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STUDIES IN THE FOUNDATIONS OF S-MATRIX THEORY

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Henry P. Stapp

August 7, 1963

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ERRATA

Professor Jost, in trying to simplify the proof of Lemma 4, produced a counter example, which has been generalized by R. Seiler* to all $r < n \leq 2$ points. The proof of Lemma 4 is valid only for the remaining points $n = 3, 4$, or r, and hence the proof of Lemma 8 is also valid only for these points. However, general proofs of Lemma 8, based on Lemmas 2 and 7, have recently been given by D. Williams and P. Minkowski,* and hence the theorems remain valid.

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* Peter Minkowski, David N. Williams, and Rudolf Seiler, "On Stapp's Theorem," E. T. H. Preprint and Proceedings of the Symposium on the Lorentz Group, Seventh Annual Summer Inst. for Theoretical Physics, 1964, University of Colorado, Boulder, Colorado (to be published).

Page 5,	line 16	read:	For any (E, r) there exists···
Page 25,	line 8	read:	$K_{2b} = \Lambda\, {}''K_{2a} \neq K_{2a}$
Page 51,	line 4	read:	vectors of K_0.
Page 51,	line 18	read:	$e_{r+1}(K_0)$, and re-orthogonalize $e_{r+2}(K_0)$··· $e_3(K_0)$, following a standard procedure.
Page 63,	line 16	read:	In case <u>all</u> of···
Page 87,	line 18	read:	$(e_1(K_0),\ e_2(K_0), \omega)$
Page 90,	line 15	read:	$\mathrm{Re}\, \Gamma_i << 0$
Page 92,	line 21,	read:	$\mathcal{L} D_r'(\bar{K}_0)$
Page 92,	line 22,	read:	$\mathcal{L} D_r'(\bar{K}_0)$
Page 109,	line 4	read:	$(1/2)v^r$ in place of v^r.
Page 109,	line 12	read:	$(1/2)v$ in place of v.

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ABSTRACT

An axiomatic approach to S-matrix theory used earlier is developed in more detail. Certain of the postulates are made weaker and are brought closer to physically verifiable propositions. The viewpoint adopted is that the axioms should, whenever possible, be assertions of specific, verifiable physical relationships rather than abstract general principles. This tends to eliminate ambiguities and superfluous entities, to cement theory to experiment, and to establish a secure logical framework. The paper is the first of a series and is devoted to detailed proofs of the fundamental consequences of the first few axioms. In particular, the unitarity of the S matrix is derived from a completeness property that incorporates the possibility of superselection rules. The relationship between spin and representations of the rotation group is derived from the defining experimental characteristics of spin. The necessary existence of an S matrix whose connected part M_c is a Lorentz invariant function is derived from the assumed relativistic invariance of experimental correlations. A general theorem is proved that asserts, roughly, that a function that is (real) Lorentz invariant anywhere is complex Lorentz invariant everywhere in its domain of regularity. Finally it is shown that the analytic continuations of M_c from regular points in the physical

region of a given process along paths generated by the complex Lorentz transformations lead to a function that is single valued in a neighborhood of all of these paths. Applications are discussed. The results derived constitute the preliminaries to a discussion of the analytic structure of the S matrix to be given in a subsequent paper.

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I. INTRODUCTION

Mandelstam's[1] success in obtaining terms of the renormalized perturbation series by methods dealing purely with scattering amplitudes at physical mass values has generated renewed interest in Heisenberg's[2] suggestion that the S matrix itself, without reference to field-theoretic ideas, may provide an adequate basis for elementary-particle physics. This idea is the basis of much current theoretical work in strong-interaction physics due in part to serious practical difficulties encountered by field-theoretic methods, in part to the apparent suitableness of S-matrix methods for dealing with the complications associated with bound states and resonances, and in part to a belief, held in some quarters, that the abstract requirements imposed by field theories may be too stringent to permit a consistent theory of interacting particles.

In an earlier paper[3] it was shown that CPT invariance and the normal connection between spin and statistics follows from S-matrix postulates. The postulates given there are, however, stronger than necessary, and fall short of achieving certain S-matrix ideals.

The present paper is the first of a series devoted to a detailed study of the axioms of S-matrix theory. The objective is to define a

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* Work done under the auspices of the U. S. Atomic Energy Commission.

set of axioms restrictive enough to permit useful practical calculations, and minimal in the sense that no superfluous elements are introduced. Also the axioms are to conform to certain S-matrix ideals, that must themselves be defined. In this first paper certain fundamental consequences of the first three axioms are derived. These results provide the basic structure of the theory.

The first postulate asserts the probability interpretation and the superposition principle. In the formulation of these assumptions given in SI the possibility of superselection rules was not admitted, and the proof of unitarity given there depended on that fact. In the following section, Section II, a reformulation of this postulate that allows for superselection rules is given. Unitarity is then proved on the basis of this weaker postulate.

The second postulate specifies that the measurement of the momentum, the spin indices, and the particle type of all particles present constitutes a complete experiment. In SI it was specified that the spin indices be connected with a representation of the rotation group in the "usual" way. Though this assumption is hardly open to question it runs counter to the general methodological principle that has been adopted here. This principle is that to promote certainty, general abstract assertions, with perhaps many far-reaching consequences, should be replaced, whenever possible, by assertions of specific physically observable relationships subject individually to direct experimental test; that the theory should, so far as possible, be built directly upon experimental fact, rather than being merely indirectly supported by these facts

a posteriori. This principle leads sometimes to a reversal of the usual order of argumentation, with what is usually regarded as a consequence being taken as the assumption. There is also a slight sacrifice of economy since broad general assumptions are replaced by limited specific ones, but the resultant gain in certainty is deemed to outweight this small aesthetic loss.

In conformity to this principle the abstract assumption regarding the connection between spin indices and rotations is replaced, in Section III, by the assumption that certain relationships between physical observables are satisfied. The usual connection is then derived from these.

In Section IV the relationship between momentum-energy and space-time is briefly discussed.

In Section V the principle of macroscopic relativistic invariance is formulated as a relationship between experimental results. From this relationship the necessary existence of a (Lorentz) invariant scattering matrix is deduced. In SI an extra ad hoc postulate was used to eliminate certain unwanted phase factors; here this result is achieved without that assumption.

In Section VI some properties of (Lorentz) invariant analytic functions are derived. A principal result is a theorem (Theorem 3) that asserts that if a function is invariant under elements of the connected component of the <u>real</u> Lorentz group over a real domain and is analytic at the points of this domain, then the complete analytic extension of the function is defined over a set of sheets each of which maps onto itself under any element of the connected component of the <u>complex</u> Lorentz group,

and the function defined (single valuedly) over each sheet is invariant under elements of this latter group. Moreover, if the real domain satisfies a certain condition that is always satisfied by a physical region, then this entire domain can be contained in a single sheet. This theorem is somewhat akin to Lemma 1 in the paper of Hall and Wightman.[4] In conjunction with a result of Hepp[5] it allows one to prove the existence of a local decomposition of an invariant function into a sum of standard polynomial invariants times holomorphic functions of scalar invariants, over domains of regularity bounded by cuts defined by equations involving only scalar invariants. This is a partial extension of the Hall-Wightman theorem to domains of this type, which is the type that will later arise from the postulate of maximal analyticity.

A second application of the theorem is to a slight weakening of the assumptions needed for the S-matrix proof of CPT invariance. In SI the existence of a physical sheet bounded by cuts defined by equations involving only scalar invariants was postulated. This assumption would be a consequence of maximal analyticity but is of course much weaker, since the boundary of the sheet is allowed to be arbitrary, subject only to the condition that it be determined by equations involving only scalars. This assumption, which disposes of certain problems regarding single-valuedness, can be weakened, for the theorem shows that the analytic continuations from various points in the physical region of a given process along paths generated by various complex Lorentz transformations lead to a function that is single valued in the neighborhood of all of these paths.

II. THE BASIC QUANTUM PRINCIPLES AND UNITARITY

Quantum theory deals with correlations between the probabilities of possible results of experiments. A class of experiments whose members are called complete experiments is characterized in the following way. Let $\mathcal{P}_r^c(E)$ be the probability that a complete experiment E has the result (E, r) in circumstance c. The $\mathcal{P}_r^c(E)$ are asserted to have the following properties:

<u>A1.</u> $\mathcal{P}_r^c(E) \geqslant 0$ for all c and r, (2.1)

<u>A2.</u> $\sum_r \mathcal{P}_r^c(E) = 1$ for all c, (2.2)

<u>A3.</u> $(\mathcal{P}_r^c(E) = 1)$ implies, for all (E_1, u) and (E_2, v),

$$\mathcal{P}_u^c(E_1) = |a_u(E_1, E, r)|^2 \tag{2.3}$$

and

$$\mathcal{P}_v^c(E_2) = |a_v(E_2, E, r)|^2 , \tag{2.4}$$

where the amplitudes are a set of complex numbers that are linearly related:

$$a_u(E_1, E, r) = \sum_v S_{uv}(E_1, E_2)\, a_v(E_2, E, r) . \tag{2.5}$$

<u>A4.</u> For any (E, r) that exists a c such that $\mathcal{P}_r^c(E) = 1$.

<u>Definition:</u> Results (E_1, u) and (E_1, t) belong to different superselection classes if and only if

$$S_{uv}(E_1, E_2)\, S_{tv}(E_1, E_2) = 0 \quad \text{for all } (E_2, v) . \tag{2.6}$$

A5 If for some E_1 there is a set of numbers $V_u(E_1)$ such that

$$V_u(E_1)\, V_t(E_1) = 0 \tag{2.7}$$

whenever (E_1, u) and (E_1, t) belong to different superselection classes, and such that

$$\sum_u |V_u(E_1)|^2 = 1 , \tag{2.8}$$

then there is some (E_2, v) such that

$$S_{uv}(E_1, E_2) \cong V_u(E_1) , \tag{2.9}$$

where $\cong$ means equal to within a phase factor that is independent of u.

The principles stated above are an extraction of elements of the usual quantum theory. They do not exhaust the usual principles, since , among other things, commutation relations and the quantum of action are not mentioned. They are expressed directly in terms of correlations between observable quantities without the introduction of such auxiliary notions as operators and states of a physical system.

In A5 the constraint (2.7) is required for consistency with (2.6). Condition (2.8) is required for consistency with A2, A3, and A4; the existence of (E_2, v) together with A4 allows the condition for A3 to be satisfied with E_2 the instance of E, and v the instance of r.

The consequence in A3 inserted into A2 with (E_1, u) the instance of (E, r) gives

$$\sum_u |S_{uv}(E_1, E_2)|^2 = 1 , \tag{2.10}$$

where the sum over v has been eliminated by combining A1 and A2 with the condition $\mathcal{P}_v(E_2) = 1$. Thus if $V_u(E_1)$ fails to satisfy (2.8) there can be no (E_2, v) such that (2.9) is valid.

The remainder of this section is devoted to the derivation of the unitarity property of the transformations $S(E_1, E_2)$. In SI the existence of an $S(E_1, E_2)$ fulfilling the conditions in Eqs. (2.1) through (2.5) for any possible $a_u(E_1)$ [or, alternatively, $a_v(E_2)$] was assumed, and unitarity followed immediately.

Given any (E_2, v) an argument essentially the same as the one just given leads to the consequence

$$\sum_u |S_{uv}(E_1, E_2)|^2 = 1 . \tag{2.11}$$

This is part of the unitarity relation. Suppose x and y label different results of E_2 that do not belong to different superselection classes. Let

$$V_v(E_2) = a \ \delta_{vx} + b \ \delta_{vy} , \tag{2.12}$$

where δ_{ij} is one for $i = j$ and is zero otherwise, and where

$$|a|^2 + |b|^2 = 1 \tag{2.13}$$

The conditions of A5 are then satisfied by $V_v(E_2)$, and there is an (E_2', w) such that

$$S_{vw}(E_2, E_2') \cong a\,\delta_{vx} + b\,\delta_{vy} . \tag{2.14}$$

By the same argument used to get (2.11) one obtains now

$$\sum_u |S_{uw}(E_1, E_2')|^2 = 1 . \tag{2.15}$$

From the requirement in A3 that the $a(E_2, E, r)$ be independent of E_1 one obtains the chain rule.

$$S_{uw}(E_1, E_2') = \sum_w S_{uv}(E_1, E_2)\, S_{vw}(E_2, E_2') . \tag{2.16}$$

This substituted into (2.15) gives, using (2.14),

$$1 = \sum_u |S_{ux}(E_1, E_2)a + S_{uy}(E_1, E_2)b|^2 . \tag{2.17}$$

Expanding and introducing (2.11) and (2.13) one obtains

$$2\,\mathrm{Re}\, ab^* \sum_u S_{ux}(E_1, E_2)\, S_{uy}^*(E_1, E_2) = 0 . \tag{2.18}$$

Since a and b are arbitrary, subject to (2.13), one obtains

$$\sum_u S_{ux}(E_1, E_2)\, S_{uy}^*(E_1, E_2) = 0 \tag{2.19}$$

for any x and y differing from each other but not belonging to

different superselection classes. If x and y do belong to different super super selection classes then (2.19) is a consequence of (2.6). Thus one obtains the unitarity relationship

$$\sum_u S^*_{uw}(E_1, E_2)\, S_{uv}(E_1, E_2) = \delta_{wv} , \tag{2.20}$$

or in matrix notation,

$$S^\dagger(E_1, E_2)\, S(E_1, E_2) = 1 . \tag{2.21}$$

Condition <u>A3</u> guarantees the existence of both a right and left inverse:

$$S^{-1}(E_1, E_2) = S(E_2, E_1) . \tag{2.22}$$

This, together with (2.21), implies the second unitarity relation

$$\sum_w S_{uv}(E_1, E_2)\, S^*_{wv}(E_1, E_2) = \delta_{uw} . \tag{2.23}$$

III. PARTICLE OBSERVABLES

B1. There are complete experiments the possible results of which may be labeled by sets of variables of the form

$$K \equiv \left\{ k_i, m_i, t_i \right\} , \tag{3.1}$$

where

$$i \in \left\{ 1, 2, \cdots, N_K < \infty \right\} , \tag{3.2}$$

$$\underset{\sim}{k}_i \in R^3 , \tag{3.3}$$

$$k_i^2 \equiv (k_i^0)^2 - \underset{\sim}{k}_i^2 = \mu_i^2 , \tag{3.4}$$

and

$$m_i \in \left\{ S_i, S_i - 1, \cdots, -S_i \right\} . \tag{3.5}$$

The finite bound N_K in (3.2) will be called the number of particles associated with the particular result K. The R^3 in (3.3) is the real three-dimensional continuum. The numbers μ_i and S_i are called the mass and spin of particle i, respectively, and are functions of t_i, the particle type. The four-vector k_i will be called the (mathematical) momentum-energy vector of the ith particle, and the indices m_i will be called the spin indices of particle i.

A "sum" over a continuum parameter $\underset{\sim}{k}_i$ is to be interpreted as an integration, and the corresponding parts of δ_{uv} interpreted as a Dirac delta function defined so that the relation

$$\sum_{\mu} \delta_{\mu v} = 1 \tag{3.6}$$

is maintained. An arbitrary nonvanishing weight factor can be introduced in the integration and compensated by a redefinition of S. It will be advantageous to use a covariant momentum space element. Specifically, a "sum" over momentum vectors $\underset{\sim}{k}_i$ will mean an integration over the covariant momentum space elements

$$\prod_i \left(\frac{d^3 \underset{\sim}{k}_i}{(2\pi)^3} \right)_c \equiv \prod_i \left(\frac{d^3 \underset{\sim}{k}_i}{(2\pi)^3} \frac{\mu_i}{\sqrt{\underset{\sim}{k}_i^{\,2} + \mu_i^{\,2}}} \right) \tag{3.7}$$

<u>Definition:</u> The set $T \equiv \left\{ t_i \right\}$ will be called the type of result labeled by $K \equiv \left\{ k_i, m_i, t_i \right\}$

<u>B2</u>. Results of the same type do not belong to different superselection classes.

<u>B3</u>. The physical interpretation of the spin indices m_i is associated with a direction $\underset{\sim}{s}_i^{\,r}$, the spin direction of particle i, which is a three-dimensional space vector in a rest frame of particle i .

<u>Definition:</u> The experiments of the type specified in <u>B1</u> will be labeled $E_\alpha(S_\beta)$, where S_β specifies the set of spin directions associated with the m_i, and α specifies other variables.

[There should be no confusion between this argument S and the transformation $S_{uw}(E_1, E_2)$]

It is convenient to adopt now the notation

$$S_{uw}(E_1(S_1); E_2(S_2)) \equiv S(K_1(S_1); K_2(S_2)) \ . \tag{3.8}$$

For experiments differing <u>only</u> in the choice of spin axes the same subscript will be used.

One particular, but arbitrary, choice of the (S_1, S_2) will be taken as a standard and the labels $K_1(S_1)$ and $K_2(S_2)$ shortened to K_1 and K_2 for it. Then one may write

$$S(K_1(S_1);\ K_2(S_2)) = \iint S(K_1(S_1);\ K_1')\ S(K_1';\ K_2')\ S(K_2';\ K_2(S_2))\ , \tag{3.9}$$

by virtue of the chain rule (2.16). The integrations are over the repeated continuum variables of K_1' and K_2'. Sums over repeated <u>spin</u> indices are always to be understood.

<u>B4</u>. Transformations associated with changes <u>only</u> of the spin directions act completely within the spin spaces. Specifically, $S(K_1(S_1);\ K_1')$ is zero unless $k_i = k_i'$ and $t_i = t_i'$, and similarly for $S(K_2';\ K_2(S_2))$.
Condition <u>B4</u> allows (3.9) to be reexpressed in the form

$$S(K_1(S_1);\ K_2(S_2)) = U(S_1, S_2)\ S(K_1;\ K_2)\ , \tag{3.10}$$

where an operator interpretation is to be made. That is, for a spin-space operator $U(S)$,

$$\begin{aligned} U(S)\ F(K) &\equiv U(S)\ F(K_0, M, T) \\ &\equiv U_{MM'}(S, K_0, T)\ F(K_0, M', T)\ . \end{aligned} \tag{3.11}$$

Here $K \equiv \left\{ k_i, m_i, t_i \right\}$ has been split into three sets K_0, M and T , in the obvious manner. The matrices, $U(S, K_0, T)$ are, according to the result proved in Section II, together with <u>B4</u>, unitary matrices in the spin space.

In terms of the U just introduced one may define spin projection operators:

$$P^K(S) = U^{-1}(S, K_0, T) D^K \; U(S, K_0, T) \; ; \tag{3.12}$$

where D^K is a diagonal matrix in the spin space specified by (K_0, T), having unity in the position specified by the M part of K and zeros elsewhere. In terms of these projection operators one may write

$$|S(K_1(S_1); \; K_2(S_2))|^2 = S^*(K') \, P^K(S) \, S(K') \; , \tag{3.13}$$

where

$$K = (K_1; K_2) \; , \qquad S = (S_1, S_2) \; , \quad K_0' = K_0 , \text{ and } T_0' = T_0 \; .$$

There will be no confusion if this is abbreviated by

$$|S(K_1(S_1); \; K_2(S_2))|^2 = S^*(K) \, P(S) \, S(K) \; , \tag{3.13a}$$

with the suppressed K_0 and T_0 of $P(S)$ being specified by the K of $S(K)$, and the suppressed superscript M of $P(S)$ being specified by the (M_1, M_2) of the left-hand side.

The quantities occurring in (3.13) are special cases of quantities

$$\mathcal{C}(E_1, u;\ E_2, w) \equiv |S_{uw}(E_1, E_2)|^2 \ . \tag{3.14}$$

The quantity $\mathcal{C}(E_1, u;\ E_2, w)$ will be called the correlation between (E_1, u) and (E_2, w). Conditions A1, A2, and A3 imply that

$$(\mathcal{P}_w(E_2) = 1) \Longrightarrow (\mathcal{P}_u(E_1) = \mathcal{C}(E_1, u;\ E_2, w)) \ , \tag{3.15}$$

where the arrow represents implication. This relationship provides the experimental significance of correlations. The special case (3.13) is

$$\mathcal{C}(K_1(S_1);\ K_2(S_2)) \equiv S^*(K)\ P(S)\ S(K) \ . \tag{3.16}$$

One may consider correlations between experiments that differ only in the choice of the orientations of spin vectors. Let $(K_2'(S_2'))$ and $(K_2''(S_2''))$ label two possible results of the same type T of two experiments related in this way. They are related to the set specified by K_2 by the transformations $S(K_2;\ K_2'(S_2'))$ and $S(K_2;\ K_2''(S_2''))$, respectively. The correlation between $(K_2'(S_2'))$ and $(K_2''(S_2''))$ is then

$$\mathcal{C}(K_2'(S_2');\ K_2''(S_2'')) = |\ S(K_2'(S_2');\ K_2''(S_2''))|^2$$

$$= \left| \int_{K_2} S^*(K_2;\ K_2'(S_2'))\ S(K_2;\ K_2''(S_2'')) \right|^2 , \tag{3.17}$$

where the chain rule (2.16) and unitarity have been used. (The subscripts on the variables of $\mathcal{C}$ and S may be considered as part of the identification of these functions.)

Let $a'(M')$ be a set of complex numbers satisfying

$$a'^*(M')\, a'(M') \equiv \sum_{M'} |a'(M')|^2 = 1 . \qquad (3.18)$$

According to A5 there is an experiment and associated result, which will be labeled by $A_2'(S_2') \equiv (a', K_{02}', T_2', (S_2'))$, such that

$$S(K_2;\, A_2'(S_2')) \cong S(K_2;\, K_2'(S_2'))\, a'(M_2') \qquad (3.19)$$

(A summation on M_2' is implied on the right.) Condition (2.8) of A5 is satisfied because of (3.18) and the unitarity of $S(K_2;\, K_2'(S_2'))$, and (2.7) is satisfied by virtue of B2 and B4 .

In the same way one may introduce the experiment and associated result labeled by $A_2''(S_2'')$. The correlation function is

$$\mathcal{C}(A_2'(S_2');\, A_2''(S_2'')) = \left| \int S^*(K_2;\, A_2'(S_2'))\, S(K_2;\, A_2''(S_2'')) \right|^2 . \qquad (3.20)$$

B5. Let $A_2(S_2)$ label a possible result of the kind just constructed. Consider a set $\Lambda \equiv \{\Lambda_i\}$ of independent rotations in the rest frames of the particles specified by $A_2(S_2)$. There exists a corresponding "rotated" result $\bar{A}_2(\bar{S}_2)$ of the rotated experiment. Correlations are invariant under this set of rotations:

$$\mathcal{C}(A_2'(S_2');\, A_2''(S_2'')) = \mathcal{C}(\bar{A}_2'(\bar{S}_2');\, \bar{A}_2''(\bar{S}_2'')) . \qquad (3.21)$$

Condition B5 expresses a purely kinematical relationship, not a dynamical one; the correlations are between experiments differing only by rotations of spin directions, not by, say, temporal displacements or in any other way. The condition is a particular case (hence weaker requirement) of the general property that kinematic correlations between the spin states of a particle are invariant under rotations in the particle rest frame.

