UCRL-10965

# University of California

# Ernest O. Lawrence Radiation Laboratory

## TRANSITION PATHS OF $\mu^-$ MESONS CAPTURED IN SOLID LITHIUM

TWO-WEEK LOAN COPY

*This is a Library Circulating Copy which may be borrowed for two weeks. For a personal retention copy, call Tech. Info. Division, Ext. 5545*

## DISCLAIMER

This document was prepared as an account of work sponsored by the United States Government. While this document is believed to contain correct information, neither the United States Government nor any agency thereof, nor the Regents of the University of California, nor any of their employees, makes any warranty, express or implied, or assumes any legal responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by its trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof, or the Regents of the University of California. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof or the Regents of the University of California.

Research and Development

UCRL-10965
UC-34 Physics
TID-4500 (19th Ed.)

UNIVERSITY OF CALIFORNIA
Lawrence Radiation Laboratory
Berkeley, California

Contract No. W-7405-eng-48

# TRANSITION PATHS OF $\mu^-$ MESONS CAPTURED IN SOLID LITHIUM

Coates R. Johnson

August 1963

(Thesis)

Reproduced by the Technical Information Department
directly from the author's copy

Printed in USA. Price $2.25. Available from the
Office of Technical Services
U. S. Department of Commerce
Washington 25, D.C.

# TRANSITION PATHS OF $\mu^-$ MESONS CAPTURED IN SOLID LITHIUM


Abstract . . . . . ii

I. Introduction . . . . . 1

II. Basic Assumptions . . . . . 4

A. The Adiabatic Approximation . . . . . 4

B. Approximations Based on the General Structure of $\mu$-Mesonic Lithium . . . . . 8

III. Energy Levels and Wave Functions . . . . . 10

A. The $\mu$-Mesonic Atom . . . . . 10

B. The Band Electrons . . . . . 27

C. The Ejected Electron . . . . . 32

IV. Transition Rates . . . . . 33

A. Radiation . . . . . 33

B. Auger Transitions Involving the Core Electrons . . 33

C. Auger Transitions Involving the Band Electrons . . 41

D. Electron Replacement Transitions . . . . . 48

E. Numerical Values . . . . . 52

V. Transition Paths . . . . . 63

VI. Conclusions . . . . . 77

Acknowledgment . . . . . 79

Appendix 1. Estimates of $\epsilon_{13}$ and $\epsilon_{23}$. . . . . 80

2. Numerical Techniques Used in Obtaining the Mesonic Energy Levels and Wave Functions. . . . . 83

3. Corrections to the Radiation Transition Rate Formulas of Section IV. . . . . 86

4. Corrections to the Auger Transition Rate Formulas of Section IV. . . . . 87

5. The Error Involved in Using Truncated Band Electron Wave Functions in $M_{1m}$. . . . . 96

6. An Estimate of the Sensitivity of the Transition Paths to Errors in the Calculated Transition Rates . . . 100

References . . . . . 102

# TRANSITION PATHS OF $\mu^-$ MESONS CAPTURED IN SOLID LITHIUM

Coates R. Johnson

Lawrence Radiation Laboratory
University of California
Berkeley, California

## ABSTRACT

This paper investigates the cascade down to the ground state of a $\mu^-$ meson captured by an atom of solid lithium. The cascade takes place predominately through three processes. These are radiation transitions, Auger transitions involving the K shell electrons of the mesonic atom, and Auger transitions involving the band electrons of the solid. It is assumed that the meson is captured in a state of high principal quantum number $(n \sim 15)$. It is found, independant of the number of K shell electrons remaining in the mesonic atom that the principal decay paths follow, at least approximately, and if the meson is captured in a high $\ell$ state, the route $\Delta n = \Delta \ell = -1$. For mesons captured in low $\ell$ states there is a tendency for the main paths to pass through states of higher $\ell$ than that given by $\Delta n = -1$, $\Delta \ell = -1$. Because of the presence of Auger transitions where $\Delta n = -1$, $\Delta \ell = 0$ and where $\Delta n = 0$, $\Delta \ell = -1$, the cascades contain a rather large dispersion away from the main paths and towards both large and small $\ell$. Reasons are given for expecting the same qualitative description of mesonic cascades to hold also for $\pi^-$ mesons captured in lithium, and for both $\mu^-$ and $\pi^-$ mesons captured in other light solids $(Z < 6)$.

# I. INTRODUCTION

The object of this paper is to investigate the cascade down to the ground state of a $\mu^-$ meson captured by an atom of solid lithium. First however something will be said about the capture process itself.

The capture of a $\mu^-$ meson in lithium is ill understood but order of magnitude calculations show that radiation capture, in which the meson releases its excess energy as photons, should be negligible compared to Auger (radiationless) capture involving the transfer of energy from the meson to an electron of the solid.

Various detailed calculations have been made, assuming Auger capture and the ejection of a K shell electron, in order to determine the initial distribution of occupied mesonic states in a newly formed mesonic atom (deBorde, 1954; Mann and Rose, 1961). None of these can be considered reliable as the capture process probably occurs at energies where standard approximations are not valid. It has usually been assumed the meson is most likely captured in a relatively high n state, $n \sim 15$ (n refers to the principal quantum number of the captured $\mu^-$ meson), for if capture does occur through the ejection of a K shell electron maximum overlap of the wave functions for the meson and electron will be secured when the radii for the meson and electron are approximately equal, that is for $n \sim \sqrt{\mu} \sim 15$ .

The probability distribution in angular momentum of the captured $\mu^-$ meson is difficult to determine, however on statistical grounds it might be expected that within a set of states the states with higher angular momentum would be more populated, as the number of independant quantum states of principal quantum number $n$ , with an angular momentum $\ell$ , is $2\ell + 1$ .

The specific question this paper will try to answer is if a $\mu^-$ meson is captured in a high $n$ and $\ell$ state what will be its principal means of losing energy and through what set of states will it pass on its way to the ground state $n = 1$, $\ell = 0$. It is hoped that from an answering of this question certain general statements can be made concerning the transitions from high energy states of both muons and pions captured in light elements $(Z < 6)$.

Previous work, relating to this question, has been done by Burbridge and de Borde (1953) and de Borde (1954) where they determined the probable number of electrons emitted during a $\mu^-$ meson cascade and compared their theoretical results with the experimental observations of emitted Auger electrons (Cosyn, et al., 1949; Fry, 1951, 1953). In 1961 Eisenberg and Kessler reinvestigated the $\mu^-$ meson cascade process in order to compare theoretical results with the more recent observations of Sterns and Sterns (1957)* and Pevsner, et al. (1961).

In all these calculations the $\mu^-$ meson was regarded as in a Coulomb field and the L shell electrons were treated as core electrons. Ruderman (1960) has observed that since the interaction of the $\mu^-$ meson with the K electrons of the mesonic atom will break the degeneracy of the mesonic energy levels with the same $n$ but with different $\ell$, and since in the light elements the L electrons should be treated

---

* It should be noted that the experimental observations of Sterns and Sterns (1957) have been questioned. See Lathrop, Lundy, Talegdi, and Winston, Phys, Rev. Letters 7, 147 (1961). Also see Eisenberg and Kessler, Phys. Rev. 130, 2349 (1963).

as band electrons, mesonic Auger transitions can occur for which $\Delta n = 0$, $\Delta \ell = -1$. The possibility that these "sliding in" transitions might dominate the cascade process in light elements, a possibility not taken into account in previous work, is the reason for this more detailed investigation.

## II. BASIC ASSUMPTIONS

### A. The Adiabatic Approximation

Solid lithium which has captured a $\mu^-$ meson will consist of $N$ nuclei, $3N$ electrons and a $\mu^-$ meson. The Schrödinger wave function $\psi$ associated with the distribution of these particles satisfies the equation

$$i\hbar \frac{\partial}{\partial t} \psi(r_{e_i}, r_{n_k}, r_m) = H(r_{e_i}, r_{n_k}, r_m)\, \psi(r_{e_i}, r_{n_k} r_m) , \tag{2.1}$$

where

$$H(r_{e_i}, r_{n_k}, r_m) = -\sum_i \nabla_{e_i}^2 - \frac{1}{M}\sum_k \nabla_{n_k}^2 - \frac{1}{\mu}\nabla_m^2 + V(r_{e_i}, r_{n_k}, r_m) , \tag{2.2}$$

$$V(r_{e_i}, r_{n_k}, r_m) = \sum_{k,\ell} \frac{Z^2}{r_{n_k n_\ell}} + \sum_{ij} \frac{1}{r_{e_i e_j}} + \sum_i \frac{2}{r_{me_i}} - \sum_{ik} \frac{2Z}{r_{e_i n_k}} - \sum_k \frac{2Z}{r_{mn_k}} , \tag{2.3}$$

with $e_i$ referring to electron coordinates, $n_k$ to nucleus coordinates, and $m$ to meson coordinates. $Z$ is the nuclear charge in atomic units, $M$ the nuclear mass in atomic units, and $\mu$ the meson mass in atomic units. For atomic units, as used here, the unit of charge is the electron charge, the unit of mass is the electron mass, the unit of distance is the Bohr radius, and the unit of energy is the Rydberg energy.

Defining a set of wave functions $\psi_A$, presumably complete,

$$\psi_A(r_{e_i}, r_{n_k}, r_m, t) = \psi_L(r_{e_i}, r_{n_k}, r_m)\, \psi_H(r_{n_k}, r_m, t) \quad , \tag{2.4}$$

where $\psi_L$ is a solution of the eigenvalue equation

$$H_L(r_{e_i}, r_{n_k}, r_m)\, \psi_L(r_{e_i}, r_{n_k}, r_m) = V_H(r_{n_k}, r_m)\, \psi_L(r_{e_i}, r_{n_k}, r_m) \quad , \tag{2.5}$$

$$H_L(r_{e_i}, r_{n_k}, r_m) = - \sum_i \nabla_{e_i}^{\,2} + V(r_{e_i}, r_{n_k}, r_m) \quad , \tag{2.6}$$

and $\psi_H$ is a solution of the equation

$$i\hbar \frac{\partial}{\partial t}\, \psi_H(r_{n_k}, r_m, t) = H_H(r_{n_k}, r_m)\, \psi_H(r_{n_k}, r_m, t) \quad , \tag{2.7}$$

$$H(r_{n_k}, r_m) = - \frac{1}{M} \sum_k \nabla_{n_k}^{\,2} - \frac{1}{\mu} \nabla_m^{\,2} + V_H(r_{n_k}, r_m) \quad , \tag{2.8}$$

and further defining the operators $H_A^{\,0}$ and $H_A^{\,1}$ by

$$H_A^{\,0}\, \psi_A = H\psi_A - \frac{2}{M} \sum_k \left\{ \vec{\nabla}_{n_k} \psi_L \cdot \vec{\nabla}_{n_k} \psi_H + \frac{1}{2} (\nabla_{n_k}^{\,2} \psi_L)\, \psi_H \right\}$$
$$- \frac{2}{\mu} \left\{ \vec{\nabla}_m \psi_L \cdot \vec{\nabla}_m \psi_H + \frac{1}{2} (\nabla_m^{\,2} \psi_L)\, \psi_H \right\} \quad , \tag{2.9}$$

$$H_A^{\,1}\,\psi_A = \frac{2}{M}\sum_k \left\{\vec{\nabla}_{n_k}\psi_L \cdot \vec{\nabla}_{n_k}\psi_H + \frac{1}{2}\left(\nabla_{n_k}^{\,2}\,\psi_L\right)\psi_H\right\}$$

$$+ \frac{2}{\mu}\left\{\vec{\nabla}_m\,\psi_L \cdot \vec{\nabla}\,\psi_H + \frac{1}{2}\left(\nabla_m^{\,2}\,\psi_L\right)\psi_H\right\}, \tag{2.10}$$

it then follows that

$$H(r_{e_i},\, r_{n_k},\, r_m) = H_A^{\,0}(r_{e_i},\, r_{n_k},\, r_m) + H_A^{\,1}(r_{e_i},\, r_{n_k},\, r_m), \tag{2.11}$$

and the set of functions $\psi_A$ are solutions of the equation

$$i\hbar\,\frac{\partial}{\partial t}\,\psi_A(r_{e_i},\, r_{n_k},\, r_m,\, t) = H_A^{\,0}(r_{e_i},\, r_{n_k},\, r_m)\,\psi_A(r_{e_i},\, r_{n_k},\, r_m, t). \tag{2.12}$$

It is to be noted that equation (2.5) can be looked on as a Schrödinger equation leading to the electron wave functions and energy levels when the meson and nuclei are considered as motionless at positions $r_m$ and $r_{n_k}$ while (2.7) is a Schrödinger equation for computing the nuclei and meson wave functions and energy levels under a potential expressing the energy of the atom, as computed through (2.5), as a function of nuclear and mesonic positions. It is thus reasonable to suppose that a wave function of the form (2.4), and thus a solution of (2.5) and (2.7), would be a good approximation to the solution of (2.1) if the "velocities" of the nuclei and meson (heavy particles) are much smaller than the "velocities" of the electrons (light particles).

The adiabatic approximation consists of neglecting the term $H_A^{\,1}$ in equation (2.1) and thus to this approximation the solutions of (2.1) are of the form (2.4). Assuming the variation of the functions $\psi_L$

and $\psi_H$ with $r_{n_k}$ and $r_m$ is not much more rapid than with $r_{e_i}$, the relatively large values of $M$ and $\mu$ guarantee that the term $H_A^1$ is small compared with $H_A^0$ and can be considered as a perturbation causing transitions between stationary states of the approximate adiabatic equation. In the case of solids, the term in $H_A^1$ involving $1/m$ is known to be very small so that only the term involving $1/\mu$ need be discussed here. Since the wave function of the meson will be spread over a distance of approximately $n^2/\mu$ times that of a core electron in the solid the fraction error involved if one neglects the term containing $1/\mu$ in equation (2.10) is of the order $1/n^2$. Thus for large $n$ the adiabatic approximation will probably be quite good and the term $H_A^1$ in equation (2.11) can be treated as a perturbation. For small values of $n$ the meson should not be treated in the same fashion as the nuclei (slow moving particle) but in the fashion of the electrons (fast moving particles) and thus the adiabatic approximation, in this different form, could also be used for small values of $n$.

A practical technique for obtaining reasonably good solutions of the Schrödinger equation of this problem then consists of solving the adiabatic equations for stationary states, which are relatively long live states of the true equation, and treating the term $H_A^1$ as a perturbation leading to transitions among these states. Thus we can follow the de-excitation of the $\mu^-$ meson until the meson is in a true stationary state of the Schrödinger equation, the ground state, from which the meson either is absorbed or decays.

B. Approximations Based on the General Structure of μ-Mesonic Lithium

The structure of solid lithium is well known. The nuclei form a body centered cubic lattice with a lattice constant of 6.52 in atomic units. This means the nuclei are well separated and the two core (1s) electrons surrounding one nucleus do not overlap significantly with the core electrons surrounding a neighboring nucleus. The remaining electrons, however, should not be considered as localized near one nucleus but are better considered as spread throughout the solid and treated as band electrons.

Of course in μ-mesonic lithium the wave functions for the core electrons and band electrons near the nucleus to which the $\mu^-$ meson is considered bound will be, because of the presence of the meson, different from those of the normal (meson free) solid. However, in light of the previous paragraph, a reasonable model of solid lithium which has captured a $\mu^-$ meson consists of one nucleus surrounded by a $\mu^-$ meson and zero, one, or two core electrons (the number depends on how many core electrons have been ejected from the solid by the meson), $N - 1$ nuclei surrounded by two core electrons, and $N$ band electrons considered as bound in the almost periodic potential formed by the nuclei, the $\mu^-$ meson, and the core electrons.

The $\mu^-$ meson, the nucleus to which it is bound, and the core electrons associated with this nucleus are collectively refered to in this paper as a μ-mesonic atom.

In considering the de-excitation of the captured $\mu^-$ meson

the energy given to the nuclei by the meson, because of the large nuclear mass, can be neglected and only the interaction of the excited meson with other electrons in the solid need be considered. Further because of the small overlap of the wave functions of the captured meson and the core electrons not associated with the mesonic atom, the interaction of the meson and these distant electrons can also be neglected. Thus when considering the transitions of the $\mu^-$ meson to its ground state within the solid the only energy levels and wave functions which need to be considreed are those associated with the band electrons and those associated with the mesonic atom in the solid.

## III. ENERGY LEVELS AND WAVE FUNCTIONS

### A. The μ-Mesonic Atom

The development of the formulism needed to compute the energy $E_M$ and associated wave functions of the mesonic atom will only be given for the case involving two core electrons. The procedure in the case of zero or one core electron is similar except for obvious simplifications. For the latter two cases only the resulting formulas will be given.

From (2.5) and (2.7) it follows for the μ-mesonic atom

$$\left[-\nabla_1^2 - \nabla_2^2 + V(r_1, r_2, r_n, r_m)\right] \psi_L(r_1, r_2, r_n, r_m) = V_H(r_n, r_m)\, \psi_L(r_1, r_2, r_n, r_m) , \tag{3.1}$$

$$\left[-\frac{1}{M}\nabla_n^2 - \frac{1}{\mu}\nabla_m^2 + V_H(r_n, r_m)\right] \psi_H(r_n, r_m) = E_M\, \psi_H(r_n, r_m) , \tag{3.2}$$

where

$$V(r_1, r_2, r_n, r_m) = \frac{2}{r_{m1}} + \frac{2}{r_{m2}} + \frac{2}{r_{12}} - \frac{2Z}{r_{nm}} - \frac{2Z}{r_{n1}} - \frac{2Z}{r_{n2}} , \tag{3.3}$$

with 1 and 2 refering to the coordinates of the two core electrons while n and m refer to the coordinates of the nucleus and $\mu^-$ meson. In equation (3.1) it has been assumed, following the arguments of Section II, that the influence of the particles external to the mesonic atom can be neglected.

Since $V_H(r_n, r_m)$ will only depend on the distance between the

meson and the nucleus, center of mass coordinates can be introduced such that

$$\left[-\nabla_1^2 - \nabla_2^2 + V(r_1, r_2, r)\right] \psi_L(r_1, r_2\ r) = V_H(r)\ \psi_L(r_1, r_2, r), \quad (3.4)$$

$$\left[\frac{1}{\bar{\mu}}\nabla_r^2 + V_H(r)\right] \psi_M(r) = \mathcal{E}_M\ \psi_M(r)\ , \quad (3.5)$$

$$-\frac{1}{\bar{M}}\ \nabla_R^2\ \psi_N\ (R) = E_c\ \ \psi_N(R)\ , \quad (3.6)$$

where

$$V(r_1, r_2, r) = \frac{2}{r_{m1}} + \frac{2}{r_{m2}} + \frac{2}{r_{12}} - \frac{2Z}{r} - \frac{2Z}{r_{ni}} - \frac{2Z}{r_{n2}}\ , \quad (3.7)$$

$$\vec{r} = \vec{r}_m - \vec{r}_n\ , \qquad \vec{R} = \frac{\mu\ \vec{r}_m + M\ \vec{r}_n}{\mu + M}\ , \quad (3.8)$$

$$\bar{\mu} = \frac{M\mu}{M + \mu}\ , \qquad \bar{M} = M + \mu\ , \quad (3.9)$$

and

$$\psi_H(r_n, r_m) = \psi_M(r)\ \psi_N(R)\ , \quad (3.10)$$

$$E_M = \mathcal{E}_M + E_c\ . \quad (3.11)$$

Because of the large value of $\bar{M}$, changes in the center of mass motion can be neglected in this problem and thus the only equations which need to be solved are (3.4) and (3.5).

Equation (3.4) can be solved, to the accuracy required in this problem, by means of perturbation theory. Introducing

$$V^0(r_1, r_2, r) = -\frac{2Z'}{r_{n1}} - \frac{2Z'}{r_{n2}} - \frac{2Z}{r} , \tag{3.12}$$

where

$$Z' = Z - 5/16 , \tag{3.13}$$

and

$$V^I(r_1, r_2, r) = V_1{}^I(r_1, r_2, r) + V_2{}^I(r_1, r_2, r) , \tag{3.14}$$

where

$$V_1{}^I(r_1, r_2, r) = \frac{2}{r_{m1}} + \frac{2}{r_{m2}} , \tag{3.15}$$

$$V_2{}^I(r_1, r_2, r) = \frac{2}{r_{12}} - \frac{5/8}{r_{n1}} - \frac{5/8}{r_{n2}} , \tag{3.16}$$

and writing

$$V(r_1, r_2, r) = V^0(r_1, r_2, r) + V^I(r_1, r_2, r) \tag{3.17}$$

it will be assumed that $V^I$ can be treated as a perturbing potential added to $V_0$. Thus for the solution of (3.4) with the core electrons

in their lowest energy state it follows

$$V_H(r) = V_0(r) + V_1(r) + V_2(r) + \cdots , \tag{3.18}$$

with

$$V_0(r) = -\frac{2Z}{r} - 2Z'^2 , \tag{3.19}$$

$$V_1(r) = \langle 00 \,|V^I|\, 00 \rangle , \tag{3.20}$$

$$V_2(r) = \sum_{n'm'} \frac{\langle 00 \,|V^I|\, n\, m\rangle\langle m\, n \,|V^I|\, 00\rangle}{E_{00} - E_{nm}} , \tag{3.21}$$

and

$$\psi_L(r_1, r_2, r) = \psi^0(r_1, r_2) + \psi^1(r_1, r_2, r) + \cdots , \tag{3.22}$$

with

$$\psi^0(r_1, r_2) = \frac{Z'^3}{\pi} e^{-Z'(r_{1n}+r_{2n})} , \tag{3.23}$$

$$\psi'(r_1, r_2, r) = \sum_{n'm'} \frac{\langle m\, n \,|V^I|\, 00\rangle}{E_{00} - E_{nm}} \,|\, n\, m\rangle , \tag{3.24}$$

where

$$\langle mn|V^I|pq\rangle = \int \psi^*_{nm}(r_1,r_2)(r_2)V^I(r_1, r_2, r)\psi_{pq}(r_1,r_2)\, d^3r_1 d^3r_2 , \tag{3.25}$$

with $\psi_{mn}$ an eigen solution and $E_{mn}$ the corresponding eigenvalue of the equation

$$\left[-\nabla_1^2 - \nabla_2^2 + V^0(r_1, r_2, r)\right] \psi_{mn}(r_1, r_2, r) = E_{mn}\psi_{mn}(r_1,r_2,r), \tag{3.26}$$

where m and n refer to the states of electrons 1 and 2 respectively. A primed summation index indicates summation over all values of the indices except $n' = m' = 0$. It will turn out that higher terms in the perturbation series than those of (3.21) and (3.24) will not be needed in this problem.

