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STUDIES OF MANY-CHANNEL SCATTERING

Berkeley, California

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Hans A. Weidenmüller

October 2, 1963

Contents

Abstract v

I. Introduction 1

II. Definition and General Properties of the Many-Channel Problem

A. Definition of the Problem 6

B. Specification of the Interaction, Explicit Solution of the Problem 9

C. General Properties of the Solutions 13

D. Connection between our Case and the More General Problem presented in II. A. 16

III. Analytical Properties of the S Matrix

A. Construction of the Riemann Surface 18

B. Analytical Properties of the S Matrix on the Riemann Surface 23

C. Other Properties of $\hat{\underset{\approx}{S}}$ and of the Solutions of (2.4) on $\hat{U}(N)$. 26

IV. Poles of the S Matrix

A. Zeros of the Determinant of $\underset{\approx}{E}$ 29

B. Zeros of D and Poles of $\hat{\underset{\approx}{S}}$ 33

C. Homogeneous and Inhomogeneous Solutions in a Neighborhood of a Pole of $\hat{\underset{\approx}{S}}$ 38

D. The Breit-Wigner Formula 42

E. Zeros of D on the Real Axis and at Branch Points . . 46

V. Generalization of Levinson's Theorem . . . . . .

A. A Relation for the Phase of D 50

B. A Relation between δ and the Scattering Phase Shifts . 53

VI. Distribution of the Poles of $\hat{\underset{\approx}{S}}$ on $\hat{U}(N)$ and its Dependence on the Potential Parameters

A. Reminder of the Single-Channel Case 58

B. The Case of N Channels 61

VII. Numerical Examples: The Two-Channel Case .

A. Parametrization 65

B. Resonances and Levinson's Theorem 67

C. Motion of the Poles close to the Real Axis; Speculations about Nuclei 71

VIII. Summary and Conclusions 73

Acknowledgments 75

Appendix 77

References 80

Footnotes 82

STUDIES OF MANY-CHANNEL SCATTERING*

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ABSTRACT

A simple case of many-channel scattering is investigated. The Riemann surface is constructed, upon which the elements of the S matrix are meromorphic functions. The structure of the S matrix near a pole is investigated in detail. It is shown that both the scattering wave functions and the elements of the S matrix are dominated by the "wave function" for the "decaying state" in the vicinity of such a pole. The physical content of the Breit-Wigner formula is discussed in the framework of these ideas.

Levinson's theorem is generalized to the case of many-channel scattering. As a by-product, it is shown that poles of the S matrix close to the positive real axis of the physical sheet cause the sum of the eigenphase shifts of the S matrix to increase approximately by π over the width of the resonance as the energy increases.

Most of the above statements can be proved with the help of a function D, holomorphic on the entire Riemann surface. Its zeros coincide with poles of the S matrix, and its phase is related to the sum of the eigenphase shifts of the S matrix. Peierls, Le Couteur, and Newton have shown that from D, it is possible to determine all elements of the S matrix. All this seems to indicate that D is a very basic function in the theory of many-channel scattering, and that its role is not yet entirely understood.

STUDIES OF MANY-CHANNEL SCATTERING*

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I. INTRODUCTION

In this paper we give a detailed analysis of the properties of an exactly soluble many-channel-scattering problem. The motivation for this investigation is threefold.

In the last twenty-five years several general theories of nuclear reactions have been developed (1). The best known of these were formulated by Kapur and Peierls (2), Wigner and collaborators (3), Humblet and Rosenfeld (4), and Feshbach (5). A very characteristic feature of all these dispersion theories of nuclear reactions is the occurrence of a Breit-Wigner formula, which describes approximately the behavior of the S matrix in the vicinity of an isolated resonance.

The conditions stated in the above-mentioned papers for the occurrence of such a resonance, as well as the expressions for the parameters describing the S matrix near a resonance, differ, however. One may ask whether the various definitions can be shown to be identical under conditions of physical relevance, as should be the case with theories that presumably have general validity; if not, whether there may occur sharp peaks in the cross section that can be accounted for by only one of the definitions and not the others, and other sharp peaks for which another definition is appropriate.

Not all of the above-mentioned papers imply, for example, that an isolated resonance is necessarily a pole of the S matrix, and although it is clear that poles of the S matrix sufficiently close to the real axis do give rise to resonance phenomena in the cross section, it should be interesting to see whether there may be sharp resonance-like peaks in the cross section not caused by poles of the S matrix.

In such a situation, it might be helpful to have an exactly soluble problem available that may serve as a test case for the general theories and may give some further insight into what happens at a resonance. For such a question, a many-channel scattering problem seems much more suitable than a single-channel one, the former being able to give sharp resonances under much less stringent and artificial conditions upon the potential parameters than the latter.

A second reason that started this investigation is this. When one analyzes the structure of the S matrix in an attempt to understand what happens at a resonance, one is inevitably led to consider the analytical continuation of the S-matrix elements to complex values of the energy. Numerous studies of the analytical properties of the S matrix have been undertaken for the single-channel case (6), but relatively little is known about the corresponding properties in the many-channel case (7). Since we propose to study a very specific case of many-channel scattering in this paper, we shall be concerned only with properties of a specific S matrix and shall not try to discuss general features of S matrices for many-channel scattering in terms of conditions imposed upon the interaction potentials. In a sense, then, our investigation will be a natural extension, to the many-channel case, of Nussenzveig's detailed discussion (8) of the properties of the S matrix for square-well potential scattering. We are, however, aware

that many of our results can be shown to be valid for wide classes of interaction potentials, and we shall try to prove this statement in a later publication.

Finally, the simple soluble case in which we are interested gives rise to sharp resonances in the scattering cross section if the potential parameters that have to be specified are suitably chosen. One may try to analyze the conditions upon the potential parameters under which sharp resonances occur. In this way, one may hope to get some better insight into the question whether the nuclear forces have to be very specific to cause the occurrence of many sharp resonances in low-energy nuclear reactions, or whether one would more or less expect any attractive and sufficiently strong force to have this effect. Whether or not one may hope to obtain an answer to this question by means of studying the present problem depends, of course, upon whether one may consider this problem as typical for what happens in nuclei. Later, we shall refer to this question in the appropriate place.

In Section II, we introduce the many-channel problem we wish to study and derive its general properties, including an explicit expression for the S matrix, the specific form of Green's theorem, etc. Our choice of interaction potentials has already been made (7, , 9), and the discovery that the many-channel problem becomes exactly soluble in this simple case is not a new one. Therefore, most of the relations we give in Section II can be found in the literature. For purposes of definition and notation we have to present them again, however. We also include a proof showing that for fixed total energy ξ we can always adjust the parameters in our model in such a way that we can exactly reproduce the elements of the scattering matrix for a much more general choice of interaction potentials.

In Section III, we study the Riemann surface upon which the S matrix becomes a single-valued function. We discuss the topological structure of

this surface, the general analytical properties of the S matrix, and the significance of analytical continuation of the unitarity relationships. A study of this kind of problem was made by Peierls (10), and subsequently by Le Couteur (11). We believe, however, that our case is the first many-channel case in which we start with a many-channel Schrödinger equation and discuss fully the analytical properties of the S matrix as consequences of the nature of the potentials contained in this equation.

The same remark applies to Section IV, where we turn our attention to the poles of the S matrix. We investigate such questions as the connection between the occurrence of poles and certain solutions of the Schrödinger equation, the behavior of the S matrix near poles, the form of the analytically continued unitarity relationships near and at a pole, the structure of the Breit-Wigner formula, the connection between total and partial widths, the relation between compound and decaying states, and the single-level approximation for the S matrix. Much of the work in this section has been stimulated by the papers of Newton and Fonda (9), of Breit (12), and of Humblet and Rosenfeld (4). The more recent paper by Davies and Baranger (13) also deals with these problems. Still, we hope we may offer new or clearer understanding of the structure of the S matrix near and at a resonance.

In Section V, we show that Levinson's theorem (14) can be extended to many-channel scattering. We shall carry through the proof in a way analogous to the original proof by Levinson for the single-channel case. The crucial point for the generalization of the theorem is that the sum of the eigenphases of the S matrix at each energy ξ is given by the phase of one analytic function D of the energy. The main difficulty is the continuation of the eigenphase shifts through thresholds (branch points), and it can be overcome by means of an analytic continuation of D in the physical sheet of the Riemann manifold

discussed in Section III. The essential part of the proof does not depend upon the specific choice of interaction potentials considered in this paper, and we intend to give a more general version of this proof in a subsequent publication.

In Section VI, we discuss the positions of the poles of the S matrix as functions of the potential parameters. We begin with a reminder of the single-channel case (8), and continue with the many-channel case without and with interaction between the channels. In Section VII, we illustrate the results obtained in the previous sections with numerical examples relating mostly to the two-channel case. A partial answer to our third question is given. We end in Section VIII with the summary and conclusions.

We finish without having related our results to the more general theories mentioned above. We think, however, that the material presented is sufficiently self-contained to merit separate publication and defer the comparison between our results and the general resonance theories to a later time.

II. DEFINITION AND GENERAL PROPERTIES OF THE MANY-CHANNEL PROBLEM

A. Definition of the Problem

We proceed in close analogy to ref. 7. We consider a nonrelativistic particle, the position of which shall be described by one coordinate x. The simplification of using only one coordinate is made to avoid the complications introduced by angular momentum and spin. This particle, confined to values of x with $x \geq 0$, so that the wave function vanishes for $x = 0$, interacts with a system that has only a finite number N of bound states. We denote the state vector of the mth bound state by $|m\rangle$ and its energy by e_m, and have the states arranged in such a way that $e_1 \leq e_2 \leq e_3 \leq \cdots \leq e_N$. For the sake of simplicity we assume $e_i \neq e_j$ for $i \neq j$. This simplifies the behavior of the scattering matrix at thresholds considerably. The system with N states is referred to as the residual system. The total system, the particle plus the residual system, is simply called the system.

Let V denote the interaction between the particle and the residual system, and let $V_{nm}(x)$ denote its matrix elements with respect to states $|n\rangle$ and $|m\rangle$:

$$V_{nm}(x) = \langle n|V|m\rangle \, . \tag{II. 1}$$

We assume that $V_{nm}(x)$ is finite and real everywhere, and that $V_{nm}(x) = 0$ for $x > a$. The constant a is called the interaction radius. Denoting the complex conjugate of a number by *, the Hermitean conjugate of a matrix by †, and the transpose of a matrix by T, we have

$$V_{nm}(x) = V^T_{nm}(x) = V^*_{nm}(x) = V^\dagger_{nm}(x). \tag{II. 2}$$

The total-state vector of the system, ψ, can be expanded into the states $|m\rangle$:

$$\psi = \sum_{m=1}^{N} f_m(x)\,|m\rangle \, , \tag{II. 3}$$

with $f_m(0) = 0$ for all m. Instead of referring to ψ, we use the N-dimensional vector of functions $\underset{\sim}{F}^T(x) = \{f_1(x), \cdots, f_N(x)\}$ to represent the system.

We introduce the symbols $\underset{\approx}{1}$ for the N-dimensional unit matrix, $\underset{\approx}{V}(x)$ for the potential matrix (II.2), $\underset{\approx}{e}$ for a diagonal matrix having the e_i as diagonal elements, and ξ for the total energy. Putting $\hbar = 1$ and the mass M of the particle with coordinate x equal to $M = 1/2$, we obtain the many-channel Schrödinger equation of the total system in a stationary state,

$$\left[\left(-\frac{d^2}{dx^2} - \xi\right) \underset{\approx}{1} + \underset{\approx}{e} + \underset{\approx}{V}(x)\right] \underset{\sim}{F}(x) = 0. \qquad \text{(II.4)}$$

Equation (II.4) will be the starting point of our investigation. Although Eq. (II.4) gives rise to inelastic scattering through the nondiagonal elements of $\underset{\approx}{V}(x)$, it introduces none of the difficulties encountered in actual inelastic-scattering processes. With inelastic scattering of two systems, rearrangement collisions become possible, too, in most cases. They introduce serious geometrical difficulties into the formulation of the problem. No expansion of the type (II.3) having only bound states $|m\rangle$ will be possible in such a case, for example. In addition, in all realistic scattering processes we encounter three- or more-particle breakup or production processes that obscure the kinematics of the reaction severely. Having avoided these difficulties, we find Eq. (II.4) sufficiently simple to allow an explicit solution if we further specify $\underset{\approx}{V}(x)$.

It is, of course, questionable whether our simplifying assumptions may introduce some false features into our results--false in the sense that they will not be present in a more realistic treatment of inelastic scattering. We believe that all the essential features of our problem are necessarily present in any more complicated problem. This is the basis of our conviction that it is useful to undertake the present study. It is clear, of course, that many

features possibly present in a more realistic treatment will be missing in our problem. In particular, the analytical properties of the S matrix are likely to be very much simpler in our problem than in any realistic case. However, the problem formulated in (II.4) is consistent with the requirements of the general theories of nuclear reactions (1, 2, 3, 4, 5) and can thus serve as a test case.

Returning to the formulation of our problem, we supplement Eq. (II.4) by imposing boundary conditions at $x \to \infty$. To do this, we observe that for $x > a$, (II.4) simplifies to

$$\left[\left(-\frac{d^2}{dx^2} - \xi\right) \underset{\approx}{1} + \underset{\approx}{e}\right] \underset{\sim}{F}(x) = 0, \tag{II.5a}$$

which is an uncoupled system of differential equations. Introducing the N wave numbers $k_1, \cdots, k_N$ with $\{k_1, \cdots, k_N\} = \underset{\sim}{K}^T$ by

$$k_i = (\xi - e_i)^{1/2} \tag{II.6}$$

so that $\qquad k_i > 0$ if $\xi > e_i$

and $\qquad \mathrm{Im}\, k_i > 0$ if $\xi < e_i$,

where Im stands for imaginary part, we can rewrite (II.5a) in the form

$$\left[+\frac{d^2}{dx^2} + k_i^2\right] f_i(x) = 0, \qquad i = 1, \cdots, N. \tag{II.5b}$$

A physical situation in which $x > a$ and the residual system is in a state $|m\rangle$ is called a channel, and in this sense we shall refer to the wave numbers k_m as the channel wave numbers. Disregarding the possibility of bound states of the total system (which we will discuss later) we define the scattering states as solutions of (II.4) with an incoming wave in one channel, ℓ, and outgoing waves in all other channels. If we introduce the velocities v_i by $v_i = 2k_i$ and normalize incoming and outgoing waves to unit flux we have for $x > a$, according to (II. 5b)

$$f_m^{(\ell)}(x) = \frac{1}{(v_m)^{1/2}} \exp(-ik_m x)\, \delta_{m\ell} - S_{m\ell} \frac{1}{(v_m)^{1/2}} \exp(+ik_m x) \,. \tag{II.7}$$

The upper index ℓ in $f_m^{(\ell)}(x)$ refers to the "incoming channel." The system of equations (II.4) together with the boundary conditions (II.7) is expected to define the solutions $f_m^{(\ell)}(x)$ uniquely.

