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CONDITIONS FOR THE DISCRETENESS OF THE SPECTRUM OF SINGULAR ELLIPTIC OPERATORS

Berkeley, California

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CONDITIONS FOR THE DISCRETENESS OF THE SPECTRUM OF SINGULAR ELLIPTIC OPERATORS

Harold Hanerfeld

(Thesis)

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1. Introduction. In this paper we consider the Dirichlet problem for singular second order elliptic operators. Our object is to formulate and prove sufficient conditions for such operators to have a discrete spectrum. Specifically, the operators are to have the form

$$l = - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i} + c(x).$$

The coefficients a_{ij}, b_i, and c are functions defined in a bounded domain $G \subset E^n$. We require that the singularities occur on ∂G; i.e., l is elliptic for $x \in G$, but we allow l to be parabolic for some $x \in \partial G$. K. O. Friedrichs* has obtained criteria for spectral discreteness of an ordinary differential operator $l = - r^{-1}(t) \frac{\partial}{\partial t} a(t) \frac{\partial}{\partial t} - q(t)$. Kreith** considered symmetric, singular, elliptic partial differential operators $l = - r(x)^{-1} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij} \frac{\partial}{\partial x_j} + c(x)$. In this paper we let $r(x) = 1$; moreover our operator l is not necessarily symmetric but $b_i^2(x)$ must be less than the least eigenvalue of the matrix $(a_{ij}(x))$.

Kreith's criteria for discreteness are developed in the following manner. The operator l majorizes an operator

$$l' = - \rho (x_n)^{-1} \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \alpha_i (x_n) \frac{\partial}{\partial x_i} + \gamma (x_n), \text{ where } x = (x_1, x_2, \ldots, x_n)$$

l and l' are respectively restricted to self adjoint operators L and L'. He shows that the spectrum of L is discrete when the spectrum of L' is discrete. By enclosing G in a parallelepiped, he is able to separate variables in the eigenvalue equation $L'u = \lambda u$. Kreith thus reduces

*see [12]
**see [11]

this equation to an ordinary differential equation to which he applies the criteria developed by Friedrichs. If the matrix $(a_{ij}(x))$ is not already diagonal, the functions $\alpha_i(x_n)$ must be less than the least eigenvalue of $(a_{ij}(x))$ at each point $x \in G$. Consequently, if L becomes parabolic at x', all $\alpha_i(x'_n) = 0$ and L' degenerates in all directions simultaneously. Kreith's criteria are governed by the direction in which L degenerates most quickly.

Our operator l is extended to a closed, densely defined, Dirichlet operator L whose resolvent set is contained in a left half-plane. We assume there exists a coordinate system in which the matrix $(a_{ij}(x))$ is diagonal. In this new coordinate system we prove an integral inequality which is related to Friedrichs' criteria for spectral discreteness in ordinary differential equations. With this integral inequality we obtain a condition for spectral discreteness of the Dirichlet operator L. When applicable, our condition has two advantages over Kreith's. L degenerates only in those directions in which the eigenvalues of $(a_{ij}(x))$ go to zero, and, secondly, our criterion is governed by the direction in which L degenerates most slowly.

The object of section 2 is to obtain the Dirichlet operator L associated with l. After formally defining l, we introduce a bilinear form $J[u,v]$ ($= RE(lu,v)$), whose domain is C_0^∞ . We show that J is closable and that its domain may be extended to a Hilbert space $\mathcal{M}_0 \subset \mathcal{L}^2$. For large positive λ , and $u, v \in \mathcal{M}_0$ we prove that $(lu,v) + \lambda (u,v)$ satisfies the condition of the Lax-Milgram Theorem. L is then obtained as a consequence of the Lax-Milgram Theorem.

In section 3 we develop a criterion for the discreteness of the spectrum of L. First we consider the problem of diagonalizing the coefficient matrix $(a_{ij}(x))$. To this end, a new condition is introduced which relates the eigenvectors of $(a_{ij}(x))$ to a curvilinear coordinates system. In the curvilinear coordinate system

$$L = - \sum_{i=1}^{n} \frac{\partial}{\partial y_i} d_{ii}(y) \frac{\partial}{\partial y_i} + \sum_{i=1}^{n} b_i(y) \frac{\partial}{\partial y_i} + c(y)$$ and some of the

$d_{ii}(y)$ go to zero at the ∂G. The rate at which the $d_{ii}(y)$ go to zero at the singular boundary will govern whether or not the spectrum of L is discrete.

In the next section we prove a corollary to the discreteness theorem of section 3. We let 1 be symmetric and obtain the existence theory of section 2 directly from the Friedrichs' extension theorem. Following Kreith, we let L majorize and operator $L' = - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} a(x) \frac{\partial}{\partial x_i} + c(x)$ and apply our discreteness criterion to L'. Thus we obtain a condition which does not depend on the diagonalization of $(a_{ij}(x))$. Next, criteria for non-discreteness are proven for the case when L' majorizes L.

Finally, in the appendix, section 6, we prove a relationship used throughout this paper. Here we develop positive lower bounds for the least eigenvalue of a positive definite hermitian matrix.

To Professor František Wolf go my wholehearted thanks for the guidance, patience and understanding given me throughout this undertaking.

2. <u>Definition and Extension of l</u>. We will consider the Dirichlet problem for the elliptic operator

$$2.01 \quad l = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i} + c(x) .$$

The following conditions are assumed to hold inside a bounded domain G.

i) $a_{ij}(x)$ are complex valued and of class C^1 in G with $a_{ij}(x) = \bar{a}_{ji}(x)$.

ii) $\sum_{i,j=1}^{n} a_{ij}(x) \zeta_i \bar{\zeta}_j > 0$ for all complex $\vec{\zeta} \neq 0$.

iii) At every point $x \in G$, the n eigenvalues of the coefficient matrix $(a_{ij}(x))$ are distinct.

iv) $c(x)$ is real continuous and bounded from below on G.

v) $b_i(x)$ are real and boundedly differentiable in $\bar{G}$ and

$$\max_{i=1,2,..n} \{b_i^2(x)\} \leq \frac{\det. (a_{ij}(x))}{\prod_{i=1}^{n} a_{ii}(x) \sum_{j=1}^{n} \frac{1}{a_{jj}}(x)} .$$

2.02 <u>Definition</u>. We say that a point x_0 on ∂G is a singular boundary point and that the ellipticity degenerates at x_0 if there exists a sequence of points $\{x_k\}$ in G for which

$$\lim_{k \to \infty} x_k = x_0 \text{ and } \lim_{k \to \infty} \sum_{i,j=1}^{n} a_{ij}(x_k) \zeta_i \bar{\zeta}_j = 0 \text{ for some fixed } \vec{\zeta} \neq 0.$$

We allow the ellipticity to degenerate on a set $S \subset \partial G$. It is

assumed that G is in canonical position,* i.e. $G \subset \{x | x_n > 0\}$ and $S \subset \{x | x_n = 0\}$, and that the boundary of G is sufficiently smooth for Green's Theorem to hold.

2.1 We will consider 1 on the space of functions which are infinitely often differentiable and which have compact support in G. With 1 we define a bilinear form $J[u,v]$ which is bounded from below.

2.11 <u>Definition</u>. $\mathcal{L}^2(G)$ denotes the Hilbert space of functions $u(x)$ for which $\int_G |u(x)|^2 \, dx < \infty$.

2.12 Definition. $C_0^\infty(G) = \{v(x) | v(x)$ is infinitely often differentiable and has compact support in $G\}$.

Let $u, v \in C_0^\infty(G)$ then,

$$(lu,v) = \int_G \left[-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij} \frac{\partial u}{\partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu \right] \bar{v} \, dx$$

$$= - \int \sum_{i,j=1}^{n} a_{ij} \underbrace{\cancel{\frac{\partial u}{\partial x_j}}}_{=0} \cos(n \cdot x_i) \bar{v} \, ds + \int_G \sum_{i,j=1}^{n} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial \bar{v}}{\partial x_i} \, dx$$

$$+ \int_G \left[\sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} \bar{v} + cu\bar{v} \right] dx$$

We define the bilinear form $J[u,v] = 1/2 \, [(lu,v) + (u,lv)]$ for all $u,v \in C_0^\infty(G)$.

*See [11] page 9

2.13 $J[u,v] =$

$$1/2\,[(lu,v) + (u,lv)] = 1/2 \int_G \left[\sum_{i,j}^{n} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial \bar{v}}{\partial x_i} + \sum_{i,j}^{n} a_{ij} \frac{\partial \bar{v}}{\partial x_j} \frac{\partial u}{\partial x_i} \right] dx$$

$$+ 1/2 \int_G \sum_{i=1}^{n} b_i \left(\frac{\partial u}{\partial x_i} \bar{v} + \frac{\partial \bar{v}}{\partial x_i} u \right) dx + \int_G cu\bar{v}\, dx$$

$$= \int_G \sum_{i,j}^{n} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial \bar{v}}{\partial x_i}\, dx + 1/2 \int_G \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} (u\bar{v})\, dx + \int_G cu\bar{v}\, dx$$

$$= \int_G \sum_{i,j}^{n} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial \bar{v}}{\partial x_i}\, dx - 1/2 \int_G \sum_{i=1}^{n} \frac{\partial b_i}{\partial x_i} u\bar{v}\, dx + \int_G cu\bar{v}\, dx$$

$$= \int_G \sum_{i,j=1}^{n} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial \bar{v}}{\partial x_i}\, dx + \int_G \left[c - 1/2 \sum_{i=1}^{n} \frac{\partial b_i}{\partial x_i} \right] u\bar{v}\, dx$$

We have assumed that c is bounded from below and $\frac{\partial b_i}{\partial x_i}$ is bounded, so that $c - 1/2 \sum_{i=1}^{n} \frac{\partial b_i}{\partial x_i}$ is bounded from below. Let $K > -\infty$ be a lower bound for $c - 1/2 \sum_{i=1}^{n} \frac{\partial b_i}{\partial x_i}$. $J[u,u]$ is a hermitian bilinear

form bounded from below, i.e. $J[u,u] \geq \int_G \sum_{i,j}^{n} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial \bar{u}}{\partial x_i} + K\|u\|^2 > -\infty$

for all $u \epsilon C_0^\infty(G)$.