With the introduction of the notation

$$S(K_2;\ A_2'(S_2')) = \psi_M\ , \tag{3.22a}$$

$$S(K_2;\ A_2''(S_2'')) = \phi_M\ , \tag{3.22b}$$

$$S(K_2;\ \bar{A}_2'(\bar{S}_2')) = \bar{\psi}_M\ , \tag{3.22c}$$

and

$$S(K_2;\ \bar{A}_2''(\bar{S}_2'')) = \bar{\phi}_M\ , \tag{3.22d}$$

condition B5 takes the form

$$|(\psi,\ \phi)|^2 = |(\bar{\psi},\ \bar{\phi})|^2\ , \tag{3.23}$$

where the bracket represents the complex inner product.

Condition B5, as expressed in (3.23), is just in the form required for an application of the argument given in the appendix to Chapter 20 of Wigner's book.[6] That is, for all vectors ψ and ϕ in a finite-dimensional space there are corresponding vectors $\bar{\psi}$ and $\bar{\phi}$, and (3.23) is valid. From Wigner's argument one concludes that it is possible to redefine the phases of the $\bar{\psi}$ and $\bar{\phi}$ such that they are obtained from ψ and ϕ by a linear unitarity transformation. That is,

$$S(K_2;\ \bar{A}_2'(\bar{S}_2')) \ = \ U_2(\Lambda)\, S(K_2;\ A_2'(S_2')) \tag{3.24}$$

where $U_2(\Lambda)$ is a <u>linear</u> unitary operator in the spin space. The linearity means that $U_2(\Lambda)$ is independent of a' and of S_2'. The linearity and unitarity ensure that the possible redefinition of phases gives a function S still satisfying the basic quantum principles.

From the linearity of $U(\Lambda)$ one concludes that the $U(\Lambda)$ are elements of a representation[7] of the product-of-rotation group;

$$U_2(\Lambda_1\, \Lambda_2) \ = \ U_2(\Lambda_1)\, U_2(\Lambda_2) \quad , \tag{3.25}$$

provided $S(K_2;\ A_2'(S_2'))$ is different from zero for each value of M', for some value of a' and S_2'. This condition is, in fact, satisfied; one may take S_2' to be the particular one chosen above Eq. (3.9), and $a'(M')$ to be $\delta_{MM'}$.

The development just carried through for $S(K_2\ (K_2');S_2')$ can be done also for $S(K_1'\ (S_1');\ K_1)$. From (3.24) and the analogous equation for $S(K_1'\ (S_1');\ K_1)$, one obtains for the projection operators defined in (3.12) the relation

$$P^K(\Lambda\ S) \ = \ U(\Lambda)\ P^K(S)\ U^{-1}(\Lambda) \tag{3.26}$$

where the $U(\Lambda)$ are unitarity, linear operators (i.e., independent of S and M).

Because Λ represents a product of independent rotations the $U(\Lambda)$ must be a direct product of representations of the rotation group:

$$U(\Lambda) \ = \ \prod_i U_i(\Lambda_i)\ .$$

B6. The representations $U_i(\Lambda_i)$ are irreducible representations whose dimensions are independent of k_i .

By virtue of the factorization property of $U(\Lambda)$ the projection operator $P^K(S)$ defined in (3.12) also factorizes:

$$P^K(S) = \prod_i P^{m_i}(\underset{\sim}{s}_i{}^r) . \tag{3.27}$$

From B6 one easily shows that $P^m(\underset{\sim}{s}_i{}^r)$ has the form

$$P^m(\underset{\sim}{s}_i{}^r) = \sum_{n=0}^{2s_i} a_{mn}(\underset{\sim}{\sigma}_i \cdot \underset{\sim}{s}_i{}^r)^n , \tag{3.28}$$

where the coefficients a_{mn} are determined by the equations

$$\sum_{n=0}^{2s_i} a_{mn}(m')^n = \delta_{mm'} \tag{3.29}$$

for $|m'| \leqslant s_i$, and where

$$I_i(\epsilon, \underset{\sim}{s}_i{}^r) = (I_i + \frac{1}{2} i \epsilon \underset{\sim}{\sigma}_i \cdot \underset{\sim}{s}_i{}^r) \tag{3.30}$$

is the operator that gives the rotation by the infinitesimal angle ϵ about the axis $\underset{\sim}{s}_i{}^r$, in the spin space of the ith particle. The label m is chosen to conform to common usuage. (The symbol $\frac{1}{2}\underset{\sim}{\sigma}$ was used rather than the more usual $\underset{\sim}{S}$ to avoid a still further proliferation of symbols S, and to emphasize the connection Pauli matrices.)

The conclusion is that, as a consequence of the assumptions made so far, the usual connection between spin projection operators and rotations in the particle rest frames must hold; that this connection is not only sufficient to give the usual experimental relationships but is also a <u>necessary</u> consequence of certain of these relationships. By the "usual" connection is meant the relations given in (3.27) through (3.30).

There is no implication in the foregoing that one must use the same representation (i.e., the same matrices $\underset{\sim}{\sigma}_i$) for spin spaces associated with different values of k_i . And indeed, since different values of k_i give distinct rest frames the analysis for each k_i is separate also in a physical sense. Of course one may change at will between representations that are unitary equivalent. This just induces a redefinition of the transformation function $S(K)$ occurring in (3.16), since the physical correlations are fixed. Stated differently, it is by specifying the representations of the spin operators that a corresponding arbitrariness is removed from $S(K)$.

It is convenient to always use the same matrices for the $\underset{\sim}{\sigma}_i$, for all values of the k_i . That is, the unitary operator $U_2(\Lambda)$ in (3.24) will be taken to be independent of K_2 , and similarly for $U_1(\Lambda)$. Accordingly, the operator $U(\Lambda)$ in (3.26) will be independent of K . As all irreducible $(2S_i+1)$-dimensional representations of the rotation group are unitary equivalent, the adoption of this convention involves no loss of generality. (Conjugate and transpose representations are included if the sense of $\underset{\sim}{s}_i^{\,r}$ is left open.)

The vectors $\underset{\sim}{s}_i^{\,r}$ used in the foregoing are three-component space vectors. They define the physical spin directions in a rest frame of the particle. Experiments for measuring spin characteristics can always be considered to measure rest-frame quantities.[8]

Of course, there are many rest frames of a particle, which differ in their orientation, and the components of $\underset{\sim}{s}_i^{\,r}$ will depend on which frame is chosen. Given a particular rest frame and the components in this frame the components in any other frame are determined by Lorentz transformations. In a general frame the spin vector will become a four-vector, which will be denoted by s_i . Since it is only the space parts in a rest frame that enter into the characterization of the experimental results the time part in such a frame is irrelevant and can be set to zero. Then the four vectors s_i will satisfy the relations

$$s_i \cdot s_i = -1 \qquad \text{and} \qquad s_i \cdot k_i = 0 \,. \tag{3.31}$$

The convenient way to specify the $\left\{\underset{\sim}{s}_i^{\,r}\right\}$ is to give the $\left\{s_i\right\}$, for these latter can all be given in a single general coordinate system, the same one used for the k_i . If the $\underset{\sim}{s}_i^{\,r}$ themselves are given then one must also specify the various particular rest frames relative to which each has physical significance. Adopting this convention we shall consider that the argument of $E(S)$ is a set $S = \left\{s_i\right\}$ of four-vectors satisfying the conditions (3.31) and specifying the spin directions associated with $E(S)$. The s_i and k_i are to be the values of the spin-four vector and the momentum-energy four-vector in some single coordinate system. For convenience the connection between the signs of the mathematical vectors s_i and k_i and the corresponding physical vectors will, however, be left open, for the moment.

IV. SPACE-TIME VARIABLES AND CONSERVATION LAWS

Space-time variables are related to momentum-energy variables by a Fourier transformation, in units where Planck's constant is 2π. This relationship, conjectured by DeBroglie, is borne out by diffraction experiments and gives the mathematical basis for the uncertainty principle. Using this connection one can deduce the conservation law of momentum-energy from translational invariance. Considerations relating to space-time variables will be discussed in detail in the following paper. For the present it will merely be remarked that the functions $S(K)$ are subject to the conservation-law constraint

$$(\sum k_i) \; S(K) = 0 . \tag{4.1}$$

Here, for simplicity, the mathematical momentum-energy vectors for one of the two experiments, say E_2, have been taken to be the negatives of the physical momentum-energy vectors.

V. MACROSCOPIC RELATIVISTIC INVARIANCE

Definition: L will denote the real proper orthochronous homogeneous Lorentz group (i.e., the connected component of the real homogeneous Lorentz group, often denoted by $L_+^{\uparrow}$) .

<u>C1</u> For every experimental result A (or B), and any $\Lambda \in L$, there is an associated transformed result ΛA (or ΛB) of the transformed experiments that is specified by vectors (perhaps tensors) related to those specifying the original result by the transformation Λ . Correlations are unchanged if each result is replaced by the associated transformed result:

$$\mathcal{C}(A, B) = \mathcal{C}(\Lambda A, \Lambda B) . \tag{5.1}$$

It will be sufficient to use three more specific assumptions, which can be regarded as particular instances of <u>C1</u>. The first of these is

<u>C1A</u>. For any $\Lambda \in L$

$$\mathcal{C}(K_1(S_1); K_2(S_2)) = \mathcal{C}(\Lambda K_1(\Lambda S_1); \Lambda K_2(\Lambda S_2)) , \tag{5.2}$$

where

$$\Lambda K = \left\{ \Lambda k_i, m_i, t_i \right\} \tag{5.3}$$

and

$$\Lambda S = \left\{ \Lambda s_i \right\} \tag{5.4}$$

Using (3.16) one may write (5.2) in the alternative form

$$S^*(K)\, P(S)\, S(K) = S^*(\Lambda K)\, P(\Lambda S)\, S(\Lambda K) , \tag{5.5}$$

where the suppressed M's on P(S) and P(ΛS) are the same. Because of the $\underset{\sim}{s}_i^{\,r}$ appearing in (3.28) the relationship between P(S) and

$P(\Lambda S)$ is relatively complicated; it depends on the particular choices of the various rest frames. However, the operator (3.28) is just the value in the rest frame of an invariant spinor function (see Appendix A)

$$P^m(k_i, s_i) = \sum_n a_{mn} P(n, k_i, s_i) \tag{5.6}$$

that satisfies, for all $\Lambda' \in L$,

$$P^m(\Lambda' k_i, \Lambda' s_i) = \Lambda'_S P^m(k_i, s_i) . \tag{5.7}$$

In particular, if $\Lambda' k_i$ and $\Lambda' s_i$ are taken to be the rest-frame values of k_i and s_i, the left-hand side of (5.7) becomes just the $P^m(\underset{\sim}{s}_i{}^r)$ of (3.28). Substituting (5.7) into (5.5) one obtains the alternative form

$$N^*(K)\, P(K, S)\, N(K) = N^*(\Lambda K)\, P(\Lambda K, \Lambda S)\, N(\Lambda K) , \tag{5.8}$$

where $P(K, S)$ is obtained from $P(S)$ by replacing each projection operator $P^m(\underset{\sim}{s}_i{}^r)$ by its covariant generalization $P^m(k_i, s_i)$, and $N(K)$ is obtained from $S(K)$ by incorporating into it the various spinor Lorentz transformations Λ'_S of (5.7). This procedure was described briefly in SIA and in more detail in SIB.

The new covariant operators $P(K, S)$ are invariant spinor functions:

$$P(\Lambda K, \Lambda S) = \Lambda_S P(K, S) . \tag{5.9}$$

The result of substituting (5.9) into the right-hand side of (5.8) can be expressed in the form

$$N^*(K)\ P(K,\ S)\ N(K) \quad \left[\Lambda_S^{-1}\ N(\Lambda K)\right]^* \quad P(K,\ S)\ \left[\Lambda_S^{-1}\ N(\Lambda K)\right] \quad , \tag{5.10}$$

provided the spin indices of the N functions are assigned a spinor index type fixed by specifying that the contractions in (5.8) connect always upper and lower indices of the same dottedness. The transformation properties of the $P(K,\ S)$ are, of course, purely algebraic consequences of their definitions. On the other hand, the transformation properties of an $N(K)$ cannot be obtained simply by examination of its specific form, since this is not given. One is, of course, free to assign any arbitrary transformation rule to any spin index of N. The important question is whether, for some particular assignment, the function $N(K)$ will be an invariant spinor function.

The continuum of operators $P^K(S)$ for various values of $S = \left\{ s_i \right\}$ span the finite $(\prod_i (2S_i + 1))$-dimensional spin space specified by K. (See Appendix A.) Thus from (5.10) one obtains immediately

$$N^*(K')\ N(K'') \quad = \quad \left[\Lambda_S^{-1}\ N(\Lambda K')\right]^* \quad \left[\Lambda_S^{-1}\ N(\Lambda K'')\right] , \tag{5.11}$$

where $k_i' = k_i''$ and $t_i' = t_i''$, but m_i' and m_i'' can differ. From this one obtains

$$N(K) \quad = \quad \Lambda_S^{-1}\ N(\Lambda K)\ \exp\left[i\ \alpha(\Lambda,\ K)\right] , \tag{5.12}$$

where $\alpha(\Lambda,\ K)$ is real and independent of the m_i indices of K.

The nondependence of $\alpha(\Lambda,\ K)$ on the indices m_i deduced above is, in effect, a consequence of the observability implied by B2 and A5 of the relative phases of spin amplitudes. According to B2 and A5

combinations of amplitudes labeled by different values of k_i are, in effect, also observables. The nondependence of the $\alpha(\Lambda, K)$ on the k_i's can be obtained by an examination of these observables, as will now be shown.

Let $(K_{2a}(S_{2a}))$ and $(K_{2b}(S_{2b}))$ specify experiments and associated possible results of the same type T. And suppose K_{2a} and K_{2b} are related by a real rotation $\Lambda'' \neq I$:

$$K_{2b} - \Lambda'' K_{2a} \neq K_{2a} \quad . \tag{5.13}$$

Let a and b be two complex numbers satisfying

$$|a|^2 + |b|^2 = 1 \tag{5.14}$$

Then according to A5 there exists an experiment and an associated possible result, which will be labeled by B, such that

$$S(K_2; B) \cong a\, S(K_2; K_{2a}(S_{2a})) + b\, S(K_2; K_{2b}(S_{2a})) \; . \tag{5.15}$$

Condition (2.8) of A5 is satisfied by virtue of (5.13), B4, (5.14), and the unitarity of $S(K_2; K_2'(S_2'))$. Condition (2.7) is satisfied by virtue of B4 and B2. By means of the chain rule, (2.16), one can also obtain

$$S(K_1(S_1); B) \cong a\, S(K_1(S_1); K_{2a}(S_{2a})) + b\, S(K_1(S_1); K_{2b}(S_{2b})) \tag{5.16}$$

If $(K_1(S_1))$ is abbreviated by A then the correlation function $\mathcal{C}(A, B)$ is

$$\mathcal{C}(A, B) = \left| a\, S(K_1(S_1); K_{2a}(S_{2a})) + b\, S(K_1(S_1); K_{2b}(S_{2b})) \right|^2$$

$$\equiv |a|^2 \; N^*(K_a) \; P(K_a, S_a) \; N(K_a)$$

$$+ \; |b|^2 \; N^*(K_b) \; P(K_b, S_b) \; N(K_b)$$

$$+ \; a^* b \; N^*(K_a) \; P(K_a, S_a; K_b, S_b) \; N(K_b)$$

$$+ \; b^* a \; N^*(K_b) \; P(K_b, S_b; K_a, S_a) \; N(K_a) \; , \tag{5.17}$$

where $K_a \equiv (K_1, K_{2a})$, etc. The operator $P(K_a, S_a; K_b, S_b)$ is the natural generalization of the operator $P(K_a, S_a) \equiv P(K_a, S_a; K_a, S_a)$ defined below (5.8). It has the property

$$P(K_a, S_a; K_b, S_b) = \Lambda_S^{-1} \, P(\Lambda K_a, \Lambda S_a; \Lambda K_b, \Lambda S_b) \; , \tag{5.18}$$

which is the generalization of (5.9).

According to <u>C1</u> correlations are invariant under the replacement of experimental results by associated transformed results. Because of this the transformation $B \to \Lambda B$ can, if the phase factors are fixed appropriately, be represented by a <u>linear</u> transformation. The proof of this, which is similar to one given earlier, will be deferred until the end of this section. This linearity, which means that the transformation is independent of the coefficients a and b , and does not change them, allows one to take as the second instance of <u>C1</u> the statement

<u>C1B</u>. For all $\Lambda \in L$ and all a and b satisfying (5.14) the right-hand side of (5.17) is numerically unchanged by the substitution

$$(K_a, S_a, K_b, S_b) \rightarrow (\Lambda K_a, \Lambda S_a, \Lambda K_b, \Lambda S_b) . \tag{5.19}$$

The restriction (5.14) is, of course, no actual limitation, due to a scaling invariance.

Experimental tests of relativistic invariance always involve interference between amplitudes of at least slightly differing momenta, and hence C1B, though more restrictive than C1A, is a special case of the type of relationship that is verified in practical tests of relativistic invariance.

The invariance of the first two terms on the right of (5.17) follows from (5.8). Consideration of the cases $a^* b = 1/2$ and $i/2$ give the separate invariance of the last two terms of (5.17). From the fact that the continuum of operators $P(K_a, K_b; S_a, S_b)$ span the spin space (see Appendix A) one obtains, using (4.18),

$$N^*(K_a)\, N(K_b) = \left[\Lambda_S^{-1}\, N(\Lambda K_a)\right]^* \left[\Lambda_S^{-1}\, N(\Lambda K_b)\right] . \tag{5.20}$$

It follows from this that the phase factor $\exp\left[i\, \alpha(\Lambda, K)\right]$ in (5.12) is the same for all sets of $K = (K_1, K_2)$ for which the K_2 are related by real rotations in accordance with (5.13). Similarly, one obtains the analogous result for the arguments K_1. If N vanishes the phase is, however, undefined.

Combining the above result with (5.12) one obtains

$$\begin{aligned} N(K) &= \Lambda_S^{-1}\, N(\Lambda K)\, \exp\left[i\, \alpha(\Lambda, K)\right] \\ &= \Lambda_S^{-1}\, N(\Lambda K)\, \exp\left[i\, \alpha(\Lambda, K')\right] \end{aligned} \tag{5.21}$$

provided K' may be carried to $K = \Lambda' K'$ by a sequence of rotations on K_1 and K_2 <u>individually</u> with the $N(K')$ nonzero at the various points on the sequence of values of K'.

Equation (5.21) may be used to establish a group property for $\exp[i\,\alpha(\Lambda, K)]$. Define K' so that $\Lambda' K' = K$, with $\Lambda' \in L_R$, the rotation subgroup of L. Then using (5.21), one obtains

$$\begin{aligned} N(\Lambda' K') \equiv N(K) &= \Lambda_S^{-1}\, N(\Lambda K)\, \exp\left[i\,\alpha(\Lambda, K')\right] \\ &= \Lambda_S^{-1}\, N(\Lambda\, \Lambda' K')\, \exp\left[i\,\alpha(\Lambda, K')\right] \\ &= \Lambda_S'\, N(K')\, \exp\left[-i\,\alpha(\Lambda', K')\right] . \end{aligned} \tag{5.22}$$

The last two lines may be rewritten

$$\begin{aligned} N(K')\, \exp\left[-i\,\alpha(\Lambda', K')\right] \exp\left[-i\,\alpha(\Lambda, K')\right] &= \Lambda_S'^{-1}\, \Lambda_S^{-1}\, N(\Lambda\Lambda' K') \\ &= (\Lambda\Lambda')_S^{-1}\, N(\Lambda\Lambda' K') \\ &= \Lambda_S''^{-1}\, N(\Lambda''\, K') \\ &= N(K')\, \exp\left[-i\,\alpha(\Lambda'', K')\right] . \end{aligned} \tag{5.23}$$

Comparison of the first and last lines gives

$$\exp\left[-i\,\alpha(\Lambda', K')\right] \exp\left[-i\,\alpha(\Lambda, K')\right] = \exp\left[-i\,\alpha((\Lambda\Lambda'), K')\right] , \tag{5.24}$$

provided the conditions for the validity of (5.21) are satisfied. Equation (5.24) is the group property. But the unique one-dimensional representation of the rotation group is the identity representation.

Hence if the conditions stated beneath (5.21) are satisfied then

$$\exp\left[-i\,\alpha(\Lambda,\,K)\right] \;=\; 1\,, \tag{5.25}$$

and the $N(K)$ are invariant spinor functions:

$$N(K) \;=\; \Lambda_S^{-1}\,N(\Lambda K)\,. \tag{5.26}$$

It is, in fact, sufficient to establish the group property over a finite but arbitrarily small neighborhood of the identity, since the entire rotation group can then be built up by multiplication. Thus a sufficient condition for the validity of (5.26) is that

$$N(\Lambda_1'\;K_1,\;\Lambda_2'\,K_2) \;\neq\; 0 \tag{5.27a}$$

for all

$$\Lambda_1' \;\epsilon\; N_r \tag{5.27b}$$

and

$$\Lambda_2' \;\epsilon\; N_r\,, \tag{5.27c}$$

where N_r is a neighborhood of the identity in L_r Hence continuity of $|N(K)|$ at K in the manifold generated by the individual rotations Λ_1' and Λ_2' is sufficient to guarantee (5.26). Indeed, continuity of $|N(\Lambda_1\,K)|$ in this manifold for any $\Lambda_1 \;\epsilon\; L$ is sufficient, since the result (5.26) with K replaced by $\Lambda_1 K$ implies (5.26) itself, by virtue of the group property of the Λ's; i.e.,

$$N(\Lambda_1\,K) \;=\; \Lambda_S^{-1}\,N(\Lambda\,\Lambda_1\,K) \tag{5.28}$$

with Λ set to $\Lambda'\,\Lambda_1^{-1}$ gives

$$N(\Lambda_1 K) = (\Lambda' \Lambda_1^{-1})_S^{-1} N(\Lambda' K)$$

$$= \Lambda_{1S} \Lambda_S'^{-1} N(\Lambda' K) ; \tag{5.29}$$

hence

$$\Lambda_{1S}^{-1} N(\Lambda_1 K) = \Lambda_S'^{-1} N(\Lambda' K) = N(K) , \tag{5.30}$$

where the special case $\Lambda' = 1$ has been used in the last step.

The conclusion is, then, that $N(K)$ is an invariant spinor function,

$$N(K) = \Lambda_S^{-1} N(\Lambda K) , \tag{5.31}$$

for all points K such that for some point $\Lambda_1 K$, $\Lambda_1 \in L$, $|N(\Lambda_1 K)|$ is continuous in the manifold generated by independent rotations of $\Lambda_1 K_1$ and $\Lambda_1 K_2$. Take Λ_1 to be the transformation to the center-of-mass frame. Then these rotations will keep the arguments in the manifold of definition defined by the conservation-law constraint. Thus $N(K)$ is an invariant spinor function wherever it is continuous within its manifold of definition.