$V_1(r)$ is easily evaluated

$$\mathcal{E}_{21}(r) = V_1(r) = \frac{4Z'}{t}\left[1 - (1 + t)\, e^{-2t}\right] , \tag{3.27}$$

where in this case

$$t = Z'r . \tag{3.28}$$

The evaluation of $V_2$ is more involved. It follows from (3.14) and (3.21)

$$V_2(r) = V^a(r) + V^b(r) + V^c , \tag{3.29}$$

with

$$V^a(r) = \sum_{m'n'} \frac{\langle 00|V_1{}^I|\, nm\rangle \langle\, mn|V_1{}^I|00\rangle}{E_{00} - E_{mn}} , \tag{3.30}$$

$$V^b(r) = 2 \sum_{m'n'} \frac{\langle 00|V_1{}^I|\ nm\rangle \ \langle mn|V_2{}^I|00\rangle}{E_{00} - E_{mn}} \quad , \tag{3.31}$$

$$V^c = \sum_{m'n'} \frac{\langle 00|V_2{}^I|\ nm\rangle \ \langle mn|V_2{}^I|00\rangle}{E_{00} - E_{mn}} \quad . \tag{3.32}$$

If wave functions $\psi_n$ are introduced which are solutions of

$$\left[ - \nabla_1{}^2 - \frac{2Z'}{r_{n1}} \right] \psi_n(r) = E_n \ \psi_n(r_1) \tag{3.33}$$

with eigenvalue $E_n$ it follows that

$$V^a(r) = 2 \sum_{n'} \frac{\langle 0|v_1|n\rangle \ \langle n|v_1|0\rangle}{E_0 - E_n} \quad , \tag{3.34}$$

$$V^b(r) = 4 \sum_{n'} \frac{\langle 0|v_1|n\rangle \ \langle n|v_2|0\rangle}{E_0 - E_n} \quad , \tag{3.35}$$

where

$$\langle m|v_\alpha|n\rangle = \int \psi_m{}^*(r_1) \ v_\alpha(r_1) \ \psi_n(r_1) \ d^3 \ r_1 \quad , \tag{3.36}$$

$$v_1(r_1) = \frac{2}{r_{m1}} \quad , \tag{3.37}$$

$$v_2(r_1) = 2Z' \left[ \frac{11/16}{Z'r_{n1}} - \left(1 - \frac{1}{Z'r_{n1}}\right) e^{-Zr'_{n1}} \right] \quad . \tag{3.38}$$

The expressions (3.34) and (3.35) can be evaluated exactly by making use of a technique discussed by Dalgarno (1955). It is easily shown that if $f$ is a solution of

$$2\vec{\nabla}_1 f(r_1) \cdot \vec{\nabla}_1 Q_0(r_1) + Q_0 \nabla_1^2 f(r_1) = \left[W(r_1) - \langle 0|W|0\rangle\right] Q_0 \quad , \tag{3.39}$$

and $Q_n$ is a solution

$$\left[-\nabla_1^2 + U(r_1)\right] Q_n(r_1) = G_n \, Q_n(r_1) \quad , \tag{3.40}$$

with eigenvalue $G_n$, then

$$\int Q_n^*(r_1) W(r_1) Q_0(r_1) d^3 r_1 = \left[G_0 - G_n\right] \int Q_n^*(r_1) f_1(r_1) Q_0(r_1) d^3 r_1 . \tag{3.41}$$

Using this fact it follows that

$$\langle n|v_1|0\rangle = (E_0 - E_n) \langle n|f_1|0\rangle \quad , \tag{3.42}$$

$$\langle n|v_2|0\rangle = (E_0 - E_n) \langle n|f_2|0\rangle \quad , \tag{3.43}$$

where

$$2\vec{\nabla}_1 f_1(r_1) \cdot \vec{\nabla}_1 \psi_0(r_1) + \psi_0(r_1) \nabla_1^2 f_1(r_1) = \left[v_1(r_1) - \langle 0|v_1|0\rangle\right] \psi_0(r_1) \quad , \tag{3.44}$$

$$2\vec{\nabla}_1 f_2(r_1) \cdot \vec{\nabla}_1 \psi_0(r_1) + \psi_0(r_1) \nabla_1^2 f_2(r_1) = \left[v_2(r_1) - \langle 0|v_2|0\rangle\right] \psi_0(r_1) \quad . \tag{3.45}$$

Thus

$$V^a(r) = 2 \sum_{n'} \langle 0|v_1|n\rangle \langle n|f_1|0\rangle , \tag{3.46}$$

$$V^b(r) = 4 \sum_{n'} \langle 0|v_2|n\rangle \langle n|f_2|0\rangle , \tag{3.47}$$

or

$$V^a(r) = 2\left[\langle 0|f_1 v_1|0\rangle - \langle 0|f_1|0\rangle \langle 0|v_1|0\rangle\right] , \tag{3.48}$$

$$V^b(r) = 4\left[\langle 0|f_2 v_2|0\rangle - \langle 0|f_2|0\rangle \langle 0|v_2|0\rangle\right] . \tag{3.49}$$

Equation (3.44) has been solved by Dagarno (1957). He finds

$$\begin{aligned} f_1(r_1) = \frac{1}{Z'}\Bigg\{ & \frac{1}{t}(t+1)\log(1-\nu) - \frac{1}{t}(t+1)e^{-2t}\log(1+\nu) \\ & - \frac{1}{t}\left[(t-1) + (t+1)e^{-2t}\right]\log(1+\mu) - \frac{1}{t}(t+1)e^{-2t}\Big[e^{t}E\left[-t(1-\nu)\right] \\ & - e^{-t}E\left[t(1+\nu)\right]\Big] + \frac{1}{2}\left[1 - (1+t)e^{-2t}\right](\mu+\nu)\Bigg\} , \end{aligned} \tag{3.50}$$

with

$$\mu = \frac{r_{nl} - r_{ml}}{r} , \qquad \nu = \frac{r_{nl} - r_{ml}}{r} , \qquad t = Z'r , \tag{3.51}$$

and

$$E\left[-t(1-\nu)\right] = \int_{-\infty}^{-t(1-\nu)} \frac{e^u}{u}\, du . \tag{3.52}$$

Equation (3.45) can be easily solved by quadrature

$$f_2(r_1) = \frac{1}{Z'}\left[\frac{3}{8}\left(E[-x] - \log x - \gamma\right) + \frac{3}{8}\,\frac{(1 - e^{-x})}{x} + \frac{5}{32}x - \frac{1}{4}e^{-x} - \frac{1}{8}\right] , \tag{3.53}$$

where

$$x = Z'r_1 , \quad \gamma = .5772157\cdots . \tag{3.54}$$

Thus

$$V^a(r) = \frac{2}{t^2}\Big[5 - (4t^2 + 8t + 10)e^{-2t} + (4t^3 + 7t^2 + 8t + 5)e^{-4t} - 2(t+1)^2e^{-2t}(1 + e^{-2t})(E[2t] - 2\log 2t - 2\gamma) - 2\left\{(t^2 + 2t - 3) + (4t + 4)e^{-2t} + (t-1)^2e^{2t}\right\}E[-2t]\Big], \tag{3.55}$$

$$V^b(r) = \frac{1}{4t}\Big[12(t-1)(E[-2t] - E[-4t]) - 12(t+1)e^{-2t}(E[-2t] - \log t - \gamma) - (10t^2 + 12t - 5)e^{-2t} + (2t - 5)e^{-4t}\Big] . \tag{3.56}$$

Defining

$$\mathcal{E}_{22}(r) = V^a(r) + V^b(r) , \tag{3.57}$$

it follows, neglecting terms dependant on $r$ which come from terms of higher order than two in perturbation theory, that

$$V_H(r) = -2Z'^2 - \frac{2Z}{r} + \mathcal{E}_{21}(r) + \mathcal{E}_{22}(r) + \mathcal{E}_c , \tag{3.58}$$

where $\mathcal{E}_c$ is independent of $r$ and in fact equal to the difference between a first order calculation of the ground state energy of $Li^+$ and the actual energy of the state. The value of $\mathcal{E}_c$ as obtained from some

recent experiments is

$$\mathcal{E}_c = 1.57 \text{ eV} . \tag{3.59}$$

The case where there is only one core electron can be analyzed in a simialr manner. In that case

$$V_H(r) = -Z^2 - \frac{2Z}{r} + \mathcal{E}_{11}(r) + \mathcal{E}_{12}(r) , \tag{3.60}$$

with

$$\mathcal{E}_{11}(r) = \frac{2Z}{t}\left[1 - (t+1)\, e^{-2t}\right] , \tag{3.61}$$

$$\begin{aligned}\mathcal{E}_{12}(r) = \frac{1}{t^2}\Big[ & 5 - (4t^2 + 8t + 10)e^{-2t} + (4t^3 + 7t^2 + 8t + 5)e^{-4t} \\ & -2(t+1)^2 e^{-2t}(1 + e^{-2t})(E\,[2t] - 2\log 2t - 2\gamma) - 2\Big\{(t^2 + 2t - 3) \\ & + (4t + 4)e^{-2t} + (t-1)^2 e^{2t}\Big\}\, E\,[-2t]\Big] ,\end{aligned} \tag{3.62}$$

where

$$t = Zr . \tag{3.63}$$

The case of no core electron is of course trivial

$$V_H(r) = -\frac{2Z}{r} . \tag{3.64}$$

For a discussion of the errors involved in (3.58) and (3.60) see Appendix 1.

In order to obtain explicitly the approximate electron wave functions perturbation formulas (3.22), (3.23), and (3.24) could be used. However

because it would be more exact to treat the unperturbed electron wave function as

$$\psi^0(r_1, r_2) = \frac{(Z' - 1)^3}{\pi} e^{-(Z'-1)(r_1+r_2)} \tag{3.65}$$

than as

$$\psi^0(r_1, r_2) = \frac{Z'^3}{\pi} e^{-Z'(r_1+r_2)} \tag{3.66}$$

for small values of the meson coordinates $(n^2 < \mu)$, the perturbation technique is best modified in such a manner as to make (3.65) the unperturbed electron wave function. This is easily done by treating

$$V_{e'}^{0}(r_1, r_2) = -\frac{2(Z' - 1)}{r_{n1}} - \frac{2(Z' - 1)}{r_{n2}} \tag{3.67}$$

as the non perturbed potential and

$$V_{e'}^{I}(r_1, r_2) = V_{e'}^{I_1}(r_1, r_2) + V_{e'}^{I_2}(r_1, r_2) \quad , \tag{3.68}$$

where

$$V_{e'}^{I_1}(r_1, r_2) = \frac{2}{r_{m1}} + \frac{2}{r_{m2}} - \frac{2}{r_{n1}} - \frac{2}{r_{n2}} \quad , \tag{3.69}$$

$$V_{e'}^{I_2}(r_1, r_2) = \frac{2}{r_{12}} - \frac{5/8}{r_{n1}} - \frac{5/8}{r_{n2}} \quad , \tag{3.70}$$

as the perturbation. Thus

$$V(r_1, r_2) = V_{e'}^{0}(r_1, r_2) + V_{e}^{I}(r_1, r_2) \quad . \tag{3.71}$$

In this case

$$\psi^0(r_1, r_2) = \frac{(Z' - 1)^3}{\pi} e^{-(Z'-1)(r_1+r_2)} , \tag{3.72}$$

$$\psi^1(r_1, r_2) = \sum_{m',n'} \frac{\langle n\, m\, | V_{e'}^{I} | 00 \rangle}{E_{00}^{e'} - E_{mn}^{e'}} | mn \rangle , \tag{3.73}$$

and

$$\langle nm | V_{e'}^{I} | 00 \rangle = \int \psi_{mn}^{e'^*}(r_1,r_2) V_{e'}^{I}(r_1,r_2) \psi_{00}^{e'}(r_1,r_2) d^3r_1\, d^3r_2 , \tag{3.74}$$

where $\psi_{mn}$ and $E_{mn}$ are the eigenfunction and corresponding eigenvalue found in solving the equation

$$\left[-\nabla_1^2 - \nabla_2^2 + V_{e'}^0(r_1,r_2)\right] \psi_{mn}^{e'}(r_1,r_2) = E_{mn}^{e'} \psi_{mn}^{e'}(r_1, r_2) , \tag{3.75}$$

with $m$ refering to the state of electron 1 and $n$ referring to the state of electron 2. Thus

$$\psi^1(r_1,r_2,r) = \sum_{m'n'} \frac{\langle nm | V_{e'}^{I_1} | 00 \rangle}{E_{00}^{e'} - E_{mn}^{e'}} | m\, n \rangle + \varphi(r_1, r_2) , \tag{3.76}$$

where $\varphi(r_1, r_2)$ is independant of $r$. Since $\varphi(r_1, r_2)$ comes from the first order perturbation terms it will be neglected in comparison to $\psi^0(r_1, r_2)$, so that to fair accuracy

$$\psi(r_1,r_2,r) = \psi^0(r_1,r_2) + \sum_{m'n'} \frac{\langle n\, m | V_{e'}^{I_1} | 00 \rangle}{E_{00}^{e'} - E_{mn}^{e'}} | m\, n \rangle . \tag{3.77}$$

Equation (3.77) can be simplified by observing

$$\psi(r_1,r_2,r) = \psi^0(r_1,r_2) + \sum_{m'} \frac{\langle m\,|v_{e_1'}|\,0\rangle}{E_0^{e'} - E_m^{e'}}\,|\,m\rangle + \sum_{n'} \frac{\langle n\,|v_{e_2'}|\,0\rangle}{E_0^{e'} - E_n^{e'}}\,|\,n\rangle, \tag{3.78}$$

where

$$\langle n\,|v_{e_1'}|\,m\rangle = \int \psi_n^{e'*}(r_1)\, v_{e_1'}(r_1)\, \psi_m^{e'}(r_1)\, d^3r_1 \quad , \tag{3.79}$$

$$\langle n\,|v_{e_1'}|\,m\rangle = \int \psi_n^{e'*}(r_2)\, v_{e_2'}(r_2)\, \psi_m^{e'}(r_2)\, d^3r_2 \quad , \tag{3.80}$$

and $\psi_n^{e'}(r_1)$ , $\psi_n^{e'}(r_2)$ , and $E_n^{e'}$ are the eigen solutions and corresponding eigen values of the equations

$$\left[ -\nabla_1^2 + \frac{2(Z'-1)}{r_{n1}} \right] \psi_n^{e'}(r_1) = E_n^{e'}\, \psi_n^{e'}(r_1) \quad , \tag{3.81}$$

$$\left[ -\nabla_2^2 + \frac{2(Z'-1)}{r_{n2}} \right] \psi_n^{e'}(r_2) = E_n^{e'}\, \psi_n^{e'}(r_2) \quad , \tag{3.82}$$

with

$$v_{e_1'}(r_1) = \frac{2}{r_{m1}} - \frac{2}{r_{n1}} \quad , \tag{3.83}$$

$$v_{e_2'}(r_1) = \frac{2}{r_{m2}} - \frac{2}{r_{n2}} \quad . \tag{3.84}$$

Making use of the fact that if $f_3$ satisfies the equation

$$2\vec{\nabla}_1 f_3(r_1)\cdot\vec{\nabla}_1 \psi_0^{e'}(r_1) + \psi_0^{e'}(r_1)\nabla_1^2 f_3(r_1) = \left[ v_{e_1'}(r_1) - \langle 0|v_{e_1'}|0\rangle \right] \psi_0^{e'}(r_1) \quad , \tag{3.85}$$

where

$$\psi_0^{e'}(r_1) = \frac{(Z'-1)^{3/2}}{\pi^{1/2}}\, e^{-(Z'-1)r_1} \quad , \tag{3.86}$$

it follows that

$$\psi(r_1,r_2,r) = \psi^0(r_1,r_2) + \left[ f_3(r_1) + f_3(r_2) - 2 \langle 0 | f_3 | 0 \rangle \right] \psi^0(r_1,r_2). \tag{3.87}$$

Solving equation (3.85) one finds

$$f_3(r_1) = \frac{1}{Z'-1} \Bigg[ \frac{1}{t}(t+1) \log (1-\nu) - \frac{1}{t}(t+1)e^{-2t} \log (1+\nu) - \frac{1}{t} \left[(t-1) + (t+1)e^{-2t}\right] \log (1+\mu) - \frac{1}{t}(t+1)e^{-2t} \left[ e^{t}E[-t(1-\nu)] - e^{-t}E[t(1+\nu)] \right] + \frac{1}{2}\left[(1-t) - (1+t)\, e^{-2t}\right] (\mu+\nu) \Bigg], \tag{3.88}$$

with in this case

$$t = (Z'-1)\, r \quad . \tag{3.89}$$

If (3.80) is expanded in terms of $r$ it is found

$$\psi(r_1, r_2, r) = \left[ 1 - r(\cos \theta_1'' + \cos \theta_2') + \cdots \right] \psi_0^{e'}(r_1)\psi_0^{e'}(r_2), \tag{3.90}$$

where $\theta_1'$ refers to the angle between the vectors $\vec{r}_{n1}$ and $\vec{r}$ while $\theta_2'$ refers to the angle between the vectors $\vec{r}_{n2}$ and $\vec{r}$ .

In the case where there is only one core electron an analogous procedure, using the same approximations, leads to the expression

$$\psi(r_1, r_2, r) = \left[ 1 - r \cos \theta_1' + \cdots \right] \psi_0^{e}(r_1) \tag{3.91}$$

with

$$\psi_0^{e}(r_1) = \frac{(Z-1)^{3/2}}{\pi^{1/2}}\, e^{-(Z-1)r_1} \quad . \tag{3.92}$$

The next step in the adiabatic approximation is to solve (3.5) using either (3.58), (3.60), or (3.64) as $V_H$ . Since $V_H$ is a spherically symmetrical potential the method of separation of variables can be used and thus the eigen solutions of (3.5) can be written in the form

$$\psi_M(r) = R_{n\ell}(r)\, Y_{\ell m}(\theta, \varphi), \tag{3.93}$$

where $Y_{\ell m}$ is the usual spherical harmonic and $R_{n\ell}$ satisfies the equation

$$\frac{1}{\bar{\mu}} \frac{1}{r^2} \frac{d}{dr}\left(r^2 \frac{d}{dr} R_{n\ell}(r)\right) + \left[\mathcal{E}_{n\ell} - V_H(r) - \frac{\ell(\ell+1)}{\bar{\mu} r^2}\right] R_{n\ell}(r) = 0. \tag{3.94}$$

The eigenvalues $\mathcal{E}_M$ of equation (3.5) have been classified in (3.94) according to the number of nodes $n_r$ in the eigensolution $R_{n\ell}$,

$$n = n_r + \ell + 1 , \tag{3.95}$$

and according to the value of $\ell$ associated with the particular eigensolution.

The eigenvalues (up to $n = 15$, $\ell = 14$) and corresponding solutions of equation (3.94), in the case where one or two core electrons are involved, were found with the help of the IBM-704 computer. The errors in the values of $\mathcal{E}_{n\ell}$ obtained due to both errors in $V_H$ and the numerical technique of solution are believed to be of the order of 0.1 eV, while the error in the difference between neighboring energy levels should be even less. Details of the calculation can be found in Appendix 2. The values of $\mathcal{E}_{n\ell}$ in the case of one or two core electrons are given in Table I and II.

Values, to be needed later, for certain matrix elements of the form

$$\overline{R^n} = \int R_{n_f \ell_f}(r)\, r^n\, R_{n_i \ell_i}(r)\, r^2\, dr, \tag{3.96}$$

Table I. The energies $-\mathcal{E}_M$ in eV when the mesonic atom contains one core electron.

| n \ ℓ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 15 | 191.74 | 191.67 | 191.53 | 191.32 | 191.04 | 190.69 | 190.25 | 189.75 | 189.12 | 188.42 | 187.62 | 186.71 | 185.69 | 184.54 | 183.28 |
| 14 | 204.59 | 204.52 | 204.36 | 204.12 | 203.81 | 203.41 | 202.91 | 202.32 | 201.60 | 200.84 | 199.92 | 198.87 | 197.69 | 196.34 | |
| 13 | 221.26 | 221.17 | 221.00 | 220.75 | 220.40 | 219.96 | 219.42 | 218.77 | 218.01 | 217.12 | 216.10 | 214.93 | 213.60 | | |
| 12 | 243.22 | 243.14 | 242.96 | 242.69 | 242.32 | 241.85 | 241.27 | 240.58 | 239.77 | 238.82 | 237.73 | 236.48 | | | |
| 11 | 272.64 | 272.56 | 272.37 | 272.10 | 271.72 | 271.24 | 270.65 | 269.95 | 269.09 | 268.15 | 267.04 | | | | |
| 10 | 312.73 | 312.64 | 312.46 | 312.19 | 311.82 | 311.36 | 310.78 | 310.10 | 309.30 | 308.36 | | | | | |
| 9 | 368.45 | 368.36 | 368.20 | 367.95 | 367.60 | 367.17 | 366.64 | 366.02 | 365.28 | | | | | | |
| 8 | 447.94 | 447.87 | 447.73 | 447.50 | 447.20 | 446.83 | 446.37 | 445.82 | | | | | | | |
| 7 | 565.45 | 565.39 | 565.27 | 565.09 | 564.84 | 564.53 | 564.16 | | | | | | | | |
| 6 | 747.90 | 747.85 | 747.76 | 747.62 | 747.43 | 747.20 | | | | | | | | | |
| 5 | 1051.67 | 1051.64 | 1051.57 | 1051.48 | 1051.35 | | | | | | | | | | |
| 4 | 1611.83 | 1611.81 | 1611.76 | 1611.70 | | | | | | | | | | | |
| 3 | 2822.73 | 2822.72 | 2822.69 | | | | | | | | | | | | |
| 2 | 6282.92 | 6282.91 | | | | | | | | | | | | | |
| 1 | 24968.28 | | | | | | | | | | | | | | |

Table II. The energies $-\mathcal{E}_M$ in eV when the mesonic atom contains two core electrons.

| n \ $\ell$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 15 | 233.11 | 233.01 | 232.80 | 232.47 | 232.03 | 231.48 | 230.79 | 229.98 | 229.02 | 227.91 | 226.63 | 225.17 | 223.49 | 221.56 | 219.35 |
| 14 | 243.06 | 242.94 | 242.69 | 242.32 | 241.82 | 241.18 | 240.40 | 239.46 | 238.36 | 237.07 | 235.58 | 233.85 | 231.85 | 229.52 | |
| 13 | 256.95 | 256.81 | 256.55 | 256.14 | 255.59 | 254.89 | 254.02 | 252.99 | 251.77 | 250.34 | 248.69 | 246.71 | 244.53 | | |
| 12 | 276.38 | 276.24 | 275.96 | 276.53 | 274.95 | 274.22 | 273.31 | 272.27 | 270.94 | 269.44 | 267.70 | 265.69 | | | |
| 11 | 303.59 | 303.46 | 303.18 | 302.75 | 302.16 | 301.43 | 300.57 | 299.43 | 298.15 | 296.66 | 294.93 | | | | |
| 10 | 341.86 | 341.72 | 341.45 | 341.04 | 340.48 | 339.78 | 338.91 | 337.88 | 336.67 | 335.26 | | | | | |
| 9 | 396.13 | 396.01 | 395.76 | 395.39 | 394.88 | 394.24 | 393.45 | 392.52 | 391.44 | | | | | | |
| 8 | 474.54 | 474.43 | 474.22 | 473.89 | 473.46 | 472.91 | 472.24 | 471.44 | | | | | | | |
| 7 | 591.26 | 591.18 | 591.01 | 590.74 | 590.38 | 589.94 | 589.39 | | | | | | | | |
| 6 | 773.19 | 773.13 | 772.99 | 772.79 | 772.51 | 772.17 | | | | | | | | | |
| 5 | 1076.63 | 1076.59 | 1076.49 | 1076.34 | 1076.15 | | | | | | | | | | |
| 4 | 1636.60 | 1636.57 | 1636.50 | 1636.41 | | | | | | | | | | | |
| 3 | 2847.40 | 2847.39 | 2847.35 | | | | | | | | | | | | |
| 2 | 6307.56 | 6307.55 | | | | | | | | | | | | | |
| 1 | -- | | | | | | | | | | | | | | |

where i and f refer to initial and final states, were also computed with the IBM-704. Where $n_i < 6$ these matrix elements were evaluated analytically, as for low values of n Coulomb wave functions are sufficiently accurate. The results of matrix element calculations in the case where the mesonic atom contains two core electrons in both the initial and final state are given in Tables III, IV, and V. The matrix elements computed with other numbers of core electrons did not deviate too greatly from those of Tables III, IV, and V. Estimated errors in the matrix elements evaluated are discussed in Appendix II.

B. The Band Electrons

It is known that wave functions of the valence electrons in lithium can be fairly well approximated by a modified plane wave of the form

$$\psi_{\vec{k}}(r_e) = \frac{1}{\sqrt{V}}\left[1 - \frac{a(\vec{k})^2}{\Omega}\right]^{-\frac{1}{2}}\left[e^{i\vec{k}\cdot\vec{r}_e} - a(\vec{k})\sum_n e^{i\vec{k}\cdot\vec{R}_n}\varphi_{1s}(\vec{r}_e - \vec{R}_n)\right] \tag{3.97}$$

with an energy

$$E_{\vec{k}} = \frac{m}{m^*}k^2 , \tag{3.98}$$

where $\vec{R}_n$ is the position vector of the nth nucleus, V is the volume of the solid, $\Omega$ is the volume per atom in the solid, $m^*$ is the effective electron mass in the solid, m is the electron mass, k is a wave vector which labels each function, and $a(\vec{k})$ is a function to be chosen such that $\psi_{\vec{k}}(r_e)$ is orthogonal to the wave functions of the core electrons (Woodruff, 1957).