2.2 By adding a suitable constant to J we define a positive form and inner product $\langle .,. \rangle$ with the same domain as J. We now extend the domain of J to a Hilbert space $\mathcal{H}_0(G) \subset \mathcal{L}^2(G)$.

Let M be a positive constant such that for $u \epsilon C_0^\infty(G)$

$$J[u,u] + M\|u\|^2 \geq \|u\|^2 .$$

Define an inner product on $C_0^\infty(G)$

$$\langle u,v \rangle \equiv J[u,v] + M(u,v).$$

We wish to show that we may complete $C_0^\infty(G)$ with $\langle u,v \rangle$ as the inner product.

2.21 <u>Definition</u>. We say a form J is closed in a Hilbert space H if $\{u_k\} \epsilon$ domain of J, and $u_k \to u \epsilon H$, as $k \to \infty$ and $J[u_k - u_s] \to 0$, for $k, s \to \infty$ implies $u \epsilon$ domain of J and $J[u_k - u] \to 0$.

2.22 <u>Definition</u>. Let J_1, J_2 be two forms such that the domain of J_1 is contained in the domain of J_2 and $J_1[u,v] = J_2[u,v]$ for $u,v \epsilon D[J_1]$. Then J_2 is called an extension of J_1.

2.23 <u>Definition</u>. A form J is said to be closable if it has a closed extension.

2.24 <u>Theorem</u>. A necessary and sufficient condition that a form J be closable is that if $u_n \epsilon$ domain of J and $u_n \to 0$ as $n \to \infty$ and $J[u_n - u_m] \to 0$ for $m, n \to \infty$ implies $J[u_n] \to 0$. $(J[u] \equiv J[u,u])$

<u>Proof</u>. See [1] page 13.

2.25 <u>Lemma</u>. J as defined in 2.13 is closable.

<u>Proof</u>. We may assume $J > 0$ or else replace J by $J + M$ in what follows. Let $u_n \in C_0^\infty(G)$ and suppose $u_n \to u$ as $n \to \infty$ and $J[u_n - u_m] \to 0$ as $n, m, \to \infty$. For every $\varepsilon > 0$ there exists an N such that if $m, n > N$

$$\varepsilon \geq J[u_n - u_m] = J[u_n] + J[u_m] - 2 \operatorname{Re} J[u_n, u_m] > 0. \text{ As } J[u_n] > 0,$$

$$\varepsilon \geq J[u_m] - 2 \operatorname{Re} J[u_n, u_m].$$

For fixed m let $n \to \infty$ then $u_n \to 0$ and

$$|J[u_n, u_m]| \leq \left| \int_G \sum_{i,j=1}^{n} a_{ij} \frac{\partial u_n}{\partial x_j} \frac{\partial \bar{u}_m}{\partial x_i} \, dx \right| + \left| \int_G \left[c - 1/2 \sum_{i=1}^{n} \frac{\partial b_i}{\partial x_i} \right] u_n \bar{u}_m \, dx \right|$$

$$= \left| -\int_G \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} a_{jj} \frac{\partial \bar{u}_m}{\partial x_i} u_n \, dx + \int_{\partial G} \sum_{i,j=1}^{n} \underbrace{\frac{\partial u_m}{\partial x_i}}_{\to 0} u_n \cos(nx_j) \, ds \right|$$

$$+ \left| \int_G [c - 1/2] \sum_{i=1}^{n} \frac{\partial b_i}{\partial x_i} u_n \bar{u}_m \, dx \right|$$

Using Schwartz's inequality, we obtain

$$|J[u_n, u_m]| \leq \| u_n \| \left[\left(\int_G \left(\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} a_{ij} \frac{\partial \bar{u}_m}{\partial x_i} \right)^2 dx \right)^{1/2} \right.$$

$$\left. + \left(\int_G \left([c - 1/2 \sum \frac{\partial b_i}{\partial x_i}] \bar{u}_m \right)^2 dx \right)^{1/2} \right] \to 0 \text{ as } n \to \infty.$$

So that $\operatorname{Re} J[u_n, u_m] \to 0$ as $n \to \infty$. Therefore $\varepsilon \geq J[u_m]$ for $m > N$.

Q.E.D.

2.26 <u>Definition</u>. $\mathcal{M}_0(G) = \{u(x) \mid u \in \mathcal{L}^2(G)$ and there exists a sequence $\{v_n\} \in C_0^\infty(G)$ for which $\langle u-v_n, u-v_n\rangle > \to 0$ as $n \to \infty\}$.

The closure of J has $\mathcal{M}_0(G)$ as its domain.

2.3 We will use the following theorem due to Lax and Milgram to show that 1 can be extended to a closed operator L densely defined in $\mathcal{L}^2(G)$.

2.31 <u>Theorem</u>. If $F(u,v)$ is a bilinear form in a Hilbert space $\mathcal{M}_0$ satisfying $|F(u,u)| \geq c_0 \|u\|^2_{\mathcal{M}_0}$, $c_0 > 0$

and $$|F(u,v)| \leq c_1 \|u\|_{\mathcal{M}_0} \|v\|_{\mathcal{M}_0}$$

then every linear functional $\langle u,v\rangle$ in $\mathcal{M}_0$ can be represented in the form $\langle u,v\rangle = F(u,w)$, where w depends only on v and is unique. Also, there exists a bounded linear operator S such that $Sv=w$.

<u>Proof</u>. See [2] page 169.

2.32 Lemma. For $u, v \in \mathcal{M}_0$, let $F(u,v) = (lu,v) + \lambda(u,v)$

$$= \int_G \sum_{i,j=1}^{n} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial \bar{v}}{\partial x_i}\, dx + \int_G \left[\sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} \bar{v} + (c+\lambda) u\bar{v} \right] dx$$

We wish to show that for λ sufficiently large

i) $|F(u,u)| \geq c_0 \| u \|^2_{\mathcal{M}_0}, c_0 > 0$

ii) $|F(u,v)| \leq c_1 \| u \|_{\mathcal{M}_0} \| v \|_{\mathcal{M}_0}$.

<u>Proof:</u> By 2.13, we have that $Re(lu,u) = J[u]$ for all $u \in C_0^\infty(G)$.

Then $|F(u,u)| = |(lu,u) + \lambda (u,u)| \geq |Re[(lu,u) + \lambda (u,u)]| =$

$|J[u] + \lambda (u,u)|$.

Choose $\lambda > M$ as M was defined 2.2. Then,

$$|F(u,u)| \geq J[u] + \lambda \| u \|^2 > J[u] + M \| u \|^2 = \| u \|^2_{\mathcal{M}_0}.$$

Therefore $c_0 = 1$.

ii) we must show

$$|(lu,v) + \lambda (u,v)| \leq c_1 \left[\left(\| u \|_1^2 + \| u \|_2^2 + M \| u \|^2 \right) \left(\| v \|_1^2 + \| v \|_2^2 + M \| v \|^2 \right) \right]^{1/2}$$

where

2.33 $$\| u \|_1^2 \equiv (u,u)_1 = \int_G \sum_{i,j=1}^{n} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial \bar{u}}{\partial x_i} dx$$

2.34 $$\| u \|_2^2 \equiv (u,u)_2 = \int_G \left[c - 1/2 \sum_{i=1}^{n} \frac{\partial b_i}{\partial x_i} \right] | u |^2 dx$$

$$= \int_G c | u |^2 dx - 1/2 \int_G \sum_{i=1}^{n} \frac{\partial b_i}{\partial x_i} | u |^2 dx.$$

Since each $\frac{\partial b_i}{\partial x_i}$ is bounded in $\bar{G}$ we can find B such that

$B \geq \frac{n}{2} \max_{\substack{i=1,\cdots,n \\ x\epsilon G}} \left\{ \frac{\partial b_i}{\partial x_i} \right\}$. Then

$$\int_G c|u|^2 dx = \|u\|_2^2 + 1/2 \int_G \sum_{i=1}^n \frac{\partial b_i}{\partial x_i} |u|^2 dx < \|u\|_2^2 + B\|u\|^2 .$$

Thus

2.35

$$\int_G (c+\lambda)|u|^2 dx \leq \|u\|_2^2 + (B+\lambda)\|u\|^2$$

For λ sufficiently large and $x\epsilon G$ $c(x) + \lambda > 0$. Then

2.36

$$\left| \int_G (c+\lambda) u\bar{v}\, dx \right|$$

$$\leq \left(\int_G (c+\lambda)|u|^2 dx \right)^{1/2} \left(\int_G (c+\lambda)|v|^2 dx \right)^{1/2}$$

$$\leq \left(\|u\|_2^2 + (B+\lambda)\|u\|^2 \right)^{1/2} \left(\|v\|_2^2 + (B+\lambda)\|v\|^2 \right)^{1/2} .$$

For $u,v\epsilon C_0^\infty(G)$, we have

$$0 \leq \int_G \sum_{i,j=1}^n a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial \bar{u}}{\partial x_i} dx = -\int_G \sum_{i,j=1}^n \frac{\partial}{\partial x_i} a_{ij} \frac{\partial u}{\partial x_j} \bar{u}\, dx$$

so that the operator $-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij} \frac{\partial}{\partial x_j}$ is positive and hermitian.