If the energy $\xi > e_i$ so that $k_i > 0$, we call the channel i open; otherwise, closed. The physically interesting ones among the solutions $\underset{\approx}{F}^{(\ell)}(x)$ of (II.4) and (II.7) are clearly those for which ℓ is open. It will turn out to be advantageous, however, to consider also the $\underset{\approx}{F}^{(\ell)}(x)$ for which ℓ is closed. For the same reason, we have in (II.7) included factors $1/(v_m)^{1/2}$ in front of exponentials that for closed channels yield zero flux, and have extended the definition of the matrix elements $S_{m\ell}$ of the scattering matrix $\underset{\approx}{S}$ to closed channels, even though cross sections can be measured only for open channels.

B. Specification of the Interaction; Explicit Solution of the Problem

We now make the assumption that for $x < a$,

$$V_{mn}(x) = V_{mn} \tag{II.8}$$

independent of x. This assumption is equivalent to the assumption that the interaction (II.1) is separable and that its x-dependent part has the form of a square-well potential. With the help of (II.8), we can easily solve the Eq. (II.4) for $x < a$. We define the matrix $\underset{\approx}{A}$ by

$$\underset{\approx}{A} = -\underset{\approx}{V} - \underset{\approx}{e} \tag{II.9}$$

for $x < a$. $\underset{\approx}{A}$ is real and symmetric because of Eq. (II.2); it can be diagonalized by an orthogonal matrix $\underset{\approx}{O}$ with $\underset{\approx}{O}\,\underset{\approx}{O}^T = \underset{\approx}{O}^T \underset{\approx}{O} = \underset{\approx}{1}$. Denoting the N eigenvalues of $\underset{\approx}{A}$ by $\lambda_1, \cdots, \lambda_N$ and introducing the diagonal matrix $\underset{\approx}{\Lambda}$ that has the λ_i as diagonal elements, we have

$$\underset{\approx}{0}\,\underset{\approx}{A}\,\underset{\approx}{0}^{T} = \underset{\approx}{\Lambda}\,. \tag{II.10}$$

When the vector of functions $\underset{\sim}{G}(x)$ is defined by

$$\underset{\sim}{G}(x) = \underset{\approx}{0}\,\underset{\sim}{F}(x), \qquad \underset{\sim}{F}(x) = \underset{\approx}{0}^{T}\,\underset{\sim}{G}(x)\,, \tag{II.11}$$

it is easily seen that $\underset{\sim}{G}(x)$ fulfills the equation

$$\left[\left(\frac{d^2}{dx^2} + \xi\right)\underset{\approx}{1} + \underset{\approx}{\Lambda}\right]\underset{\sim}{G}(x) = 0, \tag{II.12}$$

so that the ith component $g_i(x)$ of $\underset{\sim}{G}(x)$ has the form

$$g_i(x) = \bar{g}_i \times \left[\frac{\sin(\xi + \lambda_i)^{1/2}x}{(\xi + \lambda_i)^{1/2}}\right]. \tag{II.13}$$

Here the $\bar{g}_i$ are N unknown constants to be determined from the boundary conditions at $x = a$. The denominator in Eq. (II.13) has been chosen in order to make (II.13) meaningful even for $\lambda_i = -\xi$, and also in order to facilitate an analytical continuation to complex energies.

The constants $\bar{\underset{\sim}{G}}^{T} = \{\bar{g}_1, \cdots, \bar{g}_N\}$ as well as the elements of $\underset{\approx}{S}$ can now be determined by joining $\underset{\sim}{F}(x)$--as given by (II.11) and (II.13)--continuously, and with a continuous first derivative, to $\underset{\sim}{F}^{(\ell)}(x)$--as given by (II.7). For each possible choice of ℓ, $\ell = 1, \cdots, N$, we obtain a set of constants $\bar{\underset{\sim}{G}}^{(\ell)}$ and the S-matrix elements $S_{m\ell}$ for $m = 1, \cdots, N$. Thus we can determine the $2N^2$ unknowns $g_m^{(\ell)}$ and $S_{m\ell}$, with $1 \leq (m, \ell) \leq N$. Explicitly, one obtains the equations

$$\sum_{t=1}^{N} \bar{g}_t^{(\ell)}\, 0_{tm} \left\{\cos(\xi + \lambda_t)^{1/2}a - ik_m\left[\frac{\sin(\xi + \lambda_t)^{1/2}a}{(\xi + \lambda_t)^{1/2}}\right]\right\} = -i(v_m)^{1/2}\exp(-ik_m a)\,\delta_{m\ell} \tag{II.14}$$

and

$$S_{m\ell} = i\,\frac{1}{(v_m)^{1/2}}\,\exp(-ik_m a)\,\sum_s 0_{sm}\,\bar{g}_s^{(\ell)}\left\{\cos(\xi + \lambda_s)^{1/2}a + ik_m\left[\frac{\sin(\xi + \lambda_s)^{1/2}a}{(\xi + \lambda_s)^{1/2}}\right]\right\} \tag{II.15}$$

where $\bar{g}_s^{(\ell)}$ is the solution of (II.14). Obviously, the system of equations (II.14) is the more fundamental one: Whenever it has a solution, (II.15) has one, too, which can be trivially obtained by inserting the solution of (II.14). The Eq. (II.14) are a system of N inhomogeneous linear equations for the unknowns $\bar{g}_t^{(\ell)}$, $t = 1, \cdots, N$. By changing the index ℓ, we get a different inhomogeneity and different solutions. The solutions $\underset{\sim}{\bar{G}}^{(\ell)}$ all exist and are uniquely determined if the determinant of the homogeneous system does not vanish. The matrix elements of $\underset{\approx}{S}$ and the overall solutions $\underset{\sim}{F}^{(\ell)}(x)$ are then uniquely determined also.

It is this particular feature of our many-channel problem that greatly simplifies the following discussion. Instead of having to solve a coupled system of linear differential equations, we can carry through the entire investigation by studying the properties of (II.14) as a function of the energy ξ, resorting to the simple and powerful methods of linear algebra. Our solutions as functions of x are determined once and for all, and the only unknowns left are the $\bar{G}^{(\ell)}$ and the matrix $\underset{\approx}{S}$. This proves particularly advantageous for the discussion in Section IV. Aside from the energy ξ, the Eq. (II.14) depend upon the parameters $\lambda_1, \cdots, \lambda_N$, the energies $e_1, \cdots, e_N$ (which enter into the definition of the $k_1, \cdots, k_N$), and the elements of the orthogonal matrix $\underset{\approx}{O}$. From Eqs. (II.9) and (II.10) it is evident that this set of parameters is entirely equivalent to the original one, namely the elements of $\underset{\approx}{V}(x)$ and the energies $e_1, \cdots, e_N$. It is, however, more advantageous to use this set than the original one if one wants to discuss the dependence of the solutions on the parameters, and henceforth we do so.

If $\sin(\xi + \lambda_t)^{1/2} a \neq 0$ for all t, we can rewrite Eqs. (II.14) and (II.15) as follows. We define the symmetric matrix $\underset{\approx}{E}$ by

$$E_{\ell m} = \sum_{n=1}^{N} O_{n\ell} O_{nm} (\xi + \lambda_n)^{1/2} \cot(\xi + \lambda_n)^{1/2} a - ik_\ell \delta_{\ell m}. \qquad \text{(II.16)}$$

Introducing the new unknowns $\underset{\sim}{\bar{C}}^{(\ell)T} = \left\{\bar{c}_1^{(\ell)}, \cdots, \bar{c}_N^{(\ell)}\right\}$ by

$$\bar{c}_i^{(\ell)} = \sum_t \bar{g}_t^{(\ell)} 0_{ti} \left[\frac{\sin(\xi + \lambda_t)^{1/2} a}{(\xi + \lambda_t)^{1/2}}\right], \tag{II.17}$$

we obtain for (II.14)

$$\sum_m E_{\ell m} \bar{c}_m^{(s)} = -i(v_s)^{1/2} [\exp(-ik_s a)] \delta_{s\ell} . \tag{II.18}$$

If the determinant of $\underset{\approx}{E}$ does not vanish, we can easily solve (II.18) for $\underset{\sim}{\bar{C}}^{(\ell)}$ and obtain, by inserting into (II.15), the following expression for $\underset{\approx}{S}$:

$$S_{m\ell} = (k_m/k_\ell)^{1/2} [\exp(-ik_m a - ik_\ell a)] \sum_s E_{ms}^{-1}(k_1, \cdots, k_N) \times E_{s\ell}(-k_1, \cdots, -k_N). \tag{II.19}$$

Here, $\underset{\approx}{E}^{-1}$ is the inverse of $\underset{\approx}{E}$ and the matrix $\underset{\approx}{E}(-k_1, \cdots, -k_N)$ is defined in analogy to (II.16), with the signs of all the k_i's reversed. Obviously, we have

$$-E_{\ell m}(k_1, \cdots, k_N) + E_{\ell m}(-k_1, \cdots, -k_N) = 2ik_\ell \delta_{\ell m}. \tag{II.20}$$

Inspection of (II.19) shows that it remains valid even if not all $\sin(\xi + \lambda_t)^{1/2} a$ are different from zero, i.e., even if $(\xi + \lambda_t)a^2 = \pi n$ with $n = \pm 1, \pm 2, \cdots$. Therefore, we shall use this expression for $\underset{\approx}{S}$ throughout. The constants $\underset{\approx}{\bar{C}}^{(\ell)}$ become, of course, ill defined in such a case, and one has to refer back to the $\underset{\sim}{\bar{G}}^{(\ell)}$.

If the determinant of $\underset{\approx}{E}$ vanishes, then the homogeneous part of Eq. (II.14) has a nontrivial solution, and at these energies we encounter a bound state, as is shown in Section IV. Whenever this happens, at least one of the elements of the S matrix as defined by Eq. (II.19) has a pole except for very special cases discussed in Section IV. For the time being, we assume $\det \underset{\approx}{E} \neq 0$.

The coefficients $\bar{\underset{\sim}{C}}^{(\ell)}$ in (II.17) were introduced for the following purpose. We define the functions

$$X_{\ell n}(x) = \sum_{m=1}^{N} 0_{m\ell} 0_{mn} \left[\frac{\sin(\xi + \lambda_m)^{1/2} x}{\sin(\xi + \lambda_m)^{1/2} a} \right] . \tag{II.21}$$

The functions $\underset{\sim}{F}(x)$ of Eq. (II.11) can be expressed in terms of the constants $\bar{\underset{\sim}{C}}$, and the functions $X_{\ell n}(x)$ by

$$f_m^{(\ell)}(x) = \sum_s \bar{c}_s^{(\ell)} X_{sm}(x). \tag{II.22}$$

The X's have the pleasant properties that

$$X_{\ell n}(a) = \delta_{\ell n}, \frac{d}{dx} X_{\ell n}(x) \mid_{x=a} - ik_\ell X_{\ell n}(a) = E_{\ell n}. \tag{II.23}$$

In a sense, they are the correct system of functions to be used for $x < a$ if we formulate our boundary conditions in terms of the matrix $\underset{\approx}{E}$, as was done in (II.18) and (II.19). The matrix $\underset{\approx}{E}$ is the matrix of the logarithmic derivatives at the interaction radius. It is clear that, again, the X's are ill-defined if one of the $\sin(\xi + \lambda_s)^{1/2} a$ vanishes.

Finally, the cross section for a reaction leading from channel ℓ (open) to channel m (open) is, according to the normalization used in (II.7), given by

$$\sigma_{\ell \to m} = | S_{\ell m} - \delta_{\ell m} |^2 . \tag{II.24}$$

C. General Properties of the Solutions

We begin with Green's theorem. Let $\underset{\sim}{F}^{(1)}(x)$ and $\underset{\sim}{F}^{(2)}(x)$ be any two solutions of (II.4) belonging to energies ξ_1 and ξ_2, respectively, with $\underset{\sim}{F}^{(1)}(0) = 0 = \underset{\sim}{F}^{(2)}(0)$, but not necessarily subject to the boundary conditions (II.7). Then Green's theorem reads

$$(\xi_1 - \xi_2) \int_0^b \underset{\sim}{F}^{(1)}(x) \cdot \underset{\sim}{F}^{(2)}(x) dx = \underset{\sim}{F}^{(1)}(b) \cdot \frac{d}{dx} \underset{\sim}{F}^{(2)}(x) \mid_{x=b}$$

$$- \underset{\sim}{F}^{(2)}(b) \cdot \frac{d}{dx} \underset{\sim}{F}^{(1)}(x) \mid_{x=b} . \tag{II.25}$$

The S matrix is symmetric for all channels, is unitary with respect to the subspace of the open channels, and has simple time-reversal properties. By combining (II.19) and (II.20), we obtain

$$S_{m\ell} = \delta_{m\ell}[\exp(-2ik_m a)] + 2i[\exp(-ik_m a)](k_m)^{1/2} E^{-1}_{m\ell}(k_\ell)^{1/2}[\exp(-ik_\ell a)] \tag{II.26}$$

which shows that $\underset{\approx}{S}$ is symmetric because $\underset{\approx}{E}$ is symmetric:

$$S_{m\ell} = S_{\ell m} \text{ for all } m \text{ and } \ell. \tag{II.27}$$

The unitarity of $\underset{\approx}{S}$ can be shown in many ways. We prefer a direct calculation that can easily be extended to the analytical continuation of $\underset{\approx}{S}$ to complex energies. We make use of the fact that, for real energies,

$$S^*_{m\ell}(k_1, \cdots, k_N) = S_{m\ell}(-k_1^*, \cdots, -k_N^*), \tag{II.28}$$

since

$$E^*_{m\ell}(k_1, \cdots, k_N) = E_{m\ell}(-k_1^*, \cdots, -k_N^*) . \tag{II.29}$$

We notice that, according to (II.6), $-k_i^* = -k_i$ for i open, and $-k_i^* = +k_i$ for i closed. Thus, Eq. (II.20) can be rewritten in the form

$$E^*_{m\ell}(k_1, \cdots, k_N) - E_{m\ell}(k_1, \cdots, k_N) =$$

$$\delta_{m\ell} \times 2ik_m \text{ if } m \text{ is open,}$$

$$\text{or} \quad \delta_{m\ell} \times 0 \quad \text{if } m \text{ is closed.} \tag{II.30}$$

We have, from Eq. (II.19), for ℓ and t open,

$$\sum_{m \text{ open}} S_{m\ell} S^*_{mt} = \left[1/(k_\ell k_t)^{1/2}\right][\exp(-ik_\ell a + ik_t a)]$$

$$\sum_{m \text{ open}} \sum_{i,j} k_m E^{-1}_{mi}(\underset{\sim}{K}) E_{i\ell}(-\underset{\sim}{K}) \times E^{*-1}_{mj}(\underset{\sim}{K}) E^*_{jt}(-\underset{\sim}{K}) . \tag{II.31}$$

Replacing k_m by (II.30), using $E^*_{st}(-\underset{\sim}{K}) = E_{st}(\underset{\sim}{K})$ for s open, t and utilizing the symmetry of $\underset{\approx}{E}$, we find that Eq. (II.31) reduces easily to

$$\sum_{m \text{ open}} S_{m\ell} S^*_{mt} = \delta_{\ell t} \text{ if } \ell \text{ and } t \text{ are open.} \tag{II.32}$$

Equation (II.32) expresses the unitarity of the S matrix in the subspace of open channels, and Eq. (II.28) denotes the behavior of $\underset{\approx}{S}$ under time reversal.