Table III. $\overline{R^{1}}$ in atomic units when $\Delta\ell = -1$.

| $n_i$ \ $\ell_i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | $\Delta n$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 15 | 0.687 | 0.682 | 0.680 | 0.673 | 0.666 | 0.654 | 0.642 | 0.627 | 0.609 | 0.586 | 0.557 | 0.518 | 0.463 | 0.370 | 0 |
| | 0.132 | 0.140 | 0.151 | 0.159 | 0.166 | 0.180 | 0.191 | 0.205 | 0.223 | 0.240 | 0.260 | 0.288 | 0.329 | 0.384 | -1 |
| | 0.043 | 0.051 | 0.054 | 0.057 | 0.062 | 0.067 | 0.075 | 0.081 | 0.084 | 0.092 | 0.094 | 0.106 | 0.086 | | -2 |
| | 0.021 | 0.024 | 0.027 | 0.029 | 0.032 | 0.035 | 0.037 | 0.038 | 0.040 | 0.040 | 0.038 | 0.031 | | | -3 |
| 14 | 0.568 | 0.567 | 0.563 | 0.553 | 0.543 | 0.531 | 0.516 | 0.498 | 0.476 | 0.444 | 0.406 | 0.354 | 0.270 | | 0 |
| | 0.114 | 0.120 | 0.131 | 0.141 | 0.152 | 0.165 | 0.175 | 0.190 | 0.207 | 0.225 | 0.250 | 0.272 | 0.318 | | -1 |
| | 0.039[a] | 0.042 | 0.045 | 0.051 | 0.055 | 0.058 | 0.064 | 0.070[a] | 0.074 | 0.076 | 0.074 | 0.064 | | | -2 |
| | 0.018 | 0.020 | 0.023 | 0.025 | 0.027 | 0.029 | 0.030 | 0.032 | 0.032 | 0.029 | -- | | | | -3 |
| 13 | 0.470 | 0.467 | 0.462 | 0.553 | 0.443 | 0.430 | 0.411 | 0.388 | 0.361 | 0.327 | 0.276 | 0.208 | | | 0 |
| | 0.0969 | 0.104 | 0.114 | 0.123 | 0.134 | 0.147 | 0.159 | 0.181 | 0.190 | 0.210 | 0.234 | 0.262 | | | -1 |
| | 0.305 | 0.035 | 0.038 | 0.042 | 0.047 | 0.051 | 0.055[a] | 0.058 | 0.060 | 0.059 | 0.051 | | | | -2 |
| | 0.0146 | 0.018 | 0.020[a] | 0.022[a] | 0.024[a] | 0.025[a] | 0.025[a] | 0.025[a] | --- | --- | | | | | -3 |
| 12 | 0.386 | 0.382 | 0.375 | 0.367 | 0.356 | 0.341 | 0.324 | 0.298 | 0.267 | 0.227 | 0.167 | | | | 0 |
| | 0.0812 | 0.0889 | 0.0980 | 0.108 | 0.118 | 0.130 | 0.147 | 0.157 | 0.174 | 0.192 | 0.215 | | | | -1 |
| | 0.0248 | 0.0291 | 0.0323 | 0.037 | 0.040 | 0.043 | 0.046 | 0.048 | 0.048 | 0.040 | | | | | -2 |
| | 0.0121 | 0.0142 | 0.0157 | 0.018 | 0.019 | 0.020 | 0.020 | 0.018 | --- | | | | | | -3 |
| 11 | 0.314 | 0.310 | 0.304 | 0.295 | 0.283 | 0.269 | 0.248 | 0.222 | 0.187 | 0.137 | | | | | 0 |
| | 0.0679 | 0.0752 | 0.0840 | 0.0932 | 0.103 | 0.118 | 0.127 | 0.142 | 0.157 | 0.175 | | | | | -1 |
| | 0.0205 | 0.0242 | 0.0280 | 0.0305 | 0.034 | 0.037 | 0.039 | 0.038[a] | 0.033 | | | | | | -2 |
| | 0.0099 | 0.0117 | 0.0133 | 0.0145 | 0.016 | 0.016 | 0.015 | --- | | | | | | | -3 |
| 10 | 0.254 | 0.250 | 0.244 | 0.234 | 0.222 | 0.206 | 0.184 | 0.156 | 0.114 | | | | | | 0 |
| | 0.0559 | 0.0629 | 0.0708 | 0.0798 | 0.0898 | 0.100 | 0.112 | 0.126 | 0.141 | | | | | | -1 |
| | 0.0166 | 0.0200 | 0.0228 | 0.0260 | 0.0283 | 0.030 | 0.030 | 0.027 | | | | | | | -2 |
| | 0.0088 | 0.0095 | 0.0109 | 0.0120 | 0.0126 | --- | --- | | | | | | | | -3 |
| 9 | 0.203 | 0.198 | 0.192 | 0.182 | 0.170 | 0.153 | 0.129 | 0.0942 | | | | | | | 0 |
| | 0.0451 | 0.0517 | 0.0592 | 0.0675 | 0.0770 | 0.0871 | 0.0988 | 0.112 | | | | | | | -1 |
| | 0.0133 | 0.0161 | 0.0187 | 0.0210 | 0.0230 | 0.0236 | 0.0208 | | | | | | | | -2 |
| | 0.0061 | 0.0075 | 0.0087 | 0.0094 | 0.0094 | --- | | | | | | | | | -3 |
| 8 | 0.158 | 0.154 | 0.148 | 0.139 | 0.125 | 0.106 | 0.0776 | | | | | | | | 0 |
| | 0.0357 | 0.0416 | 0.0486 | 0.0563 | 0.0650 | 0.0748 | 0.0858 | | | | | | | | -1 |
| | 0.0103 | 0.0127 | 0.0149 | 0.0169[a] | 0.0179 | 0.0163 | | | | | | | | | -2 |
| | 0.0041[a] | 0.0055[a] | 0.0067[a] | 0.0070[a] | 0.0064[a] | | | | | | | | | | -3 |
| 7 | 0.120 | 0.116 | 0.109 | 0.0995 | 0.0850 | 0.0625 | | | | | | | | | 0 |
| | 0.0274 | 0.0327 | 0.0389 | 0.0461 | 0.0541 | 0.0634 | | | | | | | | | -1 |
| | | | | | | | | | | | | | | | -2 |
| | | | | | | | | | | | | | | | -3 |
| 6 | 0.00762 | 0.00697 | 0.00588 | 0.00436 | 0.00240 | | | | | | | | | | 0 |
| | 0.0200 | 0.0249 | 0.0304 | 0.0370 | 0.0449 | | | | | | | | | | -1 |
| | 0.00565 | 0.00719 | 0.00850 | 0.00860 | | | | | | | | | | | -2 |
| | 0.00226 | 0.00283 | 0.00293 | | | | | | | | | | | | -3 |
| | 0.000873 | 0.00105 | | | | | | | | | | | | | -4 |
| | 0.000254 | | | | | | | | | | | | | | -5 |
| 5 | 0.00363 | 0.00318 | 0.00242 | 0.00136 | | | | | | | | | | | 0 |
| | 0.0139 | 0.0181 | 0.0230 | 0.0290 | | | | | | | | | | | -1 |
| | 0.00370 | 0.00486 | 0.00544 | | | | | | | | | | | | -2 |
| | 0.00127 | 0.00160 | | | | | | | | | | | | | -3 |
| | 0.000344 | | | | | | | | | | | | | | -4 |
| | | | | | | | | | | | | | | | -5 |
| 4 | 0.00145 | 0.00116 | 0.000678 | | | | | | | | | | | | 0 |
| | 0.00896 | 0.0124 | 0.0167 | | | | | | | | | | | | -1 |
| | 0.00210 | 0.00280 | | | | | | | | | | | | | -2 |
| | 0.00050 | | | | | | | | | | | | | | -3 |
| 3 | 0.000436 | 0.000272 | | | | | | | | | | | | | 0 |
| | 0.00496 | 0.00778 | | | | | | | | | | | | | -1 |
| | 0.000847 | | | | | | | | | | | | | | -2 |
| | | | | | | | | | | | | | | | -3 |
| 2 | 0.0000726 | | | | | | | | | | | | | | 0 |
| | 0.00212 | | | | | | | | | | | | | | -1 |
| | | | | | | | | | | | | | | | -2 |
| | | | | | | | | | | | | | | | -3 |

[a] Interpolated. --- Indicates a value less than 10% of the value of $\overline{R^{1}}$ with the same $n_i$ and $\ell_i$, but with $\Delta n = -1$ and $\Delta\ell = -1$.

Table IV. $\overline{R^1}$ in atomic units when $\Delta\ell = +1$.

| $n_i$ \ $\ell_i$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | $\Delta n$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0.116 | 0.111 | 0.103 | 0.093 | 0.089 | 0.082 | 0.076 | 0.073 | 0.066 | 0.056 | 0.048 | 0.042 | 0.027 | | | -1 |
| 15 | 0.039 | 0.034 | 0.029 | 0.026 | 0.024[a] | 0.022 | 0.019 | --- | --- | --- | --- | --- | | | | -2 |
| | 0.017 | 0.015 | --- | --- | --- | --- | --- | --- | --- | --- | --- | | | | | -3 |
| | 0.0954 | 0.089 | 0.083 | 0.076 | 0.068[a] | 0.061 | 0.054 | 0.049 | 0.041 | 0.035 | 0.027[a] | --- | | | | -1 |
| 14 | 0.0305 | 0.026 | 0.024 | 0.021 | 0.017 | --- | --- | --- | --- | --- | --- | | | | | -2 |
| | 0.0140 | 0.012 | --- | --- | --- | --- | --- | --- | --- | --- | | | | | | -3 |
| | 0.0791 | 0.0726 | 0.065 | 0.058 | 0.053 | 0.045 | 0.038[a] | 0.032 | 0.025 | 0.019 | --- | | | | | -1 |
| 13 | 0.0243 | 0.0209 | 0.018 | 0.016 | 0.013 | --- | --- | --- | --- | --- | | | | | | -2 |
| | 0.0126[a] | 0.0102[a] | --- | --- | --- | --- | --- | --- | --- | | | | | | | -3 |
| | 0.0650 | 0.0583 | 0.0516 | 0.0447 | 0.039 | 0.033[a] | 0.027 | 0.021 | --- | --- | | | | | | -1 |
| 12 | 0.0194 | 0.0163 | 0.0145 | 0.0116 | --- | --- | --- | --- | --- | | | | | | | -2 |
| | 0.0090 | --- | --- | --- | --- | --- | --- | --- | | | | | | | | -3 |
| | 0.0526 | 0.0463 | 0.0397 | 0.0340 | 0.0285 | 0.023 | 0.018 | --- | --- | | | | | | | -1 |
| 11 | 0.0154 | 0.0134 | 0.0104 | 0.0084 | --- | --- | --- | --- | | | | | | | | -2 |
| | 0.0069 | --- | --- | --- | --- | --- | --- | | | | | | | | | -3 |
| | 0.0419 | 0.0358 | 0.0305 | 0.0251 | 0.0200 | 0.0152 | 0.011 | --- | | | | | | | | -1 |
| 10 | 0.0121 | 0.0097 | 0.0080 | --- | --- | --- | --- | | | | | | | | | -2 |
| | 0.0053 | --- | --- | --- | --- | --- | | | | | | | | | | -3 |
| | 0.0326 | 0.0274 | 0.0224 | 0.0178 | 0.0135 | 0.0094 | --- | | | | | | | | | -1 |
| 9 | 0.0092 | 0.0072 | 0.0054 | --- | --- | --- | | | | | | | | | | -2 |
| | 0.0039 | --- | --- | --- | --- | | | | | | | | | | | -3 |
| | 0.0252 | 0.0201 | 0.0158 | 0.0118 | 0.0081 | --- | | | | | | | | | | -1 |
| 8 | 0.0066 | 0.0050 | --- | --- | --- | | | | | | | | | | | -2 |
| | 0.0027[a] | --- | --- | --- | --- | | | | | | | | | | | -3 |
| | 0.0179 | 0.0139 | 0.0103 | 0.0070 | --- | | | | | | | | | | | -1 |
| 7 | | | | | | | | | | | | | | | | -2 |
| | | | | | | | | | | | | | | | | -3 |
| | | 0.00890 | 0.00599 | 0.00264 | | | | | | | | | | | | -1 |
| 6 | 0.00279 | 0.00187 | 0.000928 | | | | | | | | | | | | | -2 |
| | 0.000940 | 0.000464 | | | | | | | | | | | | | | -3 |
| | 0.00754 | 0.0050 | 0.00272 | | | | | | | | | | | | | -1 |
| 5 | 0.00155 | 0.000786 | | | | | | | | | | | | | | -2 |
| | 0.000374 | | | | | | | | | | | | | | | -3 |
| | 0.00401 | 0.00213 | | | | | | | | | | | | | | -1 |
| 4 | 0.000634 | | | | | | | | | | | | | | | -2 |
| | | | | | | | | | | | | | | | | -3 |
| 3 | 0.00154 | | | | | | | | | | | | | | | -1 |

--- Indicates a value less than 10% of the value of $\overline{R^1}$ with the same $n_i$ and $\ell_i$, but with $\Delta n = -1$ and $\Delta\ell = -1$.

[a] Interpolated

Table V. $\overline{R^2}$ in atomic units when $\Delta\ell = 0$.

| $n_i$ \ $\ell_i$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | $\Delta n$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0.141 | 0.140 | 0.140 | 0.138 | 0.136 | 0.134 | 0.131 | 0.129 | 0.127 | 0.122 | 0.113 | 0.105 | 0.0941 | 0.0717 | | -1 |
| 15 | 0.024 | 0.027 | 0.026 | 0.025 | 0.024 | 0.023[a] | 0.022[a] | 0.021[a] | 0.020 | 0.017[a] | 0.015 | 0.013[a] | --- | | | -2 |
| | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | | | | -3 |
| | 0.0980 | 0.0961 | 0.0961 | 0.0956 | 0.0941 | 0.0924 | 0.0890 | 0.0855 | 0.0832 | 0.0772 | 0.0712 | 0.0562 | 0.0471 | | | -1 |
| 14 | 0.0166 | 0.0177 | 0.0170 | 0.0175[a] | 0.0170 | 0.0151 | 0.0148 | 0.0141[a] | 0.0132 | 0.0116 | 0.0086 | --- | | | | -2 |
| | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | | | | | -3 |
| | 0.0675 | 0.0663 | 0.0662 | 0.0648 | 0.0636[a] | 0.0624 | 0.0599 | 0.0569[a] | 0.0531 | 0.0481 | 0.0918 | 0.0316 | | | | -1 |
| 13 | 0.0113 | 0.0116 | 0.0111 | 0.0108 | 0.0107 | 0.0099 | 0.0088[a] | 0.0084 | 0.0073 | 0.0061 | --- | | | | | -2 |
| | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | | | | | | -3 |
| | 0.0456 | 0.0451 | 0.0446[a] | 0.0442 | 0.0430 | 0.0416 | 0.0396[a] | 0.0368 | 0.0333 | 0.0284 | 0.0214 | | | | | -1 |
| 12 | 0.0072 | 0.0075 | 0.0075 | 0.0074 | 0.0069 | 0.0062 | 0.0056 | 0.0051 | 0.0041 | --- | | | | | | -2 |
| | --- | --- | --- | --- | --- | --- | --- | --- | --- | | | | | | | -3 |
| | 0.0307 | 0.0302 | 0.0299 | 0.0291 | 0.0281 | 0.0270 | 0.0252 | 0.0228 | 0.0196 | 0.0146 | | | | | | -1 |
| 11 | 0.0047 | 0.0048 | 0.0047[a] | 0.0045 | 0.0043 | 0.0040 | 0.0034 | 0.0028 | --- | | | | | | | -2 |
| | --- | --- | --- | --- | --- | --- | --- | --- | | | | | | | | -3 |
| | 0.0200 | 0.0196 | 0.0192 | 0.0188 | 0.0181 | 0.0169 | 0.0153 | 0.0132 | 0.00987 | | | | | | | -1 |
| 10 | 0.0029 | 0.0031[a] | 0.0029 | 0.0029 | 0.0026 | 0.0022 | 0.0018 | --- | | | | | | | | -2 |
| | --- | --- | --- | --- | --- | --- | --- | | | | | | | | | -3 |
| | 0.0126 | 0.0123 | 0.0120 | 0.0116 | 0.0110 | 0.0101 | 0.00869 | | | | | | | | | -1 |
| 9 | 0.0018 | 0.0018 | 0.0017 | 0.0016 | 0.0014 | 0.0012 | --- | | | | | | | | | -2 |
| | --- | --- | --- | --- | --- | --- | | | | | | | | | | -3 |
| | 0.00750 | 0.00735 | 0.00714 | 0.00679 | 0.00626 | 0.0544 | 0.00411 | | | | | | | | | -1 |
| 8 | 0.00010 | 0.00010 | 0.00097 | 0.00088 | 0.00074 | --- | | | | | | | | | | -2 |
| | --- | --- | --- | --- | --- | | | | | | | | | | | -3 |
| | 0.00420 | 0.00408 | 0.00392[a] | 0.00364 | 0.00319 | 0.00243 | | | | | | | | | | -1 |
| 7 | | | | | | | | | | | | | | | | -2 |
| | | | | | | | | | | | | | | | | -3 |
| 6 | 0.00170 | 0.00167 | 0.00159 | 0.00132 | | | | | | | | | | | | -1 |
| 5 | 0.000762 | 0.000738 | 0.000635 | | | | | | | | | | | | | -1 |
| 4 | 0.000278 | 0.000249 | | | | | | | | | | | | | | -1 |
| 3 | 0.0000686 | | | | | | | | | | | | | | | -1 |

--- Indicates a value less than 10% of the value of $\overline{R^2}$ with the same $n_i$ and $\ell_i$, but with $\Delta n = -1$ and $\Delta\ell = 0$.

[a] Interpolated.

Since the core electrons can be considered as represented by a wave function of the form

$$\psi_{\vec{k}}^{c}(r_e) = \frac{\Omega^{1/2}}{V^{1/2}} \sum_n e^{i\vec{k}\cdot\vec{R}_n} \varphi_{1s}(\vec{r}_e - \vec{R}_n) , \tag{3.99}$$

it follows if $a(\vec{k})$ is chosen such that

$$a(\vec{k}) = \frac{1}{N} \int e^{i\vec{k}\cdot\vec{r}_e} \sum_n e^{i\vec{k}\cdot\vec{R}_n} \varphi_{1s}(\vec{r}_e - \vec{R}_n)\, d^3r_e \tag{3.100}$$

the $\psi_{\vec{k}}(r_e)$ are orthogonal to each other and to the wave functions of the core electrons. In this problem, concerning μ-mesonic lithium, it will be assumed, at least when the mesonic atom contains only one core electron, that the same technique for approximating the wave functions and energy levels of a band electron is valid, and further that the core wave functions $\varphi_{1s}(\vec{r}_e - \vec{r}_n)$ can be taken to be of the form

$$\varphi_{1s}(\vec{r}_e - \vec{R}_n) = \frac{Z_e^{3/2}}{\pi^{1/2}} e^{-Z_e|\vec{r}_e - \vec{R}_n|} . \tag{3.101}$$

Thus

$$a(\vec{k}) = \frac{8\pi^{1/2}}{Z_e^{3/2}(1 + \frac{k^2}{Z_e^2})^2} . \tag{3.102}$$

In this paper it will be assumed

$$\Omega = 139 \text{ atomic units} , \tag{3.103}$$

and for the fermi energy

$$\epsilon_f = 0.301 \text{ atomic units} . \tag{3.104}$$

In order to make the formula

$$\epsilon_f = \frac{m}{m^*} \left[ \frac{3(2\pi)^3}{8\pi\Omega} \right]^{2/3} \tag{3.105}$$

consistent the effective mass will be chosen such that

$$\frac{m^*}{m} = 1.15 . \tag{3.106}$$

C. The Ejected Electron

The wave function of the electron ejected from the solid through an Auger transition will, in this problem, often be assumed to be a solution of the Schrodinger equation in a Coulomb field of charge $Z_e$ . Thus

$$\psi_f(r_e) = \frac{4\pi}{\sqrt{V}} \sum_{\ell,m} i^\ell \, j_\ell(k\, r_e)\, Y^*_{\ell m}(\theta_e, \varphi)\, Y_{\ell m}(\theta_{\vec{k}}\, \varphi_{\vec{k}}) , \tag{3.107}$$

$$j_\ell(k\, r_e) = \frac{2^\ell \, e^{\pi/2a} \, |\Gamma(\ell + 1 - i/a)|}{(2\ell + 1)!} (k\, r_e)^\ell \, e^{-ikr_e}$$

$$\times \; F(\ell + 1 - i/a , 2\ell + 2, 2i\, k\, r_e) , \tag{3.108}$$

$$a = k/Z_e . \tag{3.109}$$

It also is to be noted that for the energy one has

$$E_k = k^2 . \tag{3.110}$$

# IV. TRANSITION RATES

## A. Radiation

The $\mu^-$ meson interacts with the radiation field and thus through the emission of photons it can make a transition to a lower energy state. The only type of radiation transitions which need be considered in this problem are dipole transitions (see Appendix 3) with the probability of a transition per second given by

$$w_{if} = \frac{1}{12} \alpha^4 c \frac{2\ell>}{2\ell + 1} |\Delta E|^3 \bar{R}^2 \qquad \Delta\ell = \pm 1 , \tag{4.1}$$

where $\alpha$ equals the Sommerfeld fine structure constant, $c$ equals the speed of light in atomic units, $\Delta E$ equals the energy of the outgoing photon in atomic units, $\ell$ equals the angular momentum of the initial meson state in units of $\hbar$, $\ell>$ equals the greater of the initial and final meson angular momentum involved in the transition, and $\Delta\ell$ equals the change in angular momentum of the meson state.

## B. Auger Transitions Involving the Core Electrons

The interaction Hamiltonian $H_A^1$ as defined in (2.10) leads to Auger transitions in which the $\mu^-$ meson makes a transition to a lower energy state through emmiting a K electron from the mesonic atom. In the derivation of the formula for the Auger transition rate two core electrons will be assumed. The procedure in the case of one core electron is similar so that only the result will be given for that case.

The transition rate is obtained from the well known formula

$$w_{if} = \frac{2\pi}{\hbar} H_{fi}^2 \rho(E) , \tag{4.2}$$

where

$$\rho(E) = \frac{1}{2} \frac{V}{(2\pi)^3} k \, d\Omega_{\vec{k}} , \tag{4.3}$$

with $\vec{k}$ the wave vector of the ejected electron, $d\Omega_{\vec{k}}$ is the differential solid angle formed by neighboring $\vec{k}$ vectors, and $H_{fi}$ is the matrix element of the interaction Hamiltonian between the initial and final states of the system. Thus the transition rate for a meson going from one state to another is

$$w_{if} = \frac{\pi V}{(2\pi)^3 \hbar} \int H_{fi}^2 \, k \, d\Omega_{\vec{k}} , \tag{4.4}$$

with

$$E_f = E_i , \tag{4.5}$$

where $E_i$ and $E_f$ stand for the total energy of the system in the initial and final state.

From (2.10) it follows that, in the case of two core electrons and using the approximations discussed in Section II

$$H_A^1(r_1, r_2, r)\psi_A(r_1, r_2, r) = -\frac{1}{\mu}\Big[\nabla_r^2 \psi_{L_i}(r_1, r_2, r)\psi_{M_i}(r) + 2\vec{\nabla}_r \psi_{L_i}(r_1, r_2, r) \cdot \vec{\nabla}_r \psi_{M_i}(r)\Big] , \tag{4.6}$$

where in this and the following, the subscripts i and f will denote initial and final states. Thus

$$H_{fi} = -\frac{1}{\mu}\int \psi_{M_f}^*(r)\left[\int \psi_{L_f}^*(r_1, r_2, r)\nabla_r^2 \psi_{L_i}(r_1, r_2, r) d^3r_1 d^3r_2\right]\psi_{M_i}(r) d^3r - \frac{2}{\mu}\int \psi_{M_f}^*(r)\left[\int \psi_{L_f}^*(r_1, r_2, r)\vec{\nabla}_r \psi_{L_i}(r_1, r_2, r) d^3r_1 d^3r_2\right] \cdot \vec{\nabla}_r \psi_{M_i}(r) d^3r . \tag{4.7}$$

Since both $\psi_{L_f}$ and $\psi_{L_i}$ can be expanded in powers of $r$, $H_{fi}$ itself can be looked at as the matrix element of an operator which is expanded in powers of $r$ and considered taken between the initial and final wave functions of the meson. Thus to lowest order in $r$

$$H_{fi} = -\frac{1}{\mu}\int \psi^*_{M_f}(r)\left[\int \psi^{0*}_{L_f}(r_1,r_2,r)\nabla_r^2\psi^r_{L_i}(r_1,r_2,r)d^3r_1d^3r_2\right]\psi_{M_i}(r)d^3r$$

$$-\frac{2}{\mu}\int \psi^*_{M_f}(r)\left[\int \psi^{0*}_{L_f}(r_1,r_2,r)\vec{\nabla}_r\psi^r_{L_i}(r_1,r_2r)d^3r_1d^3r_2\right]\vec{\nabla}_r\psi_{M_i}(r)d^3r \quad (4.8)$$

where in the expansion of $\psi_L$ in powers of $r$, $\psi_L^0$ is the $r$ independent and $\psi_L^r$ is the first $r$ dependent part.