Therefore, by Schwartz's inequality for positive forms,

$$2.37 \quad \left| \int_G \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij} \frac{\partial u}{\partial x_j} \bar{v} \, dx \right|$$

$$\leq \left(\int_G \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij} \frac{\partial u}{\partial x_j} \bar{u} \, dx \right)^{1/2} \left(\int_G \sum_{i,j=1}^{n} a_{ij} \frac{\partial v}{\partial x_j} \bar{v} \, dx \right)^{1/2}$$

$$= \left(\int_G \sum_{i,j=1}^{n} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial \bar{u}}{\partial x_i} \, dx \right)^{1/2} \left(\int_G \sum_{j=1}^{n} a_{ij} \frac{\partial v}{\partial x_j} \frac{\partial \bar{v}}{\partial x_i} \, dx \right)^{1/2}$$

$$= \sqrt{\| u \|_1^2 \, \| v \|_1^2} = \| u \|_1 \, \| v \|_1 .$$

We assumed that $\max_{i=1,\ldots n} \{b_i^2(x)\} \leq \dfrac{\det(a_{ij}(x))}{\prod_{j=1}^{n} a_{jj}(x) \sum_{i=1}^{n} \frac{1}{a_{ii}}(x)}$, but

(see the appendix) this is a lower bound for the smallest eigenvalue of $(a_{ij}(x))$. Let $d_{ii}(x)$ be the eigenvalues of $(a_{ij}(x))$. Therefore,

$\max_{i=1\ldots n} \{b_1^2(x)\} \leq \min_{i=1\ldots n} \{d_{ii}(x)\}$. We will make use of the following

inequality $\left\| \sum_{i=1}^{n} A_i \right\|^2 \leq 2^{n-1} \sum_{i=1}^{n} \| A_i \|^2$.

Then

$$2.38 \quad \left|\left(\sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} , v \right)\right|^2 \leq \|v\|^2 \left\| \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} \right\|^2 \leq 2^{n-1} \|v\|^2 \sum_{i=1}^{n} \left\| b_i \frac{\partial u}{\partial x_i} \right\|^2$$

$$= 2^{n-1} \|v\|^2 \sum_{i=1}^{n} \int_G b_i^2 \left|\frac{\partial u}{\partial x_i}\right|^2 dx \leq 2^{n-1} \|v\|^2 \int_G \sum_{i=1}^{n} (\min d_{ii}) \left|\frac{\partial u}{\partial x_i}\right|^2 dx$$

$$\leq 2^{n-1} \|v\|^2 \int \sum_{i,j=1}^{n} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial \bar{u}}{\partial x_j} dx = 2^{n-1} \|v\|^2 \|u\|_1^2$$

Finally we combine 2.35, 2.36, 2.37 and 2.38, obtaining

$$|F(u,v)|^2$$

$$\leq 4\left[\left| \int_G \sum_{i,j=1}^{n} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial \bar{v}}{\partial x_j} dx \right|^2 + \left| \int_G \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} \bar{v} \, dx \right|^2 + \left| \int_G (c+\lambda) u \bar{v} dx \right|^2 \right]$$

$$\leq 4\left[\|u\|_1^2 \|v\|_1^2 + 2^{n-1} \|v\|^2 \|u\|_1^2 + (\|u\|_2^2 + (B+\lambda) \|u\|^2) (\|v\|_2^2 + (B+\lambda) \|v\|^2 \right]$$

$$\leq 2^{n+1} (B+\lambda)^2 \left[(\|u\|_1^2 + \|u\|_2^2 + \|u\|^2) (\|v\|_1^2 + \|v\|_2^2 + \|v\|^2) \right]$$

$$\leq 2^{n+1} (B+\lambda)^2 \left[(\|u\|_1^2 + \|u\|_2^2 + M\,\|u\|^2)\ (\|v\|_1^2 + \|v\|_2^2 + M\,\|v\|^2) \right]$$

Thus $|F(u,v)| \leq \sqrt{2^{n+1}}\ (B+\lambda) \left[(\|u\|_1^2 + \|u\|_2^2 + M\,\|u\|^2)(\|v\|_1^2 + \|v\|_2^2 + M\|v\|^2 \right]^{1/2}$

Let $c_1 > (B+\lambda)\sqrt{2^{n+1}}$. Therefore $|F(u,v)| \leq c_1\ \|u\|_{\mathcal{M}_0}\ \|v\|_{\mathcal{M}_0}$.

We have shown that the bilinear form defined by 2.32 satisfies Theorem 2.31 when $\lambda > \max.\ \{M, \min_{x \in G} |c(x)|\}$.

Q.E.D.

2.4 <u>Lemma</u>. If 1, F(u,v), and $\mathcal{M}_0$ are as previously defined, then there exists a closed operator L which is: a restriction of 1, densely defined in $\mathcal{L}^2$ and has a left half-plane in its resolvent set.

<u>Proof</u>. Let $\lambda > \max.\ \{M, \min_{x \in G} |c(x)|\}$. For $u \in \mathcal{M}_0$, each element $w \in \mathcal{L}^2(G)$ induces a linear functional (w,u) in $\mathcal{M}_0$. Since $(w,u) \leq \|w\|\ \|u\| \leq \|w\|\ \|u\|_{\mathcal{M}_0}$, (w,u) is a bounded linear functional on $\mathcal{M}_0$ for each $w \in \mathcal{L}^2(G)$. By Theorem 2.31 there exists a unique $v \in \mathcal{M}_0$ so that

2.41 $\qquad (w,u) = F(v,u) = ((1+\lambda)v,u)$ for all $u \in \mathcal{M}_0$.

2.42 Denote the dependence of v on w by $v = Sw$. S is linear and sends all of $\mathcal{L}^2(G)$ into a subset D of $\mathcal{M}_0$. S has a single valued inverse. Suppose $Sw = 0$, $w \in \mathcal{L}^2(G)$, then for all $u \in \mathcal{M}_0$, $(w,u) = F(Sw,u) = F(0,u) = 0$. Therefore $w = 0$ and S^{-1} exists and the domain of S^{-1} is D. S is bounded. For all $w \in \mathcal{L}^2(G)$, $Sw \in \mathcal{M}_0$ and

$$c_0\ \|Sw\|^2_{\mathcal{M}_0} \leq |F(Sw,Sw)| = |(w,Sw)| \leq \|w\|\ \|Sw\| \leq \|w\|\ \|Sw\|_{\mathcal{M}_0}.$$

Therefore $\|Sw\|_{\mathcal{H}_0} \leq \frac{1}{c_0}\|w\|$, and as $\|Sw\| \leq \|Sw\|_{\mathcal{H}_0}$, it follows that $\|Sw\| \leq \frac{1}{c_0}\|w\|$. S is also bounded in $\mathcal{H}_0$ since if $w \in \mathcal{H}_0$ we have that

$$\|Sw\|_{\mathcal{H}_0} \leq \frac{1}{c_0}\|w\| \leq \frac{1}{c_0}\|w\|_{\mathcal{H}_0}.$$

2.43 $D \supseteq C_0^\infty(G)$ and so D is dense in $\mathcal{H}_0$. Suppose $u, v \in C_0^\infty(G)$; we have $(1 + \lambda)\ v = w \in \mathcal{L}^2(G)$. Then $F(v,u) = ((1 + \lambda)v,u) = (w,u) = F(Sw,u)$ and $Sw = v$ i.e. $v \in D$.

2.44 S^{-1} is closed. Let $g_n \in$ domain of S^{-1} and $g_n \to g$ in $\mathcal{H}_0$ and $Sg_n^{-1} \to h$ in $\mathcal{L}^2(G)$. For $u \in \mathcal{H}_0$ and $h \in \mathcal{L}^2(G)$ we have $(h,u) = G(v,u)$. v is unique and $Sh = v \in D$. $F(v,u) = (h,u) = \lim_{n \to \infty} (Sg_n^{-1}, u) = \lim_{n \to \infty} F(g_n, u) = F(g,u)$.

Therefore $v = g$, and $g \in$ domain of S^{-1} and $S^{-1}\ g = h$.

2.45 We now define an operator L on D. Let $L \equiv S^{-1} - \lambda$ or $S^{-1} = (L + \lambda I)$, and $S = (L + \lambda I)^{-1}$. L is independent of λ and an extension of 1. For $u \in D$ and $v \in \mathcal{H}_0$, $((L + \lambda)u,v) = F(u,v) = ((1 + \lambda)u,v)$, Therefore $(Lu,v) = (1u,v)$.

L is closed and densely defined since S^{-1} is closed and densely defined. As $(L + \lambda)^{-1}$ is bounded and defined in all $\mathcal{L}^2$ we see that L has a left half-plane in its resolvent set.

3. <u>A Condition for the Discreteness of the Spectrum</u>. For the purpose of proving the main result of this section, we shall introduce a coordinate system in which the coefficient matrix $(a_{ij}(x))$ is diagonal. Although in general we cannot find such a coordinate system, we shall discuss some cases in which we can. Let

$$3.1 \qquad L = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij} \frac{\partial}{\partial x_j}$$

and $y_i = y_i(x)$ be n independent functions of x. Upon making a change of variables, the differential operator assumes the form

$$L = \sum_{k,\ell=1}^{n} \frac{\partial}{\partial y_k} \alpha_{k\ell} \frac{\partial}{\partial y_\ell}$$

where $\alpha_{k\ell} = \sum_{i,j=1}^{n} \frac{\partial y_k}{\partial x_i} a_{ij} \frac{\partial y_\ell}{\partial x_i}$. If L is to be diagonal, we must have for $k \neq \ell$

$$3.11 \qquad 0 = \alpha_{k\ell} = \bar{\alpha}_{\ell k} = \sum_{i,j=1}^{n} \frac{\partial y_k}{\partial x_i} a_{ij} \frac{\partial y_\ell}{\partial x_j} .$$

This imposes $1/2\ (n^2-n)$ conditions upon the new variables.

For $n = 2$ the system 3.11 is underdetermined, and we may impose the further condition that $y_2 = x_2$. This last condition leaves the singular boundary unchanged, and 3.11 is reduced to the homogeneous equation $a_{12} \frac{\partial y_1}{\partial x_1} + a_{22} \frac{\partial y_1}{\partial x_2} = 0$. The solution of this equation is given by $y_1 = f(\phi)$ where f is an arbitrary function and $\phi(x_1, x_2) =$ constant is a solution of the equation $a_{22}\, dx_1 = a_{12}\, dx_2$.

For $n = 3$ the system 3.11 is determined, but solution of the three equations presents a formidable problem. We may simplify the equations somewhat by first solving the system with conditions $\alpha_{13} = \alpha_{23} = 0$ and $y_3 = x_3$. The resulting first order equations can in general be solved.* However, once this is done, it is still necessary to solve a system comparable to the original system.

