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ABSTRACT

The partial-wave amplitudes are shown to have singularities along the left-hand cut, for both integral and nonintegral angular momenta, at positions where there are peaks in the third double spectral function. The nature of these singulariites and their connection with the Gribov-Pomeranchuk essential singularities in the angular-momentum plane are discussed.

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I. INTRODUCTION

The purpose of this note is to discuss the singularities produced in the energy variable of partial-wave amplitudes by the presence of a third double spectral function, and to clarify the connection between these singularities and the Gribov-Pomeranchuk singularities[1] that occur in the complex angular-momentum variable at certain negative integers. In particular, we shall be interested in those singularities that occur in the partial-wave amplitudes because of the peculiar location of the third double spectral function with respect to the direct channel. The first such singularity occurs in the partial waves at $s = s_1$ (see Fig. 1), where s is the total energy squared in the center of mass, and generates a left-hand cut that runs from $s = s_1$ to $s = -\infty$. Similar type singularities occur whenever a singular surface of the third double spectral function peaks at some value of s. (See Fig. 1)

These singularities do not occur in the full physical amplitude (at a fixed value of s) because the singularity in this case depends upon t, the crossed variable, and simply traces out the boundary of the double spectral function ρ. Moreover, these points have no connection with

physical thresholds but are essentially kinematical in nature. The importance of these singularities results from their implications for the case of complex angular momentum. Gribov and Pomeranchuk have demonstrated that the discontinuity across the singularity beginning at s_1 in the partial-wave amplitude becomes infinite when the angular momentum ℓ approaches negative integer values and, in particular, when $\ell \to -1$. This results in an accumulation of an infinite number of Regge poles at these ℓ values.[1]

We shall show that the singularity at s_1 persists even for physical ℓ values; however, in this case the discontinuity across the associated cut is finite. We shall exhibit formulas for the discontinuity across these cuts for both physical and nonphysical ℓ values. We also discuss the singularity type at $s = 0$ and $s = s_1$.

Many of the statements made herein are certainly contained implicitly in the literature on partial-wave dispersion relations. However, the points raised appear to warrant explicit discussion and clarification.

II. DEFINITION OF PARTIAL-WAVE AMPLITUDES FOR PHYSICAL AND NONPHYSICAL ANGULAR MOMENTUM

Because we are interested only in the role the third double spectral function is to play in this problem, we define an amplitude $A_3(s, t, u)$ by

$$A_3(s, t, u) = \iint dt'\, du' \, \frac{\rho_{tu}(t', u')}{(t' - t)(u' - u)} , \qquad (1)$$

where the region of integration is determined by the boundary of the double

spectral function shown in Fig. 1. For convenience, we consider the equal-mass case, so that we have

$$
\begin{aligned}
s &= 4(\nu + 1) \quad , \\
t &= -2\nu(1 - z) \, , \\
\text{and} \quad u &= -2\nu(1 + z) \\
\text{with} \quad \nu &= (s - 4)/4 \quad .
\end{aligned}
\tag{2}
$$

In the direct (or s) channel, z is the center-of-mass scattering angle.

The partial-wave projection of (1) gives

$$
A_j(s, \ell) = \int_{s_1}^{-\infty} \frac{ds'}{s' - s} \int_{t_L(s')}^{t_R(s')} \frac{dt'}{2\nu} \, Q_\ell\left(1 + \frac{t'}{2\nu}\right)
$$

$$
\times \left\{ \rho_{tu}(t', 4 - s' - t') + (-1)^\ell \, \rho_{tu}(4 - s' - t', t') \right\} , \tag{3}
$$

where the integration is carried out over the third double-spectral function (see Fig. 1).

For complex ℓ values we use the Froissart-Gribov[2] amplitudes $B_j^{\pm}(s, \ell)$ defined by

$$
(\nu)^\ell B_j^{\pm}(s, \ell) = \int_{s_1}^{-\infty} \frac{ds'}{s' - s} \int_{t_L(s')}^{t_R(s')} \frac{dt'}{2\nu} \, Q_\ell\left(1 + \frac{t'}{2\nu}\right)
$$

$$
\times \left\{ \rho_{tu}(t', 4 - s' - t') \pm \rho_{tu}(4 - s' - t', t') \right\} . \tag{4}
$$

The factor ν^ℓ makes $B^{\pm}(s, \ell)$ real in the gap $0 < s < s_0$, for all real