The behavior of the cross sections at threshold, i.e., when one of the k_i's is close to zero and positive, can be studied in analogy to reference 7. Although it is not of great physical interest, we want to give the high-energy limit for the cross sections, too, for the sake of completeness. Instead of (II.16), we define the matrix $\underset{\approx}{E}'$

$$E'_{\ell m} = \sum_{n=1}^{N} 0_{n\ell} 0_{nm} \{(\xi+\lambda_n)^{1/2} \cos(\xi+\lambda_n)^{1/2} a - ik_m \sin(\xi+\lambda_n)^{1/2} a\} , \quad \text{(II.33)}$$

and it is easy to see that $S_{m\ell}$ may be written as

$$S_{m\ell} = (k_m/k_\ell)^{1/2} [\exp(-ik_m a - ik_\ell a)] \sum_s E'^{-1}_{ms}(\underset{\sim}{K})\, E'_{s\ell}(-\underset{\sim}{K}) . \quad \text{(II.34)}$$

We expand $(\xi+\lambda_s)^{1/2} = (\xi)^{1/2} + [\lambda_s/2(\xi)^{1/2}] + \cdots$ for large ξ, and do the same with $k_i = (\xi - e_i)^{1/2} = (\xi)^{1/2} - [e_i/2(\xi)^{1/2}] + \cdots$. We calculate E' to first order in λ_s/ξ and e_i/ξ. From this expression it is easy to calculate the inverse of E' to first order in $1/\xi$, and making use of the Eqs. (II.9) and (II.10), one eventually obtains for $\xi \to \infty$,

$$S_{m\ell} \to \delta_{m\ell} - [ia/(\xi)^{1/2}]\, V_{m\ell}, \quad \text{(II.35)}$$

with $V_{m\ell}$ defined in (II.8). Correspondingly, $\sigma_{m\to\ell} \to |V_{m\ell}|^2 a^2/\xi$ according to Eq. (II.24).

D. Connection Between Our Case and the More General Problem Presented in II.A

In Section II.B, we have specialized our interaction potential greatly. We can show, however, that this specialization is not as severe as it may seem at first sight. We can prove the following statement: Given any potential matrix $\underset{\approx}{V}^1(x)$ that fulfills the requirements of Section II.A, and given the corresponding scattering matrix $\underset{\approx}{S}^1$, we can always specify the parameters $\lambda_1, \cdots, \lambda_N$; $e_1, \cdots, e_N$; and $\underset{\approx}{0}$ in our problem in order to make the S matrix of Eq. (II.19) identically equal to $\underset{\approx}{S}^1$ for fixed energy ξ. This means that we can reproduce the energy dependence of a more general scattering matrix $\underset{\approx}{S}^1$ by making the parameters in our problem energy dependent. Therefore, a study of narrow resonances in our specific case will apply to a wider class of potentials $\underset{\approx}{V}^1(x)$ defined by the requirement that the energy dependence of the parameters that reproduce $\underset{\approx}{S}^1$ exactly be small over the width of the resonance.

In order to prove the above statements we have to construct a general solution of Eq. (II.4) for $x < a$, which we join smoothly to the solution (II.7) at $x = a$. Let $\underset{\approx}{G}^{(i)}(x)$, $i = 1, \cdots, N$, be a basic set of solutions of (II.4), defined by the initial conditions $\underset{\approx}{G}^{(i)}(0) = 0$ and $\frac{d}{dx} g_\ell^{(i)}(x) \,|_{x=0} = \delta_{i\ell}$. The most general solution of (II.6) can then be written in the form

$$\sum_{i=1}^{N} \bar{G}^{(i)} \underset{\approx}{G}^{(i)}(x) \quad ,$$

with the N constants $\bar{G}^{(i)}$ as unknowns. By equating this solution and its first derivative to (II.7) at $x = a$, one can determine the elements of the S^1 matrix in analogy to the procedure in Section II.B. Again, one finds that S^1 can be written as

$$S^{1}_{m\ell} = (k_m/k_\ell)^{1/2}[\exp(-ik_m a - ik_\ell a)] \sum_s \bar{E}^{-1}_{ms}(+\underset{\sim}{K})\,\bar{E}_{s\ell}(-\underset{\sim}{K}), \qquad \text{(II.36)}$$

where $\bar{E}_{s\ell}(\underset{\sim}{K})$ is defined by

$$\bar{E}_{s\ell}(\underset{\sim}{K}) = \sum_{j=1}^{N} B^{-1}_{sj}\, C_{j\ell} - ik_\ell \delta_{s\ell}\,. \qquad \text{(II.37)}$$

Here, $B_{s\ell}$ and $C_{s\ell}$ are the matrices of the values of the functions $\underset{\sim}{G}^{(i)}(x)$ and of the derivatives of these functions at $x = a$:

$$B_{im} = g^{(i)}_m(a); \quad C_{im} = \frac{d}{dx} g^{(i)}_m(x)\,|_{x=a}\,. \qquad \text{(II.38)}$$

Both $\underset{\approx}{B}$ and $\underset{\approx}{C}$ are real from the definition of the $\underset{\sim}{G}^{(i)}(x)$, and Green's theorem can be used to show that $\underset{\approx}{\bar{E}}$ is symmetric. In (II.37) we have disregarded the possibility that $\underset{\approx}{B}^{-1}$ does not exist. We can deal with this case, however, in analogy to the situation in which $\sin(\xi+\lambda_s)^{1/2}\, a = 0$ in Section II.B. Our statement will be proved if we can show that we can choose $\lambda_1, \cdots, \lambda_N$ and $\underset{\approx}{0}$ such that

$$\sum_{j=1}^{N} B^{-1}_{sj} C_{j\ell} = \sum_{j=1}^{N} 0_{js} 0_{j\ell} (\xi+\lambda_j)^{1/2} \cot(\xi+\lambda_j)^{1/2} a\,. \qquad \text{(II.39)}$$

This follows from a comparison of (II.16) and (II.37). (The $e_1, \cdots, e_N$ must, of course, be chosen so that they reproduce the correct thresholds.) The matrix

$$\sum_{j=1}^{N} B^{-1}_{sj}\, C_{j\ell}$$

is real and symmetric; it can be diagonalized by an orthogonal matrix $\underset{\approx}{\bar{0}}$, with eigenvalues $\{\bar{\lambda}_1, \cdots, \bar{\lambda}_N\}$. Equating $\underset{\approx}{0} = \underset{\approx}{\bar{0}}$ and $\bar{\lambda}_i = (\xi+\lambda_i)^{1/2}\cot(\xi+\lambda_i)^{1/2} a$, which is always possible, we see that the required equality can indeed be achieved for a fixed energy ξ. This completes the proof.

It should be noted that the above statement has, of course, no bearing on questions like the analytical continuation of the S-matrix elements to complex energies studied in Section III.

III. ANALYTICAL PROPERTIES OF THE S MATRIX

A. Construction of the Riemann Surface (15).

The S matrix defined in (II.19) shall now be considered as a function of one of the k_i's, say k_1, and continued analytically to complex values of k_1. Two difficulties arise in this procedure: We encounter branch points whenever one of the k_i's, $i = 2, \cdots, N$, is equal to zero, because of the square-root dependence of the k_i on k_1 [this follows from (II.6)], and the elements of the S matrix will have singularities whenever the determinant of $\underset{\approx}{E}$ vanishes and at $k_1 = \infty$. We solve these difficulties in two steps. In this subsection we construct a Riemann surface that removes the difficulty of the branch points. In the next subsection, we show that the S matrix is a single-valued meromorphic function on this surface, and in Section III.C we extend some of the properties of $\underset{\approx}{S}$ derived in Sections II.B and II.C to all points of the surface.

Let us choose the zero of the energy so that $e_1 = 0$, $\xi = k_1^2 = k_2^2 + e_2 = \cdots = k_N^2 + e_N$ with $e_1 < e_2 < \cdots < e_N$. Introducing k_1 as the independent variable, we have

$$\xi = k_1^2; \quad k_i = (k_1^2 - e_i)^{1/2}. \tag{III.1}$$

Let us consider the algebraic function

$$\sum_{i=1}^{N} \beta_i k_i = t(k_1), \quad \beta_i \neq 0 \text{ for all } i, \tag{III.2}$$

for values of k_1 different from $\pm(\bar{e}_i)^{1/2}$, $i = 2, \cdots, N$, and where the β_i are constants. For each of these values of k_1, $t(k_1)$ is a 2^{N-1}-valued function of k_1. This is because we can independently choose a plus or minus sign for each k_i, $i = 2, \cdots, N$, in front of the square root in (III.1), and there are 2^{N-1} different possibilities of making such a choice. Therefore, the Riemann

surface of the function $t(k_1)$ in (III.2) has 2^{N-1} sheets. We can label these sheets in the following way: To each sheet we assign a vector τ with N-1 elements. The ith element τ_i of this vector shall be +1 or -1, according to the following rule:

If

$$\text{sign Re } k_{i+1} = \text{sign Re } k_1$$

and $\text{sign Im } k_{i+1} = \text{sign Im } k_1$,

then $\tau_i = +1$; (III.3)

if

$$\text{sign Re } k_{i+1} = -\text{sign Re } k_1$$

and $\text{sign Im } k_{i+1} = -\text{sign Im } k_1$,

then $\tau_i = -1$.

This definition suffices for all cases except when k_1 is real, $-(e_{i+1})^{1/2} < k_1 < +(e_{i+1})^{1/2}$. Here, $\text{Im } k_1 = 0$ and $\text{Re } k_{i+1} = 0$, so that both definitions (III.3) apply. In this particular case we shall choose τ_i to be +1 or -1, depending upon the sign attached to the neighboring points with $\text{Im } k_1 > 0$ on the same sheet. The 2^{N-1} vectors τ having the numbers +1 and -1 as elements in all possible combinations correspond to the 2^{N-1} sheets.

The way the sheets are connected can be seen by an extension of the well-known construction for N = 2. There, we take the two sheets, labeled $\tau_1 = \{+1\}$ and $\tau_2 = \{-1\}$, respectively, and put one on top of the other. Each sheet is a copy of the complete k_1 plane. Then we cut both sheets from $k_1 = -(e_2)^{1/2}$ to $k_1 = +(e_2)^{1/2}$ below the real axis, so that the real axis itself forms the upper shore of the cut, and the points with $\text{Im } k_1 < 0$ form the lower shore of the cut. The upper shore of the cut in the upper sheet is connected with the lower shore of the cut in the lower sheet, and vice versa. This gives the Riemann surface for N = 2. It has two simple branch points at $k_1 = \pm(e_2)^{1/2}$. If we follow a closed path on the Riemann surface that encloses one branch

point but not the other, then we need to go around this point twice in order to come back to the starting point. Having gone once around the branch point, we have moved from one of the sheets to the other one.

For $N = 3$, this procedure is repeated as follows: We first carry through the construction for $N = 2$ twice, cutting the sheets from $-(e_3)^{1/2}$ to $+(e_3)^{1/2}$ and identifying opposite shores on different sheets, as described previously. This way, we get two identical samples of the Riemann surface for $N = 2$; the only difference from the previous case is that the branch points are at $k_1 = \pm(e_3)^{1/2}$ instead of $k_1 = \pm(e_2)^{1/2}$. We complete the labeling for the two sheets in the first surface, $\tau = \{+1\}$ and $\tau_2 = \{-1\}$, by calling them $\tau_1 = \{+1, +1\}$ and $\tau_2 = \{+1, -1\}$. Correspondingly, we label the sheets τ_3 and τ_4 in the second surface by $\tau_3 = \{-1, +1\}$ and $\tau_4 = \{-1, -1\}$. We now cut all four sheets from $k_1 = -(e_2)^{1/2}$ to $k_1 = +(e_2)^{1/2}$ so that the upper shore of the cut is formed by the real axis, the lower shore of the cut by the points with $\mathrm{Im}\, k_1 < 0$. We identify opposite shores on different sheets as shown in Table I. We see that each sheet of the first surface is connected with a sheet of the second surface, the connection being the same as in the simple case $N = 2$: upper shore of $\{+1, +1\}$ with lower shore of $\{-1, -1\}$, and lower shore of $\{+1, +1\}$ with upper shore of $\{-1, -1\}$, and correspondingly with the other pair of sheets.