Finally, we return to the deferred proof that the transformation $B_2 \to \Lambda B_2$ can, by appropriate choice of phases, be represented by a <u>linear</u> transformation. Let B_2' and B_2'' label two experiments and associated possible results corresponding to linear combinations of amplitudes of the type occurring in (5.15). The $(K_0)_{2a}$, $(K_0)_{2b}$, and T_2 are to be the same for all contributions, but the a and b and the spin variables can differ. The correlation function $\mathcal{C}(B', B'')$ is

$$\mathcal{C}(B_2', B_2'') = \left| \int S^*(K_2; B_2') S(K_2; B_2'') \right|^2 . \tag{5.32}$$

The final special case of C1 is then

C1C. For experimental results B_2' and B_2'' of the type specified above, and for any $\Lambda \in L$,

$$\mathcal{P}(\Lambda B_2', \Lambda B_2'') = \mathcal{P}(B_2', B_2'') \quad . \tag{5.33}$$

This condition is kinematic, like B5, rather than dynamic, like C1B, since the experiments B_2' and B_2'' do not differ by temporal displacements.

By virtue of the covariant form of the momentum space element, (5.33) can be written as

$$\left| \int S^*(\Lambda K_2; \Lambda B_2') \, S(\Lambda K_2; \Lambda B_2'') \right|^2$$

$$= \left| \int S^*(K_2; B_2') \, S(K_2; B_2'') \right|^2 . \tag{5.34}$$

The integrands appearing in (5.34) contribute only for $(K_0)_2 = (K_0)_{2a}$ or $(K_0)_{2b}$. The equation is thus again of the form (3.23), the spin indices now covering their finite range twice, once for K_{2a} and once for K_{2b}. The same argument as before [see (3.23)] shows that the transformation $B_2 \to \Lambda B_2$ must be represented by a linear transformation, aside from certain phase factors that drop out of an expression like (5.17). This linearity ensures that the general physical requirement C1 must, for the special physical case covered in C1B, be expressed by the algebraic relationship asserted in C1B.

The conclusion, then, is that if physical phenomena satisfy the physical relationships asserted by postulates A, B, and C then there necessarily exist functions $N(K)$ related to physical correlations in

the manner specified in Section III, satisfying unitarity relations that are easily derived, and satisfying the invariance Eq. (5.31), for $\Lambda \in L$, wherever they are analytic in their manifolds of definition. From stronger assumptions these conclusions would follow more easily; the objective was to use the weakest assumptions.

The subtraction from $N(K)$ of possible parts defined over submanifolds restricted by conservation laws involving subsets of particles leaves the so-called connected part $M_c(K)$. This separation will be discussed in the following paper.

VI. ANALYTICITY AND LORENTZ INVARIANCE

In this section some consequences of assuming that $M(K)$ is regular analytic at some physical point will be examined. The main result to be established is that if an M function is regular at some physical point then the complete analytic extension of the function is defined over a multisheeted manifold each sheet of which maps onto itself under any proper complex Lorentz transformation. Furthermore, the function defined (single valuedly) and regular over any sheet is invariant under proper complex Lorentz transformations. Finally, if M is regular at each point of some real domain containing only physical points then the sheets described above can be chosen so that all the points of this domain lie in a single sheet. These results have some important consequences, which will be mentioned at the end of the section.

The initial considerations will refer to a function $F(K)$ whose domain of definition is not restricted by the mass shell and conservation-law constraints. Also the type variables $T = \{ t_i \}$ will be considered fixed. Thus the argument of $F(K)$ will be a set of the type introdiced in Section III, but with the mass constraints and type variables removed. Furthermore the <u>argument</u> (K) will be distinguished from the <u>point</u> K (without brackets). The <u>point</u> K will mean simply the set of momentum-energy vectors $\{k_i\}$. But a function at a point K will mean the set of functions whose momentum-energy variables are the point K; all spin indices are allowed.

Let the following definitions be made.

Definition: L will denote the real proper orthochronous homogeneous Lorentz group. It is continuously connected to the identity.

Definition: $\mathcal{L}$ will denote the complex proper homogeneous Lorentz group. It is continuously connected to the identity.

Definition: Λ will represent a Lorentz transformation and

$$(\Lambda K) \equiv \{\Lambda k_i , m_i\}. \tag{6.1}$$

(The t_i are temporarily either suppressed or eliminated.)

Definition: The <u>point</u> K represents the set of momentum-energy vectors $\{k_i\}$, but a function at a point means the set of functions having momentum-energy variables specified by the point K; all spin indices are allowed.

Definition: Points K_1 and K_2 related by $K_1 = \Lambda K_2$ will be said to be <u>connected by Λ</u>.

Definition: The set of points connected to K by some $\Lambda \in \mathcal{L}$ (or L) will be denoted by $\mathcal{L}K$ (or $L K$).

Definition: The set of points connected to some element of the set D by some $\Lambda \in \mathcal{L}$ (or L) will be denoted by $\mathcal{L}D$ (or $L D$).

Definition: A point K is <u>real</u> if and only if the four vectors $\{k_i\}$ are real.

Definition: A <u>real set</u> is a set of real points.

Definition: A <u>function</u> $F(K)$ is a (<u>single-valued</u>) mapping to the complex numbers.

Definition: The spin indices of (K) will be presumed to have some spinor index-type label, and $\Lambda_s F(K)$ will represent the result of the action upon $F(K)$ of the corresponding spinor transformations associated with Λ, as discussed in Appendix A.

Lemma 1. If $F(K)$ is defined (single valuedly) over a real set D and satisfies for all $\Lambda \in \mathcal{L}$ and all K such that K and ΛK are elements of D the invariance condition

$$F(K) = \Lambda_s F(\Lambda^{-1} K) , \tag{6.2}$$

then (6.2) with $\Lambda^{-1} K \in D$ and $\Lambda \in \mathcal{L}$ defines a (single-valued) function over $\mathcal{L} D$, provided any two points of D connected by a real element of $\mathcal{L}$ are also connected by an element of L.

Proof: The prescription will uniquely define $F(K)$ at K' of $\mathcal{L} D$ if for any two points K_1 and K_2 of D for which $K' = \Lambda_1 K_1 = \Lambda_2 K_2$, with Λ_1 and $\Lambda_2 \in \mathcal{L}$, one has

$$\Lambda_{1s} F(K_1) = \Lambda_{2s} F(K_2) . \tag{6.3}$$

But by the group property $K_2 = \Lambda_2^{-1} \Lambda_1 K_1 \equiv \Lambda K_1$. Thus (6.2) gives

$$F(K_2) = \Lambda_{2s}^{-1} \Lambda_{1s} F(K_1) \tag{6.4}$$

provided $\Lambda \equiv (\Lambda_2^{-1} \Lambda_1) \in L$. Hence it is sufficient to show that Λ is an element of L . If the rank $r(K_1)$ of the Gram determinant G $(G_{ij} = k_i \cdot k_j)$ at the point K_1 is four, or equivalently[4] if there are four linearly independent vectors among the vectors of K_1 , then the rank is also four at K_2 , since inner products are unchanged, and the same four vectors are also linearly independent at K_2 . In this case the linear transformation Λ is unique. Since K_1 and K_2 are real, Λ is a real element of $\mathcal{L}$. By hypothesis it is then, by virtue of its uniqueness, an element of L . This completes the proof for the case $r(K_1) = 4$. For $r(K_1) = 3$ the transformation Λ is still unique[4] and the same argument holds.

If $r(K_1) < 3$ then the transformation Λ is not always uniquely defined by the equation $K_2 = \Lambda K_1$ and it may not be real, as required for the above argument. There are several cases. If the rank $r(K_1)$ is equal to $n(K_1)$, the number of linearly independent vectors of K_1 , then the space separates into a manifold $M(K_1)$ of dimension $n(K_1) = r(K_1)$ spanned by the set K_1 and the orthogonal manifold $M^{\perp}(K_1)$. One can construct a set of real orthogonal basis vectors $e_\rho(K_1)$, each of length plus or minus one, such that the first n span $m(K_1)$ and the last $(4 - n)$ span $M^{\perp}(K_1)$. To construct such a basis one first takes $n(K_1)$ linearly independent real vectors from the set K_1 . This set is augmented by $(4 - n(K_1))$ real vectors to give a complete set of real linearly independent vectors. Because the rank $r(K_1)$ equals $n(K_1)$ the linear equations arising in the construction of $e_\rho(K_1)$ are soluble.. The

details have been given by Hall and Wightman.[4] Since the original vectors are, for us, real the coefficients in the linear equations are real and hence the solutions can be taken to be real.[9] A similar real basis, $e_\rho(K_2)$, can be constructed for K_2.

Our interest is in the various Lorentz transformations Λ' satisfying $K_2 = \Lambda' K_1$, the K_1 and K_2 being the fixed points of D connected by $\Lambda \in \mathcal{L}$; $K_2 = \Lambda K_1$. The transformations Λ' can be represented by the matrices $\Lambda''_{\rho\sigma}$ defined by

$$\Lambda' e_\rho(K_1) \equiv e^\sigma(K_2)\, \Lambda''_{\sigma\rho} \equiv e_\sigma(K_2)\, G^{\sigma\tau}(K_2)\, \Lambda''_{\tau\rho} , \tag{6.5}$$

where a summation convention is used. The labels ρ, σ, and τ specify the basis vectors, not components, and

$$G^{\sigma\rho}(K_j) \equiv e^\sigma(K_j) \cdot e^\rho(K_j) = \pm\, \delta_{\sigma\rho} \text{ for } j = 1 , 2 \tag{6.6}$$

For either value of j three of the vectors $e^\sigma(K_j)$ have length minus one and the other has length plus one.[10] That all four have length minus one is impossible because any vector v can be expanded as

$$v = v_\sigma (K_j)\, e^\sigma (K_j) \tag{6.7}$$

with

$$v_\sigma(K_j) = e_\sigma(K_j) \cdot v = e_\sigma{}^\mu(K_j)\, v_\mu , \tag{6.8}$$

where μ labels the component of the vector. Then

$$v \cdot v \equiv v_\mu G^{\mu\nu} v_\mu = v_\rho(K_j) G^{\rho\sigma}(K_j) v_\sigma(K_j) \quad . \tag{6.9}$$

If the negative sign were always to occur in (6.6) then all vectors represented by real $v_\rho(K_j)$ would have negative length. But the vector v with components $v_\mu = \delta_{\mu 0}$ has real v_ρ and positive length $[G^{\mu\nu} = (1, -1, -1, -1)]$, which is a contradiction. On the other hand if there were two real orthogonal vectors v^1 and v^2 of length plus one then

$$(v_0{}^1)^2 - |\underset{\sim}{v}^1|^2 = 1, \qquad (v_0{}^2)^2 - |\underset{\sim}{v}^2|^2 = 1, \tag{6.10}$$

and

$$v_0{}^1 v_0{}^2 = \underset{\sim}{v}^1 \cdot \underset{\sim}{v}^2 \quad . \tag{6.11}$$

From these it would follow that

$$(\underset{\sim}{v}^1 \cdot \underset{\sim}{v}^2)^2 = (1 + |\underset{\sim}{v}^1|^2)(1 + |\underset{\sim}{v}^2|^2), \tag{6.12}$$

and hence that

$$(\underset{\sim}{v}^1 \cdot \underset{\sim}{v}^2) > |\underset{\sim}{v}^1|^2 \, |\underset{\sim}{v}^2|^2 , \tag{6.13}$$

which is not possible for real vectors. Thus there is, for each j, precisely one vector $e^\sigma(K_j)$ of length plus one. Because of this the vectors $e^\sigma(K_j)$ can be generated from the original set of basis vectors by real Lorentz transformations. The transformation Λ_b connecting the two sets

$$\Lambda_b \, e^{\sigma}(K_1) \;=\; e^{\sigma}(K_2) \tag{6.14}$$

will then also be a real Lorentz transformation.

The basis set $e^{\sigma}(K_2)$ is not completely specified by this construction. It is possible to take the first n vectors (which may or may not include the one of positive length) to be given by

$$e^{\sigma}(K_2) \;=\; \Lambda \, e^{\sigma}(K_1) \qquad \sigma \;=\; (1,\, 2,\, \cdots,\, n) \;. \tag{6.15}$$

For, since $K_2 = \Lambda K_1$, these vectors span the space $M(K_2)$. They are orthogonal, since the $e^{\sigma}(K_1)$ are and $\Lambda \in \mathcal{L}$. And they are real, since Λ takes all the real vectors of K_1 into the real vectors of K_2, and hence by linearity all real vectors of $M(K_1)$ into real vectors of $M(K_2)$. Because $n(K_1) < 4$, one can by proper choice of the sense of the vectors $e^{\sigma}(K_1)$ with $\sigma > n(K_1)$ make Λ_b a real element of $\mathcal{L}$.

With the basis vectors fixed in this way it is clear that the basis vector of positive length occurs either in the first n vectors of both sets $e^{\sigma}(K_2)$ and $e^{\sigma}(K_1)$ or in the last $(4 - n)$ vectors of both sets. Also, with this choice the first n-by-n submatrix of $(\Lambda'')^{\sigma}{}_{\rho}$ is the n-by-n unit matrix. Since Λ' takes all vectors of $M(K_1)$ into vectors of $M(K_2)$ the first n columns of Λ'' have zeros except in the diagonal positions. The same property holds also for the first n rows as a consequence of the relations $\Lambda'^{-1} K_2 = K_1$ and

$$(G\,\Lambda''\,G)_{\rho\sigma} = (\Lambda''^{-1})_{\sigma\rho} \quad , \tag{6.16}$$

which is the characteristic property of Lorentz transformations. That Λ'' is a Lorentz transformation follows from (6.5) and (6.14); one obtains

$$\Lambda_b^{-1}\,\Lambda'\,e_\rho(K_1) = e_\sigma(K_1)\Lambda''^{\sigma}{}_{\rho} \qquad \Lambda''\,e_\rho(K_1) \quad , \tag{6.17}$$

which shows that $\Lambda'' \equiv \Lambda_b^{-1}\,\Lambda'$. Since Λ_b is real, the transformation Λ'' will be real if Λ' is.

The conclusion from the above remarks is that for the case $n(K_1) = r(K_1)$ all Lorentz transformations $\Lambda' \in \mathcal{L}$ satisfying $K_2 = \Lambda' K_1$ with K_1 and $K_2 \in D$, and with $K_2 = \Lambda K_1$ for some $\Lambda \in \mathcal{L}$, can be represented in the form

$$\Lambda' = \Lambda_b\,\Lambda'' \tag{6.18}$$

with a fixed real $\Lambda_b \in \mathcal{L}$ and a $\Lambda'' \in \mathcal{L}$ differing from the identity only in the $(4 - n)$-by-$(4 - n)$ subspace corresponding to $M^{\perp}(K_1)$. And conversely, for all $\Lambda'' \in \mathcal{L}$ satisfying this property, which we call P , the transformation $\Lambda' \equiv \Lambda_b\,\Lambda''$ is an element of $\mathcal{L}$ satisfying $\Lambda' K_1 = K_2$.

This result is used in the following way: The transformations $\Lambda'' \in \mathcal{L}$ satisfying P can be parameterized in such a way that the matrix elements $(\Lambda''_s)^{\sigma}{}_{\rho}$ are analytic functions of these parameters regular in a neighborhood N of the identity, and such that real parameters give there real $\Lambda'' \in L$. Such a parameterization has been given by Jost,[11] for the case with no constraint P . The restriction to a

submatrix is accomplished by setting some of his parameters to zero. Now suppose first that $\Lambda_b \in L$. Then the hypothesis of the lemma gives

$$F(K_2) = \Lambda'_s F(K_1) = \Lambda_{bs}^{-1} \Lambda''_s F(K_1) \quad , \tag{6.19}$$

for all $\Lambda'' \in L$ satisfying P. For then $K_2 = \Lambda' K_1$, with $\Lambda' \in L$, the K_1 and K_2 being the fixed points of D connected by $K_2 = \Lambda K_1$, with $\Lambda \in \mathcal{L}$. But the validity of this equation for real values of the parameters of Λ'', together with regularity in N, implies its validity throughout N. Thus (6.19) is true for $\Lambda'' \in \mathcal{L}$ satisfying P, in a neighborhood of the identity. The restriction P does not destroy the group property, since products of matrices having this property will also have it, and inverses of matrices having this property must also have it. Using the fact that the subgroups of $\mathcal{L}$ specified by the constraints P are connected, or more specifically, that any element of $\mathcal{L}$ satisfying P can be expressed as a product of a finite number of elements of $\mathcal{L}$ satisfying P from any fixed neighborhood of the origin, one obtains the result that (6.19) is true for all $\Lambda' \in \mathcal{L}$ satisfying $\Lambda' K_1 = K_2$. This ensures the validity of (6.4), from which the lemma follows, for the case $n(K_1) = r(K_1)$, provided Λ_b is an element of L.

In the above argument it was supposed that Λ_b was an element of L; then for $\Lambda'' \in L$ it followed that $\Lambda' \in L$, and (6.2) was immediately applicable. Now Λ_b is by construction a real element of $\mathcal{L}$ satisfying $\Lambda_b K_1 = K_2$. Thus, by virtue of the hypothesis of the lemma, there exists some $\Lambda' \in L$ such that $\Lambda' K_1 = K_2$. For this

Λ' the transformation $\Lambda'' = \Lambda_b^{-1} \Lambda'$ must be a real element of $\mathcal{L}$. Thus it is either an element of L or it can be written in the form $\Lambda'' = \Lambda_0 \Lambda_1''$, where Λ_1'' is an element of L and Λ_0 is the PT ($\equiv$ CPT) transformation, which is a real element of $\mathcal{L}$. Parameterizing $\Lambda_1'' \in L$ instead of Λ'' one can develop the same argument as before and prove, from the validity of (6.14) for the $\Lambda' \in L$ just introduced, its validity for all $\Lambda' \in \mathcal{L}$ satisfying $\Lambda' K_1 = K_2$. This again validates (6.4), and completes the proof of the lemma for this case $n(K^1) = r(K^1)$.

The remaining possibility is $n(K_1) > r(K_1) < 3$. For these cases the vectors of K_1 are linear combinations of $r(K_1)$ orthogonal vectors of nonzero length and a single vector of zero length orthogonal to these. The $r(K_1)$ vectors of nonzero length are obtained by first picking $r(K_1)$ of the vectors of K_1 such that the Gram determinant of these $r(K_1)$ vectors is nonvanishing. This is always possible.[4] If any one of these vectors has nonzero length then normalize it to plus or minus one, by multiplying by a real scalar, and let it be the first vector of a real basis. If on the other hand all these vectors have zero length then some real multiple of a combination of the form $(k_i + k_j)$ must have length plus or minus one, since the (Gram) determinant of the matrix $(G_{ij}) = (k_i \cdot k_j)$ is nonvanishing. Subtracting a real multiple of this normalized vector from the other vectors, in the usual way, one gets a set of $(r(K_1) - 1)$ vectors orthogonal to it. Since the Gram determinant is still nonvanishing

the process can be repeated to give a real orthonormalized (i.e. to plus or minus one) set of $r(K_1)$ vectors $(e^1(K_1), \cdots, e^{r(K_1)}(K_1))$. This same construction was used (though not described) in the case $r(K_1) = n(K_1)$.

Since in the present case $n(K_1) > r(K_1)$, there must be a vector of K_1 that is linearly independent of these first $r(K_1)$ vectors.. By subtracting from it multiples of the $e^\sigma(K_1)$ $(\sigma = 1, \cdots, r(K_1))$ a linearly independent vector w orthogonal to them can be obtained. Since the value of the Gram determinant is unaltered by adding linear combinations of certain of the vectors to others the Gram determinant of the first $r(K_1)$ vectors together with w must vanish. But then w must have zero length. The next step is to augment the set K_1 by adding $(4 - r(K_1))$ real vectors that together with the first $r(K_1)$ basis vectors give four linearly independent vectors. Since $n = 4$ implies[4] $r = 4$ one can complete the construction of a complete set of real orthonormalized basis vectors $e^\sigma(K_1)$, using the procedure just described.

The vector w is orthogonal to the first $r(K_1)$ of the $e^\sigma(K_1)$ and hence it is a linear combination of the remaining ones. Since it is real and of zero length it must, for the case $r(K_1) = 2$, be of the form

$$w = a\,(e^0(K_1) \pm e^3(K_1)) \quad , \tag{6.20}$$

where $a \neq 0$ is real and $e^0(K_1)$ is the basis vector having positive

length. That the coefficients of the $e^{\sigma}(K_1)$ are real for real w follows from the existence of the real inverse of the real Lorentz transformation generating the $e^{\sigma}(K_1)$ from the original basis vectors.

The sign of $\pm e^3(K_1)$ in (6.20) depends on the sense of the vector $e^3(K_1)$. However, only one sign is possible; if different vectors of K_1 were to give w's having different signs in (6.20), then one would have $n(K_1) = r(K_1) + 2 = 4$, whish is impossible since $n(K_1) = 4$ implies $r(K_1) = 4$.

For the case $r(K_1) = 1$ the vector w must be of the form

$$w = a\Big(e^0(K_1) + \sin\theta\, e^2(K_1) + \cos\theta\, e^3(K_1)\Big) , \qquad (6.21)$$

with a and θ real and $a \neq 0$. Moreover, for this case all vectors of K_1 must, when the part along $e^1(K_1)$ is removed, give multiples of this same vector w. To see this, note that the Gram determinant of two vectors w and w' of the form (6.21) is

$$G(w, w') = a\, a'\Big(1 - \cos(\theta - \theta')\Big)^2 , \qquad (6.22)$$

which is different from zero unless w' is a multiple of w. Thus if two vectors w and w' of the form (6.21) can be obtained as linear combinations of the vectors of (K_1), then either w' is a multiple of w or $r(K_1) \geqslant 2$. The second possibility contradicts the assumption $r(K_1) = 1$. The form (6.21) can be brought to the form (6.20) by a redefinition of the basis vectors that leaves them real and orthonormalized.

In the case $r(K_1) = 0$ all the vectors of K_1 are of zero length and they are mutually orthogonal. Expanding them in terms of an arbitrary real orthonormalized basis $e^{\sigma}(K_1)$, each one has the form

$$w = a\left(e^0(K_1) + \alpha\, e^1(K_1) + \beta\, e^2(K_2) + \gamma\, e^3(K_3)\right) \quad , \tag{6.23}$$

where a and (α, β, γ) are real and

$$\alpha^2 + \beta^2 + \gamma^2 = 1 \quad . \tag{6.24}$$

If $w \neq 0$, then any vector w' of the same form for which

$$G(w, w') = 0 \tag{6.25}$$

is, as before, a multiple of w. Thus for all the cases $n(K_k) > r(K_1)$ one can construct a real orthonormalized basis $e^{\sigma}(K_1)$ such that the vectors of K_1 are real linear combinations of a zero-length vector

$$w = e^0(K_1) + e^3(K_1) \tag{6.26}$$

and the vectors

$$e^{\sigma}(K_1) \quad : \quad (\sigma = 1, \cdots, r(K_1) < 3) \, . \tag{6.27}$$

A similar basis can be constructed for K_2. The set K_2 is related to the set K_1 by the relation $K_2 = \Lambda K_1$, $\Lambda \in \mathcal{L}$. Since Λ is not necessarily real the vectors $\Lambda\, e^{\sigma}(K_1)$ need not be real. However for $\sigma = 1, \cdots, r(K_1)$ these vectors must be real; the basis

vectors $e^{\sigma}(K_1)$ can be expressed as real linear combinations of vectors of K_1 and hence the $\Lambda\, \sigma^{\sigma}(K_1)$ will be the same linear combination of the corresponding vectors $\Lambda\, K_1$ or $\Lambda\, K_2$, and hence also real. They have a Gram determinant of rank $r(K_2) = r(K_1)$ and are orthogonal and of length minus one and hence they can be chosen to be the corresponding $e^{\sigma}(K_2)$:

$$e^{\sigma}(K_2) \equiv \Lambda\, e^{\sigma}(K_1) \qquad \left(\sigma = 1, \cdots, r(K_1)\right) . \tag{6.28}$$

The entire set of real vectors $e^{\sigma}(K_2)$, constructed in the same manner as the $e^{\sigma}(K_1)$, and using (6.28) for $\sigma = (1, \cdots, r(K_1))$, can be related to the set $e^{\sigma}(K_1)$ by the equation

$$e^{\sigma}(K_2) = \Lambda_b\, e^{\sigma}(K_1) \quad , \tag{6.29}$$

where Λ_b is a real Lorentz transformation uniquely defined by this equation, once $e^{\sigma}(K_1)$ and $e^{\sigma}(K_2)$ are picked.