ℓ (s_0 is threshold).[3] The plus and minus amplitudes coincide with the physical amplitudes for even and odd ℓ, respectively. We note that the full amplitudes giving rise to the projections in (4) can be written[4]

$$A_3^{\pm}(s,t) = \int_{s_1}^{-\infty} ds' \int_{t_L(s')}^{t_R(s')} dt' \, \frac{\rho_{tu}(t', 4-s'-t') \pm \rho_{tu}(4-s'-t', t')}{(s'-s)(t'-t)} . \quad (5)$$

III. SINGULARITY IN THE PARTIAL-WAVE AMPLITUDE AT THE POINT s_1.

Formulas (3) and (4) suggest that a singularity is present at $s = s_1$. In this section we demonstrate that this is indeed the case, and derive expressions for the discontinuity across the cut that starts at s_1. This discontinuity is shown to consist of two parts, one of which disappears for physical ℓ values [that is, it disappears for even ℓ in $B_3^{+}(s, \ell)$ and for odd ℓ in $B_3^{-}(s, \ell)$]. However, even for physical ℓ, the singularity at $s = s_1$ is still present.

We prove the existence of the singularity at $s = s_1$ by showing that the discontinuity across the left-hand cut for $s_1 < s < 0$ is singular at $s = s_1$. From (4), we compute the discontinuity across the cut for $s_1 < s < 0$ (interchanging the order of integrations),

$$\Delta B_3^{\pm}(s,\ell) = \frac{\pi}{2(-\nu)^{\ell}} \int_{t_0}^{4-s} \frac{dt'}{2\nu} \, P_{\ell}(-1 - t'/2\nu) \int_{s_B(t')}^{\infty} \frac{ds'}{s'-s}$$

$$\times \left[\rho_{tu}(t', 4-s'-t') \pm \rho_{tu}(4-s'-t', t') \right] , \quad (6)$$

where t_0 is the threshold in the t channel and $s_B(t)$ is the boundary

of the double spectral function. It is readily seen from (6) that $\Delta B_3^{\pm}(s,\ell)$ becomes complex for $s < s_1$ and hence is singular at $s = s_1$. The function $\Delta B_3^{\pm}$ is real for $s_1 < s < 0$ but has a cut running from $s = s_1$ to $s = -\infty$. The discontinuity of $\Delta B_3^{\pm}$ for $s < s_1$ is just given by the imaginary part

$$\mathrm{Im}\left[\Delta B_3^{\pm}(s,\ell)\right]_{s<s_1} = \frac{\pi}{2(-\nu)^{\ell}} \int_{t_L(s)}^{t_R(s)} \frac{dt'}{2\nu} P_{\ell}\left(-1 - \frac{t'}{2\nu}\right)$$

$$\times \left[\rho_{tu}(t', 4 - s - t') \pm \rho_{tu}(4 - s - t', t')\right] . \quad (7)$$

The important point is that the imaginary part is nonvanishing for both physical and nonphysical partial-wave amplitudes. In particular for physical amplitudes [$B_3^{+}(s,\ell)$ for even ℓ, or $B_3^{-}(s,\ell)$ for odd ℓ], (7) becomes the integral over the product of two even or two odd functions, respectively. Hence, $\Delta B_3^{\pm}(s,\ell)$, and therefore $B_3^{\pm}(s,\ell)$, have singularities at $s = s_1$ for both physical and nonphysical ℓ.

We now compute the discontinuity across the branch cut starting at $s = s_1$.

$$\Delta B_3^{\pm}(s,\ell)_{s<s_1} = \frac{\pi}{2(-\nu)^{\ell}} \int_{t_0}^{4-s} \frac{dt'}{2\nu} P_{\ell}\left(-1 - \frac{t'}{2\nu}\right) \mathrm{PV} \int_{s_B(t')}^{-\infty} \frac{ds'}{s' - s}$$

$$\times \left[\rho_{tu}(t', 4 - s' - t') \pm \rho_{tu}(4 - s' - t', t')\right]$$

$$+ \pi \int_{t_L(s)}^{t_R(s)} \frac{dt'}{2\nu} \mathrm{Re}\left\{\frac{Q_{\ell}(1 + t'/2\nu)}{(\nu)^{\ell}}\right\}\left[\rho_{tu}(t', 4 - s - t') \pm \rho_{tu}(4 - s - t', t')\right] . \quad (8)$$