We can extend the construction to arbitrary N. We take 2^{N-1} sheets (samples of the k_1 plane). Then, we construct 2^{N-2} identical samples of Riemann surfaces for $N = 2$, each having two sheets and branch points at $\pm(e_N)^{1/2}$. Pairs of such surfaces are connected in analogy to the construction for $N = 3$, cutting the sheets from $-(e_{N-1})^{1/2}$ to $+(e_{N-1})^{1/2}$. We obtain 2^{N-3} surfaces, each containing four sheets. These are again arranged in pairs; the sheets are cut from $-(e_{N-2})^{1/2}$ to $+(e_{N-2})^{1/2}$, and opposite shores are again identified. We continue until all surfaces are connected with each

other. The surface for arbitrary N will have 2^{N-1} sheets, and all its branch points will be simple. The number of branch points w(N) is given by $w(N) = (N-1)2^{N-1}$. This follows from the foregoing construction. We take two identical surfaces of the construction for N-1, and each sheet of the two surfaces gets two new branch points that are identified in pairs. Hence,

$$w(N) = 2w(N-1) + 2^{N-1}, \tag{III.4a}$$

which together with w(1) = 0 and w(2) = 2 yields

$$w(N) = (N-1)2^{N-1}. \tag{III.4b}$$

The topological structure of the Riemann surface U(N) obtained in this manner can be made transparent with the following, topologically equivalent, construction. For N = 1, the complete k_1 plane is topologically equivalent to a sphere. It has genus 0. For N = 2, we take two spheres each with a cut in its surface. They are topologically equivalent to the two cut sheets described in the construction for N = 2. The two cuts are deformed into cylinders, and the rims of the two cylinders are identified. The result is still topologically equivalent to a sphere. For N = 3, we take two pairs of spheres, each sphere having two cuts. The spheres are connected in pairs, and the result is topologically equivalent to a torus. (See Fig. 1). One can easily imagine the structures obtained by continuing this procedure, and by counting one finds that the genus p(N) of the surface U(N) is for the smallest values of N given by p(1) = 0, p(2) = 0, p(3) = 1, p(4) = 5. This can be generalized by the formula (16)

$$p(N) = \frac{w(N)}{2} - 2^{N-1} + 1 = 1 + (N-3)2^{N-2}. \tag{III.5}$$

The sheet $\{+1, \cdots, +1\}$ of U(N) is called the physical sheet. On the positive real axis of the physical sheet, the original definition of the wave

numbers k_i as given in (II.6) applies. This can be seen by taking a value of k_1 on the real axis and to the right of e_N. Then by (II.6) all k_i are positive and real, and this agrees with $\underset{\sim}{\tau} = \{+1, \cdots, +1\}$ according to (III.3). We also call this sheet the physical k_1 plane. Clearly, in all physical applications--i.e., when we are interested in the calculation of the cross sections, for example--we are interested only in the positive real axis of the physical k_1 plane. (The position of the bound states of the system is dealt with later.) Therefore, we show in Fig. 2 the physical k_1 plane with its cuts, and in Fig. 3, the physical k_1 plane for N = 2 and $e_2 = 6$. The small bold-faced numbers on the real and imaginary axis in Fig. 3 give the scale for the k_1 plane. The large italic numbers give the values of k_2 in the physical k_1 plane. The sheet $\{-1\}$ may be obtained by reversing all signs in front of the encircled numbers in Fig. 3. The two sheets are connected through the cut from $-(e_2)^{1/2}$ to $+(e_2)^{1/2}$.

We are interested in sharp peaks in the cross sections. Such peaks may be caused by poles of the S matrix lying close to the real axis. As mentioned above, the cross sections are calculated from the elements of $\underset{\approx}{S}$ for positive real values of k_1. As we shall see later, $\underset{\approx}{S}$ cannot have poles in the upper half of the physical sheet except on the imaginary axis, and the physically most interesting poles must for $k_1 < (e_N)^{1/2}$ lie on the sheets adjacent to the real axis of the physical k_1 plane, i.e., on the sheets that one reaches by crossing the real axis of the physical sheet from positive values of Im k_1. To show these sheets, we display in Fig. 4 the physical plane cut in a different fashion. (It is, of course, irrelevant which way we cut the plane.) The cut connecting $-(e_i)^{1/2}$ with $+(e_i)^{1/2}$ goes via $-a_i^*$ and a_i, $i = 2, \cdots, N$, as shown in the figure. The area outside the rectangle with the corners $\{-(e_N)^{1/2}, +(e_N)^{1/2}, a_N, -a_N^*\}$ is part of the physical k_1 plane shown in Fig. 2. The rectangle itself has been removed from the

physical k_1 plane so that one can see the next sheets adjacent to the physical k_1 axis. The U-shaped areas with corners $\{-(e_i)^{1/2}, -(e_{i-1})^{1/2}, -a^*_{i-1}, a_{i-1}, (e_{i-1})^{1/2}, (e_i)^{1/2}, a_i, -a^*_i\}$, $i = 3, \cdots, N$, and the rectangle with corners $\{-(e_2)^{1/2}, (e_2)^{1/2}, a_2, -a^*_2\}$ are parts of the sheets one enters by crossing the real k_1 axis coming from positive values of Im k_1. The vectors $\underset{\sim}{\tau}$ identifying each sheet have been written into the areas to which they belong. They can easily be determined by the following rule, which is a consequence of the construction of the Riemann surface. The vector $\underset{\sim}{\tau} = \{+1, \cdots, +1\}$ denotes the physical sheet. Its ith element $\tau_i (i = 1, \cdots, N-1)$ will change sign if we cross the real axis between $-(e_{i+1})^{1/2}$ and $+(e_{i+1})^{1/2}$. If we cross the real axis to the right of $+(e_{i+1})^{1/2}$, or to the left of $-(e_{i+1})^{1/2}$, τ_i remains unchanged.

On the surface U(N), discussed at length in this subsection, the function $t(k_1)$ [see (III.2)] is a single-valued function, holomorphic everywhere except at the 2^{N-1} points where $k_1 = \infty$, at each of which it has a simple pole unless the β_i have special properties like

$$\sum_{i=1}^{N} \beta_i = 0.$$

B. Analytical Properties of the S Matrix on the Riemann Surface

In Section III.A, we have constructed the Riemann surface U(N) for the algebraic function $t(k_1)$ [see (III.2)]. On this surface, any function depending on the variables $k_i(k_1)$, $i = 2, \cdots, N$, and k_1 itself will be a single-valued function unless it has further branch points not present in $t(k_1)$. In this section we show that the elements of $\hat{\underset{\approx}{S}}(k_1)$ are single-valued meromorphic functions on U(N) except for a finite number of points. Here, $\hat{\underset{\approx}{S}}(k_1)$ is related to $\underset{\approx}{S}(k_1)$ by the relation

$$\hat{S}_{m\ell} = (k_\ell/k_m)^{1/2} S_{m\ell} \cdot \qquad \text{(III.6)}$$

In studying $\hat{\underset{\approx}{S}}$ instead of $\underset{\approx}{S}$, we do not have to consider further branch points occurring at $k_i = 0$, $i = 1, \cdots, N$ that can be trivially excluded by (III.6).

From the definitions of $\underset{\approx}{S}$ in (II.19) and of $\underset{\approx}{E}$ in (II.16), it is clear that the elements of $\hat{\underset{\approx}{S}}$ are single-valued holomorphic functions on U(N) except for the points $k_1 = k_1^a$, where $\det \underset{\approx}{E}(k_1^a) = 0$ ($\det \underset{\approx}{E}$ means the determinant of the matrix $\underset{\approx}{E}$), and for $k_1 = \infty$, where $\underset{\approx}{E}$ and $\underset{\approx}{S}$ have an essential singularity.[1] No further branch points occur, since the functions $(\xi + \lambda_m)^{1/2} \cot (\xi + \lambda_m)^{1/2} a$ are even in k_1. The points k_1^a have $k_1 = \infty$ as their only point of accumulation: $\det \underset{\approx}{E}(k_1)$ is from (II.16) a nonconstant meromorphic function on U(N) except at $k_1 = \infty$, and its zeros are therefore isolated for all finite k_1. We exclude the 2^{N-1} points $k_1 = \infty$ (one on each sheet) from the surface U(N) and call the resulting surface $\hat{U}(N)$. On $\hat{U}(N)$, then, the elements of $\hat{\underset{\approx}{S}}$ are meromorphic functions with isolated poles of finite order whenever $\det \underset{\approx}{E} = 0$.[1] In the neighborhood of any point on $\hat{U}(N)$, we can expand each element of $\hat{\underset{\approx}{S}}$ in a Laurent series with respect to the parameter of local uniformalization. For $k_1 \neq \pm(e_i)^{1/2}$, $i = 2, \cdots, N$, the variable k_1 itself may be chosen as parameter of local uniformalization. At $k_1 = \pm(e_i)^{1/2}$, i.e., $k_i = 0$, we may choose k_i as the parameter of local uniformalization. If $k_1 = (e_i)^{1/2} + \Delta_1$, then $k_i = (k_1^2 - e_i)^{1/2} = [2(e_i)^{1/2}\Delta_1 + \Delta_1^2]^{1/2}$ can obviously not be expanded in powers of Δ_1. On the other hand, if we put $k_i = \Delta_i$ with Δ_i small in the neighborhood of $k_i = 0$, or $k_1 = (e_i)^{1/2}$, we find $k_\ell = (k_i^2 + e_i - e_\ell)^{1/2} = (e_i - e_\ell)^{1/2} + [\Delta_i^2/2(e_i - e_\ell)^{1/2}] + \cdots$ for all $\ell \neq i$. Therefore, all $k_\ell \neq k_i$ can be expanded in powers of k_i, and the same statement holds true for the quantities $(\xi + \lambda_m)^{1/2}$ for all m. It is obvious that all these expansions are even in k_i, and therefore all elements of $\underset{\approx}{E}$ are even in k_i except for the ith diagonal element that contains k_i linearly. Hence,

terms linear in Δ_i occur in $\hat{\underset{\approx}{S}}$ only through its explicit dependence on k_i.

We see that our choice of the potential matrix $\underset{\approx}{V}(x)$ in Section II.B causes the matrix $\hat{\underset{\approx}{S}}$ to have only slightly more complicated analytical properties than an algebraic function. Its elements are meromorphic functions on $\hat{U}(N)$ instead of $U(N)$. Of course, the simple formula (III.5) applies only to the genus of $U(N)$, not of $\hat{U}(N)$. However, the numbers of branch points and sheets are the same on $U(N)$ and $\hat{U}(N)$. Thus, the elements of $\hat{\underset{\approx}{S}}$ have branch points only where they are necessary for kinematical reasons, and essential singularities only at $k_1 = \infty$. Our case is only slightly more complicated than the one studied by Peierls (10) and LeCouteur (11); on the other hand, we are in the fortunate position to be able to connect the properties of $\hat{\underset{\approx}{S}}$ and, in particular, its poles with properties of the system of differential equations (II.4).

In some special cases, the elements of $\hat{\underset{\approx}{S}}$ may have the same value on two or more different sheets of $\hat{U}(N)$. By this, we mean that for k_1 fixed, $\underset{\sim}{\tau}$ changed, the elements of $\hat{\underset{\approx}{S}}$ do not change their values. If, for example, $\underset{\approx}{V}(x)$ is a diagonal matrix, then $\hat{\underset{\approx}{S}}$ is diagonal. Its ith element depends on E and k_i only. The sheets of $\hat{U}(N)$ then belong to two classes, one for which the (i-1)st element of $\underset{\sim}{\tau}$ is positive, and one for which it is negative. On all sheets of one class, $\hat{S}_{ii}$ has the same value. Similar things happen if $\underset{\approx}{V}(x)$ does not couple a whole group of channels to the rest. In these cases, it should have been sufficient to construct surfaces much simpler than $\hat{U}(N)$. Since we want to deal with the most general case, however, we shall always consider $\hat{\underset{\approx}{S}}$ on $\hat{U}(N)$--even if sometimes we face unnecessary complications.

C. Other Properties of $\hat{\underset{\approx}{S}}$ and of the Solutions of (II.4) on $\hat{U}(N)$

In this subsection, we want to define the solutions of the differential equations (II.4) on $\hat{U}(N)$ and to investigate which of their properties and which properties of $\hat{\underset{\approx}{S}}$ listed in Sections II.B and II.C remain valid on $\hat{U}(N)$.

Writing (II.7) in the form

$$\hat{f}_m^{(\ell)}(x) = (v_\ell)^{1/2} f_m^{(\ell)}(x) = [\exp(-ik_m x)]\,\delta_{m\ell} - \hat{S}_{m\ell}\,[\exp(+ik_m x)] \qquad \text{(III.7)}$$

we see that the functions $\hat{f}_m^{(\ell)}(x)$ are well defined on all points of $\hat{U}(N)$ for which the elements of $\hat{\underset{\approx}{S}}$ are holomorphic. The functions $\underset{\sim}{G}(x)$ of Eqs. (II.13) are holomorphic everywhere on $\hat{U}(N)$, with $\bar{\underset{\sim}{G}}$ constant. For any two solutions $\underset{\sim}{F}^{(1)}(x)$ and $\underset{\sim}{F}^{(2)}(x)$ of Eq. (II.4) with $\underset{\sim}{F}^{(1)}(0) = 0 = \underset{\sim}{F}^{(2)}(0)$, Green's theorem (II.25) remains valid on $\hat{U}(N)$.

Equation (II.20) can be generalized to read ($\epsilon_i = \pm 1$, $i = 1, \cdots, N$)

$$E_{\ell m}(\epsilon_1 k_1, \epsilon_2 k_2, \cdots, \epsilon_N k_N) - E_{\ell m}(+k_1, +k_2, \cdots, +k_N) = ik_m(1-\epsilon_m)\delta_{\ell m} \qquad \text{(III.8)}$$

and in this form holds on $\hat{U}(N)$ except for the points where $\sin(\xi+\lambda_m)^{1/2}a = 0$. The two functions $\hat{S}_{m\ell}$ and $\hat{S}_{\ell m}$ are, according to (II.27) and (III.6), related by

$$k_m \hat{S}_{m\ell} = k_\ell \hat{S}_{\ell m}, \qquad \text{(III.9)}$$

and this holds true for all points of $\hat{U}(N)$ for which $\hat{S}_{\ell m}$ is holomorphic. Inspection of $\underset{\approx}{E}$ and $\underset{\approx}{S}$ shows that Eq. (II.28) also remains valid at these points. Figure 3 shows

$$-k_i^*(k_1) = k_i(-k_1^*), \quad i = 2, \cdots, N, \qquad \text{(III.10)}$$

and we can rewrite Eq. (II.28) in the form

$$S_{m\ell}^*(k_1) = S_{m\ell}(-k_1^*) \text{ on the same sheet.} \qquad \text{(III.11)}$$

(Note that this relation holds only for $\underset{\approx}{S}$ and not for $\hat{\underset{\approx}{S}}$!)