All real vectors of zero length in $M(K_1)$, the manifold spanned by the vectors of K_1, are multiples of the single vector

$$w(K_1) = e^{0}(K_1) + e^{3}(K_1) , \tag{6.30}$$

since any real linear combination of the vectors of (6.27) is orthogonal to $w(K_1)$ and of nonzero length unless zero. Similarly all real zero-length vectors of $M(K_2)$ are multiples of

$$w(K_2) = e^{0}(K_2) + e^{3}(K_2) . \tag{6.31}$$

Since $w(K_1)$ is a linear combination of the vectors of K_1 the vector $\Lambda\, w(K_1)$ is in $M(K_2)$, the manifold spanned by the vectors of K_2. But then $\Lambda\, w(K_1)$ is a real nonzero vector of zero length in $M(K_2)$. Hence it is a multiple $w(K_2)$:

$$w(K_2) = c\,\Lambda\, w(K_1) \neq 0 . \tag{6.32}$$

The factor c can be taken to be unity. This follows from the fact that a real Lorentz transformation in the $(0, 3)$ subspace gives simply a scale transformation of a vector of the form (6.31):

$$\begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \cosh\alpha + \sinh\alpha \\ \sinh\alpha + \cosh\alpha \end{pmatrix} . \tag{6.33}$$

This transformation preserves the reality and orthonormality properties of the $e^{\sigma}(K_1)$. Thus it can, and will, be assumed that the basis $e^{\sigma}(K_2)$ is chosen so that

$$c = 1 , \tag{6.32a}$$

or equivalently, that

$$e^0(K_2) + e^3(K_2) = \Lambda\, e^0(K_1) + \Lambda\, e^3(K_1) . \tag{6.32b}$$

Using (6.29) one obtains, then,

$$e^0(K_1) + e^3(K_1) = \Lambda_b^{-1}\Lambda\left[e^0(K_1) + e^3(K_1)\right] . \tag{6.34}$$

The general form of the Lorentz transformation $\Lambda_b^{-1}\Lambda \in \mathcal{L}$

satisfying (6.34) is readily computed. If the rows and columns are placed in the order (0, 3, 2, 1), the general transformation matrix $(\Lambda'')_{\rho}{}^{\sigma}$ defined by

$$\Lambda'' e^{\sigma}(K_1) = e^{\rho}(K_1)(\Lambda'')_{\rho}{}^{\sigma} , \tag{6.35}$$

and consistent with (6.34), with $\Lambda'' \in \mathcal{L}$ in place of the fixed $\Lambda_b{}^{-1}\Lambda$, can be written

$1 + a$	$-a$	c	f
a	$1 - a$	c	f
$(c \cos\theta + f \sin\theta)$	$-(c \cos\theta + f \sin\theta)$	$\cos\theta$	$\sin\theta$
$(f \cos\theta - c \sin\theta)$	$-(f \cos\theta - c \sin\theta)$	$-\sin\theta$	$\cos\theta$

(6.36a)

where c, f, and θ are arbitrary complex numbers and

$$2a = c^2 + f^2 . \tag{6.36b}$$

The condition (6.34) imposes the constraint that the first two columns are the negatives of each other, aside from the unit contributions on the diagonal. This gives four conditions, only three of which are independent of Lorentz transformation condition (6.16).

Since the relations (6.28) and (6.34) are maintained if Λ is replaced by any Λ' satisfying $\Lambda' K_1 = K_2$, of which one is Λ_b, the general form of $(\Lambda'')_{\rho}{}^{\sigma} \equiv (\Lambda_b{}^{-1}\Lambda')_{\rho}{}^{\sigma}$ defined by (6.35), with $\Lambda' K_1 = K_2$, is given by (6.36) with the last $r(K_1)$ rows and columns having unity in the diagonal position and zeros elsewhere, provided $\Lambda'' \in \mathcal{L}$.

It can be assumed that $\Lambda_b \in \mathcal{L}$. If $r(K_1) < 2$ then there is freedom in the sign of at least one $e^\sigma(K_2)$, and Λ_b can be made a proper transformation. Then Λ_b will be a real element of $\mathcal{L}$. For the other case, $r(K_2) = 2$, the basis $e^\sigma(K_2)$ is uniquely specified by the conditions that have been imposed, and one cannot adjust Λ_b. However, in this case the conditions on $(\Lambda'')_\rho{}^\sigma$ require it to be unity even without the condition $\Lambda'' \in \mathcal{L}$, for one then has $c = f = \theta = 0$ from the conditions on $e^\sigma(K_1)$ for $\sigma = (1, 2)$, and condition (6.34) then gives the unique solution $\Lambda_b = \Lambda \in \mathcal{L}$.

To complete the argument for the case $n(K_1) > r(K_1)$ one first notes that Λ_b is a real element of $\mathcal{L}$ satisfying $\Lambda_b K_1 = K_2$. Thus there must, by hypothesis, exist some $\Lambda' \in L$ satisfying $\Lambda' K_1 = K_2$. But then

$$F(K_2) = (\Lambda_b \Lambda'')_s F(K_1) \tag{6.37}$$

is valid when $\Lambda'' \equiv \Lambda_b{}^{-1}\Lambda'$ corresponds to this $\Lambda' \in L$. Since Λ_b is a real element of $\mathcal{L}$ either Λ'' is an element of L or $\Lambda_0{}^{-1}\Lambda'' = \Lambda_1''$ is, where Λ_0 is the PT ($\equiv$ CPT) transformation. Then Λ'' for Λ_1'', whichever is in L, can be parameterized as in (6.36), with the appropriate constraints if $r(K_1) > 0$. For a neighborhood of real values of the parameters, subject to these constraints, one still has $\Lambda' K_1 = K_2$ with $\Lambda' \in L$. But the spinor transformation

$$\Lambda_s' \equiv \Lambda_{bs} \Lambda_s'' \qquad \text{or} \qquad (\Lambda_b \Lambda_0)_s (\Lambda_1'')_s \tag{6.38}$$

is an analytic function of these parameters, regular in a neighborhood of the origin of the free variables of (c, f, θ). Since (6.37) is true for real values of these variables it is also valid for complex values in this neighborhood. One sees by inspection of (6.36) that the set of Λ'' satisfying the conditions corresponding to $\Lambda' K_1 = K_2$, $\Lambda' \in \mathcal{L}$, is a connected set of transformations in $\mathcal{L}$. From this it follows that any element of the set can be expressed as a product of a finite number of elements of the set lying within any neighborhood of the identity, and hence that (6.37) is valid for all $\Lambda' \in \mathcal{L}$ satisfying $\Lambda' K_1 = K_2$. This validates (6.4) for this last case and completes the proof of Lemma 1.

Lemma 1A. Real points connected by a Lorentz transformation $\Lambda \in \mathcal{L}$ are connected by some real $\Lambda \in \mathcal{L}$.

Proof: The transformation Λ_b constructed in the course of the proof of Lemma 1 is the required real $\Lambda \in \mathcal{L}$.

Lemma 2. Let K_0 be a set of n linearly independent vectors. For any neighborhood N of the identity in $\mathcal{L}$ there is a neighborhood $D(N, K_0)$ of K_0 such that any two points in $D(N, K_0)$ connected by a Lorentz transformation are connected by a Lorentz transformation $\Lambda \in N$.

Proof: Suppose the rank of the Gram determinant of the vectors of the set $K_0 \equiv \left\{ k_i^{\,0} \right\}$ is $r(K_0) = r$. One can arrange the vectors of K_0

such that the rank of the Gram determinant of the first r vectors of the set is r. Using the procedure discussed in Lemma 1, but without the reality condition, a set of r orthonormal basis vectors $e_1(K_0), \cdots, e_r(K_0)$ can be constructed as linear combinations of the first r vectors of K. Completing the set K_0 to a set of four linearly independent vectors by the addition of $(4 - n)$ new vectors one can construct $(4 - n)$ more vectors $e_{n+1}(K_0), \cdots, e_4(K_0)$ that are orthonormal, and orthogonal to the first r of the basis vectors. For the case $r = n$ this gives a complete set of basis vectors $e_\sigma(K_0)$.

For the case $n = r + 1$ the subtraction from k^0_{r+1} of its components along $e_1(K_0), \cdots, e_r(K_0)$ leaves a vector $w_0 = w \neq 0$, which must be of zero length, since otherwise the rank r would be n. For some $\sigma > n$ one must have $e_\sigma(K_0) \cdot w \neq 0$, since otherwise w_0 would be a zero-length vector orthogonal to three orthonormal vectors in a four-dimensional (nondegenerate[4]) space and hence zero. Take this vector $e_\sigma(K_0)$ to be $e_4(K_0)$. Then $i\left[e_4(K_0) - w\,(e_4(K_0) \cdot w)^{-1}\right]$ is a vector of unit length orthogonal to $e_4(K_0)$ and to $e_1(K_0), \cdots, e_r(K_0)$. Take this to be the final basis vector $e_{r+1}(K_0)$.

For the case $r = n - 2$ the subtraction of components along $e_1(K_0), \cdots, e_r(K_0)$ from the vectors k^0_{r+1}, k^0_{r+2} must leave two linearly independent orthogonal vectors w_{r+1} and w_{r+2} having zero length. Otherwise there would be fewer than n linearly independent vectors, or the rank of the vectors of K_0 would be greater than r. The vectors $w \equiv w_{r+1}$ and $\bar{w}' \equiv w_{r+2}$ cannot both be orthogonal to $e_\sigma(K_0)$ for all $\sigma > n$, for then they would be orthogonal to two

orthonormal vectors. This would provide two linearly independent zero-length vectors in a two-dimensional space, which is impossible.[4] One can order the vectors of K_0 and of the $e_\sigma(K_0)$, $\sigma > n$, so that $w \cdot e_4(K_0) \neq 0$. Then the vector $i\left[e_4(K_0) - w\left(e_4(K_0)\cdot w\right)^{-1}\right]$ is a vector of unit length orthogonal to the vectors $e_1(K_0),\cdots,e_r(K_0)$, and to $e_4(K_0)$. Let it be called $e_{r+1}(K_0)$. The vectors $e_\sigma(K_0)$, $4 > \sigma > n$, can then be reorthonormalized following the standard procedure so that the $e_\sigma(K_0)$ for $\sigma \leqslant r + 1$ and $\sigma > n$ become an orthonormal set. If the original $e_\sigma(K_0)$, $\sigma > n$, are appropriately chosen the subtractions of the required vectors will not give any zero-length vectors.

From the relation $\bar{w}' \cdot e_{r+1}(K_0) = i\left(\bar{w}' \cdot e_4(K_0)\right)$ it follows that $\left[\bar{w}' - e_4(K_0)\left(e_4(K_0) \cdot \bar{w}'\right) - e_{r+1}(K_0)\left(e_{r+1}(K_0) \cdot \bar{w}'\right)\right] \equiv \bar{w}$ is a zero-length vector orthogonal to $e_4(K_0)$, $e_1(K_0),\cdots, e_{r+1}(K_0)$. It cannot vanish since $\bar{w}'$ is linearly independent of w whereas $e_4(K_0)\left(e_4(K_0) \cdot \bar{w}'\right) + e_{r+1}\left(e_{r+1}(K_0) \cdot \bar{w}'\right)$ is proportional to w . For some vector $e_\sigma(K_0)$, $\sigma > n$, one must have $e_\sigma(K_0) \cdot \bar{w} \neq 0$. Otherwise $\bar{w}$ would be a zero-length vector orthogonal to the first $r + 1$ basis vectors and the last $4 - n$ basis vectors and hence orthogonal to $4 - n + r + 1 = 3$ orthonormal basis vectors. Let this $e_\sigma(K_0)$ be $e_3(K_0)$, since it is not $e_4(K_0)$. Then the vector $i\left[e_3(K_0) - \bar{w}\left(e_3(K_0) \cdot \bar{w}\right)^{-1}\right] \equiv e$ is a vector of unit length orthogonal to all $e_\sigma(K_0)$ with $\sigma \leqslant r + 1$ or $\sigma \geqslant 3$, where these vectors are all orthonormal. This is impossible unless $r = 0$, since a vector orthogonal to four orthonormal vectors is zero.

Thus one can set $e_2(K_0) = e$. This completes the construction of the orthonormal basis $e_\sigma(K_0)$ for the case $n = r + 2$. The case $n > r + 2$ is not possible.

For K in a sufficiently small neighborhood of K_0 one can construct a basis $e_\sigma(K)$ following the procedure just described, except for the following changes: The $(4 - n)$ vectors that are added to the set K to make a linearly independent set will, for all K , be taken to be the fixed vectors $e_\sigma(K_0)$ for $\sigma > n$, constructed above. For K in a sufficiently small neighborhood $D'(K_0)$ of K_0 the augmented set will continue to have four linearly independent vectors, and one can proceed with the construction; one constructs a set $e_\sigma(K)$, $\sigma > n$, by subtracting in the standard way the components along $e_\sigma(K)$, $\sigma \leqslant r$, etc. and normalizing. For $K \in D''(K_0) \subset D'(K_0)$ the vectors arising in this procedure will have nonzero length, so that a uniform procedure can be followed for all $K \in D''(K_0)$. At the next stage the vectors $e_{r+1}(K)$ [and $e_{r+2}(K)$] can be defined in the same way as above except that additional normalization factors η (and $\bar{\eta}$) must be supplied. For K in a sufficiently small neighborhood $D'''(K_1) \subset D''(K_1)$ the various factors that are required to be nonvanishing will continue to be nonvanishing, since they will depend continuously on the vectors of K . The only ambiguity in the procedure is in the choice of sign for the normalization factors. This sign can be fixed by requiring the normalization factors to be continuous functions of K . Thus in a sufficiently small neighborhood $D(K_0)$ of K_0 a basis $e_\sigma(K)$ can be defined so that these basis vectors depend continuously on the vector K. Also, for the case

$r = n - 1$ the vector w obtained by subtracting from $k_n^0 \equiv k_{r+1}^0$ its components along $e_\sigma(K)$ $\sigma = 1, \ldots, r$, will always have the standard form $w = (e_4(K) + i\eta e_{r+1}(K))(w \cdot e_4(K))$. For the case $r = n - 2 = 0$ this vector will have the form $w = (e_4(K) + i\eta e_1(K))(w \cdot e_4(K))$, and the other vector, $\bar{w}$, will have the standard form $\bar{w} = (e_3(K) + i\bar{\eta} e_2(K))(\bar{w} \cdot e_3(K))$.

For <u>any</u> two vectors K_1 and K_2 in $D(K_0)$ a Lorentz transformation $\Lambda(K_1, K_2) \in \mathcal{L}$ is defined by the equation

$$e_\sigma(K_1) = \Lambda(K_1, K_2)\, e_\sigma(K_2). \tag{6.34}$$

If K_1 and K_2 are connected by a Lorentz transformation then $K_1^r = \Lambda(K_1, K_2) K_2^r$, where K^r is the set consisting of the first r vectors of K. This is because the vectors $e_\sigma(K)$ are constructed, following a standardized procedure, as a linear combination of the vectors of K^r, and the coefficients are given as functions only of the inner products of the vectors of K^r. For the case $r = n$, $K^r = K^n$ and this transformation connects K_1 to K_2. Since the transformation $\Lambda(K_1, K_2)$ is a continuous function of (K_1, K_2) the inverse image of any open set in N containing the identity contains a neighborhood of the point (K_0, K_0). This neighborhood must contain a neighborhood of the form $K_1 \in D(N, K_0)$, $K_2 \in D(N, K_0)$, with $D(N, K_0) \subset D(K_0)$. This $D(N, K_0)$ satisfies the requirements of the lemma for the case $r = n$.

For the cases $r < n$ any points K_1 and $K_2 \in D(K_0)$ connected by a Lorentz transformation are connected by a Lorentz transformation of the form

$$K_1 = \Lambda^w \Lambda (K_1, K_2) K_2 \equiv \Lambda^w K_2' , \tag{6.40}$$

where $\Lambda^w K_1^r = K_1^r$. For the subcase $r = n - 1$ the K_1 and K_2' differ only in the value of the vector w , and both values, w_1 and w_2' , lie in the $(e_3(K_1); e_4(K_1))$ subspace. But two vectors in a subspace connected by a Lorentz transformation are connected by a Lorentz transformation in the subspace. This is a consequence of Lemma 2 of Hall and Wightman.

The Lorentz transformations in a two-dimensional subspace can be expressed as a product of possible inversions about the space or time axis times a transformation

$$a_\pm \rightarrow (\exp \pm \Gamma) \; a_\pm = \Lambda (\Gamma) \, a_\pm \quad , \tag{6.41}$$

where Γ is a complex number and the $a_\pm$ are components along two orthogonal light-cone vectors. If two points are connected by a transformation of the form $\Lambda(\Gamma)$ then this transformation is unique.

If two points are in a neighborhood of the point $(a_+, a_-) = (1, 0)$ that contains no point with $a_+ = 0$ then if they are connected by any Lorentz transformation they are also connected by a $\Lambda (\Gamma)$. This is because for the case $a_- (0) \neq 0$ one can transform--using a $\Lambda(\Gamma)$--to a point where $a_+ = \pm a_-$. At such a point the reflections are equivalent either to the identity or to the particular $\Lambda (\Gamma)$ given by $\exp \Gamma = \exp(- \Gamma) = - 1$. As a consequence of this any sequence of

reflections and proper transformations can be reduced to a single transformation $\Lambda(\Gamma)$, for this case, by the elimination of reflections in pairs. On the other hand if $a_- = 0$ any product of reflections and $\Lambda(\Gamma)$ takes the point to a point with $a_- = 0$, which can be reached by $\Lambda(\Gamma)$ alone, or to a point with $a_+ = 0$, which by assumption is not in the original domain. Thus with the neighborhood taken small enough so that points $a_+ = 0$ are not included all points in the neighborhood connected by a Lorentz transformation are connected by a unique transformation of the form $\Lambda(\Gamma)$. One can therefore define a unique $\Lambda_1(K_1, K_2) = \Lambda(\Gamma)\Lambda(K_1, K_2)$ that satisfies

$$K_1 = \Lambda_1(K_1, K_2) K_2$$

This transformation is a uniquely defined and continuous function of the K_1 and K_2, provided the (K_1, K_2) is restricted to a sufficiently small neighborhood of (K_0, K_0).

In case K_1 and K_2 are not connected by a Lorentz transformation Eq.(6.41) can be modified by the inclusion of a scale factor λ defined by

$$a_\pm \rightarrow \lambda(\exp \pm \Gamma) a_\pm \equiv \lambda \Lambda(\Gamma) a_\pm$$

The Λ_1 is still defined to be $\Lambda(\Gamma)\Lambda(K_1, K_2)$. This Λ_1 is again continuous in K_1 and K_2.

Since $\Lambda_1(K_1, K_2)$ is continuous one can proceed just as before, and $D(N, K_0)$ can be taken to be any neighborhood of K_0 such that $D(N, K_0) \otimes D(N, K_0)$ is in the inverse image of any neighborhood of

the identity contained in N . Such a $D(N, K_0)$ must exist since the inverse image contains a neighborhood of (K_0, K_0) . The neighborhood $D(N, K_0)$ is to be restriced also by the condition that the vectors w do not have a zero component along the $w+$ axis. This is possible since for K_0 this condition is satisfied (for this case $r = n - 1$) .

For the remaining case $n = r + 2 = 2$ similar arguments apply. The vectors of K are specified by the two vectors w and $\bar{w}$. The vectors w_1 and w_2' both lie in the $(e_4(K_1); e_2(K_1))$ subspace and the vectors $\bar{w}_1$ and $\bar{w}_1'$ both lie in the $(e_3(K_1); e_1(K_0))$ subspace. Thus the transfor Λ^w will be a product of transformations in two orthogonal subspaces. The problem separates then into two disconnected parts each of which is treated in the same way as Λ^w for the $r = n - 1$ case.

Lemma 3. Let K_0 be an arbitrary set of vectors. Let the first n vectors of K_0 be linearly independent. For any neighborhood N of the identity in $\mathcal{L}$ there is a neighborhood $D(N, K_0)$ of K_0 such that if any two points K_1 and K_2 in $D(N, K_0)$ are connected by a Lorentz transformation then $K_1^n = \Lambda K_2^n$ with $\Lambda \in N$, where K^n is the set consisting of the first n vectors of K .

Proof: This is a trivial extension of the preceding lemma. The neighborhood $D(N, K_0)$ can be the intersection of any (full) neighborhood of K_0 with $D^n(N, K_0^n)$, the neighborhood in the subspace associated with the K^n specified by Lemma 2 .

Lemma 4. Let K_0 be any point and $D(K_0)$ be any neighborhood of K_0 .

Then there is a neighborhood $D_0(K_0)$ of K_0, contained in $D(K_0)$, such that any two points K_1 and K_2 in $D_0(K_0)$ connected by a $\Lambda \in \mathcal{L}$ are connected by a continuous path $K(t) = \Lambda(t) K_2$, with $K(0) = K_2$ and $K(1) = K_1$, such that $\Lambda(t) \in \mathcal{L}$ and $K(t) \in D(K_0)$ for $0 \leqslant t \leqslant 1$.

Proof: Let n be the number of linearly independent vectors of K_0 and r the rank of their Gram determinant. Arrange the vectors of K_0 so that the first n are linearly independent and the rank of the Gram determinant of the first r is r. Then according to Lemma 3 there is, for any arbitrary neighborhood N of the identity in $\mathcal{L}$, a neighborhood $D(N, K_0)$ of K_0 small enough so that if K_1 and K_2 are in $D(N, K_0)$ and are connected by a Lorentz transformation $\Lambda \in \mathcal{L}$, then there is a $\Lambda_1 \in N$ such that $K_1^n = \Lambda_1 K_2^n$, where K_1^n and K_2^n are the subsets of K_1 and K_2 consisting of their first n vectors. The neighborhood N can be taken to be a domain (i.e., connected), and hence a path $\Lambda(t)$ in N can be constructed with $\Lambda(0) = 1$, $\Lambda(\frac{1}{2}) = \Lambda_1$, and $\Lambda(t) \in N$ for $0 \leqslant t \leqslant \frac{1}{2}$. The $D_0(K_0) \subset D(N, K_0)$ and N can evidently be chosen small enough so that all points $\Lambda' K_2$ with $\Lambda' \in N$ and $K_2 \in D_0(K_0)$ are in any preassigned neighborhood of K_0, say $D_1(K_0) \subset D(K_0)$.