The second term in (8) vanishes for physical amplitudes because of the symmetry of the real part of $Q_\ell(z)$,

$$\mathrm{Re}\, Q_\ell(z) = (-1)^{\ell+1}\, \mathrm{Re}\, Q_\ell(-z) \quad . \tag{9}$$

For negative integer ℓ , the second term in (8) has poles coming from the Q_ℓ function. This infinity in the discontinuity gives rise to the Gribov-Pomeranchuk[1] essential singularities mentioned earlier. We see from (8) that these poles occur in alternate amplitudes, as follows:

$$\Delta B_3^{+}(s,\ell)_{s<s_1} \quad \text{has poles for} \quad \ell = -1,\ -3,\ -5,\ \cdots ,$$

and (10)

$$\Delta B_3^{-}(s,\ell)_{s<s_1} \quad \text{has poles for} \quad \ell = -2,\ -4,\ -6,\ \cdots .$$

We now specialize (8) to the case of physical amplitudes by writing the corresponding equation for $A_3(s,\ell)$, [see (3)] . In this case the second term from (8) vanishes, and we have

$$\Delta A_3(s,\ell)_{s<s_1} = \frac{\pi}{2} \int_{t_0}^{4-s} \frac{dt'}{2\nu}\, P_\ell\left(1 + \frac{t'}{2\nu}\right)$$

$$\times\ \mathrm{PV} \int_{s_B(t')}^{-\infty} \frac{ds'}{s'-s} \left[\rho_{tu}(t', 4-s'-t') + (-1)^\ell\, \rho_{tu}(4-s'-t', t')\right] . \tag{11}$$

If we include the other two double spectral functions and define absorptive parts in the t and u channels, $D_t(s,t,u)$ and $D_u(s,t,u)$, in the usual way, we may write

$$\Delta A_3(s,\ell)_{s<s_1} = \frac{\pi}{2} \int_{t_0}^{4-s} \frac{dt'}{2\nu} P_\ell\left(1 + \frac{t'}{2\nu}\right)$$

$$\times [\mathrm{Re}\, D_t(s,t',4-s-t') + (-1)^\ell \,\mathrm{Re}\, D_u(s,4-s-t',t')]. \quad (12)$$

We emphasize that (12) is not correct unless the real parts are taken. Further, the expression for the discontinuity for $s_1 \leq s < 0$ may be written the same way as (12), but the two expressions are not connected by analytic continuation. This is just because of the presence of the singularity at $s = s_1$.

We may determine the nature of the singularity at s_1 by considering the box diagram, which is the entire contribution at this point. This contribution is given by[5]

$$\rho_{tu}(t,u) \propto \frac{\theta[(t-u)(u-4)-16]}{\{tu[(t-4)(u-4)-16]\}^{1/2}} \quad (13)$$

in the equal-mass case.

In this case the amplitude is symmetrical, and $B_3^-(s,\ell) \equiv 0$. We may write (4) as

$$(\nu)^\ell B_3^+(s,\ell) = \int_{t_0}^{\infty} \frac{dt}{2\nu} Q_\ell\left(1 + \frac{t}{2\nu}\right) D_{3t}(s,t,4-s-t), \quad (14)$$

where D_{3t} is the absorptive part in the t channel arising from the third double spectral function. Explicitly,

$$D_{3t}(s,t,u) \propto \frac{1}{\kappa^{1/2}(t,u)} \ln \left\{ \frac{\alpha(t,u) + \frac{t-4}{4t^{1/2}} \kappa^{1/2}(t,u)}{\alpha(t,u) - \frac{t-4}{4t^{1/2}} \kappa^{1/2}(t,u)} \right\}, \quad (15)$$

where

$$\kappa(t,u) = 4tu\,[(t-4)(u-4)-4]$$

and (16)

$$\alpha(t,u) = tu - 2t - 4u + 6 \quad .$$

Equation (14) shows that the singularity at $s = 0$ arises from an end-point singularity at t_0, whereas the singularity at s_1 arises from a pinch of the contour by two coalescing singularities of D_t. We deduce

$$B_3^{+}(s,\ell) \underset{s \to 0}{\propto} (-s)^{3/2} + \text{const} \quad ,$$

(17)

$$B_3^{+}(s,\ell) \underset{s \to s_1}{\propto} \ln(s - s_1) \quad .$$

Finally, we note that the fixed singularity at s_1 does not occur in the amplitude $A_3(s,t)$, which means this singularity must cancel when the partial-wave summation is performed. The fixed singularity, however, does occur in $A_3^{\pm}(s,t)$, [see(5)], and, as we have seen, it is the partial-wave projections of this function that have the extra term in the discontinuity.