Finally, the unitarity relations can be continued analytically, as follows: For k_1 on the positive real axis of the physical sheet, (II.32) can be written as

$$\sum_{m \text{ open}} k_m \hat{S}_{m\ell} \hat{S}^*_{mt} = k_\ell \delta_{\ell t} \text{ if } \ell, t \text{ open.} \tag{III.12}$$

Equation (III.11) implies $\hat{S}^*_{mt}(k_1, \cdots, k_N) = \hat{S}_{mt}(-k_1^*, \cdots, -k_N^*)$ if k_1 is real and m and t are open. Let j be the last open channel, so that j+1 is closed. Then, $-k_i^* = -k_i$ for $i \leq j$, and $-k_i^* = +k_i$ for $i > j$ and k_1 on the positive real axis of the physical sheet. We obtain, from (III.12),

$$\sum_{m \text{ open}} k_m(k_1) \hat{S}_{m\ell}(k_1, \cdots, k_N) \hat{S}_{m\ell}(-k_1, \cdots, -k_j, +k_{j+1}, \cdots, +k_N) = k_\ell(k_1) \quad \text{for } \ell \text{ open.} \tag{III.13}$$

Now, $\hat{S}_{m\ell}(-k_1, \cdots, -k_j, +k_{j+1}, \cdots, +k_N)$ is the value of $S_{m\ell}$ at the point $-k_1$ of the real axis of the physical sheet. Having realized this fact, we may continue (III.13) to all points of $\hat{U}(N)$ for which $\hat{\underset{\approx}{S}}$ is holomorphic. If, for example, we continue k_1 along path 1, shown in Fig. 5 (which is an enlarged copy of Fig. 4), and take the values of $S_{m\ell}(k_1)$ along this path, then $-k_1$ follows the dotted path 1, and we have to take values of $S_{m\ell}(k_1)$ along this dotted path. This shows that if we continue the solid path 1 back to the real positive axis of the physical sheet where $k_1 > (e_3)^{1/2}$, the dotted path will have arrived at the negative real axis of the sheet $\{+1, -1, \cdots, -1\}$, and for such values of k_1 the Eq. (III.13)--although still valid--will no longer establish a relationship between measurable quantities. The same happens as we continue along paths 2 or 3. The fact that Eq. (III.13) can be continued to all points of holomorphy of $\hat{\underset{\approx}{S}}$ on $\hat{U}(n)$ means that there exist many relationships between the elements of $\hat{\underset{\approx}{S}}$ on various points of $\hat{U}(N)$. From the symmetry of $\underset{\approx}{S}$, we have $N(N+1)/2$ independent analytical functions $\hat{S}_{m\ell}$ on $\hat{U}(N)$. If n channels are open, the symmetric unitarity relations give $n(n+1)/2$ relations between these functions, and the total numbers of relations is equal to

$(1/2)\sum_{n=1}^{N} n(n+1) = (1/6)\,N(N+1)(N+2)$. It is clear that these relations that hold on all points of holomorphy of $\hat{\underset{\approx}{S}}$ on $\hat{U}(N)$ are very strong conditions on the analytical behavior of $\hat{\underset{\approx}{S}}$ on $\hat{U}(N)$. Equations (III.13) can be derived explicitly in analogy to the procedure in (II.31) by making use of Eq. (III.8).

For later application, let us write down the behavior of the cross sections (II.24) at threshold, i.e., in the neighborhood of $k_j = 0$, if $\det \underset{\approx}{E} \neq 0$. The influence of an opening channel upon cross sections in other channels has been studied by Newton (7) in a related case. We confine ourselves, therefore, to cross sections that have the opening channel as entrance (or exit) channel. Let $1, \cdots, j-1$ be open and $j+1, \cdots, N$ be closed channels. We expand the cross sections (II.24) in powers of k_j, for k_j small and positive. According to (II.26), we have, for $a = 0, -1, \cdots, (-j+1)$,

$$\sigma_{j\to j+a} = |\delta_{a,0}\{1-[\exp(-2ik_j a)]\} - [\exp(-ik_j a - ik_{j+a} a)]\,[2i(k_j k_{j+a})^{1/2} E^{-1}_{jj+a}]\,|^2. \tag{III.14}$$

Expanding E^{-1}_{jj+a} and the wave number k_{j+a} in powers of k_j, we obtain

$$\sigma_{j\to j+a} = \sigma_{j+a\to j} = 4\,k_j k_{j+a}\,|\,a\,\delta_{a,0} - E^{-1}_{j,j+a}(k_j = 0)\,|^2. \tag{III.15}$$

The elastic cross section at threshold is proportional to k_j^2. The inelastic cross sections are proportional to k_j unless $E^{-1}_{jj+a}(k_j = 0)$ happens to be zero, in which case they are proportional to higher powers of k_j.

IV. POLES OF THE S MATRIX

We now turn our attention to those isolated points on $\hat{U}(N)$ where $\det \underset{\approx}{E} = 0$. We generally assume that the determinant has simple zeros in the parameter of local uniformalization, and we consider the case of zeros of higher order only occasionally. If a zero is not simple, it is in general possible to change the potential parameters slightly so that it decomposes into simple zeros. We show that at least one element of $\hat{\underset{\approx}{S}}$ has a pole wherever $\det \underset{\approx}{E} = 0$, except for those points on $\hat{U}(N)$ for which $k_i = 0$ for one i, with $i = 1, \cdots, N$. In most of this section we are not concerned with these points, which will receive special attention in the last subsection. In the following subsections, we refer to the remainder of the surface $\hat{U}(N)$ simply as $\hat{U}(N)$ without having explicitly to exclude the branch points and $k_1 = 0$ each time.

A. Zeros of the Determinant of $\underset{\approx}{E}$

The system of equations (II.14) has a homogeneous solution [i.e., a solution where the right-hand side of (II.14) is identically zero] wherever its determinant vanishes. By a straightforward calculation, this determinant D' can be shown to be equal to

$$D' = \left[\prod_{i=1}^{N} \frac{\sin(\xi + \lambda_i)^{1/2} a}{(\xi + \lambda_i)^{1/2}} \right] \det \underset{\approx}{E} . \qquad \text{(IV.1a)}$$

The functions $\sin(\xi + \lambda_s)^{1/2} a$ can vanish only on the axes of the sheets of $\hat{U}(N)$. Thus, except for the rare cases in which a zero of $\sin(\xi + \lambda_s)^{1/2} a$ coincides with a zero of D' [i.e., where $\det \underset{\approx}{E}$ remains finite even though at least one of the functions $\cot(\xi + \lambda_s)^{1/2} a$ becomes infinite], the zeros of $\det \underset{\approx}{E}$ indicate the points $k_1 = k_1^a$ on $\hat{U}(N)$ for which a homogeneous solution of (II.14) exists. Still, for completeness, we henceforth consider the zeros of D', which is obviously holomorphic on $\hat{U}(N)$. For reasons

that will become apparent later, it is advantageous to consider the function

$$D = D' \prod_{j=1}^{N} \exp(+ik_j a) \qquad \text{(IV.1b)}$$

which is also holomorphic on $\hat{U}(N)$ and the zeros of which coincide with those of D'.

If $\det \underset{\approx}{E}(k_1^a) = 0$, then $\det^* \underset{\approx}{E}(k_1^a) = \det \underset{\approx}{E}^*(k_1^a) = \det \underset{\approx}{E}(-k_1^{a*}) = 0$, according to Eq. (III.11), which holds also for $\underset{\approx}{E}$. This means that whenever there is a zero of D at a point k_1^a of a sheet of $\hat{U}(N)$, there is also a zero at the point $(-k_1^a)^*$ on the same sheet. On all sheets of $\hat{U}(N)$, the zeros of D lie symmetric with respect to the imaginary axis.

Let $\underset{\sim}{H}^a(x)$ denote the solution of the differential equations (II.4) at $k_1 = k_1^a$ that fulfills the homogeneous boundary conditions resulting from (II.14) by putting the right-hand side equal to zero: With $\underset{\sim}{H}^a(x)^T = \{h_1^a(x), \cdots, h_N^a(x)\}$, we have

$$\frac{d}{dx} h_j^a(x)\Big|_{x=a} - ik_j^a h_j^a(a) = 0, \quad j = 1, \cdots, N, \qquad \text{(IV.2)}$$

which is equivalent to the homogeneous part of (II.14) if we express the $h_i(x)$ for $x < a$ in terms of the functions $g_i(x)$ defined in (II.13). Here, k_j^a is the value of k_j at $k_1 = k_1^a$. Equations (II.4) and (IV.2) imply that for $b \geq a$, $h_j^a(b)$ is given by

$$h_j^a(b) = h_j^a(a) \exp[ik_j^a(b-a)] \quad . \qquad \text{(IV.3)}$$

With $\underset{\sim}{H}^a(x)$ at k_1^a, $\underset{\sim}{H}^{*a}(x)$ is the homogeneous solution that must exist at $(-k_1^a)^*$. This can be seen by taking the complex conjugate of Eq. (II.4) and of Eq. (IV.2) and by remembering (III.10). We write $\xi^a = (k_1^a)^2$ for the energy, put $\Gamma^a = -2 \operatorname{Im} \xi^a$, and apply Green's theorem (II.25) with

$\underset{\sim}{F}^{(1)}(x) = \underset{\sim}{H}^{a}(x)$, and $\underset{\sim}{F}^{(2)}(x) = \underset{\sim}{H}^{*a}(x)$. We get for $b \geq a$, with the help of (IV.3),

$$\Gamma^{a} = \sum_{j=1}^{N} \operatorname{Re} k_j^a \, |h_j^a(b)|^2 \Big/ \int_0^b |\underset{\sim}{H}^{a}(x)|^2 \, dx \quad \text{for all } b \geq a. \qquad \text{(IV.4)}$$

From (IV.4), we can see that there cannot be any zeros of D in the upper half of the physical sheet ($\operatorname{Im} k_1 > 0$) except on the imaginary axis (4): For $\operatorname{Re} k_1 \lesseqgtr 0$ and $\operatorname{Im} k_1 > 0$, we have from $\Gamma^a = -4 \operatorname{Re} k_1^a \operatorname{Im} k_1^a$ that $\Gamma^a \gtreqless 0$. On the other hand, on the physical sheet, sign $\operatorname{Re} k_j$ = sign $\operatorname{Re} k_1$ according to (III.3), and hence from (IV.4), $\Gamma^a \lesseqgtr 0$, which is a contradiction. This conclusion does not hold anywhere else on $\hat{U}(N)$, since the factors $\operatorname{Re} k_j^a$ in (IV.4) have different signs. From $\Gamma^a = -4 \operatorname{Re} k_1^a \operatorname{Im} k_1^a$, we see that $\Gamma^a = 0$ if k_1^a is on the real and imaginary axis, $\Gamma > 0$ if k_1^a is in the second and fourth, and $\Gamma^a < 0$ if k_1^a is in the first and third quadrant of any sheet of $\hat{U}(N)$. Hence, poles of $\hat{\underset{\approx}{S}}$ on sheets adjacent to the positive real axis of the physical sheet will lie in the fourth quadrant and will give rise to resonances with positive widths, as should be expected.

We now show that all bound states of the system correspond to zeros of D on the positive imaginary or positive real axis of the physical sheet, and that in turn a zero of D on these two semiaxes implies the existence of a bound state. Indeed, a bound state has a normalizable wave function. Hence, from (IV.3), we must either have k_j^a positive imaginary, or $h_j^a(a) = 0$. If all $h_j^a(a) \neq 0$, all k_j^a, $j = 1, \cdots, N$, must be positive imaginary, and the bound state corresponds to a zero of D on the positive imaginary axis of the physical sheet. If $h_j^a(a) = 0$ for $j = j_1, \cdots, j_n$, we must have $n < N$. Otherwise $\underset{\sim}{H}^{a}(x)$ vanishes identically from (IV.2) and (II.4). The wave numbers k_j^a with $j \neq j_1, \cdots, j_n$ must again be positive imaginary;

the solution $\underset{\sim}{H}^a(x)$ depends only on these wave numbers and not on the k_j^a with $j = j_1, \cdots, j_n$. This can be seen from (IV.2): If $h_j^a(a) = 0$, then (IV.2) reads $\frac{d}{dx} h_j^a(x)\big|_{x=a} = 0$, and k_j^a does not enter the formulation of the problem at all. Hence, D vanishes either on the positive imaginary or on the real axis of the physical sheet. In the latter case, the channels j with $j \neq j_1, \cdots, j_n$ are closed, and we encounter bound states embedded in the continuum (7, 9, 17). That the zeros of D on the real or imaginary axis of the physical sheet imply the existence of a bound state can be shown if we can prove that $h_j^a(a)$ must be zero if j is open. This proof is given in another connection in Section IV. E.

Bound states embedded in the continuum cause zeros of D to occur in pairs at points k_1^a lying on the real axis symmetric to the imaginary axis. Indeed, the transformation $k_1^a \to (-k_1^a)^* = -k_1^a$ leaves -- according to (III.10) -- the k_j^a with $j \neq j_1, \cdots, j_n$ unchanged, and the same holds for the function $\underset{\sim}{H}^a(x)$, which is real. Moreover, if $h_j^a = 0$ for one j and if D = 0 on one sheet with label $\underset{\sim}{\tau}$, then D = 0 on the sheet with label $\underset{\sim}{\tau}'$, differing from $\underset{\sim}{\tau}$ only in the (j-1)st element; and correspondingly, if $h_j^a(a) = 0$ for more than one j. This follows again from (IV.2). Therefore, bound states on the real axis of the physical sheet always cause zeros of D to occur on the real axis of at least one more sheet. The number of bound states is finite. Indeed, since D has isolated zeros on $\hat{U}(N)$, their only point of accumulation could be $k_1 = \infty$. Since bound states embedded in the continuum cannot occur on the real axis of the physical sheet for $|k_1| > (e_N)^{1/2}$ (all k_i's are then real), we have only to show that there exists a finite $R > 0$, so that for $\mathrm{Im}\, k_1 > R$ there cannot be any zeros of D on the imaginary axis of the physical sheet. Let $R^2 = \max\{|\lambda_1|, \cdots, |\lambda_N|\}$ and let k_1 be purely imaginary, so that $\mathrm{Im}\, k_1 > R$. Then, $\xi < -|\lambda_s|$ for

all s and $(\xi + \lambda_s)^{1/2} a \times \cot(\xi + \lambda_s)^{1/2} a > 1$ for all s. Then $\underset{\approx}{E}$ is a real and symmetric matrix; and it is easily seen that it is positive definite. (This is true only on the positive imaginary axis of the physical sheet.) The determinant of $\underset{\approx}{E}$ thus cannot vanish for Im $k_1 > R$; hence $D \neq 0$.

All zeros of D not discussed so far do not correspond to bound states. For these zeros and for $x \geq a$, the homogeneous solution $\underset{\sim}{H}^a(x)$ does not consist only of outgoing waves. If, for example, $D = 0$ on the positive imaginary axis of the sheet $\underset{\sim}{\tau} = \{+1, \cdots, +1, -1\}$ and $h_N^a(a) \neq 0$, then, for $x \geq a$, $\underset{\sim}{H}^a(x)$ consists of purely outgoing (exponentially decreasing) waves in channels 1, ..., N-1 and of a purely incoming (exponentially increasing) wave in channel N. Therefore, the homogeneous solutions $\underset{\sim}{H}^a(x)$ are distinct only in the sense that in each channel j, there is one type of solution only, $\exp(+ik_j^a x)$ or $\exp(-ik_j^a x)$.

Those homogeneous solutions $\underset{\sim}{H}^a(x)$ that do not describe bound states are said to describe resonant or compound states. In view of the complexity of $\hat{U}(N)$, we do not want to introduce the distinction between virtual and resonant states often employed in single-channel potential-scattering theory. Not all the resonant states produce sharp peaks in the cross section, of course, but we prefer to have a common name for a certain type of mathematical behavior.