For $n > 3$ the system 3.11 is overdetermined and can only be solved under particular circumstances.

3.2 We will now develop a condition which, when satisfied, will enable us to diagonalize the coefficient matrix. From conditions i, ii of section 2, it will be shown that the eigenvalues and eigenvectors of $(a_{ij}(x))$ are continuous functions on G. We will then relate the components of the eigenvectors to the derivatives of the transformation.

The continuity of the eigenvalues of $(a_{ij}(x))$ can be deduced from the following more general theorem concerning the continuity of the roots of an equation as a function of parameters.

3.21 <u>Theorem</u>: Let A be an open set in the complex plane C, F a metric space, f a continuous complex valued function in $A \times F$, such that for each $x_0 \epsilon F$, $f(\lambda, x_0)$ is analytic in A. Let B be an open subset of A, whose closure $\bar{B}$ in C is compact and contained in A, and let $x_0 \epsilon F$ be such that no zero of $f(\lambda, x_0)$ is on the frontier of B. Then there exists a neighborhood W of x_0 in F such that:

1° for any $x \epsilon W$, $f(\lambda, x)$ has no zeros on the frontier of B.

2° for any $x \epsilon W$, the sum of the orders of the zeros of $f(\lambda, x)$ belonging to B is independent of x.

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*See [11] page 10.

Proof. See [3] page 243.

The preceding theorem is applied in the following way.

$$f(\lambda,x) = \text{determinant of } (a_{ij}(x) - \lambda I)$$

$$= \alpha_0(x) + \alpha_1(x)\lambda + \alpha_2(x)\lambda^2 + \dots + \alpha_{n-1}(x)\lambda^{n-1} + \lambda^n.$$

Since each of the $\alpha_1(x)$ are sums and products of $a_{ij}(x)$, the $\alpha_i(x)$ are continuous functions of x. Moreover as $(a_{ij}(x))$ is hermitian, the roots of the polynomial $f(\lambda, x_0)$ are all real. Consequently, we have that $\alpha_i(x)$ are real valued functions of x. Then at each x_0, $f(\lambda,x_0)$ is a polynomial in λ with real coefficients and hence is analytic.

For the open set A we choose all of C. Let $x_0 \epsilon G$ and $\lambda_1(x_0)$, $\lambda_2(x_0)$, ... , $\lambda_n(x_0)$ be the n distinct roots of $f(\lambda,x_0)$. Let $\lambda_i(x_0)$ be any root of $f(\lambda,x_0)$ and $d = \min_{j \neq k} |\lambda_j(x_0) - \lambda_k(x_0)|$. Let $\mathcal{E} > 0$ be given, define $\varepsilon' = \min \left\{ \mathcal{E}, \frac{d}{2} \right\}$ and $B = \left\{ \lambda \,\middle|\, |\lambda - \lambda_i(x_0)| < \mathcal{E}' \right\}$. Then by 2°, there is a neighborhood W of x_0 such that for any $x_1 \epsilon W$ there is one and only one $\lambda(x_1) \epsilon B$, a root of $f(\lambda,x_1)$ which we call $\lambda_i(x_1)$. Therefore for $x \epsilon W$, $|\lambda_i(x_1) - \lambda_i(x_0)| < \mathcal{E}$.

Q.E.D.

From the continuity of the eigenvalues, it is easy to show the continuity of the normalized eigenvectors. For convenience we will sometimes use the notation $A(x)$ for $(a_{ij}(x))$.

3.22 Lemma. Suppose $A(x)$ is a hermitian $n \times n$ matrix each of whose components $a_{ij}(x)$ is a continuous function of x in G. If at each point $x \epsilon G$

$A(x)$ has n distinct eigenvalues, then the components of each eigenvector are continuous functions of x in G.

Proof. We have already indicated that with the above conditions each of the eigenvalues of $A(x)$ are continuous functions of x.

Suppose $\lambda(x)$ is an eigenvalue of $A(x)$, and $(\xi_1(x),\ldots,\xi_n(x))$ is the normalized eigenvector corresponding to $\lambda(x)$. If all the eigenvalues are distinct, then the rank of $(A(x) - \lambda(x)I)$ is $n-1$, and a solution of the homogeneous system of equations $(A(x) - \lambda(x)I)\ \xi'(x) = 0$ is given by

$$\xi_1'(x) = k\ (A(x) - \lambda(x)I)_{i1}$$

$$\xi_2'(x) = k\ (A(x) - \lambda(x)I)_{i2}$$

$$\vdots$$

$$\xi_n'(x) = k\ (A(x) - \lambda(x)I)_{in}$$

for some i, $i = 1, 2,\ldots n$, where k is a constant and $(A(x) - \lambda(x)I)_{ij}$ is the cofactor of the component in the ith row and jth column of $(A(x) - \lambda(x)I)$. ♦

Since each of the cofactors $(A(x) - \lambda(x)I)_{ij}$ are sums, differences and products of continuous functions of x, each of the components $\xi_i'(x)$ are also continuous in x.

♦ See [4] page 12.

Then

$$\xi_i(x) = \frac{k\ \xi_i'(x)}{\left(\sum_{j=1} k^2\ (\xi_j'(x))^2\right)^{1/2}} = \frac{\xi_i'(x)}{\left(\sum_{j=1}^{n} (\xi_j'(x))^2\right)^{1/2}}.$$

Since not all $\xi_i'(x) = 0$, we have that $\xi_i(x)$ is the quotient of two continuous functions whose denominator is never zero. Thus $\xi_i(x)$ is continuous.

Q.E.D.

Actually, for our purpose, it is only necessary that the n eigenvalues of $(a_{ij}(x))$ be distinct in some $G_\delta = \{x = (x_1, x_2, \cdots, x_n) \mid x \epsilon G$ and $0 < x_n < \delta\}$ rather than in all G. This last statement will be proven in 3.4.

Let $A(x) = (a_{ij}(x))$, and let $S(x) = (s_{ij}(x))$ be the matrix whose columns are the normalized eigenvectors of $A(x)$. For convenience we will write A for $A(x)$ and S for $S(x)$ where the functional dependence of A and S on x is understood. As A is hermitian, $S^{-1} = S^* = (s_{ij}(x))^* = (s_{ji}(x))$. We have $S^{-1} AS = D = (d_{ii}(x))$ where D is diagonal and $d_{ii}(x)$ are the eigenvalues of A. For $k \neq \ell$

3.23 $$0 = d_{k\ell}(x) = \sum_{i,j=1}^{n} s_{ik}(x)\ a_{ij}(x)\ s_{j\ell}(x)\ .$$

At each point of G, S^{-1} is a transformation of the coordinate system to the principal axes of the ellipse $\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j = 1$.

We now introduce a new condition.

vi) At each point $x \epsilon \overline{G}$ there exists an invertible transformation of coordinates $y_i = y_i(x)$, $i = 1,2,\ldots, n$, which for $x \epsilon G$ satisfies the equations

3.24 $$\frac{\partial y_j}{\partial x_i} = s_{ij}(x), \text{ as } s_{ij}(x) \text{ was defined above.}$$

By 3.23 and 3.24, $0 = d_{k\ell} = \sum_{i,j=1}^{n} \frac{\partial y_k}{\partial x_i} a_{ij} \frac{\partial y_\ell}{\partial x_j}$. Thus condition 3.11, required for the coefficient matrix of the principal part of L to be diagonal, is satisfied. Then, in this curvilinear coordinate system,

$$L = -\sum_{i=1}^{n} \frac{\partial}{\partial y} d_{ii}(y) \frac{\partial}{\partial y} + \sum_{i=1}^{n} b_i'(y) \frac{\partial}{\partial y} + c(y) ,$$

where $b_i'(y) = \sum_{j=1}^{n} b_j(y) s_{ji}(y)$. It is evident that b_i' still satisfies condition v of section 2.02.

In order that $\frac{\partial y_i}{\partial x_j} = s_{ji}$ be satisfied exactly, it is necessary that $\frac{\partial s_{ji}}{\partial x_k} = \frac{\partial s_{ki}}{\partial x_j}$ for all combinations of x_k and x_j.

An alternate condition to 3.24 is

$$\frac{\partial y_i}{\partial x_j} = k(x)\ s_{ji}(x). \tag{3.25}$$

From the theory of exact equations, the condition for the existence of an integrating factor $k(x)$ for $y_i(x)$ is that each of the equations

$$0 = s_{ir}\left[\frac{\partial s_{im}}{\partial x_p} - \frac{\partial s_{ip}}{\partial x_m}\right] + s_{im}\left[\frac{\partial s_{ip}}{\partial x_r} - \frac{\partial s_{ir}}{\partial x_p}\right] + s_{ip}\left[\frac{\partial s_{ir}}{\partial x_m} - \frac{\partial s_{im}}{\partial x_r}\right] \tag{3.26}$$

be satisfied for all combinations of the indices m, p, r from the series 1, 2, 3,..., n and of these 1/2 (n-1) (n-2) are independent.* As we wish have the same integrating factor for each y_i , 3.26 must hold for $i = 1,2,\ldots, n$. If $k(x)$ can be found, then the transformed operator will have the form

$$L = -\sum_{i=1}^{n} \frac{\partial}{\partial y_i}\ k^2(y)\ d_{ii}(y)\ \frac{\partial}{\partial y_i} + \sum_{i=1}^{n} k(y)\ b^1(y)\ \frac{\partial}{\partial y_1} + c(y).$$

The results developed in the remainder of this section assume that 3.24 holds. However, substantially the same results are obtainable if we assume 3.25 instead of 3.24.

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*See [5] page 12.