Defining

$$g(r) = \int \psi^{0*}_{L_f}(r_1, r_2, r)\psi^r_{L_i}(r_1, r_2, r)\, d^3r_1\, d^3r_2 \quad , \qquad (4.9)$$

it follows from (4.8) that to lowest order in $r$

$$H_{fi} = \frac{1}{\mu}\int \psi^*_{M_f}(r)\nabla^2 g(r)\psi_{M_i}(r)d^3r - \frac{2}{\mu}\int \psi^*_{M_f}(r)\vec{\nabla}\, g(r)\cdot\vec{\nabla}\psi_{M_i}(r)d^3r. \qquad (4.10)$$

Note

$$\vec{\nabla}\cdot\left[\psi^*_{M_f}(r)\vec{\nabla}g(r)\ \psi_{M_i}(r)\right] = \vec{\nabla}\ \psi^*_{M_f}(r)\cdot\vec{\nabla}\ g(r)\ \psi_{M_i}(r)$$

$$+\ \psi^*_{M_f}(r)\ \vec{\nabla}\ g(r)\cdot\vec{\nabla}\ \psi_{M_i}(r) + \psi^*_{M_f}(r)\ \nabla^2 g(r)\ \psi_{M_i}(r) \quad , \qquad (4.11)$$

and since the integral over all space of the first term vanishes (as is easily seen through the use of Green's theorem and the recognition that $\psi_m$ vanishes at large distances from the nucleus ),

$$\int \psi^*_{M_f}(r)\ \vec{\nabla}\, g(r)\cdot \vec{\nabla}\, \psi_{M_i}(r) d^3r = -\int \psi^*_{M_f}(r)\, \nabla^2 g(r)\ \psi_{M_i}(r) d^3r$$

$$-\int \vec{\nabla}\, \psi^*_{M_f}(r)\cdot \vec{\nabla}\, g(r)\ \psi_{M_i}(r) d^3r \quad , \tag{4.12}$$

so that

$$H_{fi} = \frac{1}{\mu}\int \left[\vec{\nabla}\psi_{M_f}(r)\cdot\vec{\nabla}\, g(r)\psi_{M_i}(r) - \psi_{M_i}(r)\vec{\nabla}\, g(r)\cdot\vec{\nabla}\, \psi_{M_i}(r)\right] d^3r \ . \tag{4.13}$$

Since

$$\vec{\nabla}\cdot \left[\psi^*_{M_f}(r) g(r)\vec{\nabla}\psi_{M_i}(r)\right] = \vec{\nabla}\psi^*_{M_f}(r) g(r)\vec{\nabla}\psi_{M_i}(r) + \psi^*_{M_f}(r)\vec{\nabla} g(r)\cdot\vec{\nabla}\psi_{M_i}(r)$$

$$+ \psi^*_{M_f}(r)\ g\ (r)\ \nabla^2\ \psi_{M_i}(r)\ , \tag{4.14}$$

$$\vec{\nabla}\cdot \left[\vec{\nabla}\psi^*_{M_f}(r)\ g\ (r)\psi_{M_i}(r)\right] = \vec{\nabla}\ \psi^*_{M_f}(r)\ g\ (r)\ \vec{\nabla}\ \psi_{M_i}\ (r)$$

$$+ \vec{\nabla}\ \psi^*_{M_f}(r)\ \cdot \vec{\nabla}\ g\ (r)\ \psi_{M_i}(r) + \nabla^2\ \psi^*_{M_f}(r)\ g\ (r)\ \psi_{M_i}(r)\ , \tag{4.15}$$

and recognizing the fact that the integrals over all space of the terms on the left of both (4.14) and (4.15) vanish it follows

$$\int \left[\vec{\nabla}\ \psi^*_{M_f}(r)\cdot\vec{\nabla}\, g(r)\psi_{M_i}(r) - \psi^*_{M_f}(r)\ \vec{\nabla}\, g(r)\cdot \vec{\nabla}\ \psi_{M_i}(r)\right] d^3r$$

$$= \int \left[\psi^*_{M_f}(r)\ g\ (r)\ \nabla^2\ \psi_{M_i}(r) - \nabla^2\psi^*_{M_f}(r)\ g\ (r)\ \psi_{M_i}(r)\right] d^3r. \tag{4.16}$$

Thus using (4.13) it then is found

$$H_{fi} = \frac{1}{\mu}\int \left[\psi^*_{M_f}(r)\ g\ (r)\nabla^2\psi_{M_i}(r) - \nabla^2\psi^*_{M_f}(r)\ g\ (r)\psi_{M_i}(r)\ \right] d^3r, \tag{4.17}$$

so that using the equation

$$-\frac{1}{\mu}\nabla^2 \psi_M(r) + V_H(r)\,\psi_M(r) = \mathcal{E}_M\,\psi_M(r) \tag{4.18}$$

it follows

$$H_{fi} = \int \psi^*_{M_f}(r)\, g(r) \left[\Delta V_H(r) - \Delta\mathcal{E}_M\right] \psi_{M_i}(r)\, d^3r\ , \tag{4.19}$$

where

$$\Delta V_H(r) = V_{H_f}(r) - V_{H_i}(r)\ , \tag{4.20}$$

$$\Delta\mathcal{E}_M = \mathcal{E}_{M_f} - \mathcal{E}_{M_i} = k^2\ . \tag{4.21}$$

Further noting that to the lowest order in $r$

$$\Delta V_H(r) = 2(Z' - 1)^2 - (Z - 1)^2\ , \tag{4.22}$$

and defining for this case

$$Z_e = Z' - 1\ ,\quad a = k/Z_e\ ,\quad c^2 = 1 - \frac{(Z - Z')}{(Z' - 1)^2}(Z + Z' - 2), \tag{4.23}$$

it follows that to the lowest order in $r$

$$H_{fi} = (c^2 + a^2)Z_e^{\,2} \int \psi^*_{M_f}(r)\, g(r)\, \psi_{M_i}(r)\, d^3r\ . \tag{4.24}$$

To evaluate $g(r)$ note that from (3.90)

$$\psi^r_{L_i}(r_1, r_2, r) = \left[1 - r(\cos\theta'_1 + \cos\theta'_2)\right] \psi_0^{e'}(r_1)\, \psi_0^{e'}(r_2)\ , \tag{4.25}$$

and since

$$\cos\theta'_1 = \sum_m \frac{4\pi}{3}\, Y_{1m}(\theta, \varphi)\, Y^*_{1m}(\theta_1, \varphi_1)\ , \tag{4.26}$$

while it will be assumed

$$\psi^{0}_{L_f}(r_1, r_2) = \psi_f^{e'}(r_1)\,\psi_0^{e}(r_2), \tag{4.27}$$

thus it follows

$$g(r) = -\frac{4\pi}{3}\int \psi_f^{e'*}(r_1)\, r \sum_m Y_{lm}(\theta, \varphi) Y_{lm}(\theta_1, \varphi_1)\psi_0^{e'}(r_1) d^3r_1 \int \psi_0^{*e}(r_2) \times \psi_0^{e'}(r_2)\, d^3r_2 \tag{4.28}$$

if electron 1 changes state, or the same formula with $r_1$ and $r_2$ interchanged if electron 2 changes state. The use of a properly symmetrized function for $\psi_L$ does not change the results except for a negligible factor depending on an overlap integral involving $\psi_f^{e'}$ and $\psi_0^{e}$.

To the accuracy needed in this problem

$$\int \psi_0^{e*}(r_2)\, \psi_0^{e'}(r_2)\, d^3 r_2 = 1 , \tag{4.29}$$

and thus

$$g(r) = -\frac{4\pi}{3}\, r \sum_m Y_{lm}(\theta, \varphi) \int \psi_f^{e'*}(r_1) Y^*_{lm}(\theta_1, \varphi_1)\psi_0^{e'}(r_1) d^3r_1 . \tag{4.30}$$

Since

$$\psi_0^{e'}(r_1) = \frac{Z_e^{3/2}}{\pi^{1/2}}\, e^{-Z_e r_1} , \tag{4.31}$$

and using (3.107) it follows

$$g(r) = \frac{2}{3}\frac{(4\pi)^2}{\sqrt{V}}\, \frac{i}{(Z_e^3\pi)^{1/2}}\, r\, \frac{A(\rho)}{a^3}\, Y_{lm}(\theta, \varphi) Y^*_{lm}(\theta_k, \varphi_k) , \tag{4.32}$$

where

$$A(\rho) = \frac{1}{2}\int_0^\infty j_1(\rho)\, e^{-\rho/a}\, \rho^2\, d\rho , \tag{4.33}$$

$$a = k/Z_e , \tag{4.34}$$

so that

$$H_{fi} = -\frac{2}{3}\frac{(4\pi)^2}{\sqrt{V}}\frac{Z_e^{1/2}}{\pi^{1/2}}\, i\, \frac{(c^2 + a^2)}{a^3} A(\rho)\, \bar{R}\, \Omega_{fi}^{lm}\, Y_{lm}^*(\theta_k, \varphi_k) , \tag{4.35}$$

where

$$\bar{R} = \int R_{n_f \ell_f}(r)\, r R_{n_i \ell_i}(r)\, r^2 dr , \tag{4.36}$$

$$\Omega_{fi}^{lm} = \int Y_{\ell_f m_f}^*(\theta, \varphi) Y_{lm}(\theta, \varphi) Y_{\ell_i m_i}(\theta, \varphi)\, d\Omega . \tag{4.37}$$

If account is taken of

$$\sum_M \Omega_{fi}^{lm^2} = \frac{3}{8\pi}\frac{2\ell>}{2\ell + 1} , \tag{4.38}$$

it follows from (4.4) that the transition rate for a meson going from one state to another, when there are two core electrons, is given by

$$w_{if} = \frac{32\pi}{3\hbar}\frac{e^{\pi/a}}{\sinh \pi/a}\frac{2\ell>}{2\ell + 1}\left(\frac{c^2 + a^2}{1 + a^2}\right)^2 \frac{\exp|-4 \tan^{-1} a/a|}{1 + a^2} Z_e^2 R^2$$

$$\Delta\ell = \pm 1 , \tag{4.39}$$

with

$$a = k/Z_e, \quad Z_e = Z' - 1, \quad c^2 = 1 - \frac{(Z - Z')}{(Z' - 1)^2}(Z + Z' - 2). \tag{4.40}$$

An analogous procedure leads to, in the case where there is only one core electron,

$$w_{if} = \frac{16\pi}{3\hbar}\frac{e^{\pi/a}}{\sinh \pi/a}\frac{2\ell>}{2\ell + 1}\frac{\exp|-4 \tan^{-1} a/a|}{1 + a^2} Z_e^2 R^2 \qquad \ell = \pm 1, \tag{4.41}$$

with in this case

$$a = k/Z_e \quad , \quad Z_e = Z - 1 . \tag{4.42}$$

If the $\mu^-$ meson had been considered as in a coulomb potential $2Z_c/r_{nl}$ and the interaction of the electron and meson treated as the perturbation $2/r_{ml}$, then the transition rate for a mesonic change in state would be given by (to lowest order in matrix elements of $r$ )

$$w_{if} = (2) \frac{16\pi}{3\hbar} \frac{e^{\pi/a}}{\sinh \pi/a} \frac{2\ell>}{2\ell + 1} \frac{\exp|-4 \tan^{-1} a/a|}{1 + a^2} \overline{Z_e R}^{\,2} \quad \Delta\ell = \pm 1 , \tag{4.43}$$

where

$$a = k/Z_e \quad , \quad Z_e = Z_c \quad , \tag{4.44}$$

and with the factor (2) only present if the mesonic atom contains two core electrons.

Because of the similarity of (4.41) and (4.43) the assumption will be made that to sufficient accuracy formulas for monopole transitions can be derived using shielded coulomb wave functions for the initial and final states. Thus the transition rate for a monopole transition is

$$w_{if} = (2) \frac{16\pi}{3\hbar} \frac{e^{\pi/a}}{\sinh \pi/a} \frac{1}{6} \overline{Z_e^2 R^2}^{\,2} . \tag{4.45}$$

Transitions involving larger changes in meson angular momentum ($\Delta \ell = \pm 2, \pm 3$, etc.) appear to be negligible while corrections to the formulas indicated here for monopole and dipole transitions seem to be small enough so that they can be neglected in a first approximation. For a further discussion of both of these questions see Appendix 4 and 6.

C. Auger Transitions Involving the Band Electrons

Through use of formula (4.2) one finds the probability of a meson making a transition to a lower energy state and giving its energy to one of the band electrons

$$w_{if} = \frac{m^*}{m} \frac{V^2}{(2\pi)^5 \hbar} \int_0^{4\pi} \int_0^{4\pi} \int_{k_1}^{k_2} H_{fi}^2 \, k_i^{\,2} \, k_f d\Omega_f d\Omega_i dk_i \quad , \tag{4.46}$$

where $\Delta E$ equals the energy loss of the meson, where $k_i$ and $k_f$ refer to the wave vector of the interacting band electron in its initial and final state, where

$$k_f^{\,2} = k_i^{\,2} + \frac{m^*}{m} \Delta E \quad , \tag{4.47}$$

$$k_1 = \left(\frac{m^*}{m}\right)^{1/2} (\epsilon_f - \Delta E)^{1/2} \quad \Delta E < \epsilon_f , \quad k_1 = 0 \quad \Delta E > \epsilon_f \quad , \tag{4.48}$$

$$k_2 = \left(\frac{m^*}{m}\right)^{1/2} \epsilon_f^{\,1/2} \quad , \tag{4.49}$$

and where

$$H_{fi} = \int \psi^*_{M_f}(r) \, \psi^*_{e_f}(r_e) \, \frac{2}{|r - r_e|} \, \psi_{M_i}(r) \, \psi_{e_i}(r_e) d^3r \, d^3r_e \quad , \tag{4.50}$$

with $\psi_m(r)$ referring to either the initial or final mesonic wave function and $\psi_e(r_e)$ referring to either the initial or final wave function associated with the interacting band electron. Using the expansions

$$\frac{1}{|r - r_e|} = 4\pi \sum_{\ell, m} \frac{r_<^{\ell}}{r_>^{\ell+1}} \frac{1}{2\ell + 1} Y_{\ell m}(\theta, \varphi) \, Y^*_{\ell m}(\theta_e, \varphi_e), \tag{4.51}$$

$$\psi_{M_f}(r) = R_f(r)\, Y_{\ell_f m_f}(\theta, \varphi)\,, \quad Y_{M_i}(r) = R_i(r)\, Y_{\ell_i m_i}(\theta, \varphi)\,, \tag{4.52}$$

and defining

$$\Omega_{fi}^{\ell m} = \int Y^*_{\ell_f m_f}(\theta, \varphi)\, Y_{\ell m}(\theta, \varphi)\, Y_{\ell_i m_i}(\theta, \varphi)\, d\,\Omega\,, \tag{4.53}$$

it is found

$$H_{fi} = 8\pi \sum_{\ell, m} \frac{\Omega_{fi}^{\ell m}}{2\ell + 1}\, I_{\ell m}\,, \tag{4.54}$$

with

$$I_{\ell m} = \overline{R^{\ell}} \int \psi^*_{e_f}(r_e)\, \frac{Y^*_{\ell m}(\theta_e, \varphi_e)}{r_e^{\ell+1}}\, \psi_{e_i}(r_e) d^3 r_e + \int_0^{\infty} R_f(r) r^{\ell} \int_0^{4\pi} \int_0^{r}$$

$$\times \left[\frac{r_e^{2\ell+1}}{r^{2\ell+1}} - 1\right] \psi^*_{e_f}(r_e)\, \frac{Y^*_{\ell m}(\theta_e, \varphi_e)}{r_e^{\ell+1}}\, \psi_{e_i}(r_e) r_e^{\,2}\, d\Omega_e dr_e R_i(r) r^2 dr\,. \tag{4.55}$$

Since the band electrons are on the average much further from the center of the mesonic atom then the meson, as in the case of transitions involving the core electrons (See Appendix 4), the second term in (4.54), when $\ell$ is greater than zero, will be neglected. For the same reasons as for transitions involving the core electrons, the transition rates for transition involving a change in angular momentum greater than one will also be neglected. Thus with these approximations

$$I_{00} = \frac{1}{\sqrt{4\pi}} \int_0^{\infty} R_f(r) \int_0^{4\pi} \int_0^{r} \psi^*_{e_f}(r_e) \left[\frac{1}{r} - \frac{1}{r_e}\right] \psi_{e_i}(r_e) r_e^{\,2} d\Omega_e dr_e R_i(r) r^2 dr, \tag{4.56}$$

$$I_{lm} = \bar{R} \int \psi^*_{e_f}(r_e) \frac{Y_{lm}(\theta_e, \varphi_e)}{r_e^2} \psi_{e_i}(r_e)\, d^3 r_e \quad . \tag{4.57}$$

To evaluate $I_{00}$ the assumption will be made with the same justification as was used in simplifying $I_{lm}$ that the electron on the average is much further than the $\mu^-$ meson from the center of the mesonic atom. Thus under the integral in (4.56), $\psi_{e_f}(r_e)$ and $\psi_{e_i}(r_e)$ can be replaced by $\psi_{e_f}(0)$ and $\psi_{e_i}(0)$. Through energy considerations the band electron absorbing the meson's energy in a $\Delta\ell = 0$ mesonic transition is ejected from the solid. Using (3.97) and (3.107) and the approximation mentioned

$$I_{00} = \frac{\sqrt{4\pi}}{V} \frac{7}{6} \left(\frac{\pi}{a}\right)^{1/2} \frac{e^{\pi/2a}}{\left(\sinh \frac{\pi}{a}\right)^{1/2}} \left[1 - \frac{a(k_i)^2}{\Omega}\right]^{-1/2} \frac{1}{7} \left[\frac{8}{1 + \frac{k_i^2}{Z_e^2}} - 1\right] \overline{R^2},$$

$$a = k_f/Z_e \quad , \tag{4.58}$$

so that from (4.54), (4.58), and (4.57) it follows

$$H_{fi} = \frac{8\pi}{3} \frac{7}{2V} \left(\frac{\pi}{a}\right)^{1/2} \frac{e^{\pi/2a}}{\left(\sinh \frac{\pi}{a}\right)^{1/2}} \left[1 - \frac{a(k_i)^2}{\Omega}\right]^{-1/2} \frac{1}{7} \left[\frac{8}{1 + \frac{k_i^2}{Z_e^2}} - 1\right] \overline{R^2}$$

$$\Delta\ell = 0 \quad , \tag{4.59}$$

$$H_{fi} = \frac{8\pi}{3} \sum_m \Omega^{lm}_{fi} M_{lm} \bar{R} \qquad \Delta\ell = \pm 1 \ , \tag{4.60}$$

with

$$M_{lm} = \int \psi^*_{e_f}(r_e) \frac{Y_{lm}(\theta_e, \varphi_e)}{r_e^2} \psi_{e_i}(r_e)\, d^3 r_e \quad . \tag{4.61}$$

In order to evaluate $M_{lm}$ it will be assumed, when there is

only one core electron, that

$$\psi_{e_i}(r_e) = \frac{1}{\sqrt{V}} \left[ 1 - \frac{a^2(k_i)}{\Omega} \right]^{-1/2} \left[ e^{i\vec{k}_i \cdot \vec{r}_e} - a(k_i)\, \varphi_{1s}\, (r_e) \right], \quad (4.62)$$

and further if $\Delta E$ is so small that the excited band electron is not ejected from the solid that

$$\psi_{e_f}(r_e) = \frac{1}{\sqrt{V}} \left[ 1 - \frac{a^2(k_f)}{\Omega} \right]^{-1/2} \left[ e^{i\vec{k}_f \cdot \vec{r}_e} - a(k_f)\varphi_{1s}(r_e) \right] . \quad (4.63)$$

In the case where the electron is ejected from the solid the form (4.63) will continue to be used although its accuracy in the case of low energy ejected electrons may not be too great. When $\Delta E$ is above 50 eV. this wave function when placed in (4.61) gives, with a difference of only a few per cent, the same results as a normalized plane wave. A discussion of the reason why (3.97) can be simplified to the forms (4.62) and (4.63) can be found in Appendix 5.

Careful evaluation leads to

$$M^{lm^2} = \frac{1}{V^2} \left[ 1 - \frac{a^2(k_i)}{\Omega} \right]^{-1} \left[ 1 - \frac{a^2(k_f)}{\Omega} \right]^{-1} X_{fi}^{lm} , \quad (4.64)$$

with

$$X_{fi}^{lm} = \frac{16\pi^2}{K^2} \left[ Y_{lm}^*(\theta_K, \varphi_K)\, Y_{lm}(\theta_K, \varphi_K) + A(k_i, k_f)\, Y_{lm}^*(\theta_{k_i}, \varphi_{k_i}) \right.$$

$$\left. \times\, Y_{lm}(\theta_{k_i}, \varphi_{k_i}) + B(k_i, k_f)\, Y_{lm}^*(\theta_{k_f}, \varphi_{k_f})\, Y_{lm}(\theta_{k_f}, \varphi_{k_f}) \right] , \quad (4.65)$$

where

$$A(k_i,k_f) = \frac{(4\pi)^2}{k_i^{\,2}}\, b(k_f)\left[1 - \frac{Z_e}{k_i}\tan^{-1}\frac{k_i}{Z_e}\right]\left[b(k_f)\left[1 - \frac{Z_e}{k_i}\tan^{-1}\frac{k_i}{Z_e}\right]\right.$$

$$\left. - \left[1 - \frac{m^*}{2m}\,\frac{\Delta E}{k_i k_f}\,\log\frac{k_f + k_i}{k_f - k_i}\right]\right] , \qquad (4.66)$$

$$B(k_i,\ k_f) = \frac{(4\pi)^2}{k_f^{\,2}}\, b(k_i)\left[1 - \frac{Z_e}{k_f}\tan^{-1}\frac{k_f}{Z_e}\right]\left[b(k_i)\left[1 - \right.\right.$$

$$\left.\left.\frac{Z_e}{k_f}\tan^{-1}\frac{k_f}{Z_e}\right] - \left[1 + \frac{m^*}{2m}\,\frac{\Delta E}{k_i k_f}\,\log\frac{k_f + k_i}{k_f - k_i}\right]\right] , \qquad (4.67)$$

$$\vec{K} = \vec{k}_i - \vec{k}_f \ , \quad b(k_i) = 8/\left[1 + \frac{k_i^{\,2}}{Z_e^{\,2}}\right] , \quad b(k_f) = 8/\left[1 + \frac{k_f^{\,2}}{Z_e^{\,2}}\right] \qquad (4.68)$$

Terms which disappear when integrated over the angles of $\vec{k}_i$ and $\vec{k}_f$ are neglected in (4.65).

To the accuracy needed in this problem it is obvious that

$1 - \frac{a(k_i)^2}{\Omega}$ , $1 + \frac{a(k_f)^2}{\Omega}$ , and $1 + \frac{k_i^{\,2}}{Z_e^{\,2}}$ in (4.59) and (4.64)

can be replaced by $1 - \frac{a(\bar{k}_i)^2}{\Omega}$ , $1 - \frac{a(\bar{k}_f)^2}{\Omega}$ , and $1 + \frac{\bar{k}_i^{\,2}}{Z_e^{\,2}}$ where

$$\bar{k}_i^{\,2} = \frac{1}{2}\,\frac{m^*}{m}\,\epsilon_F \ , \quad \bar{k}_f^{\,2} = \bar{k}_i^{\,2} + \frac{m^*}{m}\,\Delta E \ , \qquad (4.69)$$

Thus one finally obtains using (4.46)

$$w_{if} = \frac{16\pi}{3\hbar}\,\frac{e^{\pi/a_B}}{\sinh\frac{\pi}{a_B}}\left[\frac{49}{3\pi}\left[1 - \frac{a(\bar{k}_i)^2}{\Omega}\right]^{-1}\left(\frac{m^*}{m}\right)^{5/2}\left(\frac{\epsilon_F}{Z_e^{\,2}}\right)^{3/2} h\right]\frac{1}{6}$$

$$\times\ \overline{Z_e^{\,2}R^2} \qquad \Delta\ell = 0 \ , \qquad (4.70)$$

where

$$h = \frac{1}{49}\left[\frac{8}{1+\frac{\bar{k}_i^{\,2}}{Z_e^{\,2}}} - 1\right]^2 , \qquad a_B = \left[\frac{m^*}{m}\,\frac{\Delta E}{Z_e^{\,2}}\right]^{1/2} . \qquad (4.71) \text{ and } (4.72)$$

In (4.70) use has been made of the fact that to a very good approximation for the energy changes involved in $\Delta\ell = 0$ transitions

$$\frac{e^{\pi/a}}{\sinh \pi/a} = \frac{e^{\pi/a_B}}{\sinh \pi/a_B} . \qquad (4.73)$$

It follows in the case of lithium

$$h = 0.91 \qquad Z_e = 2 , \qquad (4.74)$$

$$h = 0.88 \qquad Z_e = 1.6875 . \qquad (4.75)$$

It is to be noted that if the 2s Coulomb wave function $\varphi_{2s}(r_e)$ had been used as the initial state $\psi_{e_i}(r_e)$, the factor

$$\left[\frac{49}{3\pi}\left[1-\frac{a(k_i)^2}{\Omega}\right]^{-1}\left(\frac{m^*}{m}\right)^{5/2}\left(\frac{\epsilon_f}{Z_e^{\,2}}\right)^{3/2} h\right]$$

in (4.70) would be replaced by $1/8$. If the form

$$\frac{\Omega^{1/2}}{V^{1/2}}\, e^{i\vec{k}\cdot\vec{r}_e}\, \varphi_{2s}(r_e)$$

had been used then the factor would be replaced by

$$\frac{\Omega\, Z_e^{\,3}}{24\pi^2}\left(\frac{m^*}{m}\right)^{5/2}\left(\frac{\epsilon_f}{Z_e^{\,2}}\right)^{3/2} .$$

For $Z_e = 2$,

$$\left[\frac{49}{3\pi}\left[1-\frac{a(\bar{k}_i)^2}{\Omega}\right]^{-1}\left(\frac{m^*}{m}\right)^{5/2}\left(\frac{\epsilon_f}{Z_e^{\,2}}\right)^{3/2} h\right] = 0.161 \quad , \tag{4.76}$$

$$1/8 = 0.125 \quad , \tag{4.77}$$

$$\left[\frac{\Omega\ Z_e^{\,3}}{24\pi^2}\left(\frac{m^*}{m}\right)^{5/2}\left(\frac{\epsilon_f}{Z_e^{\,2}}\right)^{3/2}\right] = 0.133 \, . \tag{4.78}$$

When there is one core electron in the mesonic atom (4.70) will be used in computing meson transitions, however in the case where there are two core electrons in the mesonic atom, the atom will be neutral, and thus one would expect the form (4.70) to overestimate the $\Delta\ell = 0$ transition rates. This is because it was derived using band electron wave functi ons based on an overestimation of the attractive potential of the mesonic atom and thus overestimating $\psi_e(0)$ . If plane waves instead of (4.62) and (4.63) are used for $\psi_{e_i}$ and $\psi_{e_f}$ in (4.56) , one obtains for the transition rate

$$w_{if} = \left[\frac{16\pi}{3\hbar}\ \frac{1}{\pi^2}\ \frac{m^*}{m}\int_{k_1}^{k_2} k_i^{\,2}\ k_f\ dk_i\right]\frac{1}{6}\ \overline{R^2}^{\,2} \qquad \Delta\ell = 0 \ , \tag{4.79}$$

which in contrast to (4.70) probably underestimates the $\Delta\ell = 0$ transition rates when the mesonic atom contains two core electrons. This is because plane waves take no account of the presence of the attractive potential found within the K shell of the mesonic atom and thus probably underestimate $\psi_e(0)$ . Both (4.70) and (4.79) will be used in investigating transition paths when the mesonic atom contains two core electrons.

