel.[1,2,3] Such a term is generally written[4,5]

$$R(t, s) = -\pi[2\alpha(t) + 1]\beta(t) \frac{P_{\alpha(t)}(-z)}{\sin \pi \alpha(t)} , \tag{1}$$

where t is the total energy squared in the center of mass (c.m.) and z is the cosine of the c.m. scattering angle. For elastic scattering of particles of equal mass m, $z = 1 + (s/2q_t^2)$, where s is the square of the c.m. energy for the crossed-channel process and $q_t^2 = \frac{t}{4} - m^2$. The partial-wave projection of Eq. (1) is[6]

$$R(\ell, t) = \frac{\beta(t)}{\ell - \alpha(t)} \frac{2\alpha(t) + 1}{\ell + \alpha(t) + 1} . \tag{2}$$

From Eq. (2) one readily sees that bound states occur for values of t such that α equals a positive integer, and resonances occur when $\mathrm{Re}\,\alpha$ equals a positive integer and $\mathrm{Im}\,\alpha \neq 0$. Equation (1), on the other hand, embodies the result that the high-energy behavior in the crossed channel will contain a term proportional to $(z)^{\alpha(t)}$ or $(s)^{\alpha(t)}$.

While a single Regge term may represent the scattering amplitude accurately in certain regions, the analyticity of such a term in the variable s violates the known analyticity of the full amplitude $A(s, t)$. To see the

difficulty, we recall that $P_\alpha(z)$ has a cut running from $z = -1$ to $z = -\infty$. In terms of the s variable this means a cut in $R(t, s)$ running from $s = 0$ to $s = +\infty$ when $(\frac{t}{4} - m^2) > 0$, and a cut from $s = 0$ to $s = -\infty$ when $(\frac{t}{4} - m^2) < 0$. The purpose of this note is to propose a Regge pole formula that has a fixed branch cut in s starting at $s = s_0$.

To construct such a modified Regge pole term $R'(t, s)$, we consider first its partial-wave expansion,

$$R'(t, s) = \sum_{\ell=0}^{\infty} (2\ell + 1) R'(\ell, t) P_\ell(z) \quad . \tag{3}$$

The requirement that a singularity occur at $s = s_0$ means that the Lehmann ellipse for expansion (3) must pass through $z = z_0 = 1 + (s_0/2q_t^2)$. To ensure this we require that $R'(\ell, t)$ be bounded by $e^{-\rho\ell}$ as $\ell \to \infty$, where $\rho = \ln\left[z_0 + (z_0^2 - 1)^{1/2}\right]$, and also that it <u>not</u> be bounded by any smaller exponential of ℓ^7. We further require that $R'(\ell, t)$ have a pole at $\ell = \alpha(t)$ with residue $\beta(t)$. A choice of R' that fulfills these requirements is

$$R'(\ell, t) = \frac{\beta(t)\, e^{\rho\alpha(t)} e^{-\rho\ell}}{\ell - \alpha(t)} \quad . \tag{4}$$

We now determine $R'(t, s)$ from Eq. (3). First we note that with our choice of $R'(\ell, t)$, Eq. (3) can be written

$$R'(t, s) = \sum_{\ell=0}^{\infty} (2\ell + 1)\, \beta(t) \int_{\xi_0}^{\infty} dx\, x^{\alpha-\ell-1} P_\ell(z) \quad , \tag{5}$$

where $\xi_0 = e^\rho$. Reversing the order of the sum and the integral we obtain

$$R'(t, s) = \beta(t) \int_{\xi_0}^{\infty} dx\, x^{\alpha(t)} \left[-2x \frac{\partial g(x, z)}{\partial x} - g(x, z) \right] , \tag{6}$$

where[8]

$$g(x,z) = \sum_{\ell=0}^{\infty} \frac{P_\ell(z)}{x^{\ell+1}} = \frac{1}{(1 - 2zx + x^2)^{1/2}} \quad . \tag{7}$$

We integrate Eq. (6) by parts to obtain

$$R'(t,s) = \beta(t)\,[2\alpha(t) + 1] \int_{\xi_0}^{\infty} dx \frac{x^{\alpha(t)}}{(1 - 2zx + x^2)^{1/2}} + \frac{2\beta(t)\xi_0^{[\alpha(t)+1]}}{(1 - 2\xi_0 z + \xi_0^2)^{1/2}} \quad . \tag{8}$$