B. Zeros of D and Poles of $\hat{\underset{\approx}{S}}$

It is clear that the elements of $\hat{\underset{\approx}{S}}$ can have poles only if D has a zero. In this subsection, we want to show that the reverse statement is also true. If D has a simple zero at $k_1 = k_1^a$, then at least one element of $\hat{\underset{\approx}{S}}$ has a pole, and this pole is simple, too. Furthermore, the residues of all elements $\hat{S}_{i\ell}$ of $\hat{\underset{\approx}{S}}$ at $k_1 = k_1^a$ can be written as a product of two factors, one factor depending only on the entrance channel ℓ, the other only on the exit channel i. This implies that in the neighborhood of k_1^a, $\hat{S}_{i\ell}$ can be written as (4)

$$\hat{S}_{j\ell} - \delta_{j\ell} = i\left(\frac{k_\ell^a}{k_j^a}\right)^{1/2} \cdot \left[\frac{\gamma_j^a \gamma_\ell^a}{\mu_{NN}(k_1 - k_1^a)}\right] + \hat{d}_{j\ell} + \hat{e}_{j\ell}(k_1 - k_1^a) + \cdots . \quad \text{(IV.5)}$$

We exclude the points k_1^a, where $\sin(\xi + \lambda_s)^{1/2} a = 0$ for at least one s, and use Eq. (II.26) for $\underset{\approx}{S}$. (These points are considered in Section IV.E). According to Eq. (II.18), the homogeneous part of Eq. (II.14) can be written as

$$\sum_m E_{\ell m} \tilde{c}_m = 0. \quad \text{(IV.6)}$$

Since det $\underset{\approx}{E}$ has a simple zero at $k_1 = k_1^a$, the solution $\underset{\sim}{\tilde{C}}$ of (IV.6) is determined uniquely except for a factor. The constants $\tilde{c}_m$ are in general complex, and at least one of them is different from zero. We choose N-1 linearly independent vectors $\underset{\sim}{Q}^{(1)}, \ldots, \underset{\sim}{Q}^{(N-1)}$, so that with $\underset{\sim}{Q}^{(N)} = \underset{\sim}{\tilde{C}}$, the N vectors $\underset{\sim}{Q}^{(1)}, \ldots, \underset{\sim}{Q}^{(N)}$ are linearly independent vectors with determinant 1. Let the elements of the matrix $\underset{\approx}{Q}$ be defined by

$$Q_{i\ell} = Q_i^{(\ell)}, \ \det \underset{\approx}{Q} = 1. \quad \text{(IV.7)}$$

To calculate (II.26), we consider $\underset{\approx}{E}^{-1}$ for values $k_1 \neq k_1^a$, and $|k_1 - k_1^a| < \epsilon$. Here, ϵ is chosen so that for all $|k_1 - k_1^a| < \epsilon$ and $k_1 \neq k_1^a$ the function D has no zeros. Denoting the algebraic complement (18) of an element $E_{i\ell}$ of $\underset{\approx}{E}$ by $\tilde{E}_{i\ell}$, we have

$$(E^{-1})_{i\ell} = \frac{\tilde{E}_{i\ell}}{\det \underset{\approx}{E}} = \left(\underset{\approx}{Q}\, \underset{\approx}{Q}^{-1}\, \underset{\approx}{E}^{-1} (\underset{\approx}{Q}^T)^{-1}\, \underset{\approx}{Q}^T\right)_{i\ell} = \left(\underset{\approx}{Q}\left[\underset{\approx}{Q}^T\, \underset{\approx}{E}\, \underset{\approx}{Q}\right]^{-1} \underset{\approx}{Q}^T\right)_{i\ell}$$

$$= \left(\underset{\approx}{Q}\, \frac{\widetilde{\underset{\approx}{Q}^T\, \underset{\approx}{E}\, \underset{\approx}{Q}}}{\det \underset{\approx}{E}}\, \underset{\approx}{Q}^T\right)_{i\ell} . \quad \text{(IV.8)}$$

Thus, instead of calculating the inverse of $\underset{\approx}{E}$ directly, we calculate the inverse of $\underset{\approx}{Q}^T \underset{\approx}{E} \underset{\approx}{Q}$, and from it the inverse of $\underset{\approx}{E}$.

The matrix $\underset{\approx}{Q}^T \underset{\approx}{E} \underset{\approx}{Q}$ has the following properties. At $k_1 = k_1^a$, the elements in its last row and column are equal to zero. This follows from Eq. (IV.6) and from the definition of $\underset{\approx}{Q}$. Therefore, we can write

$$\left(\underset{\approx}{Q}^T \underset{\approx}{E} \underset{\approx}{Q}\right)_{iN} = \mu_{iN} \times (k_1 - k_1^a)^{\beta_{iN}} + \cdots, \text{ for } i = 1, \cdots, N, \tag{IV.9}$$

and

$$\left(\underset{\approx}{Q}^T \underset{\approx}{E} \underset{\approx}{Q}\right)_{Ni} = \mu_{Ni} \times (k_1 - k_1^a)^{\beta_{Ni}} + \cdots, \text{ for } i = 1, \cdots, N,$$

where μ_{Ni} and μ_{iN} are different from zero, and β_{iN} and β_{Ni} are positive integers. Making use of the assumption that det $\underset{\approx}{E}$ has a simple zero at $k_1 = k_1^a$ and of the fact that det $\underset{\approx}{Q}^T \underset{\approx}{E} \underset{\approx}{Q}$ = det $\underset{\approx}{E}$, we see that $\beta_{NN} = 1$ and that the determinant d of the matrix obtained from $\underset{\approx}{Q}^T \underset{\approx}{E} \underset{\approx}{Q}$ by omitting its last row and column cannot vanish at $k_1 = k_1^a$. Indeed,

$$\det \underset{\approx}{E} = \det\left(\underset{\approx}{Q}^T \underset{\approx}{E} \underset{\approx}{Q}\right) = \mu_{NN} \times d \times (k_1 - k_1^a)^{\beta_{NN}} + \cdots, \tag{IV.10}$$

where, from (IV.9), it can be seen that the dots indicate terms at least quadratic in $(k_1 - k_1^a)$. Similarly, we can infer from (IV.9) that the algebraic complements of all elements of $\underset{\approx}{Q}^T \underset{\approx}{E} \underset{\approx}{Q}$ are at least linear in $(k_1 - k_1^a)$ except for $\left(\widetilde{\underset{\approx}{Q}^T \underset{\approx}{E} \underset{\approx}{Q}}\right)_{NN}$, which by definition equals $d \neq 0$ at $k_1 = k_1^a$. We see that the only element of $\left(\underset{\approx}{Q}^T \underset{\approx}{E} \underset{\approx}{Q}\right)^{-1}$ that has a pole is the N-N element. Hence we write

$$\widetilde{\left(\frac{\underset{\approx}{Q}^T \underset{\approx}{E} \underset{\approx}{Q}}{\det \underset{\approx}{E}}\right)}_{i\ell} = \delta_{iN}\,\delta_{\ell N}\,\frac{1}{\mu_{NN}(k_1-k_1^a)} + \text{terms holomorphic at } k_1 = k_1^a. \tag{IV.11}$$

For $\underset{\approx}{E}^{-1}$, this gives, from (IV.8),

$$E_{i\ell}^{-1} = \frac{Q_{iN}\,Q_{\ell N}}{\mu_{NN}(k_1-k_1^a)} + \text{terms holomorphic at } k_1 = k_1^a, \tag{IV.12}$$

and from (II.26) and the definition of Q_{iN}, we obtain for $\hat{\underset{\approx}{S}}$ the expression (IV.5), where we have used

$$\gamma_j^a = \bar{c}_j\,[\exp(-ik_j^a a)]\,(2k_j^a)^{1/2}. \tag{IV.13}$$

According to Eqs. (II.22) and (II.23), the $\bar{c}_j$'s are just the values of the homogeneous solution $\underset{\sim}{H}^a(x)$ at $x = a$ in channel j, and we have

$$\gamma_j^a = [\exp(-ik_j^a a)]\,(2k_j^a)^{1/2}\,h_j^a(a). \tag{IV.14}$$

From Eq. (IV.3), it is clear that γ_j^a will not change its value if we replace a by $b > a$ in (IV.14). Hence, we expect μ_{NN} to have the same property. This will be borne out soon.

Equation (IV.5) shows that in the vicinity of $k_1 = k_1^a$, the matrix $\underset{\approx}{S}$ may be approximated by a simple Breit-Wigner formula. This would not hold if D had a zero of second or higher order at $k_1 = k_1^a$, since then we would in general have at least two linearly independent solutions of (IV.6), and by repeating the arguments one sees that the residues of $\underset{\approx}{S}$ could not be factorized in the form (IV.5).

It should be noted that the constants $\hat{d}_{j\ell}$, $\hat{e}_{j\ell}$, ··· defined by Eq. (IV.5) are independent of the choice of $\underset{\sim}{Q}^{(1)}$, ···, $\underset{\sim}{Q}^{(N-1)}$. The function

$$T_{j\ell} = \hat{S}_{j\ell} - \delta_{j\ell} - i\left(\frac{k_\ell^a}{k_j^a}\right)^{1/2}\left[\frac{\gamma_j^a \quad \gamma_\ell^a}{\mu_{NN}(k_1 - k_1^a)}\right] \tag{IV.15}$$

is holomorphic at $k_1 = k_1^a$ and obviously independent of $\underset{\approx}{Q}^{(1)}, \cdots, \underset{\approx}{Q}^{(N-1)}$. Its series expansion at k_1^a defines the coefficients $\hat{d}_{j\ell}$, $\hat{e}_{j\ell}, \cdots$ uniquely.

The constant μ_{NN} can be expressed in terms of the homogeneous solution by (4)

$$\frac{1}{2k_1^a}\ \mu_{NN} = -\int_0^a \left[\underset{\sim}{H}^a(x)\right]^2 dx - \sum_{j=1}^{N} i\left[h_j^a(a)\right]^2 \frac{d}{d\xi}\ k_j\ |_{\xi=\xi^a} \tag{IV.16}$$

It is defined by

$$\mu_{NN} = 2k_1^a\ \frac{d}{d\xi}\left\{\underset{\approx}{Q}^T\ \underset{\approx}{E}\ \underset{\approx}{Q}\right\}_{NN}\ |_{\xi=\xi^a}\ . \tag{IV.17}$$

Inserting $Q_{iN} = h_i^a(a)$, which is independent of the energy, and inserting (II.16) for $\underset{\approx}{E}$, one finds

$$\frac{1}{2k_1^a}\ \mu_{NN} =$$

$$\frac{1}{2}\ \sum_m \left[\sum_j 0_{mj} h_j^a(a)\right]^2 \left\{\frac{[\cos(\xi^a + \lambda_m)^{1/2} a\ \sin(\xi^a + \lambda_m)^{1/2} a - (\xi^a + \lambda_m)^{1/2} a]}{(\xi^a + \lambda_m)^{1/2}\ \sin^2(\xi^a + \lambda_m)^{1/2} a}\right\}$$

$$- i\ \sum_m \left[h_m^a(a)\right]^2\ \frac{d}{d\xi}\ k_m\ |_{\xi=\xi^a}\ . \tag{IV.18}$$

This expression is equal to (IV.16), as can be seen by inserting $\underset{\sim}{H}^a(x)$ from Eq. (II.22) into (IV.16) and carrying out the integration. The resulting

expression is very satisfactory. Namely, μ_{NN} is quadratic in $\underset{\sim}{H}^a(x)$. This is expected in view of (IV.5). The γ_j^a are linear in $\underset{\sim}{H}^a$; on the other hand, the residue of $\hat{S}_{j\ell}$ is expected to be independent of the normalization of $\underset{\sim}{H}^a$. Furthermore, μ_{NN} does not change its value if in (IV.16) we replace a by b > a. This can be shown by an explicit calculation of $\int_a^b [\underset{\sim}{H}^a(x)]^2 dx$ with the help of (IV.3); the change of the value of the first term in μ_{NN} is compensated for by the change of value of the second term. Thus μ_{NN} shares this property with the γ_j^a.

C. Homogeneous and Inhomogeneous Solutions (12) in a Neighborhood of a Pole of $\hat{\underset{\approx}{S}}$

In the preceding subsection, we have shown that poles of $\hat{\underset{\approx}{S}}$ are intimately related to the homogeneous solutions $\underset{\sim}{H}^a(x)$. We now want to elucidate the physical and mathematical background of this relationship. In particular, we want to show what happens to any one of the physically significant solutions $\underset{\sim}{F}^{(\ell)}(x)$ defined in Section II. A if we continue it analytically to a point $k_1 = k_1^a$ on $\hat{U}(N)$. Again, we disregard the points where $\sin(\xi+\lambda_m)^{1/2}a = 0$.

Our idea is the following: The differential equations (II.4) with the condition $\underset{\sim}{F}(0) = 0$ have N linearly independent solutions. For $k_1 \neq k_1^a$, the homogeneous equations (IV.6) do not have a nontrivial solution. Therefore, we can choose the inhomogeneous boundary conditions (II.14) to define the N linearly independent solutions $\underset{\sim}{F}^{(\ell)}(x)$. In other words, the solutions $\underset{\sim}{F}^{(\ell)}(x)$ defined by Eq. (II.7) can be continued analytically on the whole surface $\hat{U}(N)$ except for the points $k_1 = k_1^a$ (see Section III. C).

In order to discuss the behavior of the $\underset{\sim}{F}^{(\ell)}(x)$ as k_1 approaches k_1^a, we define a different system of linearly independent solutions $\underset{\sim}{M}^{(\ell)}(x)$, the behavior of which we know as $k_1 \to k_1^a$. We shall then express the $\underset{\sim}{F}^{(\ell)}(x)$ in terms of the $\underset{\sim}{M}^{(\ell)}(x)$ and investigate the limit $k_1 \to k_1^a$.