Consider first the case $r = n$. The neighborhood $D_1(K_0)$ will be taken small enough so that for all $K \in D_1(K_0)$, the rank $r(K)$ of the Gram determinant of the first r vectors of K remains equal to r. Then any $K \in D_1(K_0)$ can be uniquely

decomposed into a sum of two terms, $K \equiv K^r + V$, where the vectors of K^r are in the subspace spanned by the first r vectors of K, and the vectors of V lie in the subspace orthogonal to those r vectors. (Note that K^r is not the same as in Lemma 2 .)

The neighborhood $D_1(K_0)$ can be specified by conditions of the form $|| K^r - K_0^{\,r} || < \rho_r$ and $|| V || < \rho$, with ρ and $\rho_r > 0$, since this is an arbitrarily small open set containing $K_0 = K_0^{\,r}$. One can use here for instance the Euclidian norms; e.g.,

$$|| V || = \sum_i | v_i |^2 = \sum_{i,\mu} | v_i^{\mu} |^2 \tag{6.42}$$

The proof will be completed, for this case, if a continuous $\Lambda(t)$ for $\frac{1}{2} \leqslant t \leqslant 1$, with $\Lambda(\frac{1}{2}) = \Lambda_1$ and $\Lambda(1) K_2 = K_1$, can be found that acts only in the space orthogonal to the space spanned by the set $K_1^{\,n}$ and keeps $|| V || < \rho$.

The Lorentz transformation $\Lambda = \Lambda(1)\,\Lambda_1^{-1} \in \mathcal{L}$, which takes the point $\Lambda(\frac{1}{2})K_2 \equiv \Lambda_1 K_2$ to $\Lambda(1) K_2 \equiv K_1$, can, as any $\Lambda \in \mathcal{L}$, be expressed in the form[12]

$$\Lambda = R \exp A, \tag{6.43}$$

where R is a unimodular real orthogonal (hence unitary) transformation and A is Hermitian and imaginary:

$$A = - A^* = A^\dagger \tag{6.44}$$

(The metric tensor G has been converted to the unit matrix by the introduction of the appropriate imaginary units.) The required

transformation $\Lambda(t)$ for $\frac{1}{2} \leqslant t \leqslant 1$ can be taken to be defined by

$$\Lambda'(t) \equiv \Lambda(t)\,\Lambda_1^{-1} = \begin{cases} \exp\left[4(t - \frac{1}{2})\,A\right] & \text{for } \frac{1}{2} \leqslant t \leqslant \frac{3}{4} \\ R(t)\ \exp A & \text{for } \frac{3}{4} \leqslant t \leqslant 1 \end{cases} ; \qquad (6.46)$$

where $R(t)$ for $\frac{3}{4} \leqslant t \leqslant 1$ is any continuous curve from the identity E to R in the connected space of real unimodular orthogonal matrices.

The Euclidean norm $||\ V(t)\ ||$ of $V(t) \equiv \left\{\Lambda\ (t)\ v_i\right\}$ is the square root of

$$D^2(t) = \sum_i |\ \Lambda(t) v_i\ |^2 = \sum_i v_i^*\ \Lambda^\dagger\ (t)\ \Lambda\ (t)\ v_i$$

$$\equiv \left\langle \Lambda^\dagger\ (t)\ \Lambda\ (t) \right\rangle_V \qquad (6.47)$$

In the interval $\frac{3}{4} \leqslant t \leqslant 1$ the $||V(t)||$ is constant, because of the unitarity of $R(t)$:

$$R^\dagger(t)\ R(t) = \tilde{R}(t)\ R(t) = E \qquad (6.48)$$

On the other hand, in the interval $\frac{1}{2} < t < \frac{3}{4}$ one has, since $A = A^\dagger$,

$$\frac{d^2}{dt^2}\ D^2(t) = \frac{d^2}{dt^2} \left\langle \Lambda^\dagger(t)\ \Lambda(t) \right\rangle_V$$

$$= 64 \left\langle \Lambda^\dagger(t)\ A^\dagger\ A\ \Lambda(t) \right\rangle_V \geqslant 0 \qquad (6.49)$$

Because the second derivative of $||V(t)||^2$ is non-negative its maximum value must be assumed at an end point. As the end points are in $D_1(K_0)$

they satisfy $||V(t)|| < \rho$. Thus for all $0 \leqslant t \leqslant 1$ this condition is satisfied. Consequently all points $K(t) = \Lambda(t) K_2$ are in $D_1(K_0) \subset D(K_0)$. This completes the proof for the case that r , the rank of the Gram determinant of K_0 , is equal to n , the number of linearly independent vectors of K_0 .

In the case $r < n$ the first part of the transformation, $0 \leqslant t \leqslant \frac{1}{2}$, can be performed as before. For $n \geqslant 3$ this already completes the proof, since the coincidence of three linearly independent vectors ensures the coincidence of all vectors. The special form of $D_1(K_0)$ is not needed for this case.

For the case $r < n \leqslant 2$ additional arguments are required. Suppose $n = r + 1$. Then the vectors of K_0 are a linear combination of the first r vectors of K_0 , plus a linearly independent vector w_0, which must, according to the arguments given in Lemma 2 of Hall and Wightman or in Lemma 2 above, have zero length. The vector w_0 is a particular linear combination of the first n vectors of K_0. Let $w(K)$ be the same linear combination of the same vectors of K . This $w(K)$ is generally of nonzero length. As before let K^r be the part of K formed from vectors in the space spanned by its first r vectors and let $K \equiv K^r + V$. Also let $K_0 \equiv K_0^{\,r} + V_0 \equiv K_0^{\,r} + K_0^{\,w}$, where the vectors of $K_0^{\,w}$ must all be multiples of w_0 . Let K^w be the vector obtained by substituting $w(K)$ for w_0 in $K_0^{\,w}$. Finally, let the neighborhood $D_1(K_0)$ be characterized by conditions of the form $||K^r - K_0^{\,r}|| < \rho_r$, $||K^w - K_0^{\,w}|| < \rho_w < \rho$ and $||V - K^w|| < \rho$ with positive ρ's . This is evidently an arbitrarily open small set

containing K_0 .

With $D_1(K_0)$ defined in this way the same argument applies, provided $w \equiv w(K(1))$ is not a light-cone vector. For if $w \cdot w \neq 0$ the Lorentz transformation $\Lambda(t)$ for $\frac{1}{2} \leqslant t \leqslant 1$, as defined above, will act in the subspace orthogonal to the subspace spanned by the first n vectors of $K(t)$, since these n vectors are not changed by the transformation Λ . Thus the vector $K^w(t) = \Lambda(t) K^w(0)$ will be constant for $\frac{1}{2} \leqslant t \leqslant 1$ and $D(t) = ||V(t) - K^w(t)|| = ||\Lambda(t)(V(0) - K^w(0))||$ will again have, for $\frac{1}{2} < t < \frac{3}{4}$, a nonnegative second derivative, as before. Thus $D(t)$ will assume its maximum at an end point of the interval $\frac{1}{2} \leqslant t \leqslant 1$ and $K(t)$ will be in $D_1(K_0)$ for all t in this interval.

For the case that w is a light-cone vector this argument fails because the space does not separate into two orthogonal subspaces, one of the dimension n spanned by the vectors of K_1^n . Consequently, it is no longer clear that the $\Lambda'(t) = \Lambda(t)\Lambda_1^{-1}$ defined in (6.46) must leave $w(t)$ unchanged throughout the interval $\frac{1}{2} < t < 1$, just because it gives the same value $w = w(\frac{1}{2}) = w(1)$ at the two end points.

To cover the case $w \cdot w = 0$ the neighborhoods $D_1(K_0)$ associated with other orderings of the vectors of K_0 can be considered. The vector $w(K)$ was constructed as a linear combination of the first n vectors of K ; it is the $\underline{n}$th element of the set $V = \{v_i(K)\}$, the first $r = n - 1$ elements of which vanish. As a generalization, define

$$w_j^{\pm}(K) \equiv w(K) \pm \mathcal{E}_j v_j(K) \quad , \quad j \geqslant n ,$$

where the $\mathcal{E}_j$ are fixed positive numbers chosen small enough so that each $w_j^{\pm}(K)$ remains independent of the first r vectors of K over some neighborhood of K_0. The replacement of $w(K)$ by a $w_j^{\pm}(K)$ corresponds to replacing k_n by a certain fixed linear combination of the k_i. One can obtain, then, $D_1(K_{j0}^{\pm})$ analogous to the $D_1(K_0)$ above. And these domains will have their images $D_j^{\pm}(K_0)$ in the original space--the space in which the vectors K have their original meaning. Each of the $D_j^{\pm}(K_0)$ contains K_0, since this point is the image of the point $K_{j0}^{\pm}$. Also, each $D_j^{\pm}(K_0)$ will be a neighborhood of K_0, since it is the image of the neighborhood $D_1(K_{j0}^{\pm})$ of $K_{j0}^{\pm}$ under a nonsingular linear mapping. Thus the intersection of the finite number of $D_j^{\pm}(K_0)$ must contain a neighborhood $D_2(K_0)$ of K_0. If $D_2(K_0)$ is used in place of $D_1(K_0)$ in the definition of $D_0(K_0)$ then the lemma follows from the previous argument, unless for all $j \geq n$ the $w_j^{\pm}(K_1)$ have zero length.

In case of the $w_j^{\pm}(K)$ have zero length the vectors $v_i(K)$ must all be orthogonal to $w(K)$. This follows from the equations $w_j^{+}(K) \cdot w_j^{+}(K) - w_j^{-}(K) \cdot w_j^{-}(K) = 4\,\mathcal{E}_j\, v_j(K) \cdot w = 0$. Now the transformation Λ defined by $K_1 = \Lambda K_2' \equiv \Lambda \Lambda_1 K_2$ leaves $w \equiv w(K_1) = w(K_2')$ invariant. The general transformation $\Lambda \in \mathcal{L}$ that leaves a multiple of the light-cone vector $w_+ = (1, i, 0, 0)$ invariant is

$$\begin{pmatrix} 1 + a & i\,a & -b & -c \\ i\,a & 1 - a & -i\,b & -i\,c \\ b\cos\theta + c\sin\theta & i(b\cos\theta + c\sin\theta) & \cos\theta & \sin\theta \\ c\cos\theta - b\sin\theta & i(c\cos\theta - b\sin\theta) & -\sin\theta & \cos\theta \end{pmatrix}$$

where a, b, c, and θ are arbitrary complex parameters satisfying $b^2 + c^2 = -2a$. Writing an arbitrary vector v in the form

$$v = v_+ w_+ + v_- w_- + v_b e_b + v_c e_c$$

where $w_- = w_+^*$ and e_b and e_c are the third and fourth unit vectors, one finds, for $v' = \Lambda v$, that

$$v'_+ = v_+ - b\,v_b - c\,v_c + 2a\,v_-$$

$$v'_- = v_-$$

$$v'_b = \cos\theta\, v_b + \sin\theta\, v_c + 2(b\cos\theta + c\sin\theta)\, v_-\,;$$

and

$$v'_c = \cos\theta\, v_c - \sin\theta\, v_b + 2(c\cos\theta - b\sin\theta) v_-$$

¶ The transformation Λ takes $V'_2 = V$ into $V_1 = V'$; $V' = \Lambda V$. It also takes w, which can be considered a multiple of w_+, into itself. Thus Λ can be written in the above form. The conditions $v_i \cdot w = 0$ give $v_{i-} = 0$, and hence also $v'_{i-} = 0$. Thus for each value of i one must have for some single set (a, b, c, θ) the equations

$$v'_{i+} = v_{i+} - b\,v_{ib} - c\,v_{ib} = v_{i+}\,;$$

$$v'_{ib} = \cos\theta\, v_{ib} + \sin\theta\, v_{ic} ,$$

and

$$v'_{ic} = \cos\theta\, v_{ic} - \sin\theta\, v_{ib}$$

But then the equations are also satisfied with $b = c = 0$. Thus Λ can be replaced by a Λ acting in a subspace orthogonal to one containing w . This consequence of the equations $v_i \cdot w = 0$ could also be obtained using abstract arguments. Since this new Λ acts in a subspace orthogonal to one containing w , the $\Lambda^1(t) = \Lambda(t)\,\Lambda_1^{-1}$ constructed in accordance with (6.46) will act in that subspace and hence leave w invariant. Thus as before, the non-negativeness of the second derivative of $D(t)$ for $\frac{1}{2} < t < \frac{3}{4}$ ensures that $K(t)$ remains in the domain $D_1(K_0) \subset D(K_0)$ for $\frac{1}{2} \leqslant t \leqslant 1$, and hence for all $0 \leqslant t \leqslant 1$. This completes the proof for the case $r = n - 1$.

The remaining case is $r = n - 2 = 0$. In this case there are two linearly independent, orthogonal light-cone vectors $w_0 = w(K_0)$ and $\bar{w}_0 = \bar{w}(K_0)$. The related vectors $w(K)$ and $\bar{w}(K)$ are also orthogonal and they are brought into coincidence by the first part of the transformation, $\Lambda(\frac{1}{2}) \equiv \Lambda_1(K_1, K_2)$. If the $w(K_1)$ and $\bar{w}(K_2)$ are not light-cone vectors then they define a subspace which together with its orthogonal complement spans the space. And all vectors in this subspace are brought to their final values already by $\Lambda_1(K_1, K_2) = \Lambda_1$ The transformation Λ defined by $K_1 = \Lambda\,\Lambda_1\,K_2$ acts therefore only in the orthogonal subspace and hence the $\Lambda(t)$ defined by (6.46) completes the path in $\mathcal{L}$. In order to cover the case that $w(K)$ or $\bar{w}(K)$ is a

light-cone vector one takes $D_2(K_0)$ to be in the intersection of the images of the various $D_1(K^{\pm\pm}_{jk})$, where $K^{\pm\pm}_{jk}$ has $w^{\pm}_j(K) \equiv w(K) \pm \mathcal{E}_j v_j(K)$ in place of $w(K)$ and $w_k(K) \equiv \bar{w}(K) \pm \eta_k v_k(K)$ in place of $\bar{w}(K)$. If N and $D_0(K_0)$ are taken to be small enough so that $K_0 \in D_0(K_0)$ and $\Lambda' \in N$ implies $\Lambda' K_2 \in D_2(K_0)$ then the consequences needed for the lemma are satisfied unless $w^{\pm}_j(K_1)$ has zero length for all j or $\bar{w}^{\pm}_j(K_1)$ has zero length for all k, since otherwise for at least one value of j and one value of k the previous $(w \cdot w \neq 0)$ argument holds. If $w^{\pm}_j(K_1)$ has zero length for all j then $w \cdot v_j(K_1) = 0$ for all j and the argument still works because an appropriately chosen transformation $\Lambda'(t)$, $\frac{1}{2} \leqslant t \leqslant 1$, leaves w invariant. It goes similarly for $\bar{w}^{\pm}_j(K_1)$. Thus the argument for the case $r = n - 2 = 0$ is virtually the same as for the $r = n - 1$ case. This completes the proof.

Definition: A function $F(K)$ will be said to be <u>regular at a point K</u> if and only if the various functions of K corresponding to the various combinations of the spin indices are all regular analytic functions of the components of the four vectors $\{k_i\}$ at the point K.

Lemma 5. Let Λ be a fixed Lorentz transformation. Let $F_\Lambda(K)$ be defined by

$$F_\Lambda(K') = \Lambda_s F(\Lambda^{-1} K') . \tag{6.50}$$

If $F(K)$ is regular at the point $K = \Lambda^{-1} K'$ then $F_\Lambda(K)$ is

regular at the point $K = K'$.

Proof: This is an immediate consequence of the theorem in several complex variables that an analytic function of an analytic function is analytic. This well-known theorem is easily proved using the Cauchy-Riemann equations.

Corollary A. Let $F_\Lambda(K)$ be defined by (6.50), where Λ is fixed. Then $F_\Lambda(K)$ is regular at $K = K'$ if and only if $F(\Lambda^{-1} K')$ is regular at $K = \Lambda^{-1} K'$.

Proof: The first part of the corollary is just the lemma. To prove the converse apply the lemma to the function

$$F''(K) \equiv \Lambda_s^{-1} F_\Lambda(\Lambda K)$$

to show that $F''(K)$ is regular at K if $F_\Lambda(\Lambda K)$ is regular at ΛK . But $F''(K)$ is just $F(K)$. The substitution $K = \Lambda^{-1}K'$ gives the desired result. The fact that the inverses Λ^{-1} and Λ_s^{-1} exist is essential to the proof.

Corollary B. The property of being regular at a point does not depend on the choice of coordinate system relative to which the components of the vectors k are measured, provided the components in the two systems are related by a Lorentz transformation.

Proof: The proof is the same as for the lemma.

Definition: A domain in an arcwise connected open set.

Definition: A real domain is an arcwise connected real set open with respect to the set of real points.

Lemma 6. Let $F(K)$ satisfy the invariance condition

$$F(K) = \Lambda_s F(\Lambda^{-1} K) , \tag{6.53}$$

for $\Lambda \in L$, and K and $\Lambda^{-1} K$ in a real domain D containing the point K_0. Suppose $F(K)$ is defined (single valuedly) in a domain $D(K_0)$ containing K_0 and is regular at all points of $D(K_0)$. Then for each point K in $D(K_0)$ Eq. (6.53) is satisfied for $\Lambda \in N_r(K)$, where $N_r(K)$ is some neighborhood of the identity in L.

Proof: Let K_1 be a fixed arbitrary point of $D(K_0)$. Since $D(K_0)$ is a domain there exists a continuous curve $K(t)$, $0 \leqslant t \leqslant 1$, from K_0 to K_1, all points of which are in $D(K_0)$. Let the distance between two points be defined as maximum of the absolute values of the differences of the components of the vectors $\{k_i\}$. Then the distance of a point K in $D(K_0)$ to the boundary of $D(K_0)$ will be defined as the maximum (real) number $\Delta(K)$ such that every point whose distance from K is less than $\Delta(K)$ is inside $D(K_0)$. Since $D(K_0)$ is a domain $\Delta(K) > 0$ for all $K \in D(K_0)$. Moreover, $\Delta(K(t)) \geqslant a > 0$, for $0 \leqslant t \leqslant 1$. For if there were no positive lower bound $a > 0$ of $\Delta(K(t))$ one could find a sequence

t_n, $0 \leqslant t_n \leqslant 1$ with $\Delta(K(t_n)) < 2^{-n}$. These t_n would have to have an accumulation point $\bar{t}$, $0 \leqslant t \leqslant 1$. But $\Delta(K(\bar{t})) \equiv b > 0$. Hence for all t such that the distance between $K(t)$ and $K(\bar{t})$ is less than $b/2$ one would have $\Delta(K(t)) \geqslant b/2$, by the triangle inequality. Since $K(t)$ is a continuous curve the inverse map of the open set $||K(t) - K(\bar{t})|| < b/2$ contains an open interval Δt about $\bar{t}$. But since $\Delta(K(t)) > b/2$ for $t \in \Delta t$ only a finite number of the t_n can be in Δt. Hence $\bar{t}$ cannot be an accumulation point. This is a contradiction. Thus there is a positive lower bound a.

Let the maximum value of $||K(t)||$ for $0 \leqslant t \leqslant 1$ be A. Let $N(K_1)$ be a neighborhood of the identity in $\mathcal{L}$ such that if $\Lambda^{-1} \in N(K_1)$, then $||(\Lambda^{-1})_\mu{}^\nu - \delta_\mu{}^\nu|| < (a/4A)$. Then, for $\Lambda^{-1} \in N(K_1)$, it follows that $||\Lambda^{-1} K(t) - K(t)|| < a$, and the (continuous) curve $K_\Lambda(t) \equiv \Lambda^{-1}K(t)$ remains inside of $D(K_0)$ for all $0 \leqslant t \leqslant 1$.

Let N_r be a neighborhood of the identity in L such that $\Lambda^{-1}K_0 \in D$ for $\Lambda^{-1} \in N_r$. The existence of such a neighborhood follows immediately from the continuity of $\Lambda^{-1}K_0$ in Λ at the identity. For any fixed $\Lambda^{-1} \in N_r \cap N(K_1) \equiv N_r(K_1)$ there is a real domain $D(\Lambda, K_1) \subset D$, with $K_0 \in D(\Lambda, K_1)$, such that for all $K \in D(\Lambda, K_1)$ the points K and $\Lambda^{-1} K$ are in $D \cap D(K_0)$. The existence of such a $D(\Lambda, K_1)$ follows from the fact that K_0 and $\Lambda^{-1} K_0$ are in $D \cap D(K_0)$, in conjunction with the continuity of $\Lambda^{-1}K$ as a function of K. Thus (6.53) is valid for any fixed $\Lambda^{-1} \in N_r \cap N(K_1)$ for all $K \in D(\Lambda, K_1)$. The validity of (6.53), for fixed $\Lambda^{-1} \in N_r \cap N(K_1)$, for all K in the real domain $D(\Lambda, K_1)$,

together with the analyticity of both sides of the equation, as functions of K, (Lemma 5), implies the validity also at the point K_1, since one can analytically continue along $K(t)$ with the argument of the function on the right tracing simultaneously the curve $K_\Lambda(t)$, which remains inside the domain of regularity $D(K_0)$.

Lemma 6A. Lemma 6 modified by the substitution of $\mathcal{L}$ for L and of a (full complex) domain D_c for the real domain D is also valid.

Proof: Makes these substitutions throughout the proof of Lemma 6.

Lemma 7. Let $F(K)$ satisfy the invariance condition (6.53) for $\Lambda \in L$ and K and $\Lambda^{-1} K$ in a real domain D containing the point K_0. Suppose $F(K)$ is defined (single valuedly) in a domain $D(K_0)$ containing K_0 and is regular at all points of $D(K_0)$. Then (6.53) is also valid for all $K \in D(K_0)$ and $\Lambda \in \mathcal{L}$ such that there is a continuous path $\Lambda(t) \in \mathcal{L}$, $0 \leqslant t \leqslant 1$, with $\Lambda(0) = E$ and $\Lambda(1) = \Lambda$, such that $K(t) \equiv \Lambda^{-1}(t) K \in D(K_0)$ for $0 \leqslant t \leqslant 1$.