To study the analyticity of R' we first consider the last term

$$R_1(t,s) = \frac{2\beta(t)\xi_0^{\alpha(t)+1}}{(1 - 2\xi_0 z + \xi_0^2)^{1/2}} \quad . \tag{9}$$

This term can be rewritten

$$R_1(t,s) = \frac{2b(t)\left\{q_t^2 + \frac{s_0}{2} + \left[\left(\frac{s_0}{4} + q_t^2\right) s_0\right]^{1/2}\right\}^{\alpha(t)+\frac{1}{2}}}{(s_0 - s)^{1/2}} \quad , \tag{10}$$

where $b(t) = \beta(t)/(q_t^2)^{\alpha(t)}$. Assuming that $b(t)$ and $\alpha(t)$ have cuts running from $t = 4m^2$ to $t = \infty$[9], we see that $R_1(t,s)$ has cuts running from

$$\begin{aligned} s &= s_0 && \text{to} && s = \infty \ , \\ t &= 4m^2 && \text{to} && t = \infty \ , \\ t &= -s_0 + 4m^2 && \text{to} && t = -\infty \ . \end{aligned} \tag{11}$$

We note that

$$R_1(\ell,t) = \frac{\beta(t)\ \xi_0^{\alpha-\ell}}{(\ell + \frac{1}{2})} \quad , \tag{12}$$

which makes it similar to $R'(\ell,t)$ except that $R_1(\ell,t)$ has a fixed pole at $\ell = -\frac{1}{2}$ and no pole at $\ell = \alpha(t)$. A study of $R'(t,s)$, as well as the

integral term in Eq. (8) [which we denote $R_2(t,s)$], shows that they also have the cut structure (11). In the neighborhood of $s = s_0$, however,

$$R_1(t,s) \approx \frac{\text{const}}{(s_0 - s)^{1/2}} \quad , \tag{13}$$

whereas $R_2(t,s) \propto (s_0 - s)^{1/2} + \text{const.}$ (14)

Thus R_2 is finite at $s = s_0$. Chew has recently proposed using $R_2(t,s)$ as a single Regge term, instead of $R'(t,s)$, since the former, being finite, can satisfy unitarity in the crossed s-channel near $s = s_0$.[10] We note that $R_2(\ell,t)$ also has a fixed pole at $\ell = -\frac{1}{2}$, but this appears to produce no difficulty since for $\text{Re}\,\alpha > -\frac{1}{2}$, the Regge behavior in s is maintained.

The presence of the left t cut in $R'(t,s)$ may come as a slight surprise. But it can be readily understood in the following way. We may rewrite $R'(\ell,t)$ as

$$R'(\ell,t) = \frac{(q_t^{\,2})^{\ell}\, b(t) \left\{ \frac{t}{4} - m^2 + \frac{s_0}{2} + \left[s_0 \left(\frac{s_0+t}{4} - m^2 \right) \right]^{1/2} \right\}^{\alpha(t)-\ell}}{\ell - \alpha(t)} \quad . \tag{15}$$

From Eq. (15), R' is seen to have correctly located partial-wave cuts, including the left cut which starts at $t = -s_0 + 4m^2$. If R' were the exact partial-wave amplitude, then the left t cut would be canceled in doing the partial-wave sum. Since R' is only one piece of the correct partial-wave amplitude, there is no reason to expect the left cut to cancel in the sum.

Although Eq. (8) as it stands can only be defined for $\text{Re}\,\alpha < 0$, we can define it for all values of α by analytic continuation. In particular, we can utilize the fact that[11]

$$Q_{-1-\alpha(t)}(z) = \int_{\xi}^{\infty} \frac{x^{\alpha(t)}dx}{(1 - 2xz + x^2)^{1/2}} \ , \quad \xi = z + (z^2 - 1)^{1/2} \ , \tag{16}$$

to write

$$R'(t,s) = [2\alpha(t) + 1]\, \beta(t)\, Q_{-1-\alpha(t)}(z) - [2\alpha(t) + 1]\, \beta(t) \times \int_{\xi}^{\xi_0} \frac{dx\, x^{\alpha(t)}}{(1 - 2xz + x^2)^{1/2}} + R_1(t,s) \ , \tag{17}$$

which is defined for all values of α. In the neighborhood of $\alpha \approx$ integer, we notice that

$$Q_{-1-\alpha(t)}(z) \approx - \frac{\pi \cos\pi\, \alpha(t)}{\sin \pi\, \alpha(t)} P_{\alpha(t)}(z) \rightarrow \frac{-\pi P_{\alpha(t)}(-z)}{\sin \pi\, \alpha(t)} \ , \tag{18}$$

so R' looks like R near α = integer , and has the correct residue.