Using (4.46) and (4.60) it follows

$$w_{if} = \frac{1}{12\pi^4 \hbar} \left( \frac{m^*}{m} \right) \left[ 1 - \frac{a(\bar{k}_i)^2}{\Omega} \right]^{-1} \left[ 1 - \frac{a(\bar{k}_f)^2}{\Omega} \right]^{-1} \frac{2\ell >}{2\ell + 1}$$

$$\times \int_0^{4\pi} \int_0^{4\pi} X_{fi}^{lm} \; k_i^2 k_f \, d\Omega_i \, d\,\Omega_f \, dk_i \, \bar{R}^2 \qquad \Delta\ell = \pm 1 \,, \qquad (4.80)$$

The integral in (4.80) was evaluated by means of the IBM-704.

If plane waves are used for the initial and final electron wave functions, (4.80) is replaced by

$$w_{if} = \frac{8}{3\pi\hbar} \; \frac{2\ell >}{2\ell + 1} \int_{k_1}^{k_2} k_i \log \frac{k_f + k_i}{k_f - k_i} \, d\, k_i \, \bar{R}^2 \qquad \Delta\ell = \pm 1 \,. \qquad (4.81)$$

In (4.80) and (4.81) the effect of the band structure in lithium has been neglected. In the case of small energy transfers some final states with particular wave vector $\vec{k}$ assumed available for occupation in (4.80) and (4.81) will not exist thus decreasing the actual transition rates from those given by (4.80) and (4.81). Since, as will be shown later, the transition paths of the $\mu^-$ meson are very insensitive to even rather large changes in the transition rates as given by (4.80) and (4.81) the band structure of solid lithium will not be taken into account in this paper.

## D. Electron Replacement Transitions

If one of the core electrons surrounding the mesonic atom is ejected from the atom by far the most probable way for refilling this

empty 1s electron energy state is through an Auger transition in which one band electron makes a transition into the state while another band electron is ejected from the solid. The transition rate would be

$$w_{if} = \frac{2\pi}{h} \int H_{fi}^2 \frac{V^2}{(2\pi)^9} k_f \, d\Omega_f \, d^3k_a \, d^3k_b \, , \tag{4.82}$$

where $k_a$ and $k_b$ are the initial wave vectors of the band electrons involved, $k_f$ is the wave vector of the ejected electron and

$$H_{fi} = \int \psi_f^*(r_a) \, \psi_f^*(r_b) \frac{2}{|r_a - r_b|} \psi_i(r_a) \, \psi_i(r_b) \, d^3r_a d^3r_b \, , \tag{4.83}$$

where to fair approximation, at least as needed in (4.83)

$$\psi_i(r_a) \left[1 - \frac{a(\bar{k}_i)^2}{\Omega}\right]^{-\frac{1}{2}} \frac{1}{\sqrt{V}} \left[e^{i\vec{k}_a \cdot \vec{r}_a} - 8 \, e^{-Z_e r_a}\right] ,$$

$$\psi_f(r_a) = \frac{1}{\sqrt{V}} e^{i\vec{k}_f \cdot \vec{r}_a} \, , \tag{4.84}$$

$$\psi_i(r_b) = \left[1 - \frac{a(\bar{k}_i)^2}{\Omega}\right]^{-1/2} \frac{1}{\sqrt{V}} \left[e^{i\vec{k}_b \cdot \vec{r}_b} - 8 \, e^{-Z_e r_a}\right] ,$$

$$\psi_f(r_b) = \frac{Z_e^{3/2}}{\pi^{1/2}} e^{-Z_e r_b} \, . \tag{4.85}$$

In order to estimate the transition rate (4.83), $H_{fi}$ can be written in the form

$$H_{fi} = \frac{2}{V^{3/2}} \left(\frac{Z_e^3}{\pi}\right)^{1/2} \left[1 - \frac{a(\bar{k}_i)}{\Omega}\right] \left[I_1 + I_2 + I_3 + I_4\right] , \tag{4.86}$$

where

$$I_1 = \int e^{-i\vec{k}_f \cdot \vec{r}_a} \, e^{-Z_e r_b} \frac{1}{|r_a - r_b|} e^{i\vec{k}_a \cdot \vec{r}_a} \, e^{i\vec{k}_b \cdot \vec{r}_b} \, d^3r_a d^3r_b \, , \tag{4.87}$$

$$I_2 = -8\int e^{-i\vec{k}_f\cdot\vec{r}_a}\, e^{-Z_e r_b}\, \frac{1}{|r_a - r_b|}\, e^{i\vec{k}_b\cdot\vec{r}_b}\, e^{-Z_e r_a}\, d^3r_a d^3r_b\,, \tag{4.88}$$

$$I_3 = -8\int e^{-i\vec{k}_f\cdot\vec{r}_a}\, e^{-Z_e r_b}\, \frac{1}{|r_a - r_b|}\, e^{i\vec{k}_a\cdot\vec{r}_a}\, e^{-Z_e r_b}\, d^3r_a d^3r_b\,, \tag{4.89}$$

$$I_4 = 64\int e^{-i\vec{k}_f\cdot\vec{r}_a}\, e^{-Z_e r_b}\, \frac{1}{|r_a - r_b|}\, e^{-Z_e(r_a + r_b)}\, d^3r_a d^3r_b\,. \tag{4.90}$$

Since in this problem

$$k_a \ll k_f\,,\ k_a \ll Z_e\,,\ k_b \ll k_f\,,\ k_a \ll Z_e\,, \tag{4.91}$$

there follows the reasonable approximations

$$I_1 = \int e^{-Z_e r_b}\, \frac{1}{|r_a - r_b|}\, e^{-i\vec{k}_f\cdot\vec{r}_a}\, d^3r_a d^3r_b\,, \tag{4.92}$$

$$I_2 = -8\int e^{-Z_e r_b}\, \frac{1}{|r_a - r_b|}\, e^{-Z_e r_a}\, e^{-i\vec{k}_f\cdot\vec{r}_a}\, d^3r_a d^3r_b\,, \tag{4.93}$$

$$I_3 = -8\int e^{-2Z_e r_b}\, \frac{1}{|r_a - r_b|}\, e^{-i\vec{k}_f\cdot\vec{r}_b}\, d^3r_a d^3r_b\,, \tag{4.94}$$

$$I_4 = 64\int e^{-2Z_e r_b}\, \frac{1}{|r_a - r_b|}\, e^{-Z_e r_a}\, e^{-i\vec{k}_f\cdot\vec{r}_a}\, d^3r_a d^3r_b\,. \tag{4.95}$$

An evaluation of the integrals leads to

$$I_1 = \frac{(4\pi)^2}{Z_e^{\,5}}\, \frac{2}{a^2(a^2+1)^2}\,, \tag{4.96}$$

$$I_2 = -\frac{(4\pi)^2}{Z_e^5} \frac{16(a^2 + 10)}{(a^2 + 1)(a^2 + 4)^2} , \tag{4.97}$$

$$I_3 = -\frac{(4\pi)^2}{Z_e^5} \frac{32}{a^2(a^2 + 4)^2} , \tag{4.98}$$

$$I_4 = \frac{(4\pi)^2}{Z_e^5} \frac{32(a^2 + 33)}{(a^2 + 1)(a^2 + 9)^2} , \tag{4.99}$$

with

$$a = k_f/Z_e . \tag{4.100}$$

Making use of $H_{fi}$ in the form

$$H_{fi} = 2g(a) \left[1 - \frac{a(\bar{k}_i)^2}{\Omega}\right]^{-1} \left(\frac{Z_e^3}{\pi V^3}\right)^{1/2} \frac{(4\pi)^2}{Z_e^5} , \tag{4.101}$$

where

$$g(a) = \left[\frac{2}{a^2(a^2 + 1)^2} + \frac{32(a^2 + 33)}{(a^2 + 1)(a^2 + 9)} - \frac{16(a^2 + 10)}{(a^2 + 1)(a^2 + 4)} - \frac{32}{a^2(a^2 + 4)^2}\right] , \tag{4.102}$$

it is found

$$w_{if} = \frac{256}{9\pi^2\hbar} \left[1 - \frac{a(\bar{k}_i)^2}{\Omega}\right]^{-2} a\, g(a)^2 \left[\left(\frac{m^*}{m}\right) \frac{\epsilon_f}{Z_e^2}\right]^3 . \tag{4.103}$$

Since $a^2$ is expected to lie between .5 and 1.0 and noting

$$.51 < a\, g(a)^2 < 1.3 , \tag{4.104}$$

when

$$.5 < a^2 < 1.0 , \tag{4.105}$$

it will be assumed as a crude approximation

$$a\, g(a)^2 = 1.0. \tag{4.106}$$

Thus using this value it follows

$$w_{if} = \frac{16\pi}{3\hbar}\left[ \frac{16}{3\pi^3}\left[1 - \frac{a(\bar{k}_i)^2}{\Omega}\right]^{-2} \left(\frac{m^*}{m}\right)^3 \left(\frac{\epsilon_f}{Z_e^{\,2}}\right)^3 \right]. \tag{4.107}$$

Assuming another crude approximation

$$Z_e = 2.5 \quad , \tag{4.108}$$

one finds

$$w_{if} = 1.2 \times 10^{13}\ \text{sec}^{-1} \quad . \tag{4.109}$$

There exist experimental values for this transition rate in elements with $Z \geqslant 6$. The observed rates are found to be about $5 \times 10^{14}$ sec.$^{-1}$ with no $Z$ dependence. If the $L$ electrons are treated as core electrons when $Z \geqslant 6$, the calculated rates agree with the observed rates. ( Burhop, 1952; Broyles et al., 1953). Equation (4.109) indicates that in the case of light elements $Z < 6$ where the $L$ electrons act more as band electrons than as core electrons the rate of electron replacement in the $K$ shell decreases by a factor of ten or more.

## E. Numerical Values

Using the preceding formulas for mesonic transition rate along with the energy levels and matrix elements found as discussed in Section III, the transition rate for a meson out of any initial state

and into any final state (of which we are interested in this problem) can be found. See Tables VI - IX for the results of such calculations. In these tables transitions whose sum would account for less than 10% of the decays from any state have been neglected. In the case of one core electron in the mesonic atom it is to be noted that the refill transitions as computed in Section IV-D do not compete with the mesonic transitions until $n = 4$. In the case of no core electrons the competition between the two processes begins at $n = 7$. It is also to be noted from Tables VI - VIII that the ejection of a core electron does not take place with any great frequency until $n = 12$ or $n = 11$, in the case where there are two core electrons in the mesonic atom, or until $n = 9$, when the mesonic atom only contains one core electron.

Thus if the $\mu^-$ meson is captured in lithium through the ejection of a band electron the mesonic atom would most likely continue to contain two core electrons until the meson reaches the state $n = 11$ or 12, in which case one core electron would probably be ejected, at $n = 9$ the second core electron would probably be ejected, but at $n = 7$ it would most likely be replaced, and by $n = 4$ the mesonic atom would once again contain, if typical, two core electrons. The case where the $\mu^-$ meson is captured in lithium through the ejection of a core electron is similar except that initially the mesonic atom will only contain one core electron which it will not likely lose until the $\mu^-$ meson reaches the state $n = 9$.

Table VI. Transition rates in units of $10^{15}$ $sec^{-1}$ when the mesonic atom contains two core electrons and where orthogonalized plane waves are used for describing the band electrons.

| $n_i$ | $\Delta\ell$ | Particle receiving energy | $\Delta n$ | $\ell_i$ = 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 15 | 0 | Band | -1 | 5.30 | 5.30 | 5.14 | 5.00 | 4.86 | 4.64 | 4.48 | 4.35 | 4.02 | 3.46 | 2.97 | 2.39 | 1.39 | |
| | | | 0 | 0.20 | 0.39 | 0.58 | 0.72 | 0.83 | 0.92 | 0.97 | 1.01 | 1.08 | 1.06 | 0.97 | 0.87 | 0.72 | 0.475 |
| | -1 | Band | -1 | | | | | | | | | | | | | | 0.148 |
| 14 | 0 | Band | -1 | 2.49 | 2.49 | 2.46 | 2.39 | 2.30 | 2.14 | 1.92 | 1.87 | 1.61 | 1.37 | 0.853 | 0.600 | | |
| | | Core | -2 | 0.65 | 0.68 | 0.63 | 0.57 | 0.54 | 0.52 | 0.50 | 0.43 | 0.34 | 0.20 | 0.111 | | | |
| | | Band | 0 | 0.16 | 0.31 | 0.43 | 0.52 | 0.60 | 0.64 | 0.67 | 0.68 | 0.70 | 0.65 | 0.540 | 0.425 | 0.252 | |
| | -1 | | -1 | | | | | | | | | | | | | | |
| | | Core | -2 | | | | | | | | | 0.18 | 0.21 | 0.195 | 0.194 | | |
| 13 | | Band | -1 | 1.19 | 1.18 | 1.13 | 1.09 | 1.05 | 0.970 | 0.875 | 0.761 | 0.624 | 0.472 | 0.270 | | | |
| | | Core | -2 | 0.30 | 0.32 | 0.30 | 0.26 | 0.24 | 0.222 | 0.187 | 0.170 | 0.095 | | | | | |
| | | Band | 0 | 0.12 | 0.21 | 0.31 | 0.37 | 0.42 | 0.444 | 0.443 | 0.429 | 0.418 | 0.362 | 0.257 | 0.149 | | |
| | -1 | | -1 | | | | | | | | | | | | 0.080 | | |
| | | Core | -2 | | | | | | 0.109 | 0.135 | 0.160 | 0.202 | 0.185 | 0.145 | | | |
| 12 | 0 | Band | -1 | 0.548 | 0.537 | 0.527 | 0.500 | 0.467 | 0.424 | 0.365 | 0.300 | 0.218 | 0.124 | | | | |
| | | Core | -2 | 0.123 | 0.128 | 0.109 | 0.099 | 0.089 | 0.073 | 0.061 | | | | | | | |
| | | Band | 0 | 0.079 | 0.152 | 0.212 | 0.253 | 0.276 | 0.285 | 0.278 | 0.263 | 0.234 | 0.197 | 0.0775 | | | |
| | -1 | | -1 | | | | | | | | 0.043 | 0.058 | 0.067 | 0.13 | | | |
| | | | -1 | | | | | | | | | | 0.080 | 1.11 | | | |
| | | Core | -2 | | 0.029 | 0.049 | 0.065 | 0.086 | 0.101 | 0.119 | 0.136 | 0.137 | 0.101 | | | | |
| 11 | 0 | Band | -1 | 0.246 | 0.24 | 0.23 | 0.21 | 0.20 | 0.169 | 0.140 | 0.104 | | | | | | |
| | | Core | -1 | 1.27 | 1.23 | 1.19 | 1.10 | 1.04 | 0.921 | 0.770 | 0.593 | 0.325 | | | | | |
| | | Band | 0 | | 0.10 | 0.14 | 0.16 | 0.174 | 0.183 | 0.162 | 0.143 | 0.114 | | | | | |
| | -1 | | -1 | | | | | | | | 0.056 | 0.074 | 0.09 | | | | |
| | | Core | -1 | 0.070 | 0.12 | 0.15 | 0.21 | 0.29 | 0.359 | 0.473 | 0.616 | 0.768 | 1.08 | | | | |
| | | | -2 | | | | | | | | | 0.074 | | | | | |

Table VI (Continued)

| $n_i$ | $\Delta\ell$ | Particle receiving energy | $\Delta n$ | $\ell_i$ = 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 10 | 0 | Band | -1 | 0.104 | 0.100 | 0.096 | 0.089 | 0.078 | 0.064 | 0.047 | | | | | | | |
| | | Core | -1 | 0.620 | 0.610 | 0.572 | 0.522 | 0.473 | 0.387 | 0.297 | 0.153 | | | | | | |
| | -1 | Band | 0 | | 0.063 | 0.085 | 0.101 | 0.105 | 0.102 | 0.089 | 0.069 | | | | | | |
| | | | -1 | | | | | | 0.042 | 0.053 | 0.067 | 0.090 | | | | | |
| | | Core | -1 | 0.081 | 0.125 | 0.178 | 0.240 | 0.349 | 0.416 | 0.545 | 0.715 | 0.975 | | | | | |
| | | | -2 | | | 0.032 | 0.042 | 0.053 | 0.061 | 0.065 | 0.049 | | | | | | |
| | +1 | Core | -1 | 0.061 | 0.038 | | | | | | | | | | | | |
| 9 | 0 | Band | -1 | 0.042 | 0.040 | 0.037 | 0.033 | 0.028 | | | | | | | | | |
| | | Core | -1 | 0.267 | 0.258 | 0.241 | 0.211 | 0.180 | 0.134 | 0.077 | | | | | | | |
| | -1 | Band | 0 | 0.020 | 0.037 | 0.050 | 0.057 | 0.058 | 0.057 | 0.042 | | | | | | | |
| | | | -1 | | | | | 0.034 | 0.044 | 0.057 | 0.074 | | | | | | |
| | | Core | -1 | 0.075 | 0.118 | 0.167 | 0.228 | 0.307 | 0.406 | 0.536 | 0.700 | | | | | | |
| | | | -2 | | | 0.024 | 0.031 | 0.038 | | | | | | | | | |
| | +1 | Core | -1 | 0.052 | 0.031 | | | | | | | | | | | | |
| 8 | 0 | Band | -1 | 0.0151 | 0.0143 | | | | | | | | | | | | |
| | | Core | -1 | 0.0995 | 0.0933 | 0.085 | 0.073 | 0.055 | 0.077 | | | | | | | | |
| | -1 | Band | 0 | 0.0113 | 0.0202 | 0.027 | 0.030 | 0.029 | | | | | | | | | |
| | | | -1 | | 0.0115 | 0.017 | 0.023 | 0.032 | 0.043 | 0.054 | | | | | | | |
| | | Core | -1 | 0.0567 | 0.0929 | 0.135 | 0.191 | 0.264 | 0.354 | 0.475 | | | | | | | |
| | | | -2 | | | 0.015 | 0.020 | | | | | | | | | | |
| | +1 | Core | -1 | 0.0343 | 0.0192 | | | | | | | | | | | | |
| 7 | 0 | Band | -1 | 0.00481 | | | | | | | | | | | | | |
| | | Core | -1 | 0.0322 | 0.0304 | 0.0262 | 0.020 | | | | | | | | | | |
| | -1 | Band | 0 | 0.0052 | 0.0099 | 0.0129 | 0.014 | | | | | | | | | | |
| | | | -1 | 0.0053 | 0.0090 | 0.0136 | 0.020 | 0.028 | 0.039 | | | | | | | | |
| | | Core | -1 | 0.0380 | 0.0648 | 0.0987 | 0.143 | 0.204 | 0.283 | | | | | | | | |
| | +1 | Core | -1 | 0.0192 | 0.0094 | | | | | | | | | | | | |

Table VI (Continued)

| $n_i$ | $\Delta\ell$ | Particle receiving energy | $\Delta n$ | $\ell_i$ 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0 | Core | -1 | 0.0060 | 0.0058 | 0.0052 | | | | | | | | | | | |
| | | Band | 0 | 0.0022 | 0.0045 | 0.0052 | 0.0050 | | | | | | | | | | |
| 6 | -1 | | -1 | 0.0033 | 0.0062 | 0.0099 | | | | | | | | | | | |
| | | Core | -1 | 0.0209 | 0.0388 | 0.0620 | 0.0954 | 0.115 | | | | | | | | | |
| | +1 | Core | -1 | 0.0083 | 0.0034 | | | | | | | | | | | | |
| | 0 | Core | -1 | 0.0013 | | | | | | | | | | | | | |
| | | Band | 0 | 0.0008 | 0.0016 | 0.0018 | | | | | | | | | | | |
| 5 | -1 | | -1 | 0.0018 | 0.0037 | 0.0064 | 0.0106 | | | | | | | | | | |
| | | Core | -1 | 0.0096 | 0.0196 | 0.0338 | 0.0558 | | | | | | | | | | |
| | | Photon | -4 | 0.0006 | | | | | | | | | | | | | |
| | +1 | Core | -1 | 0.0025 | 0.0007 | | | | | | | | | | | | |
| | | Band | 0 | 0.0003 | 0.0005 | | | | | | | | | | | | |
| | | | -1 | 0.0008 | 0.0019 | 0.0036 | | | | | | | | | | | |
| | | Core | -1 | 0.0035 | 0.0081 | 0.0157 | | | | | | | | | | | |
| 4 | -1 | | -1 | | | 0.0002 | | | | | | | | | | | |
| | | Photon | -2 | 0.0002 | 0.0003 | | | | | | | | | | | | |
| | | | -3 | 0.0012 | | | | | | | | | | | | | |
| | +1 | Core | -1 | 0.0004 | | | | | | | | | | | | | |
| | | Core | -1 | 0.0008 | 0.0023 | | | | | | | | | | | | |
| 3 | -1 | Photon | -1 | 0.0004 | 0.0107 | | | | | | | | | | | | |
| | | | -2 | 0.0028 | | | | | | | | | | | | | |
| 2 | -1 | Photon | -1 | 0.0105 | | | | | | | | | | | | | |

Table VII. Transition rates in units of $10^{15}$ $sec^{-1}$ when the mesonic atom contains two core electrons and where plane waves are used for describing the band electrons. Transition rates out of states below those of $n = 7$ have not been computed.

| $n_i$ | $\Delta\ell$ | Particle receiving energy | $\Delta n$ | $\ell_i$ = 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 15 | 0 | Band | -1 | 0.106 | 0.106 | 0.103 | 0.100 | 0.096 | 0.092 | 0.088 | 0.085 | 0.078 | 0.067 | | | | |
| | | Core | -3 | 0.202 | 0.194 | 0.187 | 0.178 | 0.167 | 0.153 | 0.130 | 0.107 | 0.080 | | | | | |
| | -1 | Band | 0 | 0.138 | 0.291 | 0.435 | 0.547 | 0.640 | 0.739 | 0.800 | 0.854 | 0.890 | 0.905 | 0.886 | 0.834 | 0.719 | 0.494 |
| | | | -1 | 0.033 | 0.045 | 0.056 | 0.064 | 0.071 | 0.084 | 0.096 | 0.111 | 0.133 | 0.154 | 0.183 | 0.224 | 0.338 | 0.402 |
| | | Core | -3 | | | | | | | | | 0.085 | 0.085 | 0.082 | 0.054 | | |
| | +1 | Band | -1 | 0.048 | | | | | | | | | | | | | |
| 14 | 0 | Band | -1 | 0.057 | 0.057 | 0.057 | 0.055 | 0.052 | | | | | | | | | |
| | | Core | -2 | 0.646 | 0.675 | 0.634 | 0.572 | 0.535 | 0.522 | 0.497 | 0.432 | 0.340 | 0.198 | 0.111 | | | |
| | | | -3 | 0.082 | 0.077 | 0.072 | 0.067 | 0.063 | 0.051 | | | | | | | | |
| | -1 | Band | 0 | 0.110 | 0.231 | 0.326 | 0.403 | 0.477 | 0.528 | 0.566 | 0.588 | 0.598 | 0.566 | 0.520 | 0.427 | 0.268 | |
| | | | -1 | | 0.028 | 0.035 | 0.043 | 0.050 | 0.060 | 0.068 | 0.082 | 0.097 | 0.115 | 0.143 | 0.170 | 0.231 | |
| | | Core | -2 | | | 0.039 | 0.052 | 0.067 | 0.085 | 0.111 | 0.142 | 0.184 | 0.206 | 0.195 | 0.194 | | |
| | | | -3 | | | | | 0.042 | 0.048 | 0.055 | 0.061 | 0.067 | | | | | |
| 13 | 0 | Band | -1 | 0.032 | 0.031 | 0.030 | 0.029 | | | | | | | | | | |
| | | Core | -2 | 0.297 | 0.318 | 0.297 | 0.256 | 0.239 | 0.222 | 0.187 | 0.170 | 0.095 | 0.040 | | | | |
| | | | -3 | 0.030 | 0.030 | 0.027 | | | | | | | | | | | |
| | -1 | Band | 0 | 0.084 | 0.160 | 0.235 | 0.290 | 0.336 | 0.373 | 0.382 | 0.381 | 0.363 | 0.326 | 0.258 | 0.155 | | |
| | | | -1 | | | | 0.028 | 0.034 | 0.041 | 0.049 | 0.063 | 0.070 | 0.085 | 0.106 | 0.133 | | |
| | | Core | -2 | 0.023 | 0.037 | 0.051 | 0.071 | 0.085 | 0.109 | 0.135 | 0.160 | 0.202 | 0.185 | 0.145 | | | |
| | | | -3 | | | | | 0.034 | | | | | | | | | |
| | +1 | Core | -1 | 0.021 | | | | | | | | | | | | | |
| 12 | 0 | Band | -1 | 0.017 | 0.017 | 0.016 | | | | | | | | | | | |
| | | Core | -2 | 0.123 | 0.128 | 0.109 | 0.099 | 0.089 | 0.073 | 0.061 | 0.032 | 0.018 | | | | | |
| | | | -3 | 0.011 | 0.011 | 0.010 | 0.009 | | | | | | | | | | |
| | -1 | Band | 0 | 0.057 | 0.114 | 0.161 | 0.198 | 0.224 | 0.240 | 0.239 | 0.236 | 0.205 | 0.162 | 0.08 | | | |
| | | | -1 | | 0.011 | 0.015 | 0.018 | 0.022 | 0.027 | 0.035 | 0.040 | 0.050 | 0.061 | 0.08 | | | |
| | | | -1 | | | | | | | | | | 0.08 | 1.11 | | | |
| | | Core | -2 | 0.022 | 0.029 | 0.049 | 0.065 | 0.086 | 0.101 | 0.119 | 0.136 | 0.137 | 0.101 | | | | |
| | | | -3 | | 0.011 | 0.016 | 0.021 | 0.026 | 0.028 | | | | | | | | |
| | +1 | Core | -1 | 0.016 | | | | | | | | | | | | | |