Proof: The assumptions of the lemma are the same as those of Lemma 6. Thus the conclusions of Lemma 6 may be used; (6.53) is valid for every point $K \in D(K_0)$ for $\Lambda^{-1} \in N_r(K)$, a neighborhood of the

identity in L. Following Hall and Wightman,[4] and Jost,[11] the Lorentz transformations Λ in a neighborhood N of the identity in $\mathcal{L}$ can be parameterized by a continuous one-to-one mapping $\Lambda(\lambda_j)$ in such a way that the representations of Λ^{-1} and Λ_s are regular analytic functions of the λ_j for $\Lambda^{-1} \in N$; and such that for $\Lambda^{-1} \in N \cap L$ the λ_j are real; and such that the origin in λ_j maps into the identity in Λ. Such a parameterization has been given by Jost.[11]

Considered as a function of the λ_j the right-hand side of (6.53) is an analytic function regular at all points for which $\Lambda^{-1} \in N$ and $\Lambda^{-1} K \in D(K_0)$. But for Λ^{-1} in the real neighborhood of the origin $N_r(K)$ the right-hand side of the equation is, by Lemma 6, equal to the left-hand side, which is independent of λ_i. Thus the right side must be equal to the left for all $\Lambda = \Lambda(t)$ such that $\Lambda^{-1}(t') \in N$ and $\Lambda(t') K \in D(K_0)$ for $0 \leqslant t' \leqslant t$, since one can analytically continue to this point, the right-hand side remaining regular. If for all $0 \leqslant t \leqslant 1$ the $\Lambda^{-1}(t)$ are not contained in N then the continuation can be carried out stepwise by expanding $\Lambda^{-1}(t)$, in the manner specified above, about a finite sequence of intermediate points, t_n, and by using the group properties. The invariance equation is in this way validated for all points K, $\Lambda^{-1}K$ connected by a continuous path $\Lambda(t)K$ that remains always inside the domain of regularity $D(K_0)$. That only a finite number of t_n are required follows from the Heine-Borel Covering Theorem.

Lemma 7A. Lemma 7 is also true if the real D and L are replaced by complex D_c and $\mathcal{L}$.

Proof: Make these substitutions throughout the proof of Lemma 7.

Lemma 8. Let $F(K)$ be defined (single valuedly) and regular for points in a domain $D(K_0)$ containing K_0. And suppose

$$F(K) = \Lambda_s F(\Lambda^{-1} K) \tag{6.54}$$

for $\Lambda \in L$ and Λ and $\Lambda^{-1} K$ in a real domain D containing K_0. Then for every point $K_1 \in D(K_0)$ there is a domain $D_0(K_1)$ containing K_1 such that the equation

$$F(K; D_0(K_1)) \equiv \Lambda_s F(\Lambda^{-1} K) \tag{6.55}$$

with $\Lambda^{-1} K \in D_0(K_1)$ and $\Lambda \in \mathcal{L}$ defines a (single valued) function $F(K; D_0(K))$ over the points $K \in \mathcal{L}\, D_0(K_1)$. This function is regular throughout its domain of definition and coincides with $F(K)$ in the domain $D_0(K_1) \subset D(K_0)$.

Proof: The assumptions are the same as those of Lemma 7. Thus the invariance equation (6.54) holds for all K and $\Lambda^{-1} K$ connected by a path $\Lambda(t) K$, $0 \leqslant t \leqslant 1$, that is everywhere in $D(K_0)$. Consider an arbitrary point $K_1 \in D(K_0)$. According to Lemma 4, there is a domain $D_0(K_1)$ containing K_1 such that the points of every pair of points in $D_0(K_1)$ connected by a Lorentz transformation are connected

by a continuous path $\Lambda(t)K$, $0 \leqslant t \leqslant 1$, that is everywhere in $D(K_0)$. Lemma 7 then ensures that the invariance equation (6.54) is valid for all K, $\Lambda^{-1}K \in D_0(K_1)$. This in turn ensures that (6.55) defines a (single valued) function $F(K; D_0(K_1))$. To show this suppose for some $K \in \mathcal{L}D_0(K_1)$ the points $\Lambda_1^{-1}K$ and $\Lambda_2^{-1}K$ are both in $D_0(K_1)$. Then one can write

$$F_1(K; D_0(K_1)) = \Lambda_{1s} F(\Lambda_1^{-1} K) \tag{6.56}$$

and

$$F_2(K; D_0(K_1)) = \Lambda_{2s} F(\Lambda_2^{-1} K) . \tag{6.57}$$

That these are equal follows from Eq.(6.54) expressed in the form

$$F(\Lambda_1^{-1} K) = \Lambda_{1s}^{-1} \Lambda_{2s} F((\Lambda_2^{-1} \Lambda_1) \Lambda_1^{-1} K) , \tag{6.58}$$

which is true because both arguments are in $D_0(K_1)$.

Since $F(K; D_0(K_1))$ is independent of the particular Λ used on the right of (6.55), so long as $\Lambda^{-1}K \in D_0(K_1)$, the values of $F(K; D_0(K_1))$ in some neighborhood of any $K \in \mathcal{L}D_0(K_1)$ can be generated from a fixed Λ, as a consequence of the continuity of $\Lambda^{-1}K$ as a function of K, for fixed Λ. That is, the inverse map of the open set $D_0(K_1)$ of $\Lambda^{-1}K$'s is an open set $D_\Lambda(K_1)$ of K's. But for fixed Λ the regularity of the left-hand side of (6.55) is ensured by Lemma 5, since $\Lambda^{-1}K \in D_0(K_1) \subset D(K_0)$. Finally, that $F(K; D_0(K_1))$ coincides with $F(K)$ for $K \in D_0(K_1)$ is true by virtue of (6.55) with $\Lambda = I$.

Lemma 8A. The lemma remains valid if the real D and L are replaced by a complex D and $\mathcal{L}$.

Some concepts from the theory of functions of several complex variables will now be introduced.[13]

Definition: A <u>regular function element e</u> is a triple $\left[K_e;\ D_e;\ F_e(K)\right]$ consisting of a base point K_e, a domain D_e containing K_e, and an associated function $F_e(K)$ defined (single valuedly) and regular in D_e.

Definition: Two regular function elements will be called <u>equivalent</u> if and only if they have the same base point and their functions coincide in some neighborhood of this point.

Definition: A <u>germ</u> is a set of regular function elements such that

(1) any two elements of the set are equivalent

and

(2) any regular function element equivalent to an element of the set is also in the set.

Definition: A <u>germ neighborhood</u> $N(D_N,\ F_N(K))$ is the set of all germs containing a regular function element $\left[K;\ D_N;\ F_N(K)\right]$. The domain D_N and the function $F_N(K)$ are called the <u>base domain</u> and the <u>characteristic function</u> of the germ neighborhood, respectively.

Definition: The topological (Hausdorff) space with germs as points and

germ neighborhoods as neighborhoods will be called the <u>germ space</u>.

Definition: The <u>domain of regularity</u> of a function $F(K)$ defined (single valuedly) and regular in a domain D is the set of all germs connected to any germ of $N(D, F(K))$ by a continuous curve in the germ space.

Definition: The unique germ $g\left[e\right]$ containing e is called the germ <u>specified</u> by e. (Uniqueness is easily proved.)

Definition: The <u>base point $K(g)$</u> of a germ g is the common base point of the $e \in g$.

Definition: $\overline{F}(g) \equiv F_e(K(g))$, with $e \in g$. ($\overline{F}(g)$ is independent of the choice of $e \in g$.)

Definition: Let $N \equiv N(D_N, F_N(K))$ be a germ neighborhood. Then, for $K \in D_N$, define $g_N(K) \equiv g\left[e\right]$, where $e = \left[K; D_N; F_N(K)\right]$.

Remark: $g_N(K)$ is the unique $g \in N$ such that $K(g_N(K')) = K'$. Restated, $g_N(K)$ is the unique inverse of $K(g)$, subject to the condition that $g \in N$.

Lemma 9. If the characteristic functions of two germ neighborhoods N and N' coincide in a domain $D \subset (D_N \cap D_{N'})$, then $g_N(K) = g_{N'}(K)$, for $K \in D$.

Proof: The associated function of any element e of $g_N(K)$ coincides with $F_N(K)$ for K in some neighborhood $N(K)$ of $K \in D$. Thus it must coincide with $F_{N'}(K)$ in $N(K) \cap D$ and hence in some neighborhood of K. Thus e is in $g_{N'}(K)$. Conversely every $e \in g_{N'}(K)$ is in $g_N(K)$.

Some terminology associated with Lorentz invariant analytic functions will now be introduced.

Definition: A function will be called <u>$\mathcal{L}$ (or L)-invariant</u> over a set of points S if and only if it satisfies

$$F(K) = \Lambda_s F(\Lambda^{-1} K)$$

for any K and Λ such that Λ is in $\mathcal{L}$ (or L) and both K and $\Lambda^{-1} K$ are in S.

Definition: An <u>orbit</u> is a set of points K all connected to a single point by a Lorentz transformation $\Lambda \in \mathcal{L}$.

Definition: A <u>regular orbit</u> is a set of germs whose base points cover exactly once the points of an orbit, and such that the image in the germ space of any continuous curve in the orbit is a continuous curve in the germ space.

Definition: Let $g(K)$ for $K \in \mathcal{L} K_0$, $K(g(K')) = K'$, be the germs of a regular orbit. This regular orbit will be called <u>$\mathcal{L}$-invariant</u> if and only if the function $F(K) \equiv \bar{F}(g(K))$ is

$\mathcal{L}$-invariant over the orbit $\mathcal{L}\, K_0$.

Definition: A domain of regularity will be called <u>$\mathcal{L}$-invariant</u> if and only if it is a union of $\mathcal{L}$-invariant regular orbits.

Theorem 1. A function defined (single valuedly) and regular in a domain containing a point and L-invariant over a real domain containing the point has an $\mathcal{L}$-invariant domain of regularity.

Proof: Let K_0 be the point in the real domain and let the function be called $F(K)$. There is a domain $D(K_0)$ containing K_0 on which $F(K)$ is defined and regular. Thus the set $e_0 \equiv [K_0; D(K_0); F(K)]$ constitutes a regular function element. Let g_0 be the germ specified by e_0 . This g_0 is in $N \equiv N(D(K_0), F(K))$. Let g_1 and g_2 be any two germs in N . Then there is a continuous curve in the germ space connecting g_1 and g_2 . In particular, if $K(t)$ is a continuous curve in $D(K_0)$ connecting $K(g_1)$ and $K(g_2)$ then $g_N(K(t))$ will be a continuous curve in the germ space connecting g_1 and g_2 . For consider any germ neighborhood $N' \equiv N(D', F'(K))$ that contains a germ $g_N(K\,(t_0))$, where t_0 is some fixed value of t , $0 \leqslant t \leqslant 1$. Let D'' be a domain in $D' \cap D(K_0)$ containing $K(t_0)$. Any germ of N' with base point in D'' is identical to the germ of N with the same base point. For D'' is a domain and hence

the function $F'(K)$ must be identical with $F(K)$ for $K \in D''$. This is true because $F(K)$ and $F'(K)$ are both regular over the domain D'' and they coincide over some neighborhood of $K(t_0) \in D''$, since $g(K(t_0))$ contains both $\left[K(t_0);\ D;\ F\right]$ and $\left[K(t_0);\ D';\ F'\right]$. Since the functions $F'(K)$ and $F(K)$ are identical for $K \in D''$ the germs of N' and N with base points in D'' must be identical, by virtue of Lemma 9. Because $K(t_0)$ is in the domain D'', and $K(t)$ is a continuous curve, the inverse image of the points $K(t) \in D''$ contains an interval Δt that contains t_0 and is open with respect to the set $0 \leqslant t \leqslant 1$. The germs $g_{N'}(K(t))$ with t in the interval Δt are all in the arbitrary neighborhood N' containing $g_{N'}(K(t_0))$. Thus, this curve $g_N(K(t)) \equiv g(t)$ is continuous. Hence any two germs in N can be connected by a continuous curve. This means that the word "any" in the definition of domain of regularity can be replaced by "every" with no change in the meaning. (That two continuous curves joined at their end points give a continuous curve follows easily.)

Consider now an arbitrary germ g in the domain of regularity of $F(K)$. It is connected to g_0 by a continuous curve $g(t)$ in the germ space. Since $g(t)$ is continuous the inverse image of any germ neighborhood containing a germ $g(t_0)$ contains an interval Δt containing t_0 that is open with respect to the set $0 \leqslant t \leqslant 1$. By the Heine-Borel theorem, the closed bounded set $0 \leqslant t \leqslant 1$ is covered by a finite number of these intervals, Δ_i, with $i = 1, 2, \cdots, n$. Associated with these intervals are corresponding germ neighborhoods N_i, with $i = 1, 2, \cdots, n$,

such that for $t \in \Delta_i$, $g(t) \in N_i$. And there is then a sequence $\{t_i\}$ so that $g(t_i)$ is in both N_i and N_{i+1}.

The assumptions of the theorem are a paraphrasing of the assumptions of Lemma 8. Thus for each point K_1 of $D(K_0)$ there is a domain $D_0(K_1) \subset D(K_0)$ containing K_1 such that $F(K)$ is $\mathcal{L}$-invariant in $D_0(K_1)$. The first N_i can be taken to be $N_1 = N$. Take $K_1 = K(t_1)$. Then K_1 will also lie in the domain D_2, in which lie the base points of the germs of N_2. The germ neighborhood N_2 is characterized by the requirement that each of its germs has an element having the domain D_2 and the function $F_2(K)$. Also, N_2 contains the germ $g(t_1)$, which is also in $N_1 = N$, and which therefore has the element $\left[K(t_1) ; D(K_0) ; F(K)\right]$. But then $F_2(K)$ and $F(K) \equiv F_1(K)$ must coincide with each other in some neighborhood of K_1. But since $F(K)$ is $\mathcal{L}$-invariant in $D_0(K_1)$ the function $F_2(K)$ is $\mathcal{L}$-invariant in some domain containing K_1. Thus the conditions for Lemma 8A are satisfied for $F_2(K)$. Hence for any point K_2 in D_2 there is a domain containing K_2 such that $F_2(K)$ is $\mathcal{L}$-invariant in this domain. Take $K_2 = K(t_2)$. The argument may then be repeated to give $\mathcal{L}$-invariance in a domain about $K_i = K(t_i)$ for $i = 3$, and by iteration, for $i = n - 1$. In particular, there is a point K_{n-1} of the domain D_n, in which lie the base points of N_n, such that $F_n(K)$ is $\mathcal{L}$-invariant in some domain containing K_{n-1}. Lemma 8A now shows that there is a domain $D_n(K_g)$ containing K_g, the base point of the germ g, such that there is a function $F_g(K)$ defined (single

valuedly) over $\mathcal{L}D_n(K_g)$, where it is regular and $\mathcal{L}$-invariant, and which coincides with $F_n(K)$ in $D_n \cap \mathcal{L}D_n(K_g)$, which contains K_g . The germ neighborhood $N_g \equiv N(\mathcal{L}D_n(K_g), F_g(K))$ contains g , by virtue of Lemma 9, since $F_g(K)$ coincides with $F_n(K)$ in a neighborhood of K_g .

The set of germs $g' \in N_g$ with $K(g') \in \mathcal{L}K_g$ constitute an $\mathcal{L}$-invariant regular orbit containing g . Let $g(K(g')) = g'$ for $g' \in N_g$. That any continuous $K(t) \in \mathcal{L}K_g$ has a continuous image $g(K(t))$ follows from the argument given earlier, since $g(K(t)) \in N_g$. (See Lemma 10.) The $\mathcal{L}$-invariance of the set $g \in N_g$ with $K(g) \in \mathcal{L}K_g$ follows from the $\mathcal{L}$-invariance of $F_g(K)$ over $\mathcal{L}D_n(K_g) \supset \mathcal{L}K_g$. Thus each germ g in the domain of regularity of $F(K)$ is on an $\mathcal{L}$-invariant regular orbit. Since all pints of this orbit are connected to g by a continuous path they are also contained in the domain of regularity of $F(K)$. Thus each germ g in the domain of regularity of $F(K)$ is a member of an $\mathcal{L}$-invariant regular orbit each of whose members is also in the domain of regularity of $F(K)$. This is what was to be proved.

Theorem 1A: Theorem 1 is also true if "L-invariant" is replaced by "$\mathcal{L}$-invariant", and the real domain is replaced by a (complex) domain.

Definition: A germ neighborhood will be said to be <u>$\mathcal{L}$-invariant</u> if and only if its base domain is of the form $\mathcal{L}D$ and its characteristic function is $\mathcal{L}$-invariant over $\mathcal{L}D$.

Theorem 1': The domain of regularity of a function satisfying the conditions of Theorem 1 is a union of $\mathcal{L}$-invariant germ neighborhoods.

Proof: In the course of proving Theorem 1 it was shown that each g in the domain of regularity of such a function is in an $\mathcal{L}$-invariant germ neighborhood N_g. All the points of this neighborhood are in the domain of regularity since one is, by virtue of the following lemma, which was also proved in the course of proving Theorem 1.

Lemma 10: The image in a germ neighborhood of a continuous curve in its base domain is a continuous curve in the germ space.

The converse of this lemma is:

Lemma 10': The image $K(g(t))$ of a continuous curve $g(t)$ in the germ space is continuous.

Proof: A continuous function of a continuous function is continuous. But $K(g)$ is continuous, since given any domain D containing $K(g)$ one can take a germ neighborhood N_g containing g specified by a function element whose domain D', which contains $K(g)$, is contained in D. Then for all $g' \in N_g$, $K(g) \in D$.

Lemma 11: Let D be a real domain satisfying the condition of Lemma 1 that points of D connected by a real $\Lambda \in \mathcal{L}$ are connected by a $\Lambda \in L$. Let there be two converging sequences $K_i \to K_0$ and

$\bar{K}_i \to \bar{K}_0$ whose limit points K_0 and $\bar{K}_0$ are in D. And suppose $\bar{K}_i \in \mathcal{L} K_i$. Then $\bar{K}_0 \in \mathcal{L} K_0$.

Proof: The scalar and pseudoscalar invariants formed from corresponding vectors of K_i and $\bar{K}_i = \bar{\Lambda}_i K_i$ are equal. Thus these points map into the same points in the space of scalar and pseudoscalar invariants. As the mapping from K to the space of invariants is continuous, the converging sequences $K_i \to K_0$ and $\bar{K}_i \to \bar{K}_0$ map into converging sequences in the space of the invariants. Thus K_0 and $\bar{K}_0$ have the same scalar and pseudoscalar invariants.

In case r, the rank of the Gram determinant of K_0 or $\bar{K}_0$, is greater than two it follows from a trivial generalization of Lemma 2 of Hall and Wightman that K_0 and $\bar{K}_0$ are connected by a Lorentz transformation $\Lambda \in \mathcal{L}$; that the transformation is <u>proper</u> in the case $r = 4$ follows from the invariance of the pseudoscalar invariants, and for $r = 3$ there is sufficient freedom to allow Λ to be made proper. Thus the lemma is proved for the case $r > 2$.

Let $n(K)$ be the number of linearly independent vectors in the set K. And let $n = \max(n(K_0), n(\bar{K}_0))$. The above argument works equally well for all the cases $r = n$. One constructs the the orthonormalized basis vectors $e_\sigma(K_0)$ and $e_\sigma(\bar{K}_0)$ in the manner specified in Lemma 1 above and obtains $\bar{K}_0 = \Lambda_b K_0$, where Λ_b is the real $\Lambda \in L$ defined by $e_\sigma(\bar{K}_0) = \Lambda_b e_\sigma(K_0)$. Thus $\bar{K}_0$ and K_0 are connected by an element of $\mathcal{L}$. This completes the proof for the case $r = n$.

Because K_0 and $\bar{K}_0$ are real, the only other cases are $n = r + 1 < 4$. Suppose $n(K_0) = r + 1 < 4$. Then, as in Lemma 1, one can construct a set $e_1(K_0), \cdots, e_r(K_0)$, $e_0(K_0) + e_3(K_0)$, which spans the space of the vectors of K_0. The combination $e_0(K_0) + e_3(K_0)$ is chosen to be equal to some vector ω of zero length formed as a linear combination of vectors of K_0. Such a vector must exist in this case. If $\bar{\omega}$, the same linear combination of the corresponding vectors of $\bar{K}_0$, is not zero then one can construct a set $e_1(\bar{K}_0), \cdots, e_r(\bar{K}_0)$, $\pm e_0(\bar{K}_0) \pm e_3(\bar{K}_0)$, by means of the same operations as before, but with the corresponding vectors of $\bar{K}_0$. The two $\pm$ signs are independent and will be specified by the condition that the Λ_b defined by $e_\sigma(\bar{K}_0) = \Lambda_b\, e_\sigma(K_0)$ is in L. For $r(K_0) < 2$ the sign of $e_3(\bar{K}_0)$ is not determined by this condition and it can, and will, be taken positive.

The points $\bar{K}_0$, $\bar{K}_i$, and $\bar{\omega}$ can be represented by the transformed quantities $K_0' = \Lambda_b^{-1}\, \bar{K}_0$, $K_i' = \Lambda_b^{-1}\, \bar{K}_i$, and $\omega' = \Lambda_b^{-1}\, \bar{\omega}$. This, in effect, refers the barred points to the same coordinate system, $e_\sigma(K_0)$, used for the unbarred points K_0, K_i, and ω. In particular $\omega' = \pm e_0(K_0) \pm e_3(K_0)$, where the $\pm$ signs are the same as the corresponding ones in $\bar{\omega}$. The vectors ω (or ω') are what is left after removing from some vector of K_0 (or the corresponding vector of K_0') the parts along $e_1(K_0), \cdots, e_r(K_0)$. In this same way one constructs from the sets $K_0 \equiv \{k_{0\alpha}\}$ and $K_0' \equiv \{k_{0\alpha}'\}$ the sets of light-zone vectors $\{\omega_\alpha\} \equiv \{a_\alpha\, \omega\}$ and $\{\omega_\alpha'\} = \{a_\alpha'\, \omega'\}$ by removing the parts along $e_1(K_0), \cdots, e_r(K_0)$.

That the vectors of these sets are collinear follows from the condition $n = r + 1 < 4$. In the special case that $\omega' = \omega$ and $a_\alpha = a'_\alpha$ one has again $\bar{K}_0 = \Lambda_b K_0$ with $\Lambda_b \in L$. But if $\omega' \neq \omega$ or $a_\alpha \neq a'_\alpha$, for some α, then $\bar{K}_0$ and K_0 are not connected by a $\Lambda \in L$. However these cases cannot occur. This will now be shown by an examination of points in D near K_0 and $\bar{K}_0$. ¶ In the real $0 - 3$ plane consider a set of small circles $\{C(\omega_\alpha)\}$ drawn around the points $\{\omega_\alpha\}$, and a set of small circles $\{C(\omega'_\alpha)\}$ drawn around the points $\{\omega'_\alpha\}$. A set of points with one in each $C(\omega_\alpha)$ corresponds to a real point near K_0. And a set of points with one in each $C(\omega'_\alpha)$ corresponds to a real point near $\bar{K}_0$. By taking the circles sufficiently small these two points near K_0 and $\bar{K}_0$ respectively will be constrained to lie in arbitrarily small real neighborhoods about K_0 and $\bar{K}_0$, and hence in D.

The plan is to show that there is a real point arbitrarily close to K_0 connected to a point arbitrarily close to $\bar{K}_0$ by a real $\Lambda \in \mathcal{L}$, but not by a $\Lambda \in L$. The sets of points in the real $0 - 3$ plane connected by $\Lambda \in L$ lie on the various hyperbolas having the light-cone lines as asymptotes. The circles are centered on these light-cone lines, the $C(\omega_\alpha)$ lying on the line with positive slope and the $C(\omega'_\alpha)$ lying either on this line or on the other one, depending on the signs in $\omega' = \pm e_0(K_0) \pm e_3(K_0)$.