By transforming variables, we can write Eq. (17) in a somewhat more convenient form,

$$R'(t,s) = [2\alpha(t) + 1]\, b(t)\, (q_t^{\,2})^{\alpha(t)}\, Q_{-1-\alpha(t)}\left(1 + \frac{s}{2q_t^{\,2}}\right) + \frac{[2\alpha(t) + 1]\, b(t)}{2} \times \int_{s_0}^{s} \frac{ds'}{\left[s'(\frac{s'}{4} + q_t^{\,2})\right]^{1/2}} \frac{\left\{q_t^{\,2} + \frac{s'}{2} + [s'(\frac{s'}{4} + q_t^{\,2})]^{1/2}\right\}^{\alpha(t)+\frac{1}{2}}}{(s' - s)^{1/2}} + R_1(t,s) \ . \tag{19}$$

To summarize, we have found a modified formula for a single Regge pole in the t channel, that has the correct analyticity and asymptotic behavior in s, produces resonances and bound states in the t channel with correct residues, and - if Chew's suggestion[10] is incorporated - does not

violate unitarity in the s channel. Such a form for a Regge pole term appears to have application in dynamical calculations (see, in particular, reference 12) and perhaps in semiphenomenological studies as well.[13]

ACKNOWLEDGEMENT

The author would like to express his appreciation for the encouragement and helpful suggestions of Professor Geoffrey F. Chew.

FOOTNOTES AND REFERENCES

* Work done under the auspices of the U. S. Atomic Energy Commission.

1. G. F. Chew and S. C. Frautschi, Phys. Rev. Letters 7, 394 (1961).
2. G. F. Chew and S. C. Frautschi, Phys. Rev. Letters 8, 41 (1962).
3. V. N. Gribov, J. Exptl. Theoret. Phys. (U.S.S.R.) 14, 478 and 1395 (1962).
4. T. Regge, Nuovo Cimento 14, 951 (1959).
5. T. Regge, Nuovo Cimento 18, 947 (1960).
6. Bateman Project Staff, Higher Transcendental Functions (McGraw-Hill Book Company, Inc., New York, 1953), Vol. I, p. 167, Eq. (6).
7. For a discussion of the convergence of expansions of functions in terms of Legendre polynomials see
   E. T. Whittaker and G. N. Watson, A Course of Modern Analysis (Cambridge at the University Press, Cambridge, 1958), 4th ed., Chap. 15, p. 322 and G. Szegö, Orthogonal Polynomials (American Mathematical Society, New York, 1939), Chap. 10.
8. See reference 6, p. 154, Eq. (33).
9. For discussions of the analyticity of $\alpha(t)$ and $\beta(t)$ in the potential and relativistic cases, see A. O. Barut and D. Zwanziger, Phys. Rev. 127, 974 (1962) and J. R. Taylor, Phys. Rev. 127, 2257 (1962).
10. G. F. Chew, An Analogue in Potential Scattering for the New Form of Strip Approximation, Cambridge preprint (1963).
11. See E. T. Whittaker and G. N. Watson (reference 7), p. 334.
12. G. F. Chew, The Self-Consistent S Matrix with Regge Asymptotic Behavior, CERN preprint TH/292 (1962).
13. It has been pointed out to the author that there is in circulation a preprint by N. Khuri, Institute for Advanced Study, Princeton, N. J. (1962) which gives a discussion closely related to the one presented herein.

This report was prepared as an account of Government sponsored work. Neither the United States, nor the Commission, nor any person acting on behalf of the Commission:

A. Makes any warranty or representation, expressed or implied, with respect to the accuracy, completeness, or usefulness of the information contained in this report, or that the use of any information, apparatus, method, or process disclosed in this report may not infringe privately owned rights; or

B. Assumes any liabilities with respect to the use of, or for damages resulting from the use of any information, apparatus, method, or process disclosed in this report.

As used in the above, "person acting on behalf of the Commission" includes any employee or contractor of the Commission, or employee of such contractor, to the extent that such employee or contractor of the Commission, or employee of such contractor prepares, disseminates, or provides access to, any information pursuant to his employment or contract with the Commission, or his employment with such contractor.