We note that condition vi holds, whenever s_{ij} is constant, for all i and j. For example, consider the following matrix:

$$(a_{ij}(x)) = \begin{pmatrix} 1 + x_2 & \frac{1}{1 + x_2} \\ \frac{1}{1 + x_2} & 1 + x_2 \end{pmatrix}$$

$(a_{ij}(x))$ is positive definite for $x_2 > 0$, and by some simple calculations,

it follows that $d_{11} = \dfrac{1 + 2x_2 + x_2^2}{1 + x_2}$, $d_{22} = \dfrac{2x_2 + x_2^2}{1 + x_2}$

and $(s_{ij}(x)) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

3.3 In order to simplify the notation used while singling out a particular coordinate, we will denote the point $y = (y_1, y_2, \ldots y_n)$ by $(\bar{y}_i, y_i)$, where $\bar{y}_i = (y_1, y_2, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n)$. Accordingly, the volume element dy is $d\bar{y}_i \, dy_i$ where $d\bar{y}_i = dy_1 \ldots dy_{i-1} \, dy_{i+1} \ldots dy_n$,

and $\displaystyle\int_G f(y)\, dy = \int_{G_{\bar{y}_i}} \left[\int_{a_i}^{b_i} f(y)\, dy_i \right] d\bar{y}_i$ where a_i and b_i are the appropriate limits of integration along the curve y_i.

3.31 <u>Lemma</u>. Let $d_{ii}(y) > 0$ in G and $d_{ii}(y) \geq 0$ for y on ∂G and $u(y) = 0$ for y on ∂G. Then there exists a positive function $\tau_i(y)$ such that

$$\int_G d_{ii}(y) \left|\frac{\partial u}{\partial y_i}\right|^2 dy \geq 1/4 \int_G \frac{|u|^2\, dy}{\tau_i^2(y)\, d_{ii}(y)}$$

<u>Proof</u>. With the above notation,

$$3.32 \qquad \int_G d_{ii}(y) \left|\frac{\partial u}{\partial y_i}\right|^2 dy = \int_{G_{\bar{y}_i}} \left[\int_{a_i}^{b_i} d_{ii}(y) \left|\frac{\partial u}{\partial y_i}\right|^2 dy_i\right] d\bar{y}_i \, .$$

As the y coordinates are curvilinear, the singularities which occur for $x_n = 0$ may appear in either a_i, b_i or both a_i and b_i. Let $a_i < m_i < b_i$.

$$3.33 \ \underline{\text{Definition}}. \ \tau_i^-(y) = -\int_{m_i}^{y_i} \frac{d\xi_i}{d_{ii}(\bar{y}_i, \xi_i)} + k_i \qquad \text{For } a_i \leq y_i \leq m_i \, .$$

$$\tau_i^+(y) = \int_{m_i}^{y_i} \frac{d\xi_i}{d_{ii}(\bar{y}_i \xi_i)} + k_i \qquad \text{For } m_i \leq y_i \leq b_i \, .$$

where $k_i > 0$ is such that when $a_i \leq y_i \leq m_i$, $\tau_i^- > 0$ and when $m_i \leq y_i \leq b_i$, $\tau_i^+ > 0$. $\tau_i^-(y)$ is a decreasing function of y_i, while $\tau_i^+(y)$ is an increasing function of y_i. $\tau_i^-(\bar{y}_i, m_i) = \tau_i^+(\bar{y}_i, m_i) = k_i$, and $\tau_i^-(\bar{y}_i, a_i) > k_i$, $\tau_i^+(\bar{y}_i, b_i) > k_i$.

3.34 <u>Definition</u>.

$$\tau_i(y) = \begin{cases} \tau_i^-(y) & \text{for } a_i \leq y_i \leq m_i . \\ \tau_i^+(y) & m_i \leq y_i \leq b_i . \end{cases}$$

3.35 <u>Definition</u>. $\eta_i(y) = \log \tau_i(y)$. $\eta_j = y_j$, for $j \neq i$.

For $a_i \leq y_i \leq m_i$, $\eta_i(y)$ is a decreasing function of y_i, while for $m_i \leq y_i \leq b_i$, $\eta_i(y)$ is an increasing function of y_i. Also

$$\eta_i(\bar{y}_i, a_i) > \log k_i , \quad \text{and} \quad \eta_i(\bar{y}_i, b_i) > \log k_i .$$

Let $v_i = u\tau_i^{-1/2}$. Then $\dfrac{\partial \eta_i}{\partial \tau_i} = \dfrac{1}{\tau_i}$,

3.36 $$\frac{\partial \eta_i}{\partial y_i} = \frac{\partial \eta_i}{d\tau_i} \frac{\partial \tau_i}{\partial y_i} = \frac{1}{\tau_i} \left(\mp \frac{1}{d_{ii}} \right) = \mp \frac{1}{\tau_i d_{ii}} \quad \text{and}$$

$$\frac{\partial v_i}{\partial y_i} = \frac{\partial v_i}{\partial \eta_i} \frac{\partial \eta_i}{\partial y_i} = \mp \frac{1}{\tau_i d_{ii}} \frac{\partial v_i}{\partial \eta_i}$$

where the minus sign refers to τ in the range $a_i \leq y_i \leq m_i$ and the plus sign refers to τ in the range $m_i \leq y_i \leq b_i$.

$$3.37 \quad \frac{\partial u}{\partial y_i} = \tau_i^{1/2} \frac{\partial v_i}{\partial y_i} + 1/2 \; \tau_i^{-1/2} \frac{\partial \tau_i}{\partial y_i} v_i = \mp \frac{1}{\tau_i^{1/2} d_{ii}} \frac{\partial v_i}{\partial \eta_i} \mp \frac{v_i}{2\tau_i^{1/2} d_{ii}} =$$

$$\mp \frac{1}{d_{ii}\tau^{1/2}} \left(\frac{\partial v_i}{\partial \eta_i} + \frac{v_i}{2}\right) ,$$

From 3.32, 3.36 and 3.37 we have

$$\int_{G_{\bar{y}_i}} \left[\int_{a_i}^{b_i} d_{ii}(y) \left|\frac{\partial u}{\partial y_i}\right|^2 dy_i \right] d\bar{y}_i = \int_{G_{\bar{y}_i}} \left[\int_{a_i}^{b_i} d_{ii} \left(\frac{\tau_i^{1/2}}{d_{ii}}\right)^2 \left(\frac{\partial v_i}{\partial \eta_i} + \frac{v_i}{2}\right)\left(\frac{\partial \bar{v}_i}{\partial \eta_i} + \frac{\bar{v}_i}{2}\right) dy_i \right] d\bar{y}_i$$

$$= \int_{G_{\bar{y}_i}} \left[\int_{a_i}^{b_i} \frac{\left|\frac{\partial v}{\partial \eta_i}\right|^2}{\tau_i d_{ii}} dy_i + 1/4 \int_{a_i}^{b_i} \frac{|v_i|^2}{\tau_i d_{ii}} dy_i + 1/2 \int_{a_i}^{b_i} \frac{\frac{\partial |v_i|^2}{\partial \eta_i}}{\tau_i d_{ii}} dy_i \right] d\bar{y}_i$$

$$= \int_{G_{\bar{y}_i}} \left[\int_{a_i}^{b_i} \frac{\left|\frac{\partial v_i}{\partial \eta_i}\right|^2}{\tau_i d_{ii}} dy_i + 1/4 \int_{a_i}^{b_i} \frac{|v_i|^2}{\tau_i d_{ii}} dy_i - 1/2 \int_{a_i}^{b_i} \frac{\partial |v_i|^2}{\partial \eta_i} \frac{\partial \eta_i}{\partial y_i} dy_i \right] d\bar{y}_i$$

$$= \int_G \frac{\left|\frac{\partial v_i}{\partial \eta_i}\right|^2}{\tau_i d_{ii}} dy + 1/4 \int_G \frac{|u|^2}{\tau_i^2 d_{ii}} dy + 1/2 \int_{G_{\bar{y}_i}} \left[\left|\frac{u(a_i)}{\tau_i^{1/2}(a_i)}\right|^2 - \left|\frac{u(b_i)}{\tau_i^{1/2}(b_i)}\right|^2 \right] d\bar{y}_i .$$

a_i and b_i are on ∂G so that $\dfrac{u(a_i)}{\tau_i^{1/2}(a_i)} = \dfrac{u(b_i)}{\tau_i^{1/2}(b_i)} = 0$. For $y \epsilon G$,

$\dfrac{\left|\dfrac{\partial v_i}{\partial \eta_i}\right|^2}{\tau_i d_{ii}} \geq 0$, and $\displaystyle\int_G \frac{\left|\dfrac{\partial v_i}{\partial \eta_i}\right|^2}{\tau_i d_{ii}}\, dy \geq 0$. Consequently,

$$\int_G d_{ii}(y) \left|\frac{\partial u}{\partial y_i}\right|^2 dy_i \geq 1/4 \int_G \frac{|u(y)|^2}{\tau_i^2(y)\, d_{ii}(y)}\, dy .$$

Q.E.D.

When we apply the inequality to each of the terms of

$\displaystyle\int_G \sum_{i=1}^{n} d_{ii}(y) \left|\frac{\partial u}{\partial y_i}\right|^2 dy$, we get:

3.38 $$\int_G \sum_{i=1}^{n} d_{ii}(y) \left|\frac{\partial u}{\partial y_i}\right|^2 dy \geq 1/4 \int_G \sum_{i=1}^{n} \frac{|u|^2}{\tau_i^2\, d_{ii}}\, dy.$$

3.4 Definition. The essential spectrum of an operator A, $\sigma_e(A)$, consists of the points λ for which the range of $A-\lambda I$ is not closed and of eigenvalues of infinite multiplicity.

3.41 Definition. An operator A has a singular sequence $\{u_n\}$ if the sequence $\{u_n\}$ is normalized and has no convergent subsequence and there is a λ such that $\lim_{n\to\infty}(A-\lambda I)u_n = 0$.

3.42 Theorem. (Wolf) Let A be closed and densely defined. Then A has a singular sequence corresponding to $\lambda = 0$ iff the range of A is not closed or the null space of A is infinite.

Proof. See [6] page 215.

In section 2 we proved that the operator l defined by 2.01 and 2.02 has an extension L. L is called the Dirichlet operator associated with l, and L is closed and densely defined on $\mathcal{L}^2(G)$. For λ real and sufficiently large, $(L+\lambda I)^{-1}$ exists and is bounded and defined on all $\mathcal{L}^2(G)$.