If $C(\omega_\alpha)$ and $C(\omega'_\alpha)$ lie on the positively and negatively sloped light-cone lines, respectively, then there is always a $\Lambda \in L$ connecting some point of $C(\omega_\alpha)$ to some points of $C(\omega'_\alpha)$. Moreover,

there are then also points in these circles connected by any still "larger" $\Lambda \in L$ The magnitude of the Lorentz transformation is measured by the quotient of the initial over the final (Euclidean) distances of the point from the negatively sloped light-cone line. From these facts it follows that some set of points, one in each of a given set of circles along the positively sloped light-cone line, can be taken into some set of points, one in each of any given set of corresponding circles along the negatively sloped light-cone line, by a single Lorentz transformation $\Lambda \in L$. Thus for the cases $\omega' = \pm (e_0(K_0) - e_3(K_0))$ one can find a $\Lambda \in L$ connecting some real point in any real neighborhood of K_0 to some real point in any neighborhood of $\bar{K}_0$, even though the points themselves cannot be so connected.

The same conclusion holds if one uses instead of $\Lambda \in L$ the real $\Lambda \in \mathcal{L}$ obtained by multiplying the $\Lambda \in L$ by a reflection through the origin in the $0 - 3$ plane. However, as will soon be shown, the points connected in this way cannot be connected by any $\Lambda \in L$. Since by taking the neighborhoods of K_0 and $\bar{K}_0$ small enough the points will be in D, one obtains a contradiction with the assumed property of D. Thus this case $\omega' = \pm (e_0(K_0) - e_3(K_0))$ can, in fact, not occur.

To see that there would be points in D connected by real $\Lambda \in \mathcal{L}$ but not by $\Lambda \in L$, consider first the case $\omega' = -e_0(K_0) + e_3(K_0)$. A time-like point in the circle $C(\omega)$ will be carried to a time-like point in the corresponding circle $C(\omega')$ by the real $\Lambda \in \mathcal{L}$. Since these two points are in the forward and backward light cones respectively

they cannot be connected by an $\Lambda \in L$. The other case, $\omega' = e_0(K_0) - e_3(K_0)$, occurs only if $r = 2$, as previously mentioned. But now a space-like point in $C(\omega)$ is taken to a space-like point in $C(\omega')$ by the real $\Lambda \in \mathcal{L}$. However, transformations involving the first two vectors, $e_1(K_0)$ and $e_2(K_0)$, are not allowed, because the components of vectors of K_0 and K_0' in these subspaces are fixed and equal, and hence these two space-like vectors, which lie in the right and left space cones, respectively, cannot be connected by a $\Lambda \in L$.

The remaining cases are $\omega' = \pm \omega$, or zero. If $\omega' = \pm \omega$ and $r < 2$ then the construction used above again allows certain points near K_0 to be connected to corresponding points near K_0' . One first uses a $\Lambda \in L$ in the $0 - 3$ plane to take the points of the $C(\omega_\alpha)$ to points near the negatively sloped light-cone line, and then uses a rotation through π in the $2 - 3$ plane to bring the points to the desired positions in the $0 - 3$ plane. In particular if ω_α and ω_α' have the same sense, certain time-like vectors near ω_α can be taken to time-like vectors near ω_α' . If ω_α and ω_α' have opposite senses then space-like vectors can be connected. However, if ω_α and ω_α' have the same (opposite) sense a space-like (time-like) point near ω_α can be carried to a space-like (time-like) point near ω_α' by a real $\Lambda \in \mathcal{L}$. But these points cannot be connected by a $\Lambda \in L$ unless $\omega = \omega'$ and $a_\alpha = a_\alpha'$. In that case $K_0' = K_0$ and $\overline{K}_0 = \Lambda_b K_0$, as asserted by the lemma.

The next case is $\omega' = \omega$ and $r = 2$. If $\omega' = \omega$ and $a_\alpha = a_\alpha'$ for all α then $K_0' = K_0$ and $\overline{K}_0 = \Lambda_b K_0$, which proves

the lemma. If $a_\alpha \neq a'_\alpha$ for some α then $\overline{K}_0$ and K_0 are, in fact, not connected by a $\Lambda \in L$. In any event it is sufficient to show that $\omega' = \omega$ and $r = 2$ imply $a_\alpha = a'_\alpha$ for all α.

The conditions $K_i \to K_0$ and $K'_i = \Lambda_i K_i \to K'_0$ are now involved, for the first time. Let $e_1(K)$, $e_2(K)$, and $\omega(K)$ be the linear combinations of the vectors of K that become $e_1(K_0)$, $e_2(K_0)$, and $\omega(K_0) \equiv \omega$ when K becomes K_0. The $e_i(K)$ are then generally not orthonormalized, and $\omega(K)$ is not a null vector. The Λ_i are specified by the conditions $\Lambda_i \in \mathcal{L}$ and by the quantities $e_1(K_i) \equiv e_{i1}$, $e_2(K_i) = e_{i2}$, and $\omega(K_i) \equiv \omega_i$; and

$$e'_{i1} \equiv e_1(K'_1) \equiv e_1(\Lambda_i K_i) = \Lambda_i e_1(K_i) \equiv \Lambda_i e_{i1}, \quad e'_{i2} = \Lambda_i e_{i2},$$

and $\omega'_i = \Lambda_i \omega_i$, at least for sufficiently large i, where the e_{i1}, e_{i2}, and ω_i are linearly independent. For these quantities give the effect of Λ_i on three linearly independent vectors. But since $e'_{i1} \to e_{i1}$, $e'_{i2} \to e_{i2}$, and $\omega'_i \to \omega_i$, it follows from Lemma 3 that $\Lambda_i \to 1$. For Lemma 3 says that given any neighborhood N of the identity in $\mathcal{L}$ one can find a neighborhood N' of $(e_1(K_0, e_2(K_0), \omega)$ such that any points in N' connected by a $\Lambda \in \mathcal{L}$ are connected by a $\Lambda \in N$. Since for the case of three linearly independent vectors the $\Lambda \in \mathcal{L}$ is uniquely defined by these points one concluded that since the sets $(e_{i1}, e_{i2}, \omega_i)$ and $(e'_{i1}, e'_{i2}, \omega'_i)$ both converge to $(e_1(K_0), e_2(K_0), \omega)$, the $\Lambda_i \in \mathcal{L}$ connecting them must approach the identity. But if $\Lambda_i \to 1$ and $K_i \to K_0$ then $\Lambda_i K_i \to K_0$. Thus $\overline{K}_0 = \Lambda_b K_0$, which proves the lemma for this case.

If $\omega' = -\omega$ a reflection through the origin in the $0 - 3$ plane takes one to the previous case $\omega' = \omega$. Because of the condition on D this case is then ruled out, since $\overline{K}_0$ is connected to K_0 by a real $\Lambda \in \mathcal{L}$ but not by a $\Lambda \in L$.

Next there is the case $\omega' = 0$. If all of the $\omega'_\alpha = 0$, [i.e. if $n(\overline{K}_0) = r$], then this case is ruled out by the same argument that was used in the case $\omega' = -e_0(K_0) + e_3(K_0)$; there are points of D connected by real $\Lambda \in \mathcal{L}$ but not by $\Lambda \in L$. (The possibility $n(K_0) = n(\overline{K}_0) = r$ with $e_1(K_0), \cdots, e_r(K_0)$ all space-like is also ruled out in this way, it might be added.) If $\omega' = 0$ but some ω'_α is a nonzero vector lying along the negatively sloped light-cone line one may again use the same argument as was used for the case $\omega' = -e_0(K_0) + e_3(K_0)$ case; the $C(\omega')$ is simply centered at the origin instead of at its former position.

For the case $r < 2$, $\omega' = 0$, and $\omega'_\alpha = a'_\alpha \omega \neq 0$ for some α, the argument used in the case $r < 2$, $\omega' = \pm\omega$, goes through without any change.

Finally there is the same case but with $r = 2$. Every ω_α and ω'_α is either zero or on the positively sloped light-cone line. For every α either ω_α or ω'_α is zero; otherwise it can be made into the case $\omega' = \pm\omega$. And not every ω'_α is zero; otherwise it is the previously considered case $n(\overline{K}_0) = r$. This means that the Λ_i are such that the following conditions can be satisfied:

$$(e_{i1},\ e_{i2},\ \omega_i) \longrightarrow \big(e_1(K_0),\ e_2(K_0),\ \omega\big)\ ,$$

$$\Lambda_i\ (e_{i1},\ e_{i2},\ \omega_i) \longrightarrow \big(e_1(K_0),\ e_2(K_0),\ 0\big)\ ,$$

$$(e''_{i1},\ e''_{i2},\ \omega''_i) \longrightarrow \big(e_1(K_0),\ e_2(K_0),\ \omega_1\big)\ ,$$

and

$$\Lambda_i^{-1}\ (e''_{i1},\ e''_{i2},\ \omega''_i) \longrightarrow \big(e_1(K_0),\ e_2(K_0),\ 0\big)\ .$$

Here the double-primed quantities are a particular set of primed quantities, the ω''_i being an $\omega'_\alpha(K_i)$ whose limit is $\omega_1 \neq 0$.

These two conditions on the set Λ_i are incompatible. The first two equations imply that, for sufficiently large i , the points $\Lambda_i\ \omega$ must lie in a narrow cone-like region about the negatively sloped light-cone line, whereas the second two imply that $\Lambda_i\ \omega$ must lie far from the origin in some narrow cone-like region about the positively sloped light-cone line. The incompatibility of these conditions rules out this last possiblity.

The consequences for the $\Lambda_i\ \omega$ asserted above follow from a detailed examination of the converging sequences. A general description of the argument should be sufficient. Since $(e_{i1},\ e_{i2}) \to \big(e_1(K_0),\ e_2(K_0)\big)$ one can choose basis vectors e_{0i} and e_{3i} in the subspace orthogonal to the one spanned by the $(e_{i1},\ e_{i2})$ in such a way that $(e_{0i},\ e_{3i}) \to \big(e_0(K_0),\ e_3(K_0)\big)$. The $(e_{0i},\ e_{3i})$, unlike the $(e_{i1},\ e_{i2})$ are to be parts of an orthonormal basis. A set $(e'_{0i},\ e'_{3i})$ similarly related to the $(e'_{i1},\ e'_{i2}) \equiv \Lambda_i(e_{i1},\ e_{i2})$ is constructed. Then $\Lambda_i^{\ r}$ is defined by the conditions $\Lambda_i^{\ r}(e_{i1},\ e_{i2}) = \Lambda_i(e_{i1},\ e_{i2})$ and $\Lambda_i^{\ r}(e_{0i}, e_{3i}) = (e'_{0i},\ e'_{3i})$.

Since $(e_{i1}, e_{i2}, e_{3i}, e_{0i})$ and $(e'_{i1}, e'_{i2}, e'_{3i}, e'_{0i})$ both converge to $(e_1(K_0), e_2(K_0), e_3(K_0), e_0(K_0))$ it follows that $\Lambda_i^r \to 1$, by Lemma 3.

Since $\Lambda_i^r(e_{i1}, e_{i2}) = \Lambda_i(e_{i1}, e_{i2})$ it follows that $\Lambda_i^\omega \equiv (\Lambda_i^r)^{-1}\Lambda_i$ acts only in the (e_{0i}, e_{3i}) subspace. Also since $(e_{0i}, e_{3i}, \omega_i) \to (e_0(K_0), e_3(K_0), \omega)$, with $\omega = e_0(K_0) + e_3(K_0)$, one has $\omega_i \to (e_{0i} + e_{3i})$.

Since $\Lambda_i^r \to 1$ and $\Lambda_i = \Lambda_i^r \Lambda_i^\omega$ the condition $\Lambda_i \omega_i \to 0$ implies $\Lambda_i^\omega \omega_i \to 0$. Since Λ_i^ω acts only in the (e_{0i}, e_{3i}) subspace the problem is reduced now to a problem in this two-dimensional space. The two conditions $\Lambda_i^\omega \omega_i \to 0$ and $\omega_i \to e_{0i} + e_{3i}$ imply that $\Lambda_i^\omega(e_{0i} + e_{3i}) \to 0$; the general Lorentz transformation in this two-dimensional space is represented by

$$(e_{0i} + e_{3i}) \longrightarrow (\exp \Gamma_i)(e_{0i} + e_{3i})$$

and

$$(e_{0i} - e_{3i}) \longrightarrow [\exp(-\Gamma_i)](e_{0i} - e_{3i}) \quad ,$$

and hence one cannot transform a point near $(e_{0i} + e_{3i})$ to a point near the origin unless $\mathrm{Re}\,\Gamma_i \gg 0$ But in this case the point $(e_{0i} + e_{3i})$ is also brought close to the origin. Moreover any point is brought closer to the line $\lambda(e_{0i} - e_{3i})$. Thus the point ω will be brought closer to the line $\lambda(e_{0i} - e_{3i})$. As i increases, the lines $\lambda(e_{0i} - e_{3i})$ are constrained to lie in smaller and smaller cones about the line $\lambda(e_0(K_0) - e_3(K_0))$. Thus for sufficiently large i the point ω must be taken by Λ_i closer to a point near some small cone about $\lambda(e_0(K_0) - e_3(K_0))$, the cone becoming narrower with increasing i. Thus for sufficiently large i the

$\Lambda_i\,\omega$ are constrained to lie in a cone-like region about the $e_0(K_0) - e_3(K_0)$ axis.

If, on the other hand, Λ_i^{-1} takes a point near the $(e''_{0i} + e''_{3i})$ axis to a point near the origin then $\mathrm{Re}\,\Gamma_i \gg 0$. But then under Λ_i all points are moved further from the line $\lambda\,(e''_{0i} - e''_{3i})$ and closer to the line $\lambda\,(e''_{0i} + e''_{3i})$. Thus $\Lambda_i\,\omega$ must for sufficiently large i be far from the origin in a narrow cone-like region about the line $\lambda\,(e_0(E_0) + e_3(K_0))$ By taking i large enough these two cones can be made arbitrarily narrow. Hence the allowed regions will not overlap. This gives the contradiction.

Theorem 2 . Let D be a real domain satisfying the conditions of Lemma 1 . Let $F(K)$ be defined (single valuedly) and L-invariant over D, and be regular at points of D, in the (weak) sense that for any point $K' \in D$ there is a domain $D(K')$ containing K', and a function $F(K, K')$ that is regular at points $K \in D(K')$ and which coincides with $F(K)$ at points $D_r(K')$, some real domain contained in $D \cap D(K')$ and containing K'. Let C be any closed, bounded subset of D. Then there is an $\mathcal{L}$-invariant germ neighborhood whose base domain $B = \mathcal{L}B$ contains C and whose characteristic function coincides with $F(K)$ for $K \in C$.

Proof: Let K_0 be any point of C. Let $C(K_0, \rho)$ be a polysphere of radius ρ centered at K_0. Let $\rho_i \to 0$ be a monotonically decreasing set of radii converging to zero. And let the first ρ_i be small

enough so that $C(K_0, \rho_i) \subset D(K_0)$, for all i. Suppose K_i is an infinite sequence of points in $\mathcal{L}C$ such that $K_i \in C(K_0, \rho_i)$ and such that $F(K_i, K_0) \neq F'(K_i)$, where $F'(K_i)$ is the (single-valued) $\mathcal{L}$-invariant extension of $F(K)$ to $\mathcal{L}D$, which according to Lemma 1 exists. For each point $K_i \in \mathcal{L}C$ there is a point $\bar{K}_i \in C \cap \mathcal{L}K_i$. Since C is closed and bounded the $\bar{K}_i$ have a limit point $\bar{K}_0 \in C$. And one can find a subsequence $\bar{K}_i \longrightarrow \bar{K}_0 \in C$.

The point $\bar{K}_0$ cannot be on $\mathcal{L}K_0$. If it were there would, according to Lemma 1A and the property of D, be a $\Lambda \in L$ such that $\bar{K}_0 \in \Lambda K_0$. This Λ would map the real domain $D_r(K_0)$ containing K_0 into some real domain containing $\bar{K}_0 \in C$. The intersection of this domain $\Lambda D_r(K_0)$ with D contains a real domain $D_r'(\bar{K}_0)$ containing $\bar{K}_0$. At points of $D_r'(\bar{K}_0)$ the value of $F(K)$ is given in terms of $F(K)$ at points of $D_r(K_0)$ by the L-invariance condition. Now according to Lemma 8 there is an $\mathcal{L}$-invariant germ neighborhood, with a base domain $\mathcal{L}D_0(K_0)$, having a characteristic function that coincides with $F(K; K_0)$ for $K \in D_0(K_0) \subset D(K_0)$. The value of $F(K)$ at points of $D_r'(\bar{K}_0) \cap \mathcal{L}D_0(K_0)$ must coincide with the value of the characteristic function at these points, since both are given in terms of $F(K)$ at $K \in D_r(K_0)$ by the L-invariance condition. But then $F'(K)$ must coincide with this characteristic function for all points of $\mathcal{L}D_r'(K_0) \cap \mathcal{L}D_0(K_0)$ Therefore $F'(K_i) = F(K_i, K_0)$ for all $K_i \in \mathcal{L}D_r'(K_0) \cap D_0(K_0)$. This precludes the possibility that a subsequence of the $\bar{K}_i \in C$ converge to $\bar{K}_0$. Thus the limit point $\bar{K}_0$ cannot lie on $\mathcal{L}K_0$.

But according to Lemma 11 the point $\bar{K}_0$ must lie on $\mathcal{L} K_0$, since $K_i \to K_0$, $\bar{K}_i \to \bar{K}_0$, K_0 and $\bar{K}_0$ are in D, and $\bar{K}_i \in \mathcal{L} K_i$. Thus there can be no infinite sequence of K_i with the specified properties. In particular for some $\rho_0 > 0$ there can be no points $K_i \in (C(K_0, \rho_0) \cap \mathcal{L} C)$ with $F'(K_i) \neq F(K_i, K_0)$.

Take some ρ_0' with $\rho_0 > \rho_0' > 0$ such that $C(K_0, \rho_0') \subset D_0(K_0)$. Then the restriction of the $\mathcal{L}$-invariant germ neighborhood over $\mathcal{L} D_0(K_0)$ to the $\mathcal{L}$-invariant germ neighborhood over $\mathcal{L} C(K_0, \rho_0')$ is an $\mathcal{L}$-invariant germ neighborhood whose characteristic function coincides with $F'(K)$ for $K \in (\mathcal{L} C \cap \mathcal{L} C(K_0, \rho_0'))$.

The point K_0 was an arbitrary point of C. This construction can be carried through for every point $K' \in C$. Let the radius corresponding to ρ_0', but for the general $K' \in C$, be denoted by $\rho(K')$. One can take $\rho(K') < A$, some positive upper bound.

Let $r_i \longrightarrow 0$ be an infinite sequence of positive numbers that decrease monotonically to zero. Let K_0 be an arbitrary point of C and let $C(K_0, r(K_0))$ be a polysphere of radius $r(K_0)$ about the point K_0. Let $r_i(K\ ) > 0$ be less than $\rho(K\ )$ and less than r_i. Let K_i be a new set of points such that for each K_i there is a $K_i' \in C$ such that $K_i \in C(K_0, r_i(K_0)) \cap \mathcal{L} C(K_i', r_i(K_i'))$ and such that the characteristic functions constructed above for K_0 and K_i' fail to coincide at $K = K_i$. Either an infinite sequence of K_i can be found or there is some $a(K_0)$ such that for $r_i < a(K_0)$ no such K_i exists.

Suppose there is an infinite sequence of K_i . For each K_i there is a $\bar{K}_i \in \mathcal{L} K_i$ that is in $C(K_i', r_i(K_i'))$. Since the union of the $C(K', \rho(K'))$, $K' \in C$, is a bounded set the $\bar{K}_i$ must have an accumulation point $\bar{K}_0$. This point must be in C , since the $r_i(K_i') \longrightarrow 0$. This point $\bar{K}_0$ is a limit point for a subsequence of the $\bar{K}_i$. The other $\bar{K}_i$ can be omitted. This limit point must, according to Lemma 11 lie on $\mathcal{L} K_0$. By virtue of the property of D there must then be a $\Lambda \in L$ such that $\bar{K}_0 = \Lambda K_0$. Thus $\bar{K}_0$ is in $\mathcal{L} C(K_0, \rho(K_0))$. But since $\bar{K}_i \longrightarrow \bar{K}_0$ and $K_i' \longrightarrow \bar{K}_i$, also $K_i' \longrightarrow \bar{K}_0$, and the $K_i' \in C$ must be in $\mathcal{L} C(K_0, \rho(K_0))$, except for a finite few which can be omitted. Then also the $C(K_i', r_i(K_i'))$ will be completely inside $\mathcal{L} C(K_0, \rho(K_0))$, except for a finite few, which can be omitted. But then the characteristic functions over $\mathcal{L} C(K_0, \rho(K_0))$ and $\mathcal{L} C(K_i', \rho(K_i'))$ must coincide at the points in $C(K_i', r_i(K_i'))$ since they coincide over points of C contained in this polysphere, whose intersection with $\mathcal{L} C(K_0, \rho(K_0))$ is a domain, $C(K_i', r_i(K_i'))$. But then the two characteristic functions must coincide at $\bar{K}_i$, and hence also at points of $\mathcal{L} \bar{K}_i$, and hence at K_i . This contradicts the assumption concerning the K_i . Thus there cannot be an infinite sequence of K_i satisfying those conditions, and hence there is an $a(K_0)$ such that for $r_i < a(K_0)$ the characteristic function over $\mathcal{L} C(K_0, \rho(K_0))$ coincides with the characteristic function over $\mathcal{L} C(K', \rho(K'))$ for all $K' \in C$, at all points $K \in C(K_0, r_i(K_0)) \cap \mathcal{L} C(K', r_i(K'))$ and hence at all points $K \in \mathcal{L} C(K_0, r_i(K_0)) \cap \mathcal{L} C(K', r_i(K'))$, where $r_i(K) < \min(r_i, \rho(K))$

The point K_0 was an arbitrary point of C . Thus there is for every $K' \in C$ a characteristic radius $a(K') > 0$. If there is no

lower bound $\bar{a} > 0$ such that $a(K') \geqslant \bar{a} > 0$ for all $K' \in C$ then one can find a sequence of $K_i \in C$ such that $a(K_i) \longrightarrow 0$. These K_i must have an accumulation point $\bar{K} \in C$, though $a(\bar{K}) > 0$. But such an abrupt jump in $a(K)$ at $K = \bar{K}$ is not possible, for if $b(\bar{K}) = \min \left\{ a(\bar{K}) , \rho(\bar{K}) \right\}$ then certainly $a(K) \geqslant \frac{1}{3} b(\bar{K}) > 0$ for $K \in C(\bar{K}, \frac{1}{2} b(\bar{K})) \cap C$, since for these K all points of $C(K, \frac{1}{3} b(\bar{K}))$ are in $C(\bar{K}, b(\bar{K}))$, where the various characteristic functions coincide even with the weaker limit $a(\bar{K})$ on the r_i, and hence certainly for $r_i < \frac{1}{3} b(\bar{K})$. Thus there must be an $\bar{a} > 0$ such that $a(K') > \bar{a}$ for all $K' \in C$. Thus the union of the $\mathcal{L}$-invariant germ neighborhoods over the base domains $\mathcal{L}C(K', b'(K'))$, with $K' \in C$ and $b'(K') = \min (\bar{a}, \rho(K'))$, satisfies the required conditions; its base domain contains all points $K' \in C$, it has an $\mathcal{L}$-invariant characteristic function defined (single valuedly) over its base domain $B = \mathcal{L}B$, and this characteristic function coincides with $F'(K)$ for $K \in B \cap \mathcal{L}C$.