Table VII (Continued)

| $n_i$ | $\Delta\ell$ | Particle receiving energy | $\Delta n$ | $\ell_i$ = 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 11 | 0 | Core | -1 | 1.27 | 1.23 | 1.19 | 1.10 | 1.04 | 0.921 | 0.770 | 0.593 | 0.325 | | | | | |
| | | | -2 | 0.053 | 0.05 | | | | | | | | | | | | |
| | | Band | 0 | | 0.08 | 0.11 | 0.13 | 0.14 | 0.154 | 0.140 | 0.129 | 0.100 | 0.059 | | | | |
| | -1 | | -1 | | | | | | | | | | 0.043 | | | | |
| | | Core | -1 | 0.070 | 0.12 | 0.15 | 0.21 | 0.29 | 0.359 | 0.473 | 0.616 | 0.768 | 0.108 | | | | |
| | | | -2 | | | | | | | | | | | | | | |
| | +1 | Core | -1 | 0.026 | | | | | | | | | | | | | |
| 10 | 0 | Core | -1 | 0.620 | 0.610 | 0.572 | 0.522 | 0.473 | 0.387 | 0.297 | 0.153 | | | | | | |
| | | Band | 0 | | 0.048 | 0.066 | 0.078 | 0.085 | 0.085 | 0.077 | 0.061 | | | | | | |
| | -1 | Core | -1 | 0.081 | 0.125 | 0.178 | 0.240 | 0.349 | 0.416 | 0.545 | 0.714 | 0.975 | | | | | |
| | | | -2 | | | 0.032 | | | | | | | | | | | |
| | +1 | Core | -1 | 0.031 | 0.038 | | | | | | | | | | | | |
| 9 | 0 | Core | -1 | 0.267 | 0.258 | 0.241 | 0.211 | 0.180 | 0.134 | 0.077 | | | | | | | |
| | | Band | 0 | | 0.028 | 0.038 | 0.044 | 0.047 | 0.044 | 0.035 | | | | | | | |
| | -1 | Core | -1 | 0.075 | 0.118 | 0.167 | 0.228 | 0.307 | 0.406 | 0.536 | 0.699 | | | | | | |
| | | | -2 | | | 0.024 | | | | | | | | | | | |
| | +1 | Core | -1 | 0.026 | 0.031 | | | | | | | | | | | | |
| 8 | 0 | Core | -1 | 0.100 | 0.093 | 0.085 | 0.073 | 0.055 | 0.032 | | | | | | | | |
| | | Band | 0 | | 0.015 | 0.020 | 0.023 | 0.023 | | | | | | | | | |
| | -1 | Core | -1 | 0.057 | 0.093 | 0.135 | 0.191 | 0.264 | 0.354 | 0.475 | | | | | | | |
| | | | -2 | | | 0.015 | 0.020 | 0.023 | | | | | | | | | |
| | +1 | Core | -1 | 0.017 | 0.019 | | | | | | | | | | | | |
| 7 | 0 | Core | -1 | 0.032 | 0.030 | | | | | | | | | | | | |
| | | Band | 0 | | 0.006 | 0.026 | 0.020 | | | | | | | | | | |
| | -1 | | -1 | | 0.006 | | | | | | | | | | | | |
| | | Core | -1 | 0.038 | 0.065 | 0.099 | 0.143 | 0.204 | 0.283 | | | | | | | | |
| | +1 | Core | -1 | 0.010 | 0.009 | | | | | | | | | | | | |

Table VIII. Transition rates in units of $10^{15}$ $sec^{-1}$ when the mesonic atom contains one core electron.

| $n_i$ | $\Delta\ell$ | Particle receiving energy | $\Delta n$ | $\ell_i$ = 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 15 | 0 | Band | -1 | 4.32 | 4.32 | 4.17 | 4.11 | 4.05 | 3.90 | 3.63 | 3.36 | 3.06 | 2.65 | 2.17 | 1.69 | 0.920 | |
| | -1 | Band | 0 | | 0.20 | 0.29 | 0.36 | 0.42 | 0.46 | 0.49 | 0.53 | 0.52 | 0.48 | 0.46 | 0.40 | 0.307 | 0.177 |
| | | | -1 | | | | | | | | | | | | | | 0.089 |
| 14 | 0 | Band | -1 | 2.30 | 2.26 | 2.24 | 2.12 | 2.00 | 1.92 | 1.82 | 1.63 | 1.38 | 1.12 | 0.846 | 0.480 | | |
| | -1 | Band | 0 | | 0.16 | 0.23 | 0.28 | 0.32 | 0.32 | 0.38 | 0.38 | 0.36 | 0.34 | 0.30 | 0.22 | 0.131 | |
| | | | -1 | | | | | | | | | | | | | 0.045 | |
| 13 | 0 | Band | -1 | 1.17 | 1.15 | 1.12 | 1.07 | 1.03 | 0.942 | 0.843 | 0.705 | 0.600 | 0.444 | 0.242 | | | |
| | -1 | Band | 0 | 0.06 | 0.12 | 0.17 | 0.22 | 0.24 | 0.26 | 0.269 | 0.264 | 0.244 | 0.212 | 0.162 | 0.093 | | |
| | | | -1 | | | | | | | | | | | | 0.030 | | |
| 12 | 0 | Band | -1 | 0.570 | 0.553 | 0.537 | 0.513 | 0.474 | 0.429 | 0.375 | 0.290 | 0.223 | 0.124 | | | | |
| | | Core | -2 | 0.123 | 0.123 | 0.116 | 0.107 | 0.093 | 0.077 | 0.050 | 0.036 | | | | | | |
| | -1 | Band | 0 | 0.041 | 0.089 | 0.123 | 0.154 | 0.173 | 0.181 | 0.179 | 0.169 | 0.149 | 0.113 | 0.064 | | | |
| | | | -1 | | | | | | | | | | | 0.030 | | | |
| | | Core | -2 | | 0.035 | 0.052 | 0.067 | 0.086 | 0.106 | 0.125 | 0.138 | 0.139 | 0.131 | | | | |
| 11 | 0 | Band | -1 | 0.274 | 0.268 | 0.254 | 0.237 | 0.218 | 0.191 | 0.156 | 0.115 | 0.064 | | | | | |
| | | Core | -2 | 0.052 | 0.050 | 0.047 | 0.041 | 0.034 | 0.026 | 0.016 | | | | | | | |
| | -1 | Band | 0 | 0.028 | 0.062 | 0.084 | 0.103 | 0.114 | 0.117 | 0.111 | 0.100 | 0.075 | 0.044 | | | | |
| | | | -1 | | | | | | | 0.018 | 0.023 | 0.028 | 0.035 | | | | |
| | | Core | -2 | | 0.029 | 0.040 | 0.053 | 0.067 | 0.080 | 0.090 | 0.090 | 0.068 | | | | | |
| 10 | 0 | Band | -1 | 0.116 | 0.111 | 0.106 | 0.099 | 0.086 | 0.070 | 0.052 | 0.029 | | | | | | |
| | | Core | -2 | 0.018 | 0.017 | 0.016 | 0.013 | 0.010 | | | | | | | | | |
| | -1 | Band | 0 | 0.021 | 0.041 | 0.056 | 0.067 | 0.072 | 0.072 | 0.064 | 0.051 | 0.030 | | | | | |
| | | | -1 | | 0.006 | 0.008 | 0.011 | 0.015 | 0.018 | 0.023 | 0.030 | 0.038 | | | | | |
| | | Core | -2 | 0.011 | 0.020 | 0.029 | 0.038 | 0.047 | 0.054 | 0.055 | 0.042 | | | | | | |
| 9 | 0 | Band | -1 | 0.046 | 0.044 | 0.041 | 0.037 | 0.031 | 0.023 | 0.013 | | | | | | | |
| | | Core | -1 | 0.239 | 0.226 | 0.210 | 0.192 | 0.162 | 0.123 | 0.047 | | | | | | | |
| | -1 | Band | 0 | 0.013 | 0.024 | 0.035 | 0.039 | 0.040 | 0.038 | 0.030 | 0.018 | | | | | | |
| | | | -1 | | | | | | 0.022 | 0.030 | 0.037 | | | | | | |
| | | Core | -1 | 0.640 | 0.485 | 0.367 | 0.274 | 0.205 | 0.132 | 0.101 | 0.062 | | | | | | |
| | | | -2 | | | 0.020 | 0.026 | 0.031 | 0.034 | 0.027 | | | | | | | |
| | +1 | Core | -1 | 0.040 | 0.023 | | | | | | | | | | | | |

Table VIII (Continued)

| $n_i$ | $\Delta\ell$ | Particle receiving energy | $\Delta n$ | $\ell_i$ 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 8 | 0 | Band | -1 | 0.029 | 0.016 | 0.014 | 0.012 | | | | | | | | | | |
| | | Core | -1 | 0.092 | 0.087 | 0.079 | 0.068 | 0.051 | 0.030 | | | | | | | | |
| | -1 | Band | 0 | 0.007 | 0.013 | 0.018 | 0.020 | 0.020 | 0.017 | | | | | | | | |
| | | | -1 | | | 0.009 | 0.013 | 0.018 | 0.024 | 0.032 | | | | | | | |
| | | Core | -1 | 0.047 | 0.078 | 0.114 | 0.163 | 0.218 | 0.304 | 0.409 | | | | | | | |
| | | | -2 | | | 0.009 | 0.013 | 0.017 | 0.019 | | | | | | | | |
| | +1 | Core | -1 | 0.028 | 0.015 | | | | | | | | | | | | |
| 7 | 0 | Core | -1 | 0.031 | 0.028 | 0.025 | 0.019 | | | | | | | | | | |
| | -1 | Band | 0 | 0.003 | 0.006 | 0.008 | | | | | | | | | | | |
| | | | -1 | | 0.006 | 0.008 | 0.012 | 0.017 | 0.024 | | | | | | | | |
| | | Core | -1 | 0.0311 | 0.053 | 0.081 | 0.117 | 0.167 | 0.232 | | | | | | | | |
| | | | -2 | | 0.005 | 0.006 | 0.008 | 0.010 | | | | | | | | | |
| | +1 | Core | -1 | 0.016 | 0.006 | | | | | | | | | | | | |
| 6 | 0 | Core | -1 | 0.0056 | 0.0054 | 0.0049 | | | | | | | | | | | |
| | -1 | Band | 0 | 0.0016 | 0.0027 | | | | | | | | | | | | |
| | | | -1 | 0.0022 | 0.0040 | 0.0064 | 0.0099 | 0.015 | | | | | | | | | |
| | | Core | -1 | 0.0167 | 0.0311 | 0.0495 | 0.0760 | 0.115 | | | | | | | | | |
| | | | -2 | 0.0013 | 0.0024 | 0.0036 | 0.0039 | | | | | | | | | | |
| | +1 | Core | -1 | 0.0066 | 0.0027 | | | | | | | | | | | | |
| 5 | 0 | Core | -1 | 0.0012 | 0.0012 | | | | | | | | | | | | |
| | -1 | Band | 0 | 0.0007 | 0.0010 | | | | | | | | | | | | |
| | | | -1 | 0.0012 | 0.0025 | 0.0043 | 0.0071 | | | | | | | | | | |
| | | Core | -1 | 0.0076 | 0.0155 | 0.0268 | 0.0441 | | | | | | | | | | |
| | | | -2 | | 0.0009 | 0.0012 | | | | | | | | | | | |
| | | Photon | -4 | 0.0006 | | | | | | | | | | | | | |
| | +1 | Core | -1 | 0.0018 | 0.0005 | | | | | | | | | | | | |

Table VIII (Continued)

| $n_i$ | $\Delta\ell$ | Particle receiving energy | $\Delta n$ | $\ell_i$ 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0 | Core | -1 | 0.0002 | | | | | | | | | | | | | |
| | | Band | 0 | 0.0002 | | | | | | | | | | | | | |
| | | | -1 | 0.0005 | 0.0012 | 0.0024 | | | | | | | | | | | |
| 4 | -1 | Core | -1 | 0.0028 | 0.0064 | 0.0123 | | | | | | | | | | | |
| | | | -1 | | | 0.0002 | | | | | | | | | | | |
| | | Photon | -2 | 0.0002 | 0.0003 | | | | | | | | | | | | |
| | | | -3 | 0.0012 | | | | | | | | | | | | | |
| | +1 | Core | -1 | 0.0003 | | | | | | | | | | | | | |
| | | Core | -1 | 0.0006 | 0.0019 | | | | | | | | | | | | |
| 3 | -1 | Photon | -1 | 0.0004 | 0.0011 | | | | | | | | | | | | |
| | | | -2 | 0.0028 | | | | | | | | | | | | | |
| 2 | -1 | Photon | -1 | 0.0105 | | | | | | | | | | | | | |

Table IX. Transition rates in units of $10^{15}$ $sec^{-1}$ when the mesonic atom contains no core electrons.

| $n_i$ | $\Delta\ell$ | Particle receiving energy | $\Delta n$ | $\ell_i$ = 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 10 | 0 | Band | -1 | 0.116 | 0.111 | 0.107 | 0.099 | 0.0865 | 0.0707 | 0.0525 | 0.0294 | |
| | -1 | Band | -1 | | 0.007 | 0.009 | 0.0118 | 0.0153 | 0.0193 | 0.0244 | 0.0312 | 0.0394 |
| | | | -2 | | | | | 0.0039 | 0.0044 | 0.0045 | 0.0036 | |
| 9 | 0 | Band | -1 | 0.0458 | 0.0463 | 0.0409 | 0.0367 | 0.0309 | 0.0229 | 0.0127 | | |
| | -1 | Band | -1 | 0.00405 | 0.00638 | 0.0090 | 0.0121 | 0.0161 | 0.0210 | 0.0273 | 0.0352 | |
| | | | -2 | | | | 0.0026 | 0.0032 | 0.0034 | 0.0027 | | |
| | +1 | Band | -1 | 0.00298 | | | | | | | | |
| 8 | 0 | Band | -1 | 0.0165 | 0.0156 | 0.0141 | 0.0120 | 0.00905 | 0.00517 | | | |
| | -1 | Band | -1 | 0.00398 | 0.00651 | 0.00952 | 0.0132 | 0.0181 | 0.0243 | 0.0323 | | |
| | | | -2 | | | 0.00153 | 0.00204 | 0.00234 | 0.00198 | | | |
| | +1 | Band | -1 | 0.00253 | 0.00140 | | | | | | | |
| 7 | 0 | Band | -1 | 0.00518 | 0.00480 | 0.00412 | 0.00318 | 0.00184 | | | | |
| | -1 | Band | -1 | 0.00319 | 0.00546 | 0.00826 | 0.0121 | 0.0170 | 0.0236 | | | |
| | | | -2 | | 0.000715 | 0.00106 | 0.00134 | 0.00125 | | | | |
| | +1 | Band | -1 | 0.00164 | 0.000811 | | | | | | | |

## V. TRANSITION PATHS

Assuming either one or two core electrons bound to the mesonic atom, and through the use of Tables VI, VII, and VIII, the transition paths of mesons captured in states $n = 15$, $\ell = 1, 5, 10$, and 14 have been followed. These are shown in Figures 1 - 12. For the case of two core electrons the Figures include paths found both using transitions computed with orthogonalized plane waves as the band electron wave functions, which probably overestimate the $\Delta n = -1$ $\Delta\ell = 0$ transitions, and those found using unmodified plane waves as the band electron wave functions, which probably underestimate the $\Delta n = -1$, $\Delta\ell = 0$ transitions Both cases were discussed in Section IV - C. A more accurate representation of this cascade (2 core electrons) presumably lies between the two extremes (between the paths given in Figures 1 - 4 and 5 - 8).

Figures 1 - 12 assume that the initial state contained 100 mesons and show the number of mesons passing through each state as these mesons move down to lower energies. If a meson moved into an $\ell = 0$ state its path was not continued. All numbers in the Tables have been rounded off to the nearest integer. A discussion of the inaccuracies of the computed transition paths due to the inaccuracies of the transition rates as computed from the formulas of Section IV can be found in Appendix 6. The results of that discussion seem to be that although a more accurate investigation of the problem of $\mu^-$ mesonic transition rates would most likely lead to a slight quantitative change in the

| n \ $\ell$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 15 | | | | | | | | | | 1 | 5 | 19 | 26 | $76^a$ | 100 |
| 14 | | | | | | | | | 1 | 5 | 15 | 36 | $62^a$ | 74 | |
| 13 | | | | | | | | 2 | 6 | 17 | 36 | $60^a$ | 49 | | |
| 12 | | | | | | 1 | 3 | 8 | 17 | 28 | $34^a$ | 44 | | | |
| 11 | | | | | | 1 | 2 | 6 | 10 | 19 | $48^a$ | | | | |
| 10 | | | | | | 1 | 6 | 13 | 23 | $59^a$ | | | | | |
| 9 | | | | | | 5 | 11 | 21 | $63^a$ | | | | | | |
| 8 | | | | | 4 | 10 | 21 | $67^a$ | | | | | | | |
| 7 | | | | 4 | 9 | 19 | $70^a$ | | | | | | | | |
| 6 | | | 4 | 9 | 20 | $70^a$ | | | | | | | | | |
| 5 | | 4 | 10 | 19 | $70^a$ | | | | | | | | | | |
| 4 | 3 | 10 | 19 | $70^a$ | | | | | | | | | | | |
| 3 | 8 | 18 | $70^a$ | | | | | | | | | | | | |
| 2 | 5 | $70^a$ | | | | | | | | | | | | | |
| 1 | $85^a$ | | | | | | | | | | | | | | |

Fig. 1. Transition paths when two core electrons are present in the mesonic atom and when orthogonalized plane waves are used for describing the band electrons. The initial state into which the meson is assumed captured is $n = 15$, $\ell = 14$. The superscript a indicates the $\ell$ state of the mean meson (with respect to $\ell$ distribution) passing through each n state. The dotted line shows the route $\Delta n = \Delta\ell = -1$ .

| n \ ℓ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 15 | | | | | | | | 1 | 5 | 23 | $100^a$ | | | | |
| 14 | | | | | | | 1 | 4 | 14 | 39 | $77^a$ | | | | |
| 13 | | | | | | 1 | 3 | 9 | 25 | $44^a$ | 43 | | | | |
| 12 | | | | | 1 | 4 | 10 | 22 | $36^a$ | 42 | 26 | | | | |
| 11 | | | | | 1 | 3 | 7 | 15 | $24^a$ | 24 | 6 | | | | |
| 10 | | | | | 3 | 7 | 15 | $29^a$ | 35 | 17 | | | | | |
| 9 | | | | 2 | 5 | 13 | 24 | $35^a$ | 23 | | | | | | |
| 8 | | | 2 | 4 | 11 | 22 | $36^a$ | 29 | | | | | | | |
| 7 | | 1 | 3 | 9 | 20 | $32^a$ | 36 | | | | | | | | |
| 6 | 1 | 3 | 9 | 21 | $34^a$ | 36 | | | | | | | | | |
| 5 | 3 | 9 | 21 | $34^a$ | 36 | | | | | | | | | | |
| 4 | 7 | 21 | $34^a$ | 36 | | | | | | | | | | | |
| 3 | 15 | $32^a$ | 38 | | | | | | | | | | | | |
| 2 | 9 | $39^a$ | | | | | | | | | | | | | |
| 1 | $66^a$ | | | | | | | | | | | | | | |

Fig. 2. Transition paths when two core electrons are present in the mesonic atom and when orthogonalized plane waves are used for describing the band electrons. The initial state into which the meson is assumed captured is $n = 15$, $\ell = 10$. The superscript a indicates the $\ell$ state of the mean meson (with respect to $\ell$ distribution) passing through each n state. The dotted line shows the route $\Delta n = \Delta \ell = -1$.

| n \ ℓ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 15 | | | | 2 | 15 | $100^a$ | | | | | | | | | |
| 14 | | | 1 | 6 | 27 | $85^a$ | | | | | | | | | |
| 13 | | | 3 | 11 | 33 | $57^a$ | | | | | | | | | |
| 12 | | 1 | 7 | 19 | $40^a$ | 49 | | | | | | | | | |
| 11 | | 1 | 7 | 16 | $35^a$ | 38 | | | | | | | | | |
| 10 | | 3 | 12 | 27 | $40^a$ | 28 | | | | | | | | | |
| 9 | | 6 | 17 | $32^a$ | 36 | 15 | | | | | | | | | |
| 8 | 1 | 10 | 26 | $35^a$ | 25 | 5 | | | | | | | | | |
| 7 | 3 | 19 | $37^a$ | 32 | 11 | 1 | | | | | | | | | |
| 6 | 10 | 34 | $39^a$ | 18 | 2 | | | | | | | | | | |
| 5 | 20 | $37^a$ | 27 | 5 | | | | | | | | | | | |
| 4 | 26 | $28^a$ | 11 | | | | | | | | | | | | |
| 3 | $20^a$ | 11 | 2 | | | | | | | | | | | | |
| 2 | $3^a$ | 2 | | | | | | | | | | | | | |
| 1 | $16^a$ | | | | | | | | | | | | | | |

Fig. 3. Transition paths when two core electrons are present in the mesonic atom and when orthogonalized plane waves are used for describing the band electrons. The initial state into which the meson is assumed captured is $n = 15$, $\ell = 5$. The superscript a indicates the $\ell$ state of the mean meson (with respect to $\ell$ distribution) passing through each n state. The dotted line shows the route $\Delta n = \Delta \ell = -1$ .

| n \ $\ell$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 15 | 4 | 100 | | | | | | | | | | | | | |
| 14 | 5 | $96^a$ | | | | | | | | | | | | | |
| 13 | 5 | $72^a$ | | | | | | | | | | | | | |
| 12 | 8 | $72^a$ | | | | | | | | | | | | | |
| 11 | | $67^a$ | | | | | | | | | | | | | |
| 10 | 3 | $75^a$ | | | | | | | | | | | | | |
| 9 | 10 | $63^a$ | 5 | | | | | | | | | | | | |
| 8 | 12 | $45^a$ | 11 | | | | | | | | | | | | |
| 7 | 14 | $30^a$ | 12 | | | | | | | | | | | | |
| 6 | 13 | $19^a$ | 8 | 1 | | | | | | | | | | | |
| 5 | 12 | $9^a$ | 6 | | | | | | | | | | | | |
| 4 | 6 | $7^a$ | 1 | | | | | | | | | | | | |
| 3 | $6^a$ | 1 | | | | | | | | | | | | | |
| 2 | | | | | | | | | | | | | | | |
| 1 | 2 | | | | | | | | | | | | | | |

Fig. 4. Transition paths when two core electrons are present in the mesonic atom and when orthogonalized plane waves are used for describing the band electrons. The initial state into which the meson is assumed captured is $n = 15$, $\ell = 1$. The superscript a indicates the $\ell$ state of the mean meson (with respect to $\ell$ distribution) passing through each n state. The dotted line shows the route $\Delta n = \Delta\ell = -1$ .

| n \ $\ell$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 15 | | 1 | 1 | 2 | 3 | 4 | 6 | 8 | 11 | 16 | 22 | 28 | 37 | 55 | $100^a$ |
| 14 | | | | | 1 | 2 | 4 | 7 | 11 | 16 | 21 | 30 | $42^a$ | 45 | |
| 13 | | | | | 1 | 1 | 2 | 4 | 6 | 10 | 15 | $20^a$ | 21 | | |
| 12 | | 2 | 2 | 4 | 6 | 8 | 12 | 15 | $19^a$ | 19 | 17 | 25 | | | |
| 11 | | | | | 1 | | 3 | 5 | 9 | $20^a$ | 31 | | | | |
| 10 | | 2 | 2 | 2 | 5 | 7 | 11 | 17 | $24^a$ | 32 | | | | | |
| 9 | | 2 | 4 | 5 | 7 | 11 | 17 | $23^a$ | 36 | | | | | | |
| 8 | | 3 | 3 | 7 | 10 | 16 | $23^a$ | 39 | | | | | | | |
| 7 | 1 | 3 | 6 | 8 | 14 | $23^a$ | 41 | | | | | | | | |
| 6 | | | | | | | | | | | | | | | |
| 5 | | | | | | | | | | | | | | | |
| 4 | | | | | | | | | | | | | | | |
| 3 | | | | | | | | | | | | | | | |
| 2 | | | | | | | | | | | | | | | |
| 1 | | | | | | | | | | | | | | | |