Let $G_\delta = \{x = (x_1, x_2, \ldots, x_n) \mid x \in G \text{ and } 0 < x_n < \delta\}$. Let L_δ be the Dirichlet operator associated with l on G_δ. The following theorem is a restatement of a theorem by Wolf.

3.43 Theorem. Let L and L_δ be the Dirichlet operators associated with l on G and G_δ respectively. λ_0 is in the essential spectrum of L_δ on G_δ iff λ_0 is in the essential spectrum of L on G.

Proof. See [7] pages 168 and 176.

Because of this theorem, condition iii of section 2.01 and condition vi of 3.2 will be required to hold in some region G_δ rather than in all G.

3.44 <u>Theorem</u>. Let L be the Dirichlet operator associated with 1 on G. The spectrum of L is discrete if

$$\lim_{x_n \to 0} \sum_{i=1}^{n} \frac{1}{\tau_1^2 \, (y(\bar{x}_n, x_n)) \; d_{ii} \; (y(\bar{x}_n, x_n))} = \infty \text{ uniformly in } \bar{x}_n .$$

<u>Proof</u>. Let $\frac{\partial(x)}{\partial(y)}$ be the Jacobian of the transformation of coordinates

$y_i = y_i(x)$. Then $\frac{\partial(x)}{\partial(y)} = \frac{1}{\frac{\partial(y)}{\partial(x)}} = \frac{1}{\det \left(\frac{\partial y_j}{\partial x_i}\right)} = \frac{1}{\det (s_{ij})}$.

But $\det (s_{ij}) = \det (s_{ji}) = \det (s_{ij})^{-1}$, so that $\det (s_{ij}) = 1$. Let $u \in C_0^\infty (G)$.

3.45 Then $|(Lu,u)| \geq \operatorname{Re} [(Lu,u)]$

$$= - \int_G \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij} \frac{\partial u(x)}{\partial x_j} \bar{u}(x) \, dx + \operatorname{Re} \int_G \left[\sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} \bar{u} + c \, |u|^2 \right] dx$$

$$= - \int_{G'} \sum_{i=1}^{n} \frac{\partial}{\partial y_i} d_{ii} \frac{\partial u(y)}{\partial y_i} \overline{u(y)} \frac{\partial(x)}{\partial(y)} \, dy + \int_G \left[\sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} |u|^2 + c \, |u|^2 \right] dx$$

$$= - \int_{G'} \sum_{i=1}^{n} \frac{\partial}{\partial y_i} d_{ii}(y) \frac{\partial u(y)}{\partial y_i} \bar{u}(y) \, dy + \int_G \left[- 1/2 \sum_{i=1}^{n} \frac{\partial b_i}{\partial x_i} + c \right] |u|^2 \, dx$$

$$\geq - \int_{G'} \sum_{i=1}^{n} \frac{\partial}{\partial y_i} d_{ii}(y) \frac{\partial u(y)}{\partial y_i} \bar{u}(y)\, dy - K^{\dagger} \| u \|^2_{\mathcal{L}^2(G)}$$

$$= - |K| \, \|u\|^2_{\mathcal{L}^2(G)} + \int_{G'} \sum_{i=1}^{n} d_{ii}(y) \left|\frac{\partial u(y)}{\partial y_i}\right|^2 dy - \underbrace{\int_{\partial G'} \sum_{i}^{n} d_{ii}(y) \frac{\partial u(y)}{\partial y_i} \bar{u}(y) \cos(n.y) dy}_{=0}$$

$$\geq 1/4 \int_{G'} \sum_{i=1}^{n} \frac{|u(y)|^2}{\tau_i^2(y)\, d_{ii}(y)} dy - |K| \, \|u\|^2_{\mathcal{L}^2(G)} .$$

Let $\lambda_0 \epsilon \sigma_e(L)$. Then there exists a normalized singular sequence $\{u_n\} \epsilon D(L)$ on G such that $\lim_{n \to \infty} Lu_n - \lambda_0 u_n = 0$. For each n there is a sequence of normalized functions

$$\{u'_{kn}\} \epsilon C_0^{\infty}(G) \text{ such that } \langle u_n - u'_{nk}, u_n - u'_{nk} \rangle = J[u_n - u'_{nk}] + M\|u_n - u'_{nk}\|^2$$

$$\to 0 \text{ as } k \to \infty .$$

Construct the sequence $\{u'_n\}$ where for each $n, u'_n \epsilon \{u'_{kn}\}$ and

$$J[u_n - u'_n] + M\| u_n - u'_n\|^2 \leq 1/n .$$

---

† K is defined in section 2.1

$\{u_n'\}$ has no convergent subsequence (otherwise so would $\{u_n\}$).

Also

$$|((L-\lambda_0)u_n' , u_n')| \leq |((L-\lambda_0)u_n , u_n)| + |((L-\lambda_0)u_n, u_n' - u_n)|$$

$$+ |((L-\lambda_0)(u_n' - u_n), u_n)| + |((L-\lambda_0)(u_n' - u_n), u_n' - u_n)| .$$

Let $(L-\lambda_0)u_n = v_n$ and $u_n' - u_n = w_n$. By the definition of a singular sequence $\|v_n\| \to 0$ as $n \to \infty$. By 3.46 $\|w_n\| \to 0$ as $n \to \infty$.

3.47 $\quad |(v_n, u_n)| \leq \|v_n\| \, \|u_n\| = \|v_n\| \to 0$ as $n \to \infty$.

3.48 $\quad |(v_n, w_n)| \leq \|v_n\| \, \|w_n\| \to 0$ as $n \to \infty$.

3.49 $\quad |((L-\lambda_0)w_n, u_n)| \leq |(Lw_n, u_n)| + |\lambda_0| \, |(w_n, u_n)|$.

$|\lambda_0| \, |(w_n, u_n)| \leq |\lambda_0| \, \|w_n\| \, \|u_n\| \to 0$ as $n \to \infty$. By theorem 2.31, $|(Lw_n, u_n) + \lambda(w_n, u_n)| \leq c_1 \, \|w_n\|_{\mathfrak{m}_0} \, \|u_n\|_{\mathfrak{m}_0}$.

$$|(Lw_n, u_n)| \leq c_1 \, (J[w_n] + M \, \|w_n\|^2)^{1/2} \, (J[u_n] + M \, \|u_n\|^2)^{1/2} + |\lambda| \, |(w_n, u_n)|.$$

By 2.13, $J[u_n] = \mathrm{Re}(Lu_n, u_n) \to \lambda_0$ as $n \to \infty$. Thus, $(J[u_n] + M \, \|u_n\|^2)^{1/2} \to (\lambda_0 + M)^{1/2}$ as $n \to \infty$. By 3.46, $(J[w_n] + M \, \|w_n\|^2)^{1/2} \to 0$ as $n \to \infty$. Also, $|\lambda| \, |(w, u_n)| \to 0$ as $n \to \infty$. Consequently, $|(Lw_n, u_n)| \to 0$ as $n \to \infty$.

3.50 If in 3.49 we replace u_n by w_n, we have that $|((L-\lambda_0)\, w_n\, ,\, w_n)| \to 0$ as $n \to \infty$.

Combining 3.47, 3.48, 3.49 and 3.50, we obtain $|((L-\lambda_0)u_n', u_n')| \to 0$ as $n \to \infty$. Thus, $\lim_{n \to \infty} (Lu_n'\, u_n') = \lambda_0$.

Let $N > 0$ be given. There is a $\delta > 0$ such that if $0 < x_n \leq \delta$

$$\sum_{i=1}^{n} \frac{1}{\tau_i^2\, (y(\bar{x}_n, x_n))\, d_{ii}(y(\bar{x}_n, x_n))} \geq N + |K| .$$

Then $$|(L_\delta\, u_n', u_n')| > 1/4 \int_{G'_\delta} \sum_{i=1}^{n} \frac{|u_n'\, (y)|^2}{\tau_i^2\, (y)\, d_{ii}\, (y)}\, dy - |K|\, \|u_n'\|^2_{\mathcal{L}^2(G_\delta)}$$

and as $\| u_n' \|_{\mathcal{L}^2(G_\delta)}$ is normalized, $|(L_\delta\, u_n'\, ,\, u_n')| \geq N + |K| - |K| = N.$

Therefore $|(L_\delta\, u_n', u_n')| \geq N$ for every n; consequently $\lambda_0 \geq N$. As N was arbitrary, $\sigma_e(L_\delta)$ is empty, and the spectrum of L_δ is discrete. By Theorem 3.4.3 the spectrum L is discrete.

Q.E.D.

4. <u>The Method of Majorization</u>. If the operator l is symmetric, the existence theory for l is obtained directly from the Friedrichs extension. As a corollary to theorem 3.44, a condition for discreteness is developed which eliminates the necessity to diagonalize $(a_{ij}(x))$.

4.1 Consider the elliptic operator

$$4.11 \quad l = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j} + c(x).$$

The following conditions are assumed to hold inside a bounded domain G:

4.12 i) $a_{ij}(x)$ are complex valued and of class C^1 in G with $a_{ij}(x) = \overline{a_{ji}(x)}$.

ii) $$\sum_{i,j=1}^{n} a_{ij}(x)\, \xi_i \bar{\xi}_j > 0 \quad \text{for all complex } \vec{\xi} \neq 0 .$$

iii) $c(x)$ is real, continuous and bounded from below on G. Without loss of generality, we may assume $c(x) \geq 0$.

We continue to assume that $G \subset \left\{ x \,|\, x_n > 0 \right\}$, $S \subset \left\{ x \,|\, x_n = 0 \right\}$, and that the boundary of G is sufficiently smooth for Green's Theorem to hold. We will redefine $\mathcal{M}_0(G)$.

4.13 <u>Definition.</u> $\mathcal{M}_0(G) = \Big\{ u(x) \,|\, u \in \mathcal{L}^2(G)$, and there exists a sequence $\{u_n\} \in C_0^\infty(G)$ for which $(l(u-v_n), u-v_n) + (u-v_n, u-v_n) \to 0$ as $n \to \infty \Big\}$.

For $u \in C_0^\infty(G)$, the operator l is non negative and symmetric. The discreteness theory for l follows from the theorem of Friedrichs.