Definition: An <u>enlargement</u> of a germ neighborhood N is a germ neighborhood containing N but not contained in N.

Definition: A germ neighborhood N will be called <u>maximal</u> if and only if no enlargement of N exists.

Lemma 12. Every germ neighborhood is contained in a maximal germ neighborhood.

Proof: Let N be an arbitrary germ neighborhood. A maximal germ neighborhood $N_M \supset N$ can be constructed as follows: Let $\{K_i\}$ be a denumerable sequence of points that is everywhere dense in the space in which lie the base points of the germs of the germ space. Let the K_i be enumerated. If a point K_i is reached that is in the base domain of an enlargement of N then replace N by this enlargement (probably one of many possible enlargements) and proceed iteratively with the enumeration of the points of the sequence $\{K_i\}$. Because the union of a (finite or infinite) set of open sets is an open set the result of this denumerable sequence of operations is a germ neighborhood N_a, since the base domain D_a is certainly connected and the function $F_a(K)$ is defined (single valuedly) over D_a and is regular at any point in D_a.

Let $\bar{D}_a$ be the set of accumulation points of the points $K_i \in D_a$. No enlargement of N_a can contain a point whose base point $\bar{K}$ is not in $\bar{D}_a$. For any such point $\bar{K}$ must be an accumulation point of points K_i not in D_a. Hence any enlargement containing a point with such a base point $\bar{K}$ would also contain a point with base point K_i not in D_a. This is impossible, for if there were such a K_i then when this K_i was reached in the enumeration it could have been included in the base domain of an enlargement of the then current germ neighborhood, since enlargements of enlargements are themselves also enlargements. But the construction was such that

if any K_i can be included in the base domain of any enlargement of the then-current germ neighborhood then it is in fact included in the enlargement associated with this K_i. Thus this K_i would be in D_a. Thus no K_i not in D_a and no accumulation point $\vec{K}$ of these K_i can be the base point of a point in any enlargement of N_a; the base points of all points of every enlargement of N_a are in $\bar{D}_a$.

If a point with base point $\bar{K} \in \bar{D}_a$ is in an enlargement of N_a then the value of the characteristic function of the enlargement at $K = \bar{K}$ is unique; it is the same for any enlargement. For in order that a point with base point $\bar{K} \in \bar{D}_a$ be in an enlargement of N_a the corresponding characteristic function must be defined (single valuedly) and regular in a neighborhood $N(\bar{K})$ of $\bar{K}$, and it must coincide with $F_a(K)$ for $K \in D_a \cap N(\bar{K})$. Thus it must coincide with $F_a(K)$ at the points $K_i \in D_a \cap N(\bar{K})$, which are dense in a neighborhood of $\bar{K}$ But the value of $F_a(K)$ at these points then determines the function at $K = \bar{K}$ by virtue of the continuity requirement implied by the regularity at $\bar{K}$ of the characteristic function of the enlargement.

Let D_M be the subset of $\bar{D}_a$ consisting of all the points of $D_a \subset \bar{D}_a$ and of all the base points of the points of any enlargement of N_a. Since the D_M is a union of domains each of which has a point in common with D_a the set D_M is a domain. Since the value of the characteristic function of any enlargement of N_a is uniquely defined for every $K \in D_M$ one may denote it by $F_M(K)$. This function is regular at every $K \in D_M$ because it is defined for $K \in D_M$ by an enlargement of N_a. Thus one may define a germ neighborhood

$N_M \equiv N(D_M, F_M)$. This germ neighborhood contains N_a and hence N . Moreover, this germ neighborhood N_M is maximal. For any enlargement of N_M would also be an enlargement of N_a . But no enlargement of N_a exists that is also an enlargement of N_M because N_M contains every point of every enlargement of N_a .

Lemma 12A . Every $\mathcal{L}$-invariant germ neighborhood is contained in a maximal germ neighborhood that is $\mathcal{L}$-invariant.

Proof. Let $N = N(D, F)$ be an $\mathcal{L}$-invariant germ neighborhood. If an enlargement of N exists then an $\mathcal{L}$-invariant enlargement also exists. To prove this, note first that any enlargement of N is a domain containing a point of N and some point not in N . By connecting these with a continuous curve one can, by a simple construction, find, in the enlargement, a point P_0 not in N such that any neighborhood of P_0 contains a point of N . Let the base point of P_0 be K_0 . According to the Corollary to Lemma 8 there is a domain $D_0(K_0)$ containing K_0 such that the function defined in $D_0(K_0)$ as the characteristic function of the enlargement of N can be extended to a function $F'(K)$ that is $\mathcal{L}$-invariant throughout $\mathcal{L}\, D_0(K_0)$ and and regular there. It must coincide with the characteristic function of the original $\mathcal{L}$-invariant germ neighborhood, wherever both are defined, since both functions are $\mathcal{L}$-invariant over their domains of definition and they coincide in $D_0(K_0) \cap D$, which contains a point of every orbit common to both domains. Thus the union of the original

$\mathcal{L}$-invariant given neighborhood N with the $\mathcal{L}$-invariant germ neighborhood $N' \equiv N(\mathcal{L}D_0, F')$ constitutes an enlargement of the original one, and this enlargement is $\mathcal{L}$-invariant. Thus if an $\mathcal{L}$-invariant germ neighborhood has an enlargement it has an $\mathcal{L}$-invariant enlargement.

By virtue of this, one may proceed just as in Lemma 12 , using however only $\mathcal{L}$-invariant enlargements. After running through the denumerable set K_i one has an $\mathcal{L}$-invariant germ neighborhood $N_a = N(D_a, F_a)$. Now, no point not in $\bar{D}_a$ can be the base point of an $\mathcal{L}$-invariant enlargement. The set $D_M \subset \bar{D}_a$ is defined by using only $\mathcal{L}$-invariant enlargements. Thus $N_M = N(D_M, F_M)$ is a germ domain that is maximal with respect to $\mathcal{L}$-invariant enlargements. But then according to the first paragraph N_M is also maximal. Thus it is a maximal germ neighborhood that is $\mathcal{L}$-invariant.

Definition: The base domain of a maximal germ neighborhood will be called a <u>sheet</u>.

Theorems 1A and 2 , in conjunction with Lemma 12A, are summarized in

Theorem 3. Let $F(K)$ be a function defined (single valuedly) over a real domain D . For every Λ in the real proper orthochronous homogeneous Lorentz group L and every K such that K and ΛK are in D let $F(K)$ satisfy the Lorentz invariance condition

$$F(K) = \Lambda_s^{-1} F(\Lambda K)$$

If $F(K)$ is regular at some point $K \in D$ then the analytic continuation of $F(K)$ from the neighborhood of this point is defined over a manifold covered by a set of sheets each of which maps onto itself under any element of the proper homogeneous complex Lorentz group $\mathcal{L}$. And for any sheet the associated function defined (single valuedly) and regular at all points of this sheet satisfies the Lorentz invariance condition for all $\Lambda \in \mathcal{L}$.

Moreover, if every point of D is a regular point of $F(K)$ and D has the property, specified in Lemma 1, that any points of D connected by a real $\Lambda \in \mathcal{L}$ are connected by a $\Lambda \in L$, then any closed bounded subset C of D can be completely contained in a single $\mathcal{L}$-invariant sheet, with $F(K)$ coinciding with the function defined over that sheet for $K \in C$.

Definition: The <u>restricted mass shell</u> is the subset W in the space of points $K \equiv \{k_1, \cdots, k_n\}$ that satisfy the n mass constraints

$$k_i^2 \equiv \sum_\mu (k_i^\mu)^2 = m_i^2 \qquad (i = 1, \cdots, n),$$

the four conservation laws

$$\sum_i k_i^\mu = 0 \qquad \mu = 0, 1, 2, 3$$

and the condition that the set $K \in W$ have more than one linearly independent vector. The m_i are fixed positive numbers and $n \geq 4$.

Lemma 13 . The restricted mass shell W is a $(3n - 4)$ complex-dimensional manifold.

Proof: Consider any point $\bar{K} \in W$. Let the k_i be ordered so that the last two are linearly independent. Let $\Lambda(\bar{K})$ be a Lorentz transformation that is such that the energy components of the vectors of $\bar{K}'(\bar{K}) \equiv \Lambda(\bar{K})\bar{K}$ are all nonzero. Such a $\Lambda(\bar{K})$ surely exists since the $\bar{K}$ are a finite set of nonzero vectors. Let the components 1, 2, 3, be numbered so that $\bar{k}'^0_{n-1}/\bar{k}'^0_n \neq \bar{k}'^3_{n-1}/\bar{k}'^3_n$ This is possible because $\bar{k}_{n-1}$ and $\bar{k}_n$ are linearly independent. By a small change in $\Lambda(\bar{K})$ that does not upset the above inequalities one can also ensure that $(\bar{k}'^0_{n-1} + \bar{k}'^0_n)^2 \neq (k'^3_{n-1} + k'^3_n)^2$, since $k_{n-1} + k_n \neq 0$.

With $\Lambda(\bar{K})$ fixed in this way the set of vectors $K'(\bar{K}, K)$ is defined by $K'(\bar{K}, K) \equiv \Lambda(\bar{K})K$. The set $Z(\bar{K}, K)$ is then defined as the set of $(3n - 4)$ complex variables consisting of the three space components of the first $(n - 2)$ vectors of $K'(\bar{K}, K)$ and the first two components of the $(n - 1)$st vector of $K'(\bar{K}, K)$. The set of functions $Z(\bar{K}, K)$ are analytic functions (in fact linear functions) of the vectors of K . They define a set of mappings of K space onto Z space.

By virtue of the conditions that have been imposed on the vectors of K' the inverse transformation, $K'(\bar{K}; Z)$, that maps Z back into $K' \in W$ is uniquely defined for $Z \in U(\bar{K})$, a domain containing $Z \equiv Z(\bar{K}, \bar{K})$, and is an analytic function of Z there.

This follows from simple algebra or from the implicit function theorem,[14] the conditions of which are easily verified.

The set W can be made into a topological (Hausdorff) space by defining the open sets in W to be the restriction of open sets in K space to W. The topology in K space and Z space will be taken as the usual one induced by the Euclidian norm. With the topology of W defined in this way the continuity of the functions $K(\overline{K}; Z)$ and $Z(\overline{K}; K)$, considered as mappings between K space and Z space, which follows from their analyticity, implies that these mappings are continuous mappings between $U(\overline{K})$ and its image $U_W(\overline{K}) \subset W$. For if a neighborhood of a point $Z \in U(\overline{K})$ maps into a K-space neighborhood of its image $K = K(\overline{K}; Z)$ then it must also map into a W-space neighborhood of $K = K(\overline{K}; Z)$, since it maps into W. And conversely, if a neighborhood of $K \in W$ in K space maps into a neighborhood in Z space, then its restriction to W also maps into this neighborhood. Thus the transformation $K(\overline{K}; Z)$ defines a one-to-one continuous mapping of neighborhoods of $\overline{K} \in W$ contained in $U_W(\overline{K})$ onto neighborhoods of $\overline{Z}$ contained in $U(\overline{K})$. Since the inverse is also continuous the transformation is, by definition, a homeomorphism and the open sets in $U_W(\overline{K})$ and $U(\overline{K})$ are homeomorphic images of each other. Since $\overline{K}$ was an arbitrary point of W the set W has an open covering by sets homeomorphic with open sets of $C^{(3n-4)}$, and hence W is a $(3n-4)$ (complex)-dimensional manifold.[15]

Definition: The functions $K(\bar{K}; Z)$ and $Z(\bar{K}; K)$ will denote the functions introduced in the proof of Lemma 13. The function $Z(\bar{K}; K)$ is defined for $\bar{K} \in W$ and for all K, and for each $\bar{K} \in W$ it is an analytic function of K. The function $K(\bar{K}; Z)$ is defined for $\bar{K} \in W$ and $Z \in U(\bar{K})$, a domain containing $Z = Z(\bar{K}, \bar{K})$, and for each $\bar{K} \in W$ it is an analytic function of Z for $Z \in U(\bar{K})$. The function $K(\bar{K}; Z)$ maps points $Z \in U(\bar{K})$ into $U_W(\bar{K}) \subset W$. Its reciprocal is $Z(\bar{K}, K)$ in the sense that $Z(\bar{K}; K(\bar{K}; Z')) = Z'$ for $Z' \in U(\bar{K})$ and $K(\bar{K}; Z(\bar{K}; K')) = K'$ for $K' \in U_W(\bar{K}) \subset W$.

Remark: The set $U_W(\bar{K})$, as a homeomorphic image of the domain $U(\bar{K})$, is a domain.

Definition: The mapping $\phi(\bar{K})$ is a mapping of $K \in U_W(\bar{K})$ to $Z \in U(\bar{K})$ defined by $\phi(\bar{K})\, K = Z(\bar{K}; K)$ for $\bar{K} \in W$ and $K \in U_W(\bar{K})$.

Definition: The restricted mass shell W together with the complex structure induced by the collection $\left\{ U_W(\bar{K}),\ \phi(\bar{K}) \right\}$, $\bar{K} \in W$, is called the complex <u>analytic manifold $\tilde{W}$ of W</u>.

Definition A: A function $M(K)$ defined on a restricted mass shell W will be called <u>regular</u> at $\bar{K} \in W$ if and only if $M(\phi^{-1}(\bar{K})\, Z) \equiv (M \circ \phi^{-1}(K))\, Z$ is a regular function of Z at $Z = \phi(\bar{K})\, \bar{K}$.

Definition A': A function $M(K)$ defined on a restricted mass shell W will be called <u>regular</u> at $\bar{K} \in W$ if and only if $M \circ \phi^{-1}$ is regular

at $Z = \phi\,\bar{K}$ for <u>every</u> one-to-one mapping ϕ, such that $\phi^{-1} Z \equiv K(Z) \in W$ is an analytic function at $Z = \phi\,\bar{K}$.

Lemma 14. Definitions A and A' are equivalent.

Proof: If $M(K)$ is regular (A') at $\bar{K} \in W$ it is certainly regular (A) at $\bar{K} \in W$ since $\phi(\bar{K})$ is a particular ϕ. If $M(K)$ is regular (A) at $\bar{K} \in W$ and ϕ is a one-to-one mapping such that $\phi^{-1} Z \equiv K(Z) \in W$ is an analytic function at $Z = \phi\,\bar{K}$, then $(M \circ \phi^{-1})Z \equiv M(K(Z)) \equiv M(\phi^{-1}(\bar{K})\ Z(\bar{K};\ K(Z)))$. But $M \circ \phi^{-1}(\bar{K})$ is an analytic function of its argument Z for $Z = Z(\bar{K};\ \bar{K})$, and $Z(\bar{K};\ K)$ is an analytic function of K for $K = \bar{K}$, and $K(Z)$ is an analytic function at $Z = \phi\,\bar{K}$. Thus $M \circ \phi^{-1}$ is an analytic function of Z at $\phi\,\bar{K}$, since it is an analytic function of an analytic function of an analytic function.

Theorem 4. The preceding theorems and lemmas remain valid if $F(K)$ is replaced by $M(K)$ defined on a restricted mass shell W, and all domains are taken to be domains relative to W.

Proof: The mass shell contains all points having the same scalar invariants as any point on it, and in particular all points on any orbit intersecting it. This is the only global property of the K space that was used in any of the above proofs. For local properties one replaces the topology of K space by the topology of W space. Some of the proofs become vastly simplified because for real $K \in W$ one has $n = r$.

Remark 1. Any real domain of W satisfies the condition of Lemma 1; two real points of W connected by a real $\Lambda \in \mathcal{L}$ that is not a $\Lambda \in L$ must have opposite energy components and hence they cannot both be in a real domain in W. The M_c functions have been shown to satisfy the L-invariance condition at regular physical points. Thus if D_r is a real (physical) domain of regularity of M_c (defined over W) then, by Theorem 3, any closed bounded set $C \subset D_r$ is contained in a sheet S that maps onto itself under any $\Lambda \in \mathcal{L}$, and the function M_c has a single-valued analytic continuation throughout S, and is $\mathcal{L}$ invariant there.

Remark 2. One consequence of the above remark is a slight weakening of the assumptions needed for the S-matrix proof of CPT invariance. In the original proof[3] the postulate of minimal analyticity required the existence of a physical sheet that was bounded by cuts defined by equations involving only scalar invariants. This condition on the boundary was imposed specifically to eliminate problems associated with a possible multivaluedness in the continuation to the CPT image point. However, consequence of Theorem 3 drawn in the above remark is the existences of the single-valued $\mathcal{L}$-invariant continuation to the CPT-image point. The proof of CPT invariance in this way is similar to the field-theoretic proof of Jost;[11] that proof rested heavily on Lemma 1 of Hall and Wightman, which is rather analogous to Theorem 3.

Remark 3. In the construction of the decomposition of the analytic M_c functions into analytic functions of scalar invariants time standard

(polynomial) invariants,[16,17] the $\mathcal{L}$-invariance of the domains of regularity is a basic ingredient. A fundamental result that can be drawn from this paper (Theorems 1 and 3, and the L-invariance at physical points established in previous sections) is that any domain of regularity of M_c containing a physical point is $\mathcal{L}$-invariant. Since M_c is defined by analytic continuation from physical points, any domain of regularity of M_c is $\mathcal{L}$-invariant.

APPENDIX A. GENERALIZED SPINOR CALCULUS

The Lorentz transformations $\Lambda^{\mu}{}_{\nu}(A, B)$ are defined by the equation

$$A\,\sigma_{\nu}\,B = \sigma_{\mu}\,\Lambda^{\mu}{}_{\nu}(A, B)\ , \tag{A.1}$$

where $\sigma_{\mu} = (\sigma_0, \underset{\sim}{\sigma})$ are the usual Pauli matrices, and A and B are unimodular two-by-two matrices. The unimodular two-by-two matrices form a group. The canonical irreducible representations of this group of dimension $(2a + 1)$ are generated by the recursion relation

$$A^{(a)\,\alpha'}{}_{\alpha} = C_{bc}(a,\alpha;\ \beta,\gamma)\ C_{bc}(a,\alpha';\ \beta',\gamma')\ A^{(b)\,\beta'}{}_{\beta}\ A^{(c)\,\gamma'}{}_{\gamma}\ , \tag{A.2}$$

where the coefficients C are the usual Clebsch-Gordan coefficients. The $A^{(1/2)}$ is identified with A .

Generalized spinor indices of order $(2a + 1)$ are introduced. They can be either upper or lower and either dotted or undotted. The distinction between indices of these various types is with respect to the effect upon them of the operator Λ_s . The action of this operator is defined as follows:

$$\begin{aligned}
\Lambda_s\,\xi_{\alpha} &= A^{(a)\,\alpha'}{}_{\alpha}\ \xi_{\alpha'} \\
\Lambda_s\,\xi_{\dot\alpha} &= \xi_{\dot\alpha'}\ B^{(a)\dot\alpha'}{}_{\dot\alpha} \\
\Lambda_s\,\xi^{\alpha} &= \xi^{\alpha'}\left(A^{(a)}\right)^{-1}{}^{\alpha}{}_{\alpha'} \\
\Lambda_s\,\xi^{\dot\alpha} &= \left(B^{(a)}\right)^{-1}{}^{\dot\alpha}{}_{\dot\alpha'}\ \xi^{\dot\alpha'}
\end{aligned} \tag{A.3}$$

Here $B^{(a)}$ is defined by the analog of (A.2) with B's in place of A's. If a function has several spinor indices then Λ_s acts individually on each in the manner given by (A.3).

Let $f(V)$ be a function of a set $V = \left\{ v \cdots v_n \right\}$ of four-vectors. Let $\Lambda V = \left\{ \Lambda v_1, \cdots, \Lambda v_n \right\}$ where

$$(\Lambda v)^\mu = \Lambda^\mu{}_\nu(A, B)\, v^\nu \quad . \tag{A.4}$$

If $f(V)$ carries spinor indices and satisfies the equation

$$\Lambda_s\, f(V) = f(\Lambda V) \quad . \tag{A.5}$$

Then f will be called an invariant spinor function. The Pauli matrices σ_μ will be considered to have matrix elements $\sigma_{\mu\alpha\dot\beta}$. Then the function

$$g(v) \equiv \sigma \cdot v \tag{A.6}$$

is, by virtue of the conventions adopted, an invariant spinor function.

A generalization of the Pauli $\sigma_{\mu\alpha\dot\beta}$ to higher dimension is defined by the recursion formula

$$\left(\sigma^{a}_{(\mu)}\right)_{\alpha\dot\alpha'} = C_{bc}\left(a, \alpha; \beta, \gamma\right) C_{bc}(a, \dot\alpha'; \dot\beta', \dot\gamma')\left(\sigma^{b}_{(\mu')}\right)_{\beta\dot\beta'} \left(\sigma^{c}_{(\mu'')}\right)_{\gamma\dot\gamma'} \tag{A.7}$$

where $a = b + c$. Here $(\mu) = \left\{ \mu_1, \cdots, \mu_{2a} \right\}$ is a set of $2a$ vector indices and $(\mu) = (\mu') + (\mu'')$. The $\left(\sigma^{a}_{(\mu)}\right)$ is symmetric and traceless in each pair of tensor indices.[17]

The projection operator on the spin state of quantum number m with respect to a spin direction (axis of quantization) s is given in terms of these $\left(\sigma^{a}_{(\mu)}\right)$ by the equation

$$P^{m}(s) = \sum_{h} \sigma^{a}_{(\mu)} \prod_{j=1}^{2a} \left(v^{r} + m_{hj}\, s^{r}\right)^{\mu_j} , \tag{A.8}$$

where the summation over h is subject to the constraints

$$\sum_{h} m_{hj} = m ,$$

where

$$|m_{hj}| = \frac{1}{2} .$$

The s^{r} and v^{r} are rest frame values of s and v, where v is the covariant velocity. The projection operator $P^{m}(s)$ is the rest frame value of the general invariant spinor function

$$P^{m}(k_i, s_i) = \sum_{h} \sigma^{a}_{(\mu)} \prod_{j=1}^{2a} (v + m_{hj}\, s) . \tag{A.9}$$

The multiple continuum of matrices $P^{m}(s)$ span the $(2a + 1)^2$ dimensional space in which they are contained. In particular, if

$$\mathrm{Tr}\, P^{m}(s)\, M = 0$$

for all m and s, then $M = 0$. Taking s^{r} to be along the z axis one immediately finds that the diagonal elements of M are zero. A consideration of infinitesimal rotations of s^{r} then shows that the first off-diagonal elements vanish and one can proceed in steps to show that $M = 0$.

FOOTNOTES AND REFERENCES

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