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FOOTNOTES AND REFERENCES

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FIGURE CAPTIONS

Fig. 1. Singularity produced by the peaking of the third double spectral function.

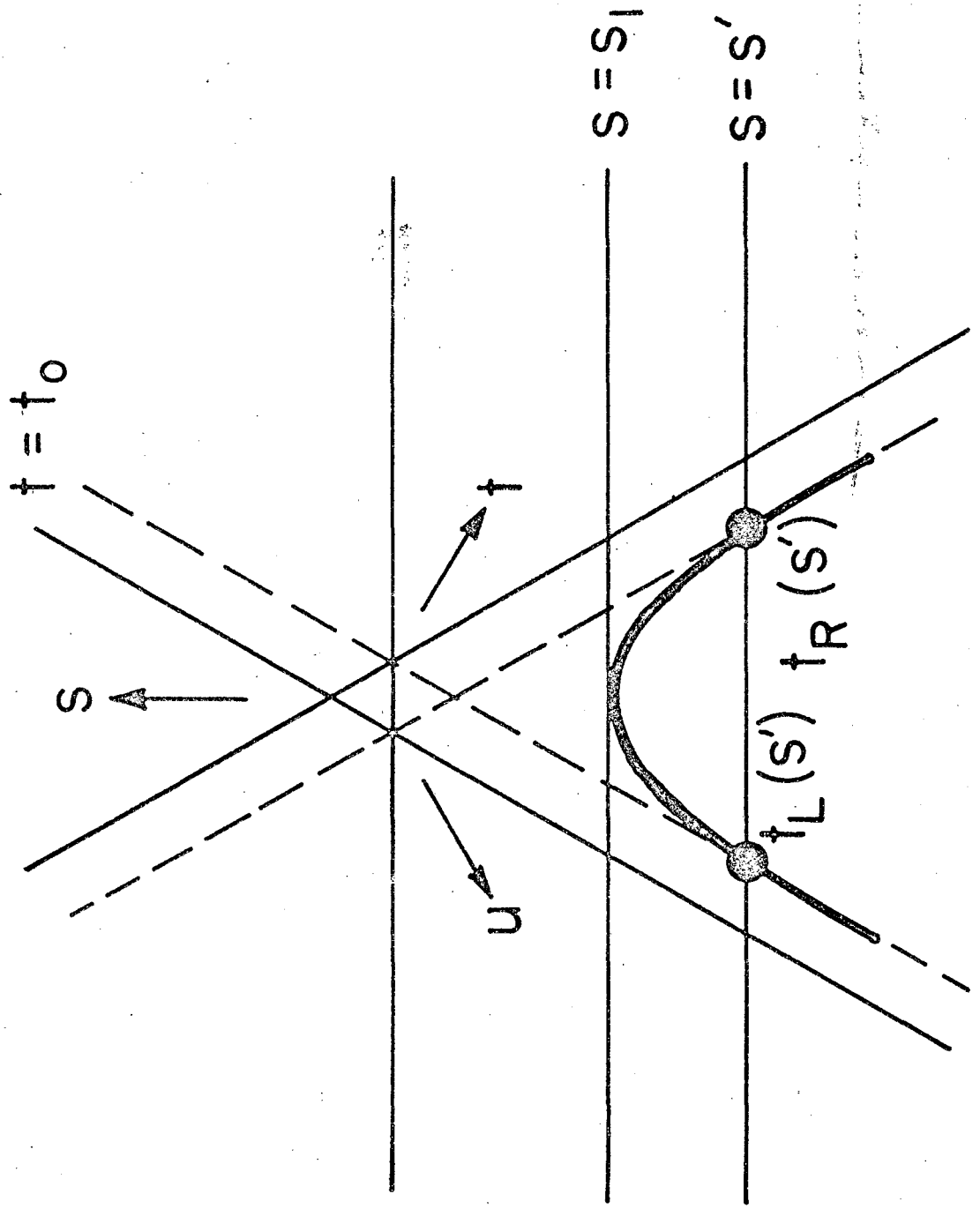

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Fig. 1.

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