The solutions $\underset{\sim}{M}^{(\ell)}(x)$ are defined as follows: Since at $k_1 = k_1^a$ Eqs. (IV.6) have a solution, the inhomogeneous equations

$$\sum_j E_{\ell j} \bar{m}_j = q_\ell \tag{IV.19}$$

will be soluble only if $\sum_\ell q_\ell \bar{c}_\ell = 0$. We choose the N-1 linearly independent vectors $\{q_1^t, \cdots, q_N^t\}$, $t = 1, \cdots, N-1$ to be given by the first N-1 column vectors of the matrix $\underset{\approx}{Q}^{-1}$, where $\underset{\approx}{Q}$ is as defined in (IV.7),

$$q_\ell^t = Q_{t\ell}^{-1}, \; t = 1, \cdots, N-1. \tag{IV.20}$$

They will by definition be perpendicular to $\underset{\sim}{\bar{C}}$. Hence, because of (IV.19) they define a set of N-1 linearly independent solutions $\underset{\sim}{M}^{(\ell)}(x)$. By keeping the boundary conditions (IV.19) constant, these solutions can be defined in a neighborhood of k_1^a. In this neighborhood and for $k_1 \neq k_1^a$, they can be made a complete system of N linearly independent solutions by adding the function $\underset{\sim}{M}^{(N)}(x)$, which one obtains from (IV.19) by putting $q_\ell = Q_{N\ell}^{-1}$. Now $\underset{\sim}{M}^{(N)}(x)$ is not defined at $k_1 = k_1^a$. We propose to show, however, that

$$\lim_{k_1 \to k_1^a} (k_1 - k_1^a) \underset{\sim}{M}^{(N)}(x) = \frac{1}{\mu_{NN}} \underset{\sim}{H}^a(x). \tag{IV.21}$$

Indeed, defining the coefficients

$$\bar{\bar{m}}_j^{(N)} = \sum_s Q_{js}^{-1} \bar{m}_s^{(N)}, \tag{IV.22}$$

one can, for $k_1 \neq k_1^a$, rewrite Eqs. (IV.19) for $\bar{m}_j^{(N)}$ as

$$\sum_j \left(\underset{\approx}{Q}^T \underset{\approx}{E} \underset{\approx}{Q} \right)_{sj} \bar{\bar{m}}_j^{(N)} = \delta_{sN} . \tag{IV.23}$$

We make the further substitution

$$w_i = \overline{\overline{m}}_i^{(N)} \text{ for } i \neq N, \quad w_N = \overline{\overline{m}}_N^{(N)} \times (k_1 - k_1^a). \tag{IV.24}$$

Insertion of (IV.24) into (IV.23) shows that the coefficients $\{w_1, \cdots, w_N\}$ satisfy a system of inhomogeneous equations, the determinant of which is differe from zero in a neighborhood of k_1^a, this point k_1^a included. This inclusion follows from the properties (IV.9) and (IV.10), and from $\beta_{NN} = 1$. Therefore, these coefficients are uniquely determined and finite in this neighborhood. In particular, after substitution of w_i for $\overline{\overline{m}}_i^{(N)}$, the Nth of the equations (IV.23) reads at $k_1 = k_1^a$

$$\mu_{NN}\, w_N = 1. \tag{IV.25}$$

Therefore, we have $w_N = \frac{1}{\mu_{NN}}$ at $k_1 = k_1^a$ and, since all w_i are finite, from (IV.24)

$$\lim_{k_1 \to k_1^a} (k_1 - k_1^a)\, \overline{\overline{m}}_j^{(N)} = \frac{1}{\mu_{NN}}\, \delta_{jN}\,. \tag{IV.26}$$

With the definition (IV.22) and $Q_{jN} = h_j^a(a)$, this yields (IV.21).

Our last step consists in establishing the connection between the N linearly independent solutions $\underset{\sim}{F}^{(\ell)}(x)$ and $\underset{\sim}{M}^{(j)}(x)$ in the neighborhood of k_1^a but for $k_1 \neq k_1^a$ and in taking the limit $k_1 \to k_1^a$ with the help of (IV.21). By comparing the boundary conditions (II.18) for the $\underset{\sim}{F}^{(\ell)}(x)$ with the conditions (IV.19) for the $\underset{\sim}{M}^{(j)}(x)$, one easily sees that

$$\underset{\sim}{F}^{(s)}(x) = -i(2k_s)^{1/2}\,[\exp(-ik_s a)] \sum_j Q_{sj}\; \underset{\sim}{M}^{(j)}(x), \tag{IV.27a}$$

and that in particular

$$\lim_{k_1 \to k_1^a} (k_1 - k_1^a)\, \underset{\sim}{F}^{(s)}(x) = -i(2k_s)^{1/2}\, [\exp(-ik_s a)]\, h_s^a(a)\, \frac{1}{\mu_{NN}}\, \underset{\sim}{H}^a(x). \quad \text{(IV.27b)}$$

With the help of (IV.3) and (IV.14), this can, for $x \geqslant a$, be written as

$$\lim_{k_1 \to k_1^a} (k_1 - k_1^a)\, f_\ell^{(s)}(x) = -i\, \gamma_s^a\, \frac{1}{\mu_{NN}}\, \gamma_\ell^a\, \frac{1}{(v_\ell)^{1/2}}\, [\exp(ik_\ell^a x)], \quad \text{(IV.27c)}$$

and thus agrees with (IV.5) because of the definitions (II.7) and (III.6).

We believe that the equations (IV.27) show the Breit-Wigner formula (IV.5) and its implications in a new light. In the vicinity of a pole of $\hat{\underset{\approx}{S}}$ at k_1^a, the physically significant solutions $f_\ell^{(s)}(x)$ are dominated by the homogeneous solution $\underset{\sim}{H}^a(x)$, which is the function describing the "compound state."[2] This "state" is independent of formation and decay because the proportionality constant connecting $\underset{\sim}{F}^{(s)}(x)$ and $\underset{\sim}{H}^a(x)$ is independent of the properties of $\underset{\sim}{F}^{(s)}(x)$ and dependent only upon the properties of $\underset{\sim}{H}^a(x)$. If $h_s^a(a) = 0$ for one or several s's, which is perfectly possible (for examples, see section IV. E) then according to (IV.27b), $\underset{\sim}{F}^{(s)}(x)$ will be holomorphic in the neighborhood of k_1^a. In this case, the peak caused by the pole of $\hat{\underset{\approx}{S}}$ at k_1^a will not show up in the cross sections having s as incoming or outgoing channel, no matter how close the pole is to the real axis of the physical sheet. Then $\lim_{k_1 \to k_1^a} \underset{\sim}{F}^{(s)}(x)$ exists and is different from $\underset{\sim}{H}^a(x)$.

D. The Breit-Wigner Formula

In this subsection, we discuss some of the consequences of the formulas derived in IV. A, B, C. We are interested only in poles of $\hat{\underset{\approx}{S}}$ close to the real axis of the physical sheet. If the pole is at $k_1 = k_1^a$, then the channels i for which $\mathrm{Re}\, k_1^a \geq (e_i)^{1/2}$ shall be called open, the other ones closed. From the sign conventions made in (III.3) we know that $\mathrm{Re}\, k_m^a > 0$ if m is open, and $\mathrm{Re}\, k_m^a < 0$ if m is closed on sheets adjacent to the positive real axis of the physical sheet. Therefore, Eq. (IV.4) can be written as

$$\Gamma^a = \frac{\sum_{m\ \mathrm{open}} |\mathrm{Re}\, k_m^a|\ |h_m^a(b)|^2 - \sum_{m\ \mathrm{closed}} |\mathrm{Re}\, k_m^a|\ |h_m^a(b)|^2}{\int_0^b |\underset{\sim}{H}^a(x)|^2\, dx} \geq 0 \text{ for all } b \geq a. \tag{IV.4}$$

This equation gives the total width as a <u>sum</u> of contributions of the open and closed channels. The latter contribute, since the energy ξ^a is not real, and therefore the current in these channels is not zero. The contribution of the closed channels tends toward zero, of course, as k_1^a approaches the real axis. But then Γ^a is zero, too. Owing to the position of the pole, the contribution of the closed channels to Γ^a is negative. However, from (IV.3) we see that, with increasing b, the contribution from the closed channels becomes smaller and smaller, so that we may write

$$\Gamma^a = \lim_{b \to \infty} \frac{\sum_{m\ \mathrm{open}} \mathrm{Re}\, k_m^a\ |h_m^a(b)|^2}{\int_0^b |\underset{\sim}{H}^a(x)|^2\, dx} . \tag{IV.28}$$

This is the usual formula for Γ^a. For applications in nuclear physics, we expect to obtain very good answers by taking b to be of the order of atomic

distances. It must be stressed, however, that the calculation of widths of decaying states with the help of (IV.28), and with b taken at the nuclear surface, may lead to wrong answers.

The functions $\underset{\sim}{H}^{a}(x)$ are the true "eigenfunctions" of a "decaying state", the width of which is given by (IV.28). The time dependence of $\underset{\sim}{H}^{a}(x)$ is, of course, given by $\exp(-i\xi^{a}t)$ if we continue the time-dependent Schrödinger equation analytically. Therefore,

$$\underset{\sim}{H}^{a}(x,t) \sim \exp[-(\Gamma^{a}/2)t].$$

In the study of decay processes, like α decay, one would like to deal with a state having a purely exponential time dependence like $\underset{\sim}{H}^{a}(x,t)$. It has been shown (19) that this is not possible. In the framework of the present studies, we may rephrase this result by saying that any physical state can be described by a wave packet (19). Such a wave packet is defined as a linear superposition of solutions of Eqs. (II.4) with real energies ξ. But $\underset{\sim}{H}^{a}(x)$, being an eigenfunction to complex energy ξ^{a}, can never be expanded into such solutions.

In the single-channel problem, a pole of the scattering function $S(k_1)$ at $k_1 = k_1^{a}$ implies the existence of a zero of $S(k_1)$ at $k_1 = -k_1^{a}$. This follows from unitarity. The generalization of this statement to the many-channel case is the following: If, in Fig. 5, we follow the solid line for path 3 down to a pole at k_1^{a}, then the expansion of the $\hat{\underset{\approx}{S}}$ matrix is given by (IV.5). In order to write down the analytically continued unitarity relationships (III.13), we have to expand the elements of $\hat{\underset{\approx}{S}}$ at $-k_1^{a}$, which one reaches by following the broken line 3 in Fig. 5. We know that $\hat{\underset{\approx}{S}}$ cannot have a pole at $-k_1^{a}$, hence we write, in a neighborhood of $-k_1^{a}$,

$$\hat{\underset{\approx}{S}} = \tilde{\underset{\approx}{d}} + (k_1 - k_1^{a})\,\tilde{\underset{\approx}{e}} + \cdots \qquad \text{(IV.29)}$$

By using (IV.5) and (IV.29), we find that the unitarity relations (III.13) yield

$$\sum_{j\ \mathrm{open}} \left(k_j^a\right)^{1/2} \gamma_j^a\, \tilde{d}_{j\ell} = 0 \quad \text{or} \quad \det_{\mathrm{open}} \underset{\approx}{\tilde{d}} = 0, \tag{IV.30}$$

where $\det_{\mathrm{open}}$ means the subdeterminant with respect to the open channels. Other relations follow by inserting the expansions (IV.29) and (IV.5) into (III.13), and comparing equal powers in $(k_1 - k_1^a)$. Thus, the above statement for the single-channel case can be generalized and requires that $\det_{\mathrm{open}} \underset{\approx}{\hat{S}} = 0$ at $k_1 = -k_1^a$. Since $\det \underset{\approx}{\hat{S}} = \det \underset{\approx}{S}$ from (III.6), this means that if $\underset{\approx}{S}$ has a pole at k_1^a, its subdeterminant must have zeros at $-k_1^a$ and at $(+k_1^a)^*$ on the same sheet because of (III.11).

This fact has some bearing on the possibility of a single-level approximation for $\underset{\approx}{\hat{S}}$ if the pole at k_1^a is isolated and very close to the real axis. (In the remainder of this subsection, we shall use the word isolated not in its mathematical sense, but in the sense in which it is used in nuclear-reaction theories (1).) We define $\bar{\mu}_{NN}$ by the right-hand side of (IV.16) and write in the vicinity of $\xi = \xi^a$, approximating (IV.5):

$$\hat{S}_{j\ell} \cong \delta_{j\ell} + i\left(\frac{k_\ell^a}{k_j^a}\right)\left[\frac{\gamma_j^a\, \gamma_\ell^a}{\bar{\mu}_{NN}\,(\xi - \xi^a)}\right] . \tag{IV.31}$$

A simple calculation shows that $\underset{\approx}{\hat{S}}$ approximated by (IV.31) does not fulfill the unitarity relations (III.13) unless the following two conditions are fulfilled:

$$\bar{\mu}_{NN} \text{ and the } \gamma_j^a,\ j = 1, \cdots, N, \text{ are real}$$

$$\text{and} \qquad \sum_{j\ \mathrm{open}} (\gamma_j^a)^2 = -\,\bar{\mu}_{NN}\,\Gamma^a . \tag{IV.32}$$

If (IV.32) holds, then $\det_{\mathrm{open}} \underset{\approx}{\hat{S}} = 0$ at $k_1 = (k_1^a)^*$. It is clear from the definitions (IV.14) and (IV.16) that we cannot expect the first of conditions (IV.32) to be

fulfilled in general for complex k_1^a. Similarly, we know of no relationship that would enable us to derive the second of the relations (IV.32).[3] Indeed, the definition (IV.14) of γ_j^a is independent of the choice of a, as pointed out below this equation, whereas Eq. (IV.4), the only expression for Γ^a we have, although independent of b invokes quantities like $|h_j^a(b)|^2$, which themselves depend very strongly upon b. This means that we cannot in general expect the single-level approximation (IV.31) to hold even if k_1^a is very close to the real axis. On the other hand, the conditions (IV.32) are fulfilled, for example, in the Wigner-Eisenbud treatment (3) of nuclear reactions.

We interpret this apparently contradictory situation as follows. If one observes experimentally an isolated resonance with negligible background, then the approximation (IV.31) is known to be good, and hence (IV.32) must be fulfilled approximately. This possibility we cannot exclude, of course. If one observes an isolated resonance with nonnegligible background, then the approximation (IV.31) will not be valid, and one shall have to consider the next-order terms in (IV.5). Hence, the experiments can still be fitted with all γ_j^a real, since the only phases that enter the expressions for the cross sections (II.24) are the relative phases between the pole term and the next-order terms, and by choosing the γ_j^a real, one simply redefines the phases of $\hat{\underset{\approx}{d}}$ and $\hat{\underset{\approx}{e}}$. A possible deviation from (IV.32) will be practically undetectable, since with increasing background it becomes increasingly more difficult to determine the γ_j^a precisely. A similar situation occurs when one investigates the case of two overlapping resonances, and it seems therefore impossible to decide, on experimental grounds, between the description of resonances with a set of γ_j^a not obeying relations (IV.32) and with another set of real γ_j^a fulfilling (IV.32). This, of course, means that the Breit-Wigner formula in the usual treatment (3) with parameters obeying (IV.32) provides

for an excellent parametrization of experimentally observed resonances, inasmuch as they are caused by nearby lying poles of the S matrix and, in our opinion, explains its success. In Section VII, we shall, with the help of numerical examples, investigate how far conditions (IV.32) are fulfilled in special cases.