Fig. 5. Transition paths when two core electrons are present in the mesonic atom and when plane waves are used for describing the band electrons. The initial state into which the meson is assumed captured is $n = 15$, $\ell = 14$. The transition paths are only followed down to $n = 7$. The superscript a indicates the $\ell$ state of the mean meson (with respect to $\ell$ distribution) passing through each n state. The dotted line shows the route $\Delta n = \Delta\ell = -1$ .

| n \ ℓ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 15 | | 2 | 4 | 8 | 13 | 19 | 27 | 38 | 52 | $75^a$ | 100 | | | | |
| 14 | | 2 | 3 | 4 | 5 | 9 | 13 | $18^a$ | 22 | 21 | 5 | | | | |
| 13 | | | 1 | 2 | 2 | 3 | $3^a$ | 4 | 5 | | | | | | |
| 12 | 1 | 6 | 8 | 12 | 16 | 20 | $25^a$ | 29 | 27 | 19 | 1 | | | | |
| 11 | | 1 | 2 | 2 | 3 | 4 | $6^a$ | 5 | 4 | | | | | | |
| 10 | 1 | 5 | 8 | 9 | 14 | $16^a$ | 19 | 18 | 10 | 1 | | | | | |
| 9 | 1 | 8 | 10 | 13 | 17 | $19^a$ | 19 | 14 | 3 | | | | | | |
| 8 | 2 | 10 | 12 | 17 | $19^a$ | 20 | 17 | 5 | | | | | | | |
| 7 | 3 | 11 | 15 | $19^a$ | 20 | 19 | 6 | | | | | | | | |
| 6 | | | | | | | | | | | | | | | |
| 5 | | | | | | | | | | | | | | | |
| 4 | | | | | | | | | | | | | | | |
| 3 | | | | | | | | | | | | | | | |
| 2 | | | | | | | | | | | | | | | |
| 1 | | | | | | | | | | | | | | | |

Fig. 6. Transition paths when two core electrons are present in the mesonic atom and when plane waves are used for describing the band electrons. The initial state into which the meson is assumed captured is $n = 15$, $\ell = 10$. The transition paths are only followed down to $n = 7$. The superscript a indicates the $\ell$ state of the mean meson (with respect to $\ell$ distribution) passing through each n state. The dotted line shows the route $\Delta n = \Delta\ell = -1$ .

| n \ $\ell$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 15 | 3 | 11 | 23 | 41 | $66^a$ | 100 | | | | | | | | | |
| 14 | 2 | 6 | 12 | $16^a$ | 18 | 10 | | | | | | | | | |
| 13 | | | 1 | $3^a$ | 1 | | | | | | | | | | |
| 12 | 4 | 19 | 30 | $36^a$ | 33 | 21 | | | | | | | | | |
| 11 | | 3 | 5 | 8 | $22^a$ | 1 | | | | | | | | | |
| 10 | 2 | 18 | 25 | $28^a$ | 29 | 5 | | | | | | | | | |
| 9 | 2 | 23 | 31 | $33^a$ | 21 | 3 | | | | | | | | | |
| 8 | 4 | 28 | $36^a$ | 29 | 11 | 1 | | | | | | | | | |
| 7 | 9 | 33 | $34^a$ | 20 | 4 | | | | | | | | | | |
| 6 | | | | | | | | | | | | | | | |
| 5 | | | | | | | | | | | | | | | |
| 4 | | | | | | | | | | | | | | | |
| 3 | | | | | | | | | | | | | | | |
| 2 | | | | | | | | | | | | | | | |
| 1 | | | | | | | | | | | | | | | |

Fig. 7. Transition paths when two core electrons are present in the mesonic atom and when plane waves are used for describing the band electrons. The initial state into which the meson is assumed captured is $n = 15$, $\ell = 5$. The transition paths are only followed down to $n = 7$. The superscript a indicates the $\ell$ state of the mean meson (with respect to $\ell$ distribution) passing through each n state. The dotted line shows the route $\Delta n = \Delta\ell = -1$.

| n \ $\ell$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 15 | 27 | $100^a$ | | | | | | | | | | | | | |
| 14 | 9 | $22^a$ | 9 | | | | | | | | | | | | |
| 13 | | $1^a$ | | | | | | | | | | | | | |
| 12 | 15 | $67^a$ | 6 | | | | | | | | | | | | |
| 11 | | $8^a$ | 1 | | | | | | | | | | | | |
| 10 | 7 | $43^a$ | 8 | | | | | | | | | | | | |
| 9 | 5 | $39^a$ | 2 | | | | | | | | | | | | |
| 8 | 8 | $28^a$ | 9 | | | | | | | | | | | | |
| 7 | 9 | $20^a$ | 9 | 1 | | | | | | | | | | | |
| 6 | | | | | | | | | | | | | | | |
| 5 | | | | | | | | | | | | | | | |
| 4 | | | | | | | | | | | | | | | |
| 3 | | | | | | | | | | | | | | | |
| 2 | | | | | | | | | | | | | | | |
| 1 | | | | | | | | | | | | | | | |

Fig. 8. Transition paths when two core electrons are present in the mesonic atom and when plane waves are used for describing the band electrons. The initial state into which the meson is assumed captured is $n = 15$, $\ell = 1$. The transition paths are only followed down to $n = 7$. The superscript a indicates the 1 state of the mean meson (with respect to $\ell$ distribution) passing through each n state. The dotted line shows the route $\Delta n = \Delta\ell = -1$.

| n \ $\ell$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 15 | | | | | | | | | | | 1 | 3 | 17 | 67 | $100^a$ |
| 14 | | | | | | | | | | 2 | 7 | 25 | $75^a$ | 83 | |
| 13 | | | | | | | | 1 | 3 | 11 | 35 | $74^a$ | 74 | | |
| 12 | | | | | | | 1 | 4 | 10 | 29 | $67^a$ | 63 | | | |
| 11 | | | | | | 1 | 3 | 7 | 16 | $37^a$ | 43 | | | | |
| 10 | | | | | 2 | 5 | 11 | 23 | $53^a$ | 53 | | | | | |
| 9 | | | | | 2 | 3 | 10 | 22 | $40^a$ | | | | | | |
| 8 | | | | 1 | 5 | 11 | 27 | $56^a$ | | | | | | | |
| 7 | | | 1 | 3 | 10 | 25 | $58^a$ | | | | | | | | |
| 6 | | 1 | 2 | 9 | 25 | $59^a$ | | | | | | | | | |
| 5 | 1 | 2 | 8 | 26 | $60^a$ | | | | | | | | | | |
| 4 | 2 | 7 | 26 | $61^a$ | | | | | | | | | | | |
| 3 | 4 | 25 | $63^a$ | | | | | | | | | | | | |
| 2 | 6 | $64^a$ | | | | | | | | | | | | | |
| 1 | $86^a$ | | | | | | | | | | | | | | |

Fig. 9. Transition paths when one core electron is present in the mesonic atom. The initial state into which the meson is assumed captured is $n = 15$, $\ell = 14$. The superscript a indicates the $\ell$ state of the mean meson (with respect to $\ell$ distribution) passing through each n state. The dotted line shows the route $\Delta n = \Delta\ell = -1$.

| n \ $\ell$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 15 | | | | | | | | | 2 | 15 | $100^a$ | | | | |
| 14 | | | | | | | | 2 | 9 | 33 | $85^a$ | | | | |
| 13 | | | | | | | 2 | 8 | 21 | $47^a$ | 65 | | | | |
| 12 | | | | | | 1 | 5 | 13 | 28 | $46^a$ | 44 | | | | |
| 11 | | | | | 1 | 3 | 7 | 14 | $22^a$ | 28 | 15 | | | | |
| 10 | | | | 1 | 3 | 8 | 14 | 27 | $40^a$ | 31 | | | | | |
| 9 | | | | 1 | 3 | 6 | 13 | $23^a$ | 33 | | | | | | |
| 8 | | | | 4 | 8 | 14 | $30^a$ | 45 | | | | | | | |
| 7 | | | 2 | 7 | 12 | $28^a$ | 49 | | | | | | | | |
| 6 | | 2 | 5 | 12 | 30 | $50^a$ | | | | | | | | | |
| 5 | 2 | 5 | 11 | 30 | $51^a$ | | | | | | | | | | |
| 4 | 4 | 9 | 32 | $52^a$ | | | | | | | | | | | |
| 3 | 6 | 31 | $54^a$ | | | | | | | | | | | | |
| 2 | 8 | $55^a$ | | | | | | | | | | | | | |
| 1 | $80^a$ | | | | | | | | | | | | | | |

Fig. 10. Transition paths when one core electron is present in the mesonic atom. The initial state into which the meson is assumed captured in $n = 15$, $\ell = 10$. The superscript a indicates the $\ell$ state of the mean meson (with respect to $\ell$ distribution) passing through each n state. The dotted line shows the route $\Delta n = \Delta\ell = -1$.

| n \ $\ell$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 15 | | | | 1 | 9 | $100^a$ | | | | | | | | | |
| 14 | | | | 4 | 21 | $91^a$ | | | | | | | | | |
| 13 | | | 1 | 10 | 33 | $78^a$ | | | | | | | | | |
| 12 | | | 3 | 16 | 40 | $63^a$ | | | | | | | | | |
| 11 | | 1 | 6 | 19 | $34^a$ | 36 | | | | | | | | | |
| 10 | 1 | 4 | 13 | 27 | $39^a$ | 25 | | | | | | | | | |
| 9 | | 4 | 14 | $23^a$ | 28 | 13 | | | | | | | | | |
| 8 | 1 | 8 | 24 | $36^a$ | 26 | 6 | | | | | | | | | |
| 7 | 3 | 16 | $34^a$ | 34 | 13 | 1 | | | | | | | | | |
| 6 | 7 | 27 | $36^a$ | 21 | 3 | | | | | | | | | | |
| 5 | 17 | $34^a$ | 29 | 7 | | | | | | | | | | | |
| 4 | 24 | $30^a$ | 14 | 1 | | | | | | | | | | | |
| 3 | $19^a$ | 15 | 3 | | | | | | | | | | | | |
| 2 | $5^a$ | 4 | | | | | | | | | | | | | |
| 1 | 23 | | | | | | | | | | | | | | |

Fig. 11. Transition paths when one core electron is present in the mesonic atom. The initial state into which the meson is assumed captured in $n = 15$, $\ell = 5$. The superscript a indicates the $\ell$ state of the mean meson (with respect to $\ell$ distribution) passing through each n state. The dotted line shows the route $\Delta n = \Delta\ell = -1$.

| n \ $\ell$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 15 | | $100^a$ | | | | | | | | | | | | | |
| 14 | | $100^a$ | | | | | | | | | | | | | |
| 13 | 5 | $100^a$ | | | | | | | | | | | | | |
| 12 | 6 | $95^a$ | | | | | | | | | | | | | |
| 11 | 6 | $73^a$ | | | | | | | | | | | | | |
| 10 | 9 | $72^a$ | | | | | | | | | | | | | |
| 9 | 2 | $62^a$ | | | | | | | | | | | | | |
| 8 | 15 | $53^a$ | 6 | | | | | | | | | | | | |
| 7 | 13 | $35^a$ | 10 | | | | | | | | | | | | |
| 6 | 15 | $20^a$ | 10 | | | | | | | | | | | | |
| 5 | 11 | $11^a$ | 5 | 1 | | | | | | | | | | | |
| 4 | 8 | $5^a$ | 4 | | | | | | | | | | | | |
| 3 | $4^a$ | 4 | | | | | | | | | | | | | |
| 2 | 1 | | | | | | | | | | | | | | |
| 1 | 5 | | | | | | | | | | | | | | |

Fig. 12. Transition paths when one core electron is present in the mesonic atom. The initial state into which the meson is assumed captured in $n = 15$, $\ell = 1$. The superscript a indicates the $\ell$ state of the mean meson (with respect to $\ell$ distribution) passing through each n state. The dotted line shows the route $\Delta n = \Delta\ell = -1$.

transition paths of the mesons, the qualitative results shown in Figures 1 - 12 would probably not be modified.

It is seen that in both cases (one or two core electrons) the general path followed by a $\mu^-$ meson captured in a high $\ell$ state, as it loses energy, can be described as $\Delta n = \Delta \ell = -1$ with a dispersion towards both large and small $\ell$. For a meson captured in a low $\ell$ states there is a tendency for the main decay paths to pass through states of higher $\ell$ than that given by $\Delta n = \Delta \ell = -1$. This general description of mesonic transition paths also applies to the case of no core electron ( at least between $n = 10$ and $n = 7$, the only region important in this problem) as can be seen from Table IX. Thus one arrives at a general result independant of the details of the core electronic replacement transitions.

In addition it should be remarked that the use of the formulas of Section IV along with the tables of matrix elements and energy levels of Section III allow one to calculate the transition rate for the $\mu^-$ meson out of various states $\ell = 0$, and for these transitions it is found $\Delta \ell = 0$ predominates for $n > 8$ while $\Delta \ell = +1$ is dominate for $n < 7$.

## CONCLUSIONS

If a $\mu^-$ meson is captured in lithium, in a high n state, its principal means of losing energy will vary as it makes transitions to lower and lower energy states. Between $n \sim 15$ and $n \sim 10$ Auger transitions involving the band electrons will generally dominate with the transitions $\Delta n = 0$, and $\Delta \ell = -1$ and $\Delta n = 1$, $\Delta \ell = 0$ most prominate, between $n \sim 10$ and $n \sim 4$ Auger transitions involving the core electrons dominate where $\Delta n = -1$, $\Delta \ell = 0$ and $\Delta n = -1$, $\Delta \ell = -1$ are prominate with $\Delta n = -1$, $\Delta \ell = -1$ becoming the most important in the lower end of this range. Below $n \sim 4$ radiation transitions are the most important with $\Delta \ell = n - \ell$, $\Delta \ell = -1$ as the major transition. The most probable transition paths can be described as following, if the meson is captured in a high $\ell$ state, the route $\Delta n = -1$, $\Delta \ell = -1$, with a dispersion towards both large and small $\ell$ For mesons captured in low $\ell$ states there is a tendency for the main routes to pass through states of higher $\ell$ than that given by $\Delta n = -1$, $\Delta \ell = -1$.

What can be said about the transition paths involving $\mu^-$ mesons captured in other light elements $(Z < 6)$ ? As long as the model used in this paper can be considered valid it is evident that there should be no qualitative difference in the transition paths. This is evident upon investigating the transition rate formulas of Section IV. They all are relatively insensitive to changes in nuclear charge. Recognizing, as discussed in Appendix 6 the insensitivity of the transition paths on charges in transition rates one can say that for the range of nuclear

charge $Z < 6$ no major qualitative change in transition paths can be expected using the approximations of this paper.

In the case $\pi^-$ meson capture, because the ratio of the mass of the $\pi^-$ meson and the $\mu^-$ meson is so close to one $\left( \frac{m_\pi}{m_\mu} \sim 1.3 \right)$ and because of the insensitivity of the transition paths to transition rate changes, the same qualitative description for the transition paths holds as for $\mu^-$ meson capture.

Acknowledgment

The author would like to thank Professors M. Ruderman and C. Schwartz for their helpful advice and comments with regard to this paper.

APPENDIX 1

In order to estimate the error involved in neglecting third and higher order terms in the perturbation theory solution for $V_H$ of equation (3.4), applied to $\mu$- mesonic lithium with one or two core electrons, use will be made of a paper by Bates and Carson (1956). In this paper Bates and Carson solve an equation similar to (3.4) by numerical methods maintaining five decimal place accuracy in the energy level determination. Their equation differs from (3.4) as applied to $\mu$- mesonic lithium in that the nucleus is considered to have a charge $Z = 2$ and the $\mu^-$ meson is replaced by a positive charged particle. They restrict their consideration to the case of one core electron. If this equation (Bate's and Carson's equation) is solved by perturbation theory one obtains

$$E(t) = V_H(r) - \frac{4}{r} = -4 - \frac{2}{3}\,\mathcal{E}_{11}(t) + \mathcal{E}_{12}(t)$$

$$- \frac{3}{2}\left[\mathcal{E}_{13}(t) - \frac{3}{2}\,\mathcal{E}_{14}(t) + \left(\frac{3}{2}\right)^2 \mathcal{E}_{15}(t) + \cdots\right], \quad \text{(A.1)}$$

where $\mathcal{E}_{11}$ and $\mathcal{E}_{12}$ are given by (3.61) and (3.62) while the expressions of the form $\mathcal{E}_{1n}$ in the parenthesis of (A.1) are <u>n</u>th order corrections to (3.60). Using the values obtained by Bates and Carson for $E$, the expression

$$\overline{\mathcal{E}}_{13}(t) = \mathcal{E}_{13}(t) - \frac{3}{2}\,\mathcal{E}_{14}(t) + \left(\frac{3}{2}\right)^2 \mathcal{E}_{15}(t) + \cdots \quad \text{(A.2)}$$

can be estimated (see Table A.I). $\overline{\mathcal{E}}_{13}$ can be regarded as an estimate of the order of magnitude of $\mathcal{E}_{13}$.

In the case of two core electrons $\mathcal{E}_{23}$, the third order correction to (3.58), would be approximately twice as large as $\mathcal{E}_{13}$ since contributions due to electron-electron correlation are quite a bit smaller than electron-meson interaction contributions.

Table A.I. $\mathcal{E}_{13}$ as obtained from the data of Bates and Carson. All values are in atomic units except where indicated.

| t | $E(t)$ | $\mathcal{E}_{11}(t)$ | $\mathcal{E}_{12}(t)$ | $\bar{\mathcal{E}}_{13}(t)$ | $\mathcal{E}_{11}$(eV) | $\mathcal{E}_{12}$(eV) | $\bar{\mathcal{E}}_{13}$(eV) |
|---|---|---|---|---|---|---|---|
| 0 | -9.00000 | 6.00000 | -1.0000 | 0.000 | 81.5 | -13.6 | 0.0 |
| 0.5 | -8.26730 | 5.37817 | -0.6965 | -0.009 | 73.1 | - 9.5 | -0.1 |
| 1.0 | -7.33109 | 4.37598 | -0.3986 | 0.010 | 59.5 | - 5.4 | 0.1 |
| 1.5 | -6.60274 | 3.50213 | -0.2340 | 0.023 | 47.6 | - 3.2 | 0.3 |
| 2.0 | -6.06671 | 2.83516 | -0.1405 | 0.026 | 38.6 | - 1.9 | 0.4 |
| 3.0 | -5.39092 | 1.98017 | -0.0501 | 0.014 | 26.9 | - 0.7 | 0.2 |
| 4.0 | -5.02439 | 1.49748 | -0.0187 | 0.005 | 20.4 | - 0.3 | 0.1 |

APPENDIX 2

Introducing the variable $s$ where

$$s^2 = t \tag{A.3}$$

equation (3.94) can be put in the form

$$\frac{dX(s)}{ds} = \left[2 s \epsilon - 2s v(s)\right] Y(s) - \frac{4(\ell + 1)}{s} X(s) , \tag{A.4}$$

$$\frac{dY(s)}{ds} = 2 s X(s) , \tag{A.5}$$

with

$$Y(s) = t^{-(\ell+1)} r R_{n\ell}(r) . \tag{A.6}$$

In the case of two core electrons

$$v(s) = \frac{\bar{\mu}}{Z'^2} V_H(r) , \qquad \epsilon = \frac{\bar{\mu}}{Z'^2} \mathcal{E}_{n,\ell} , \tag{A.7}$$

while in the case of one or no core electron

$$v(s) = -\frac{\bar{\mu}}{Z^2} V_H(r) , \qquad \epsilon = \frac{\bar{\mu}}{Z^2} \mathcal{E}_{n,\ell} . \tag{A.8}$$

In solving (A.4) and (A.5) the initial values of $X(s)$ and $Y(s)$ were obtained through a power series expansion in $s$, and then the equation was integrated on the IBM-704 using the fourth order Runge-Kutta-Gill method of numerical integration. Because of the form of equation (A.4) the mesh spacing used in the numerical integration had to be varied in order to prevent loss of significant figures due to cancellation. By a method of trial and error combined

with graphical interpolation and extrapolation it was found that usually after only four tries the value of $\epsilon$ could be obtained to 0.01 . The accuracy was judged through seeing how close, at a significantly distant point the asymptotic condition of the true solution

$$X(s) + \sqrt{\epsilon}\ Y(s) = 0 \tag{A.9}$$

was satisfied.

In order to test the accuracy of the numerical method itself in approximating the differential equation the Runge-Kutta-Gill method was applied to the case of a pure coulomb field. Table A.II contains the results of this application and it is easy to see that the accuracy obtained for $\epsilon$ is within 0.01 . Thus Tables I and II contain values of $\mathcal{E}_{n\ell}$ which as eigensolutions of (3.94) can be considered as accurate to .01 eV.

The needed values of $\overline{R^n}$ were obtained by the IBM-704 using Simpson's rule. In order to test the accuracy of the method some values for $\overline{R}$, when the potential was a pure coulomb potential, were obtained through use of the IBM-704 and compared with the known values. See Table A.III . The accuracy seems sufficient for the purposes of this problem.

Table A.II. Comparison of the values obtained through numerical integration with the known energies of certain states. Energies are in atomic units.

| n | $\ell$ | $\mathcal{E}_M$ | $\mathcal{E}_M$(numerical integration) | n | $\ell$ | $\mathcal{E}_M$ | $\mathcal{E}_M$(numerical integration) |
|---|---|---|---|---|---|---|---|
| 15 | 14 | 184.00 | 184.00 | 6 | 5 | 1150.00 | 1150.00 |
| 15 | 0 | 184.00 | 184.00 | 6 | 0 | 1150.00 | 1150.00 |
| 11 | 10 | 342.15 | 342.15 | 5 | 4 | 1656.00 | 1656.00 |
| 11 | 0 | 342.15 | 342.14 | 5 | 0 | 1656.00 | 1656.00 |
| 10 | 9 | 414.00 | 414.00 | 2 | 1 | 10350.01 | 10350.01 |
| 10 | 0 | 414.00 | 414.00 | 2 | 0 | 10350.01 | 10350.02 |

Table A.III. Comparison of the values obtained through numerical integration with the known values of certain matrix elements. Matrix elements are in atomic units.

| Initial state | | Final state | | $\bar{R}$ | $\bar{R}$ (numerical integration) |
|---|---|---|---|---|---|
| n = 15 | $\ell$ = 14 | n = 15 | $\ell$ = 14 | 0.19850 | 0.19849 |
| n = 15 | $\ell$ = 14 | n = 14 | $\ell$ = 13 | 0.33207 | 0.33180 |

# APPENDIX 3

The question of whether the dipole approximation is to be used in the case of radiation transitions hinges on whether

$$(\vec{k} \cdot \vec{r})_{fi} \ll 1 \quad , \tag{A.10}$$

where $(\vec{k} \cdot \vec{r})_{fi}$ is the matrix element of the wave vector of the outgoing photon times the position vector, with origin at the nucleus, taken between the initial and final meson states. This value can be estimated as follows

$$k = \frac{\alpha}{2} \Delta E \quad , \tag{A.11}$$

$$\Delta E \approx \mu Z_m^2 \left( \frac{1}{n_f^2} - \frac{1}{n_i^2} \right) < \frac{\mu Z_m^2}{n_f^2} \quad , \tag{A.12}$$

$$(r)_{fi} \approx \frac{n_f^2}{\mu Z_m} \quad , \tag{A.13}$$

thus

$$(\vec{k} \cdot \vec{r})_{fi} \approx \frac{n_f^2}{\mu Z_m} \frac{\alpha}{2} \Delta E < \frac{\alpha}{2} Z_m \ll 1 \quad . \tag{A.14}$$

# APPENDIX 4

In order to estimate the transition rates of Auger transitions involving $\Delta\ell = 0, \pm 1, \pm 2$, a model consisting of a $\mu^-$ meson and an electron surrounding the lithium nucleus will be assumed. This was the model, in conjunction with further simplifications to be discussed, used in deriving (4.43) and (4.45). The potential will then be

$$V(r_e, r) = -\frac{2Z}{r_e} - \frac{2Z}{r} + \frac{2}{|r - r_e|} \quad . \tag{A.15}$$

Defining

$$V_0(r_e, r) = -\frac{2Z_e}{r_e} - \frac{2Z_m}{r} \quad , \tag{A.16}$$

$$V_1(r_e, r) = \frac{2}{|r - r_e|} - \frac{2(Z - Z_e)}{r_e} - \frac{2(Z - Z_m)}{r} \quad , \tag{A.17}$$

it follows

$$V(r_e, r) = V_0(r_e, r) + V_1(r_e, r) \quad , \tag{A.18}$$

where $V_1$ is to be treated as a perturbation. Because of the orthogonality relations among electron and meson states associated with the potential $V_o$, in the following calculations for transition rates only the first term of (A.17) gives any contribution.