4.14 <u>Theorem (Friedrichs).</u> If l is a positive, symmetric operator on a dense subset C_0^∞ of the Hilbert space $\mathcal{L}^2(G)$, then l has a unique self adjoint extension, L, whose domain is contained in $\mathcal{M}_0(G)$.

<u>Proof.</u> See [8] p. 330

From the Friedrichs Theorem it follows that the self adjoint operator L is a Dirichlet operator. If $(a_{ij}(x))$ is diagonal, then L satisfies the hypothesis of theorem 3.44, and hence that theorem holds for L. If $(a_{ij}(x))$ is not diagonal and:

iv) At every point $x \in G$ the n eigenvalues of the coefficient matrix $(a_{ij}(x))$ are distinct;

v) At every point $x \in \overline{G}$ there exists an invertible transformation of coordinates $y_i = y_i(x)$; $i = 1, \ldots, n$ which for $x \in G$ satisfies the equations

$$\frac{\partial y_j}{\partial x_i} = s_{ij}(x) \text{ as } s_{ij}(x) \text{ was defined in section 3.2;}$$

then L again satisfies the hypothesis of theorem 3.44.

4.2 We now define a second operator on G.

4.21 $$\text{Let } l' = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} a(x) \frac{\partial}{\partial x_i} + c(x).$$

The following conditions are assumed to hold in G.

4.22 i) $a(x)$ is real valued and of class C^1 for all $x \in G$. Furthermore $a(x) > 0$ for $x \in G$.

ii) $$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\bar{\xi}_j \geq a(x)\sum_{i=1}^{n} |\xi_i|^2 \text{ for all } x \in G \text{ and } \vec{\xi} \neq 0.$$

$a_{ij}(x)$ are the same as the functions in 4.12.

iii) $c(x)$ is the same function as found in 4.12.

Denote by L' the Dirichlet operator associated with l' on G.

4.23 Lemma (Rellich). Suppose A is a non-negative self adjoint operator in a Hilbert space H. The spectrum of A is discrete iff every sequence of functions $\{u_i\}$ in the domain of A which satisfies $(Au_i,u_i) + (u_i,u_i) \leq C < \infty$ has a subsequence which is convergent in H.

Proof. See [9] p. 343.

4.24 Lemma If the spectrum of L' is discrete, then the spectrum of L is discrete.

Proof. Suppose the spectrum of L is not discrete. By 4.23 we can choose a sequence $\{u_k\}$ in the domain of L which has no subsequence convergent in $\mathcal{L}^2(G)$ and for which $(Lu_k,u_k) + (u_k,u_k) \leq C$. By the definition of the domain of L we can choose a sequence $\{v_k\} \in C_0^\infty(G)$ for which $(L(u_k-v_k), (u_k-v_k)) + (u_k - v_k, u_k - v_k)) \leq \frac{1}{k}$. $\{u_k\}$ has no convergent sequence in $\mathcal{L}^2(G)$. By ii of section 4.22, $(L'v_k, v_k) + (v_k, v_k) \leq (Lv_k, v_k) + (v_k, v_k) \leq C + 1$. By 4.23, the spectrum of L' is not discrete.

Q.E.D.

In the appendix two positive lower bounds for the least eigenvalue of a positive definite hermitian matrix are derived. We may take either lower bound as $a(x)$, for both satisfy conditions i and ii of 4.22

Let the coefficient matrix $(a_{ij}(x))$ be diagonal for $x \in \overline{G}$, and suppose G is in the canonical position of section 2. In this situation we may simplify the definition of $\tau_i(x)$, which was previously defined in section 3.3.

4.25 <u>Definition</u>. $\tau_i(x) = -\int_{b_i}^{x_i} \frac{1}{a_{ii}(\overline{x}_i, x_i)}\, dx_i + k_i$, for

$a_i \leq x_i \leq b_i$. Where $k_i > 0$ is such that when $a_i \leq x_i \leq b_i$, $\tau_i > 0$.

4.26 <u>Corollary (to theorem 3.44)</u> Let $a(x)$ be either of the lower bounds for the least eigenvalue of $(a_{ij}(x))$ derived in the appendix. Let $\tau_i(x)$ be the function defined by 4.25 where for all i, $a_{ii}(x) = a(x)$.

If $\lim\limits_{x_n \to 0} \frac{1}{a(x)} \sum\limits_{i=1}^{n} \frac{1}{\tau_i^2(x)} = \infty$ uniformly in $\overline{x}_n$ then the spectrum of L is discrete.

<u>Proof</u>. As $a(x)$ is defined, it satisfies the conditions 4.12. By the hypothesis and theorem 3.44, the spectrum of L' is discrete. By Lemma 4.24, the spectrum of L is discrete.

Q.E.D.

4.27 <u>Definition</u>. $\Gamma = \{x \mid -\infty < f_i < x_i < g_i < \infty \ , \ i < n; \ 0 < x_n < g_n < \infty\}$.

4.28 <u>Lemma</u>. Suppose $l' = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} a_i(x) \frac{\partial}{\partial x_i} + c(x)$ is defined in a region $\Gamma \subset G$ where $a_i(x)$ and $c(x)$ satisfy the conditions of 4.12. Suppose also that

$$\text{i)} \qquad \sum_{i=1}^{n} a_i(x) |\xi_i|^2 \geq \sum_{i,j=1}^{n} a_{ij}(x) \xi_i \bar{\xi}_j \quad \text{in } \Gamma \text{ for all } \vec{\xi} .$$

If the spectrum of L' is not discrete then the spectrum of L is not discrete.

<u>Proof</u>. See [11] page 27.

4.29 <u>Theorem</u>. Suppose l' satisfies the conditions of Lemma 4.28 with $\bar{\Gamma} \subset G$ and that for all $i \leq n$, $a_i(x) \equiv a(x)$. Define $\tau(x) \equiv \tau_n(x)$ as $\tau_n(x)$ was defined in 4.25. Suppose further that for $x \in \bar{\Gamma}$:

$$\text{i)} \qquad \lim_{x_n \to 0} a(\bar{x}_n, x_n) = 0 \quad \text{uniformally in } \bar{x}_n .$$

$$\text{ii)} \qquad \frac{\partial a(\bar{x}_n, x_n)}{\partial x_i} \leq C_2 \, a(\bar{x}_i, x_i) \ , \quad i < n.$$

$$\text{iii)} \qquad |c(x)| + \frac{1}{a(x) \, \tau^2(x)} \leq C_3 < \infty :$$

then the spectrum of L is not discrete.

Proof. We will use Lemma 4.23 to prove that the spectrum of L' is not discrete. For this purpose we shall construct a sequence of functions $\{u_k\}$ in the domain of L' which has no convergent subsequence and for which $(L'u_k, u_k) + (u_k, u_k) \leq C < \infty$.

Define the coordinate transformation: $\eta_i = x_i$, $i < n$; $\eta_n = \log \tau$. For $x \in \overline{\Gamma}$ the Jacobian of the transformation is $= -\dfrac{1}{\tau(x)a(x)} \neq 0$. The region Γ is transformed into a region Γ', where $\Gamma' =$

$\{\eta \mid -\infty < f_i \leq \eta_i < g_i < \infty,\ i < n;\ 0 < \log \tau(\overline{x}_n, g_n) < \eta_n < \infty\}$.

Let $b = \sup\limits_{\overline{x}_n \in \overline{\Gamma}} \tau(\overline{x}_n, g_n)$.

Define a sequence of functions $\{w_j\}$ by:

$$
\begin{aligned}
&w_j = 0, && \eta_n \leq j, \\
&w_j = A_j(\eta_n - j), && j \leq \eta_n \leq 2j, \\
4.30 \qquad &w_j = A_j j, && 2j \leq \eta_n \leq 3j, \\
&w_j = A_j(4j - \eta_n), && 3j \leq \eta_n \leq 4j, \\
&w_j = 0, && 4j \leq \eta_n,
\end{aligned}
$$

where A_j is a constant to be defined. Let s be an integer such that $s \geq b$, let $\{w_k\}$ be that subsequence of $\{w_j\}$ for which $k = 1,2,3,\ldots$, and $j = s4^{k-1}$. Note that

$$\int_b^\infty |w_j|^2 \, d\eta_n = \frac{5}{3} A_j^2 j^3. \equiv \underline{4.31}$$

Let $d = \prod_{i=1}^{n-1} (g_i - f_i)$. Define $u_k(\eta) = w_k(\eta_n) \left[\frac{2^{n-1}}{d}\right]^{1/2} \prod_{i=1}^{n-1} \sin \pi \left(\frac{\eta_i - f_i}{g_i - f_i}\right)$.

Define $v_k = \tau^{1/2} u_k$. $v_k \in C_0^\infty(\Gamma')$.

4.32 Let A_k, in w_k, be such that $\int_{\Gamma'} a\tau^2 |u_k|^2 \, d\eta = 1$.

By iii and 4.32

$$1 = \int_{\Gamma'} a\tau^2 |u_k|^2 d\eta \geq \frac{1}{C_3} \int_{\Gamma'} |u_k|^2 d\eta \geq \frac{1}{C_3} \int_{2k}^{3k} A_k^2 k^2 \, d\eta_n = \frac{A_k^2 k^3}{C_3}$$

We now consider $\{v(x)\}$ for $x \in \Gamma$. We shall show $\{v_k(x)\}$ has no convergent subsequence in Γ. By 4.30 and iii,

$$\|v_i - v_j\|^2 = \int_\Gamma |v_i - v_j|^2 dx = \int_{\Gamma'} \tau^2 a |u_i - u_j|^2 d\eta = \int_{\Gamma'} \tau^2 a [|u_i|^2 + |u_j|^2] d\eta \geq \frac{2}{C_3}.$$

Therefore $v_k(x)$ has no convergent subsequence. For convenience the index k will be used only where necessary in the rest of this proof.

$$(L'v, v) + (v, v) = \int_\Gamma \left[\sum_{i=1}^{n} a(x) \left| \frac{\partial v}{\partial x_i} \right|^2 + (c+1)|v|^2 \right] dx .$$

4.33 $\int_\Gamma (c+1)|v|^2 dx = \int_{\Gamma'} a\tau^2 (c+1)|u|^2 \, d\eta \leq (C_3+1) \int_{\Gamma'} a\tau^2 |u|^2 d\eta = C_3 + 1$.

Note that $\frac{\partial \tau}{\partial \eta_n} = \frac{\partial e^{\eta_n}}{\partial \eta_n} = e^{\eta_n} = \tau$. Then, $\frac{\partial v}{\partial x_n} \quad \frac{\partial v}{\partial \eta_n} \frac{\partial \eta_n}{\partial x_n} = -\frac{1}{\tau a} \frac{\partial v}{\partial \eta_n} =$

$= -\frac{1}{\tau a} \left[\tau^{1/2} \frac{\partial u}{\partial \eta_n} \quad \frac{u\tau^{1/2}}{2} \right] = -\frac{1}{a\tau^{1/2}} \left[\frac{\partial u}{\partial \eta_n} + \frac{u}{2} \right]$. We use the inequality

4.35 $|a + b|^2 \leq 2[|a|^2 + |b|^2]$.

By 4.35, 4.31, 4.32 and the definition of u,

$$4.36 \quad \int_{\Gamma} a\left|\frac{\partial v}{\partial x_n}\right|^2 dx = \int_{\Gamma'} \frac{a}{a^2\tau}\left|\frac{\partial u}{\partial \eta_n} + \frac{u}{2}\right|^2 a\tau \, d\eta = 2\int_{\Gamma'}\left|\frac{\partial u}{\partial \eta_n}\right|^2 d\eta + 2\int_{\Gamma'}|u|^2 d\eta$$

$$= 2\left[A_k^2 k^3 \frac{5}{3} + \int \left|\frac{\partial w}{\partial \eta_n}\right| d\eta_n\right] = \frac{10A_k^2 k^3}{3} + 4 A_k^2 k \leq \left[\frac{10}{3} + \frac{4}{k^2}\right] C_3 \leq 8C_3 .$$

By 4.25 and i ,

$$4.37 \quad \tau a = a\int_{x_n}^{g_n} \frac{1}{a(x_n, \xi)} d\xi + ak_n \leq (g_n - x_n) + ak_n \to g_n \quad \text{as} \quad x_n \to 0$$

Consequently $a\tau$ is bounded in $\bar{\Gamma}$. Let $a\tau < C_5$ for $x \epsilon \bar{\Gamma}$. For $i<n$,

$$4.38 \quad \frac{\partial v}{\partial x_i} = \sum_{j=1}^{n} \frac{\partial v}{\partial \eta_j}\frac{\partial \eta_j}{\partial x_i} = \frac{\partial v}{\partial \eta_i} + \frac{\partial \eta_n}{\partial x_i}\frac{\partial v}{\partial \eta_n} = \tau^{1/2}\left[\frac{\partial u}{\partial \eta_i} + \frac{\partial \eta_n}{\partial x_i}\left(\frac{\partial u}{\partial \eta_n} + \frac{u}{2}\right)\right] .$$

By ii, and 4.25

$$4.39 \quad \left|\frac{\partial \eta_n}{\partial x_i}\right| = \left|\frac{1}{\tau}\frac{\partial \tau}{\partial x_i}\right| = \left|\frac{1}{\tau}\int_{x_n}^{g_n} \frac{\partial a}{\partial x_i}\frac{1}{a^2} dx_n\right| = \left|\frac{1}{\tau}\int_{x_n}^{g_n} \frac{C_2}{a}\right| dx_n$$

$$\leq C_2\left|1 - \frac{k_n}{\tau}\right| \leq 2 C_2$$

For $i < n$, by 4.38

$$4.40 \quad \int_{\Gamma} a\left|\frac{\partial v}{\partial x_i}\right|^2 dx = \int_{\Gamma'} a^2 \tau^2\left|\frac{\partial u}{\partial \eta_i} + \frac{\partial \eta_n}{\partial x_i}\left(\frac{\partial u}{\partial \eta_n} + \frac{u}{2}\right)\right|^2 d\eta \leq$$

by 4.35 , 4.37 , and 4.39 ,

$$\leq 2C_5^2 \int_{\Gamma'}\left|\frac{\partial u}{\partial \eta_i}\right|^2 d\eta + 8C_5^2 C_2^2 \int_{\Gamma'}\left[\left|\frac{\partial u}{\partial \eta_n}\right|^2 + \left|\frac{u}{2}\right|^2\right] d\eta$$

by the definition of u and 4.31,

$$\leq 2C_5^2\left[\pi^2(g_i-f_i)^2 \int |w|^2 \, d\eta_n + 4\, C_2^2 \int \left|\frac{\partial w}{\partial \eta_n}\right|^2 d\eta_n + \frac{5\, A_k^2\, k^3}{3}\, C_2^2\right]$$

$$= 2C_5^2\left[\frac{5\, A_k^2\, k^3}{3}\left(\pi^2(g_i-f_i)^2 + \frac{24}{5}\, C_2^2\, \frac{1}{k^2} + C_2^2\right)\right] \leq \frac{10}{3}\, C_5^2 C_3\left(\pi^2(g_i-f_i)^2 + 6\, C_2^2\right) = C_6 .$$

C_6 is independent of k. Combining 4.34, 4.36 and 4.40 $(L'v,v) + (v,v) \leq 8C_3 + (n-1)\, C_6 + C_3 + 1 = C$ which is independent of k.

By Lemma 4.23, the spectrum of L' is not discrete. By Lemma 4.28 the spectrum of L is not discrete.

Q.E.D.

<u>Remark</u>. If $a(x) = a(x_n)$, then our result is equivalent to one obtained by Kreith.*

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* See 11 page 28

5. Appendix: Lower Bounds for Positive Definite Matrices. The inequality which is developed here is used in several of the preceding sections.

5.1 Let $A = (a_{ij})$ be a positive definite hermitian matrix of order n. Define $A^{-1} \equiv B' = (b_{ij})$, $|A| \equiv$ determinant of A, and A_{ii} the cofactor of a_{ii}. Let λ_n the minimum eigenvalue of A, and λ_m the maximum eigenvalue of B.

5.12 Theorem. $$\lambda_n > \frac{|A|}{\prod_{j=1}^{n} a_{jj} \sum_{i=1}^{n} \frac{1}{a_{ii}}}$$

Proof. As the eigenvalues of B are the reciprocals of the eigenvalues of A, they must all be positive. Therefore

$$\lambda_m < \text{trace } B = \sum_{i=1}^{n} b_{ii} = \frac{\sum_{i=1}^{n} A_{ii}}{|A|} .$$

We will show that A_{ii} is also positive definite. $A_{ii} = (-1)^{2i} \times$ (determinant formed from A by eliminating the ith row and column).

Let $\vec{\xi} \neq 0$ be any vector with $\xi_i = 0$. Since A is positive definite the quadratic form $\sum_{k,\ell=1}^{n} a_{k\ell} \xi_k \bar{\xi}_\ell > 0$. This last condition and

the even exponent 2i make A_{ii} positive definite. For positive definite hermitian matrices $|A| \leq \prod_{j=1}^{n} a_{jj}$.† Then for each i,

†See [10] page 126

$$A_{ii} \leq \frac{\prod_{j=1}^{n} a_{jj}}{a_{ii}} .$$

Thus $\lambda_m < \frac{1}{|A|} \sum_{i=1}^{n} A_{ii} \leq \frac{1}{|A|} \sum_{i=1}^{n} \frac{\prod_{j=1}^{n} a_{ii}}{a_{ii}} = \frac{1}{|A|} \prod_{j=1}^{n} a_{jj} \sum_{i=1}^{n} \frac{1}{a_{ii}}$.

as $\lambda_n = \frac{1}{\lambda_m}$ we have

5.13 $$\lambda_n > \frac{|A|}{\prod_{j=1}^{n} a_{jj} \sum_{i=1}^{n} \frac{1}{a_{ii}}} .$$

Q.E.D.

5.2 Consider the characteristic equation of a positive definite hermitian matrix A of order N.

$$0 = f(\lambda) = a_n - a_{n-1} \lambda + a_{n-2} \lambda^2 - \dots + (-1)^{n-1} a_1 \lambda^{n-1} + (-1)^n \lambda^n ,$$

where the a_i are real. That the a_i are also positive can be deduced from the knowledge that all the roots of $f(\lambda)$ are positive and from Descarte's law of signs.

5.21 <u>Descarte's law states</u>: $f(\lambda)$ has n or $n-2k$ positve real roots if there are n variations in sign of the terms of $f(\lambda)$, n and k are non-negative integers.

Let the roots of $f(\lambda)$ be $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n > 0$. As is well known a_n may be expressed in terms of $(a_{ij}(x))$ by

$a_n = \prod_{i=1}^{n} \lambda_i = |A|$. a_{n-1} cannot in general be simply expressed in terms of the components of (a_{ij}). However, when we express a_{n-1} in terms of the sums and products of a_{ij} the term, $\prod_{j=1}^{n} a_{jj} \sum_{i=1}^{n} \frac{1}{a_{ii}}$, found in Theorem 5.12 is always also found in the expression for a_{n-1} and is always positive. This last remark and the prededing theorem suggest that $\frac{a_n}{a_{n-1}}$ may be a lower bound for λ_n.

5.22 <u>Lemma</u>. $\lambda_n > \frac{a_n}{a_{n-1}} = \frac{|A|}{a_{n-1}}$.

<u>Proof</u>. $\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \dots + \frac{1}{\lambda_n} > \frac{1}{\lambda_n}$.

Therefore

5.23 $$\frac{\prod_{j=1}^{n} \lambda_j \sum_{i=1}^{n} \frac{1}{\lambda_i}}{\prod_{j=1}^{n} \lambda_j}$$

If we invert 5.23, we have

$$\lambda_n > \frac{\prod_{j=1}^{n} \lambda_j}{\prod_{j=1}^{n} \lambda_j \sum_{i=1}^{n} \frac{1}{\lambda_i}} = \frac{a_n}{a_{n-1}} = \frac{A}{a_{n-1}}.$$

Q.E.D.

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