It follows from first order perturbation theory that

$$w_{if} = \frac{\pi V}{(2\pi)^3 \hbar} \int H_{fi}^2 \, k \, d\Omega_k \tag{A.19}$$

where

$$H_{fi} = 8\pi \sum_{\ell,m} \frac{\Omega_{fi}^{\ell m}}{2\ell + 1} I_{\ell m} , \tag{A.20}$$

and

$$I_{\ell m} = \overline{R^{\ell}} \int \psi_f^* (r_e) \frac{Y_{\ell m}^* (\theta_e, \varphi_e)}{r_e^{\ell+1}} \psi_i (r_e) d^3 r_e$$

$$+ \int_0^{\infty} R_f(r) r^{\ell} \left[ \int_0^{4\pi} \int_0^{r} \left[ \frac{r_e^{2\ell+1}}{r^{2\ell+1}} - 1 \right] \psi_{e_f}(r_e) \frac{Y_{\ell m}^*(\theta_e, \varphi_e)}{r_e^{\ell+1}} \psi_{e_i}(r_e) \right.$$

$$\left. \times \, d\Omega_e \, r_e^2 \, d r_e \right] R_i(r) \, r^2 \, d r , \tag{A.21}$$

$$\overline{R^{\ell}} = \int R_f (r) \, r^{\ell} \, R_i(r) \, r^2 \, d r . \tag{A.22}$$

Making use of the fact that

$$\psi_i(r_e) = \frac{Z_e^{3/2}}{\pi^{1/2}} e^{-Z_e r_e} , \tag{A.23}$$

$$\psi_f(r_e) = \frac{4\pi}{\sqrt{V}} \sum_{\ell,m} i^{\ell} j_{\ell}(k r_e) Y_{\ell m}^* (\theta_e, \varphi_e) Y_{\ell m} (\theta_k , \varphi_k) , \tag{A.24}$$

with

$$j_{\ell}(k r_e) = \frac{2^{\ell} e^{\pi/2a} |\Gamma(\ell + 1 - i/a)|}{(2\ell + 1)!} (k r_e)^{\ell} e^{-ikr_e}$$

$$\times \, F(\ell + 1 + i/a , 2\ell + 2 , 2i k r_e) \tag{A.25}$$

$$a = k/Z_e , \tag{A.26}$$

one arrives at

$$I_{\ell m} = \frac{4\pi}{\sqrt{V}} \; (-i)^{\ell} \left(\frac{Z_e^{\,3}}{\pi}\right)^{1/2} \; {}^{\ell}R_{fi} \; Y^{*}_{\ell m}(\theta_k, \varphi_k) , \tag{A.27}$$

with

$$^{\ell}R_{fi} = \int_0^{\infty} R_f(r) r^{\ell} \left[ \int_0^{\infty} r_e^{1-\ell} \, j_{\ell}(k\, r_e) e^{-Z_e r_e} dr_e + \int_0^{r} r_e^{1-\ell} \right.$$

$$\left. \times \left[ \frac{r_e^{2\ell+1}}{r^{2\ell+1}} - 1 \right] j_{\ell}(k\, r_e)\, e^{-Z_e r_e} \, d\, r_e \right] R_i(r)\, r^2 \, d\, r . \tag{A.28}$$

Thus

$$w_{if} = \sum_{\ell} w^{\ell}_{if} , \tag{A.29}$$

where

$$w^{\ell}_{if} = \frac{128\,\pi}{h} \; Z_e^{\,4} \; a \sum_{m} \frac{\Omega^{\ell m^2}_{fi}}{(2\ell + 1)^2} \; {}^{\ell}R^2_{fi} . \tag{A.30}$$

Explicitly

$$^{0}R_{fi} = \int_0^{\infty} R_f(r) \left[ \int_0^{r} \left[ \frac{r_e^{\,2}}{r} - r_e \right] j_o(k\, r_e)\, e^{-Z_e r_e} \, dr_e \right] R_i(r) r^2 \, d\, r , \tag{A.31}$$

$$^{1}R_{fi} = \int_0^{\infty} R_f(r) \left[ \int_0^{\infty} j_1(k\, r_e) e^{-Z_e r_e} \, dr_e + \int_0^{r} \left[ \frac{r_e^{\,3}}{r^3} - 1 \right] \right.$$

$$\left. \times \; j_1(k\, r_e) e^{-Z_e r_e} \, d\, r_e \right] R_i(r)\, r^2 \, d\, r , \tag{A.32}$$

$$^{2}R_{fi} = \int_0^{\infty} R_f(r) r^2 \left[ \int_0^{\infty} r_e^{-1} \, j_2(k\, r_e) e^{-Z_e r_e} \, d\, r_e \right] R_i(r) r^2 \, d\, r . \tag{A.33}$$

Since the core electrons are on the average, at least when Auger transitions of this problem are energetically possible, much further from the center of the mesonic atom than the $\mu^-$ meson, the second term in (A.28) has been neglected in (A.33). This should not change the order of magnitude of the results of this calculation.

Thus as regards to order of magnitude

$$ {}^1R_{fi} = \left(\frac{2\pi}{a}\right)^{1/2} (1 + a^2)^{-1/2} (1 - e^{-2\pi/a})^{-1/2} \frac{\overline{R}}{Z_e} e^{-\frac{2}{a}\tan^{-1}a} , \quad (A.34) $$

$$ {}^2R_{fi} = \left(\frac{2\pi}{a}\right)^{1/2} \left(\frac{1 + a^2}{1 + 4a^2}\right)^{1/2} (1 - e^{-2\pi/a})^{-1/2} \frac{3}{4} \overline{R^2} \left(e^{-\frac{2}{a}\tan^{-1}a} - \frac{1}{9}\right) , \quad (A.35) $$

so that

$$ \frac{w^2}{w^1} = L(\ell_i, \ell_f)\, O(a) \frac{\overline{Z_e^2 R^2}^{\,2}}{\overline{Z_e R}^{\,2}} , \quad (A.36) $$

with

$$ L(\ell_i, \ell_f) = \frac{2\ell_i - 2}{2\ell_i - 1} \qquad \Delta\ell_2 = -2 , \quad (A.37) $$

$$ L(\ell_i, \ell_f) = \frac{(2\ell_i + 4)(\ell_i + 1)}{(2\ell_i + 3)\ \ell_i} \qquad \Delta\ell_2 = 2 , \quad (A.38) $$

$$ L(\ell_i, \ell_f) = \frac{(2\ell_i + 1)}{2\ell_i} \left[ \frac{2\ell_i(2\ell_i + 2)}{{}_i - 1)^2 (2\ell_i + 3)^2} \left[ \frac{8}{3}\ell_i(\ell_i + 1) - \frac{5}{2} \right] \right] \quad (A.39) $$

$$ \Delta\ell_2 = 0 , $$

and

$$O(a) = \frac{81}{320} \frac{(1 + a^2)^2}{1 + 4a^2} \left(1 - \frac{1}{9} e^{\frac{2}{a} \tan^{-1} a}\right)^2 , \tag{A.40}$$

where $w_1$ and $\overline{Z_e R}^2$ are connected with the transitions $\Delta\ell_1 = -1$, while $w_2$ and $\overline{Z_e^2 R^2}^2$ are connected with the transitions $\Delta\ell_2 = 0, \pm 2$ .

Table A.IV presents $O(a)$ for various values of $\Delta E$ . It is immediately clear that the transitions smaller than those involving $\Delta\ell = \pm 1$ and thus, assuming the model of this appendix sufficiently accurate, can be neglected in a first approximation. The error should usually be less than a few percent.

In order to estimate the needed corrections to formulas (4.43) and (4.45) due to the neglect of higher powers of $r$ in their matrix elements, it is to be noted expressions (A.31) and (A.32) can be written in the form

$$^0R_{fi} = -\left(\frac{2\pi}{a}\right)^{1/2} (1 - e^{-2\pi/a})^{-1/2} \frac{1}{6} \int R_f(r)\, r^2 \left[1 - K_0(r)\right] R_i(r) r^2\, dr , \tag{A.41}$$

$$^1R_{fi} = \left(\frac{2\pi}{a}\right)^{1/2} (1 - e^{-2\pi/a})^{-1/2} (1 + a^2)^{-1/2} \frac{e^{-\frac{2}{a}\tan^{-1} a}}{Z_e}$$

$$\times \int R_f(r)\, r \left[1 - K_1(r)\right] R_i(r) r^2\, d r , \tag{A.42}$$

with

$$K_0 = \left(\frac{a}{2\pi}\right)^{1/2} (1 - e^{-2\pi/a})^{1/2} \left[1 + 6 \int_0^r \left[\frac{r_e^2}{r^3} - \frac{r_e}{r^2}\right] j_0(k\, r_e)\right.$$

$$\left. \times\, e^{-Z_e r_e}\, d\, r_e\right] , \tag{A.43}$$

Table A.IV. Values of $O(a)$.

| $\Delta E$(eV) | $O(a)$ |
|---|---|
| 10 | 0.01 |
| 25 | 0.04 |
| 50 | 0.08 |
| 500 | 0.38 |
| 1000 | 0.80 |

Table A.V. The dependence of $K_0$ and $K_1$ on $\Delta E$ and t. The values of t are in atomic units while those of $\Delta E$ are in eV.

| | $K_0$ | | | $K_1$ | | |
|---|---|---|---|---|---|---|
| t \ $\Delta E$ | 10 | 100 | 1000 | 10 | 100 | 1000 |
| 0.1 | 0.09 | 0.09 | 0.09 | 0.01 | 0.01 | 0.03 |
| 0.2 | | | 0.18 | | | 0.12 |
| 0.4 | | 0.33 | | | 0.13 | |
| 0.5 | 0.38 | | | 0.13 | | |
| 0.7 | | 0.51 | | | 0.32 | |
| 1.0 | 0.60 | | | 0.35 | | |

$$K_1(r) = \left(\frac{a}{2\pi}\right)^{1/2} \left(1 - e^{-2\pi/a}\right)^{1/2} \left(1 + a^2\right)^{1/2} e^{\frac{2}{a}\tan^{-1} a} Z_e$$

$$\times \int_0^r \left[1 - \frac{r_e^3}{r^3}\right] j_1(k r_e)\, e^{-Z r_e}\, d r_e \,. \tag{A.44}$$

In the case where $k r < 1$ and $Z r < 1$ the integrals in (A.43) and (A.44) can be evaluated as a power series. This is no restriction in this problem as

$$k \lesssim \sqrt{\Delta E} \quad , \tag{A.45}$$

$$\Delta E \lesssim \frac{\mu Z_m^2}{n_f^2} \quad , \tag{A.46}$$

$$r \lesssim \frac{n_f^2}{\mu Z_m} \quad . \tag{A.47}$$

Thus

$$k r \lesssim \left(\frac{n_f^2}{\mu}\right)^{1/2} \leqslant 1 \quad , \tag{A.48}$$

$$Z_e r \lesssim Z_m r \leqslant \frac{n_f^2}{\mu} \leqslant 1 \,. \tag{A.49}$$

For the case $k r < 1$, $Z_e r < 1$,

$$K_0(r) = t \left[1 - \frac{1}{20}(11 + a^2)t + \frac{1}{90}(19 - 5a^2)t^2 - \frac{1}{2520}(157 - 80a^2 + 3a^4)\, t^3 + \cdots\right] , \tag{A.50}$$

$$K_1(r) = \frac{(1 + a^2)}{10}\, e^{\frac{2}{a}\tan^{-1} a}\, t^2 \left[1 - \frac{5}{6} t + \frac{1}{28}(11 - a^2)\, t^2 - \frac{1}{144}(19 - 5a^2)\, t^3 + \frac{1}{4536}(157 - 80a^2 + 3a^4)\, t^4 + \cdots\right] \tag{A.51}$$

where

$$t = Z_e r \tag{A.52}$$

Formulas (4.43) and (4.45) were derived assuming that the values of $K_1$ and $K_0$ are negligible compared to one, at least for values of $r$ for which the mesonic wave functions give significant contributions to the integrals in (A. 41) and (A. 42). Table A.V contains $K_0$ and $K_1$ for various values of $\Delta E$ .

In order to estimate the correction which should be applied to (4.43) and (4.45) in order to take into account the neglected factors $K_0(r)$ and $K_1(r)$, this very crude procedure might be followed. Assume the principal contribution to the integrals (A.41) and (A.42) comes from a region very near $t = Z_e r = \frac{n}{\mu}$, where $n$ is the smaller of $n_i$ and $n_f$, for this value of $t$ estimate $K_0(t)$ and $K_1(t)$ from Table A.V, then assume the corrections to (4.43) and (4.45) are $C_1 = (1 - K_1(t))^2$ and $C_0 = (1 - K_0(t))^2$ respectively. Values for $C_1$ and $C_0$ obtained in this way are given in Table A.VI . Presumably $C_1$ can also be considered as an estimate of corrections which should be applied to (4.39) and (4.41) .

Table A.VI. The values of $C_0$ and $C_1$ for different meson states. Atomic units are used for $t$.

| n | t | $K_0$ | $K_1$ | $C_0$ | $C_1$ |
|---|---|---|---|---|---|
| 15 | 1.10 | 0.62 | 0.40 | 0.14 | 0.36 |
| 14 | 0.96 | 0.58 | 0.34 | 0.18 | 0.44 |
| 13 | 0.83 | 0.53 | 0.28 | 0.22 | 0.52 |
| 12 | 0.71 | 0.48 | 0.25 | 0.27 | 0.56 |
| 11 | 0.59 | 0.43 | 0.20 | 0.32 | 0.64 |
| 10 | 0.49 | 0.38 | 0.15 | 0.38 | 0.72 |
| 9 | 0.40 | 0.32 | 0.12 | 0.46 | 0.77 |
| 8 | 0.31 | 0.25 | 0.09 | 0.56 | 0.83 |
| 7 | 0.24 | 0.20 | 0.06 | 0.64 | 0.88 |
| 6 | 0.18 | 0.15 | 0.04 | 0.72 | 0.92 |
| 5 | 0.12 | 0.10 | 0.02 | 0.81 | 0.96 |
| 4 | 0.08 | 0.07 | 0.02 | 0.86 | 0.96 |
| 3 | 0.04 | 0.04 | 0.01 | 0.92 | 0.98 |
| 2 | 0.02 | 0.02 | 0.01 | 0.96 | 0.98 |
| 1 | 0.00 | 0.00 | 0.00 | 1.00 | 1.00 |

APPENDIX 5

It is to be shown that the terms $\frac{a(k)}{\sqrt{V}} \sum_{n=1}^{\infty} e^{i\vec{k}\cdot\vec{R}_n} \varphi_{1s}(\vec{r}_e - \vec{R}_n)$

in the wave functions

$$\psi(r_e) = \left[1 - \frac{a(k)^2}{\Omega}\right]^{-1/2} \frac{1}{\sqrt{V}} \left[e^{i\vec{k}\cdot\vec{r}_e} - a(k) \sum_{n=0}^{\infty} e^{i\vec{k}\cdot\vec{R}_n} \varphi_{1s}(\vec{r}_e - \vec{R}_n)\right] \tag{A.53}$$

give a negligible contribution to the integral

$$M_{1m} = \int \psi_f^*(r_e) \frac{Y_{1m}^*(\theta_e, \varphi_e)}{r_e^2} \psi_i(r_e)\, d^3 r_e \tag{A.54}$$

compared to the contribution of other terms and thus can be, to a good approximation, neglected. It follows from (A. 53) and (A.54)

$$M_{1m} = \left[1 - \frac{a(k_i)^2}{\Omega}\right]^{-1/2} \left[1 - \frac{a(k_f)^2}{\Omega}\right]^{-1/2} \frac{1}{V} \left\{ \int e^{i(\vec{k}_i - \vec{k}_f)\cdot\vec{r}_e} \right.$$

$$\times \frac{Y_{1m}(\theta_e, \varphi_e)}{r_e^2} d^3 r_e - \sum_{n=0}^{\infty} e^{i(\vec{k}_i - \vec{k}_f)\cdot\vec{R}_n} \int \left[a(k_i)\, e^{-i\vec{k}_f\cdot\vec{\rho}} \varphi_{1s}(\rho)\right.$$

$$\left. + a^*(k_f) e^{i\vec{k}_i\cdot\vec{\rho}} \varphi_{1s}(\rho) - a^*(k_f) a(k_i) \varphi_{1s}^2(\rho)\right] \frac{Y_{1m}(\theta_e, \varphi_e)}{|\vec{\rho} + \vec{R}_n|^2} d^3 \rho\,, \tag{A.55}$$

where

$$\vec{\rho} = \vec{r}_e - \vec{R}_n \quad . \tag{A.56}$$

Since from (3.100)

$$a(k) = \int e^{i\vec{k}\cdot\vec{r}} \varphi_{1s}(r)\, d^3 r \quad , \tag{A.57}$$

and because for terms $n \geqslant 1$, $Y_{1m}(\theta_e, \varphi_e)$ and $|\vec{\rho} + \vec{R}_n| \sim \vec{R}_n$ can be considered constant under the integral, the contribution to (A.54) of terms of the form

$$\frac{a(k)}{\sqrt{V}} \sum_{n=1}^{\infty} e^{i\vec{k}\cdot\vec{R}_n} \varphi_{1s}(\vec{r}_e - \vec{R}_n)$$

can be estimated as

$$\delta M_B = \left[1 - \frac{a(k_i)^2}{\Omega}\right]^{-1/2} \left[1 - \frac{a(k_f)^2}{\Omega}\right]^{-1/2} \frac{1}{V} \Big[- a(k_i)a^*(k_f)$$

$$+ a^*(k_f)a(k_i) - a^*(k_f)a(k_i)\Big] \sum_{n=1}^{\infty} \frac{e^{i(\vec{k}_i - \vec{k}_f)\cdot\vec{R}_n}}{R_n^{\,2}} Y_{1m}(\theta_n, \varphi_n) \, , \tag{A.58}$$

or approximately

$$\delta M_B = - \left[1 - \frac{a(k_i)^2}{\Omega}\right]^{-1/2} \left[1 - \frac{a(k_f)^2}{\Omega}\right]^{-1/2} \frac{a(k_i)\, a^*(k_f)}{\Omega V}$$

$$\times \int_d^{\infty} e^{i(\vec{k}_i - \vec{k}_f)\cdot\vec{R}} Y_{1m}(\theta, \varphi)\, d R\, d \Omega \quad , \tag{A.59}$$

where $d$ is the distance between neighboring atomic centers in the solid. The contribution due to the first term in (A.55) is

$$\mathcal{E}M_A = \left[1 - \frac{a(k_i)^2}{\Omega}\right]^{-1/2} \left[1 - \frac{a(k_f)^2}{\Omega}\right]^{-1/2} \frac{1}{V}$$

$$\times \int_0^\infty e^{i(\vec{k}_i - \vec{k}_f)\cdot\vec{r}_e} \; Y_{lm}(\theta_e, \varphi_e) \; d\, r_e \; d\, \Omega_e \quad . \tag{A.60}$$

Thus

$$\frac{\mathcal{E}M_B}{\mathcal{E}M_A} = \frac{a^*(k_f)\; a(k_i)}{\Omega} \quad \frac{\int_d^\infty e^{i\vec{K}\cdot\vec{r}_e} \; Y_{lm}(\theta_e, \varphi_e) \; dr_e d\Omega e}{\int_0^\infty e^{i\vec{K}.\vec{R}} \; Y_{lm}(\theta, \varphi\;) \; dRd\Omega} \quad , \tag{A.61}$$

where

$$\vec{K} = \vec{k}_i - \vec{k}_f \quad , \tag{A.62}$$

In evaluating (A.61) it is found

$$\frac{\mathcal{E}M_B}{\mathcal{E}M_A} = \frac{a^*(k_f)\; a(k_i)}{\Omega} \quad j_o(K\, d) \quad , \tag{A.63}$$

and since

$$a(k) \leqslant 8 \left(\frac{\pi}{Z_e^3}\right)^{1/2} \quad , \tag{A.64}$$

$$j_o(K\, d) \leqslant 1 \quad , \tag{A.65}$$

it follows

$$\frac{\mathcal{E}M_B}{\mathcal{E}M_A} \leqslant \frac{64\pi}{\Omega\, Z_e^3} \quad . \tag{A.66}$$

Thus for lithium

$$\frac{\delta M_B}{\delta M_A} \leq .07 . \tag{A.67}$$

# APPENDIX 6

In order to get an idea of the sensitivity of the calculated transition paths to errors in the calculated transition rates of Section V, corrections will be applied to these rates.

The corrections will be an attempt to take into account the factors $K_1$ and $K_0$, discussed in Appendix 5, which were neglected in computing Tables VI-IX. Using the factors $C_0$ and $C_1$ as given in Appendix 5, revised transition rates are computed for the case of two core electrons in the mesonic atom and using orthogonalized plane waves as the wave functions of the band electrons. Because of the relative magnitudes of $C_0$ and $C_1$ as given in Table A.VI these corrections will increase the frequency of the sliding transitions $\Delta n = 0$, $\Delta \ell = -1$. Figure A.1 shows the revised transition paths if $n_i = 15$ $\ell_i = 10$. It does not differ qualitatively from that of Figure 2 where these corrections were not taken into account.

| n \ $\ell$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 15 | | | | | | | 1 | 4 | 13 | 38 | $100^a$ | | | | |
| 14 | | | | | 1 | 2 | 5 | 12 | 26 | $48^a$ | 62 | | | | |
| 13 | | | | 1 | 3 | 4 | 9 | 16 | $25^a$ | 31 | 25 | | | | |
| 12 | | | 1 | 3 | 5 | 7 | 12 | 20 | $30^a$ | 29 | 11 | | | | |
| 11 | | | 1 | 2 | 3 | 5 | 9 | 16 | $22^a$ | 18 | 1 | | | | |
| 10 | | | 2 | 4 | 5 | 10 | 21 | $31^a$ | 28 | 7 | | | | | |
| 9 | | | 5 | 5 | 7 | 18 | $27^a$ | 29 | 11 | | | | | | |
| 8 | | 2 | 6 | 8 | 17 | $26^a$ | 33 | 14 | | | | | | | |
| 7 | 1 | 4 | 8 | 16 | $23^a$ | 33 | 18 | | | | | | | | |
| 6 | 2 | 7 | 18 | 23 | $36^a$ | 18 | | | | | | | | | |
| 5 | 4 | 17 | 24 | $38^a$ | 18 | | | | | | | | | | |
| 4 | 13 | 24 | $39^a$ | 19 | | | | | | | | | | | |
| 3 | 13 | $36^a$ | 21 | | | | | | | | | | | | |
| 2 | 11 | $22^a$ | | | | | | | | | | | | | |
| 1 | $55^a$ | | | | | | | | | | | | | | |

Fig. A.1. Transition paths when two core electrons are present in the mesonic atom. The initial state into which the meson is assumed captured is $n = 15$, $\ell = 10$. The superscript a indicates the $\ell$ state of the mean meson (with respect to $\ell$ distribution) passing through each n state. The dotted line shows the route $\Delta n = \Delta\ell = -1$.

## REFERENCES


1. Bates, D. R., and Carson, T. R., Proc. Roy. Soc. A, 234, 419 (1956).
2. Broyles, Thomas, and Haynes, Phys. Rev. 89, 715 (1953).
3. Burbridge, G. R., and de Brode, A. H., Phys. Rev 89, 189 (1953) .
4. Burhop, E. H. S., The Auger Effect, Cambridge Univ. Press (1952).
5. Cosyns, Dilworth, Occhialini, Schoenberg, and Page, Proc. Phys. Soc. A, 62, 801 (1949).
6. Dalgarno, A., and Lewis, J. T., Proc Roy. Soc. A, 233 70 (1955).
7. Dalgarno, A., and Lynn, N., Proc. Phys. Soc. A, 70, 223 (1957).
8. De Brode, A. H., Proc. Phys. Soc. A., 67, 55 (1954).
9. Eisenberg, V., and Kessler, D., Il Nuovo Cimento 19, 1195 (1961).
10. Fry, W. F., Phys Rev. 83, 594 (1951).
11. Fry, W. F., Nuovo Cimento 10, 490 (1953).
12. Mann, R. A., and Rose, M. E., Phys, Rev., 121, 293 (1961).
13. Pevsner, Strand, Madansky, and Toohig, Il Nuovo Cimento 19, 409 (1961).
14. Ruderman, M. A., Phys. Rev., 118, 1932 (1960).
15. Sterns, M. B., and Sterns, M., Phys. Rev. 105, 1573 (1957).
16. Woodruff, T. O., Solid State Phys. 4, 367 (1957).

This report was prepared as an account of Government sponsored work. Neither the United States, nor the Commission, nor any person acting on behalf of the Commission:

A. Makes any warranty or representation, expressed or implied, with respect to the accuracy, completeness, or usefulness of the information contained in this report, or that the use of any information, apparatus, method, or process disclosed in this report may not infringe privately owned rights; or

B. Assumes any liabilities with respect to the use of, or for damages resulting from the use of any information, apparatus, method, or process disclosed in this report.

As used in the above, "person acting on behalf of the Commission" includes any employee or contractor of the Commission, or employee of such contractor, to the extent that such employee or contractor of the Commission, or employee of such contractor prepares, disseminates, or provides access to, any information pursuant to his employment or contract with the Commission, or his employment with such contractor.