E. Zeros of D on the Real Axis and at Branch Points

In Sections IV. B through IV. D, we have excluded the points k_1^a where at least one of the functions $\sin(\xi + \lambda_m)^{1/2}a = 0$. Also, in the previous part of this section we have excluded the branch points $k_i = 0$, $i = 2, \cdots, N$, and the points $k_1 = 0$. The behavior of $\hat{\underset{\approx}{S}}$ at these points shall now receive special attention. Furthermore, we want to study the conditions under which a pole of $\hat{\underset{\approx}{S}}$ may lie on the real axis of the physical sheet and thus give rise to a bound state embedded in the continuum, and what form $\hat{\underset{\approx}{S}}$ has at such a pole. The latter question has been dealt with in refs. 7, 9, and 17, and we shall pursue it only to the extent to which we need the results in Section V.

Let us begin with zeros of D on the real axis of the physical sheet, $k_1 = k_1^a$ real and such that neither any of the functions $\sin(\xi^a + \lambda_m)^{1/2}a$ vanishes, nor any of the $k_i^a = 0$, $i = 1, \cdots, N$. Writing out in full the condition for outgoing waves only and using (II.11), we have

$$\sum_j 0_{jm}\, g_j(a) = d_m\, [\exp(ik_m a)] \tag{IV.33}$$

and

$$\sum_j 0_{jm} \frac{d}{dx} g_j(x) \Big|_{x=a} = ik_m d_m [\exp(ik_m a)]. \tag{IV.34}$$

Multiplying the complex conjugate of (IV.33) by (IV.34), summing over m, taking the imaginary part of the resulting expression, and using (II.13), one obtains

$$\sum_{m \text{ open}} (-i) \left|d_m\right|^2 \operatorname{Re} k_m \left[\exp(-2 \operatorname{Im} k_m a)\right] = 0. \qquad \text{(IV.35)}$$

This implies $d_m = 0$ for all open channels, or

$$\sum_j 0_{jm}\, g_j(a) = 0 \quad \text{and} \quad \sum_j 0_{jm} \frac{d}{dx} g_j(x) \Big|_{x=a} = 0 \text{ if m is open.} \qquad \text{(IV.36a)}$$

This is the proof lacking for the statement in Section IV. A. The conditions (IV.36a) imply that the Q_{iN} in Section IV. B are given by

$$Q_{iN} = h_i^a(a) = \sum_m 0_{mi}\, g_m(a) = 0 \text{ if i is open;} \qquad \text{(IV.36b)}$$

hence $\gamma_i^a = 0$ for i open, and the only elements of $\hat{\underset{\approx}{S}}$ that will have poles are the elements $\hat{S}_{im}$ with i and m closed, i. e., the elements in the submatrix of $\hat{\underset{\approx}{S}}$ relating to closed channels. This means that a pole on the real positive axis of the physical sheet will not influence the scattering cross sections at all. (For the phase shifts, see, however, Sec. V.)

The conditions for the occurrence of a pole on the real axis are $\det \underset{\approx}{E} = 0$ for real k_1^a, and $h_1^a(a) = 0 = \cdots = h_n^a(a)$, all together (n + 1) conditions. We see that with increasing number n of open channels we are less and less likely to find poles on the real axis, and they cease to exist for $k_1 > (e_N)^{1/2}$. However, it is obvious that we can always adjust our potential parameters so as to give poles on the real axis for $k_1 < (e_N)^{1/2}$.

In the branch points $k_i = 0$, $i = 2, \cdots, N$, and for $k_1 = 0$, all our previous formulae remain valid if we replace the expansions in terms of $k_1 - k_1^a$ by the expansion in terms of the parameters of local uniformalization,

k_i, $i = 1, \cdots, N$. The reason that we have excluded these points so far is not this trivial point, but the threshold behavior of the cross sections (III.15), which is changed. Let us again exclude the points where at least one of the functions $\sin(\xi^a + \lambda_m)^{1/2} a = 0$, and let us assume that $\det \underset{\approx}{E}$ has a simple root in k_i at $k_i = 0$, $i = 1, \cdots, N$. Then, Eq. (IV.5) will hold with $\gamma_j^a = 0$ for $j < i$, and γ_j^a possibly different from zero for $j \geq i$. From (IV.35) we cannot deduce that $\gamma_i^a = 0$. Therefore, Eq. (III.15) has to be modified for the elastic cross sections. Indeed, using (IV.5) and keeping only the lowest-order term in a power-series expansion in k_i, one finds, for $\gamma_i^a \neq 0$,

$$\sigma_{i \to i} = 4 . \tag{IV.37}$$

This follows from $\hat{S}_{ii} = S_{ii}$, from $\mu_{NN} = -i\left|h_i^a(a)\right|^2$ [Eq. (IV.16)], and from Eq. (IV.14). We see that under such circumstances the elastic cross section at threshold is not zero, but reaches its maximum possible value. This is of importance for understanding the results of Section V. If $\det \underset{\approx}{E}$ has a zero of higher order than the first at $k_i = 0$, then from the discussion in Section IV. A it follows that $h_i^a(a) = 0$, and Eq. (III.15) will apply.

Finally, we turn to the points k_1^a where $\sin(\xi^a + \lambda_m)^{1/2} a = 0$ for at least one m, and for which $\det \underset{\approx}{E}(k_1^a)$ remains finite. Let $m_1, \cdots, m_i$ denote the set of m's for which $\sin(\xi^a + \lambda_m)^{1/2} a = 0$. It is clear that instead of working with the matrix $\underset{\approx}{E}$, we should in (II.19) and (II.26) have employed the matrix

$$\sum_n 0_{n\ell} 0_{nm} f_n - ik_m \delta_{m\ell} + ik_m \sum_{j=m_1}^{m_i} 0_{jm} 0_{j\ell} , \tag{IV.38}$$

where $f_n = (\xi^a + \lambda_n)^{1/2} \cot(\xi^a + \lambda_n)^{1/2} a$ if $n \neq m_1, \cdots, m_i$, and $f_n = (\xi^a + \lambda_n)^{1/2}$ if $n = m_1, \cdots, m_i$. With the help of this matrix, we

can again define the homogeneous equations (IV.6), and again we can show that (IV.35) follows. Also, we can construct a matrix $\underset{\approx}{Q}$ to obtain the expression (IV.5) with the γ_i^a defined by (IV.14). This shows that again the S-matrix elements that are singular must relate to closed channels only. Thus, the main results obtained in the previous sections remain valid in this case, too. The reason that we did not use the matrix (IV.38) from the very beginning is, of course, that it does not have the pleasant property (II.30).

V. GENERALIZATION OF LEVINSON'S THEOREM

Rather than stating the generalized version of this theorem at the beginning of this section and then proving it, we prefer to start with the proof and state the theorem at the end because we have to introduce some new symbols and concepts. We shall proceed as follows: In Section V.A, we prove what amounts to a Levinson theorem for the phase of the function D defined in (IV. 1b). In Section V. B, we show that for values of k_1 on the positive real axis of the physical sheet, the phase of D is related to the eigenphase shifts of the scattering matrix. This completes the proof.

A. A Relation for the Phase of D

We know that in the upper half plane of the physical sheet, D can have zeros only on the positive imaginary and on the real axis. Its zeros indicate bound states, and a zero of D on the positive real axis at $k_1 = k_1^a$ implies the existence of a zero of D on the negative real axis. Both zeros correspond to the same bound state. Zeros of D at the branch points correspond to bound states if the zero is of higher order than the first in the parameter of local uniformalization, and to states that might be considered either bound or scattering states if the zero is simple. From what was said in Section IV, it is also clear that if D has a zero of nth order at $k_1 = k_1^a$ the bound state is not more than n-fold degenerate. At the end of this subsection, we show that (7) $D \to +1$ as $|k_1| \to \infty$, with $\mathrm{Im}\, k_1 \geq 0$ on the physical sheet. Finally, we have $D^*(k_1) = D(-k_1^*)$.

We define the phase δ of D by writing

$$D(k_1) = |D(k_1)| \{\exp[-i\delta(k_1)]\}. \tag{V.1}$$

We now integrate the logarithmic derivative of $D(k_1)$ along the contour shown in Fig. 6. This contour is defined as follows: Starting at $k_1 = R_0$ on the

positive real axis of the physical sheet, where $R_0 = \max\{[(e_N)^{1/2}+1], R\}$ with R as defined in Section IV.A, we follow a semicircle with radius R_0 around the point $k_1 = 0$ until we come to the point $k_1 = -R_0$. Then, we follow the real axis in the direction of increasing values of k_1. We avoid all branch points and zeros of D on the real axis by deforming the path into the upper half plane. The contour is closed when we reach $k_1 = R_0$. The contour C defined in this way encloses all zeros of D on the positive imaginary axis ($\mathrm{Im}\, k_1 > 0$) and no other ones. We consider the integral (15) along C,

$$\frac{1}{2\pi i} \oint_C \left[\frac{d}{dz} D(z)\right] \frac{1}{D(z)}\, dz \equiv I . \tag{V.2}$$

Since we have avoided the branch points, the differential dz and the derivative of $D(z)$ are well defined, and a simple application of the theorem of the residues (15) gives

$$I = m , \tag{V.3}$$

where m is the number of bound states (zeros of D) on the imaginary axis, each bound state counted n-fold if the zero of D is of nth order. On the other hand, since $D \to 1$ for $|k_1| \to \infty$, we can deform C by letting $R_0 \to \infty$, and obtain

$$I = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \left[\frac{d}{dz} D(z)\right] \frac{1}{D(z)}\, dz,$$

or, with (V.1),

$$2\pi m = \delta(-\infty) - \delta(+\infty). \tag{V.4}$$

It is understood, of course, that both $\delta(-\infty)$ and $\delta(+\infty)$ must be multiples of 2π. The fact that $D^*(k_1) = D(-k_1^*)$ implies that on the real axis

$$\delta(-k_1) + \delta(+k_1) = 2\pi\nu, \quad \nu \text{ integer}, \tag{V.5}$$

whenever $D(k_1) \neq 0$. In particular, (V.5) applies for $k_1 \to \infty$, and we call this particular fixed integer ν_0:

$$\delta(-\infty) + \delta(+\infty) = 2\pi\nu_0 \quad . \tag{V.6}$$

We now continue (V.6) with the help of (V.5) towards smaller values of $|k_1|$ along the real axis. So long as we do not cross a zero of D, or a branch point, ν_0 cannot change its value. Therefore, if we follow the contour C along the real axis, it is easy to see that we will also have

$$\delta(-0) + \delta(+0) = 2\pi\nu_0, \tag{V.7}$$

where $\delta(\pm 0) = \lim_{k_1 \to \pm 0} \delta(k_1)$, the limits being taken along the real axis. If D does not vanish at $k_1 = 0$, then obviously $\delta(-0) = \delta(+0)$. If, on the other hand, D has a zero of order σ in k_1 at $k_1 = 0$, it is a straightforward matter to see that

$$\delta(-0) = \delta(+0) - \pi\sigma \tag{V.8}$$

according to (V.1), if we follow the contour C. Hence, $2\pi\nu_0 = 2\delta(+0) - \pi\sigma$, and using (V.4) and (V.6), we have

$$\delta(+0) - \delta(+\infty) = \pi(m + \tfrac{1}{2}\sigma). \tag{V.9}$$

This equation is a Levinson theorem for the phase of D. In the derivation of (V.9) we have made use of the asymptotic behavior of D, $D \to 1$. This can be shown as follows: We write

$$D = \left(\prod_{i=1}^{N} [\exp(ik_i a)] \sin(\xi + \lambda_i)^{1/2} a \; \frac{1}{(\xi + \lambda_i)^{1/2}} \right)$$

$$\times \det\left[(\xi + \lambda_j)^{1/2} \cot(\xi + \lambda_j)^{1/2} a \; \delta_{j\ell} - \sum_s ik_s 0_{js} 0_{\ell s}\right]. \tag{V.10}$$

Now, we expand for large k_1, on the physical sheet

$$k_s = k_1 - \frac{e_s}{2k_1} + \cdots, \tag{V.11a}$$

and have, to lowest order in powers of k_1^{-1},

$$D = \prod_{i=1}^{N} \left\{ [\exp(ik_1 a)] \sin (k_1^2 + \lambda_i)^{1/2} a \; \frac{1}{(k_1^2 + \lambda_i)^{1/2}} \right.$$

$$\left. \times [(k_1^2 + \lambda_i)^{1/2} \cot (k_1^2 + \lambda_i)^{1/2} a - ik_1] \right\} + 0 \left(\frac{1}{k_1}\right) . \qquad \text{(V.11b)}$$

Each of the N factors in (V.11b) has the limit + 1 if $|k_1| \to \infty$ and $\operatorname{Im} k_1 \geq 0$. This completes the proof of (V.9).

B. Relation Between δ and the Scattering Phase Shifts

The unitarity relations (II.32) are valid on the real axis of the physical sheet. They imply that a submatrix $\underset{\approx}{W}$ of $\underset{\approx}{S}$ is unitary. Here $\underset{\approx}{W}$ is composed of all elements S_{im} of $\underset{\approx}{S}$ $\cdots$ for which $i, m < j$ if $(e_{j-1})^{1/2} < k_1 < (e_j)^{1/2}$. Hence, all eigenvalues of $\underset{\approx}{W}$ have absolute magnitude one, and their product can be written as $e^{2i\delta'_s(k_1)}$. In other words, $\delta'_s(k_1)$ is the sum of the eigenphase shifts of the scattering matrix, and $2\delta'_s(k_1)$ is equal to the phase of det $\underset{\approx}{W}$. Of course, $\delta'_s(k_1)$ is determined only modulo π. We propose to show that we can always add a multiple of π to $\delta'_s(k_1)$ so that for the resulting value $\delta_s(k_1)$ the relation

$$\delta_s(k_1) = \delta(k_1) , \quad \text{for } k_1 \geq 0, \qquad \text{(V.12)}$$

is fulfilled for all values of k_1 for which neither $D = 0$ nor $k_i = 0$, $i = 2, \cdots, N$, nor $\sin(\xi + \lambda_m)^{1/2} a = 0$ for any m. These points are considered later.

The proof of (V.12) is trivial for $k_1 > (e_N)^{1/2}$. A comparison of the definitions of D in (IV.1b) and of $\underset{\approx}{S}$ in (II.19) shows

$$\det \underset{\approx}{W} = \det \underset{\approx}{S} = \det \hat{\underset{\approx}{S}} = \frac{D^*(k_1)}{D(k_1)} = \exp[2i\delta(k_1)] \text{ for } k_1 > (e_N)^{1/2}, \qquad \text{(V.13)}$$

because of Eqs. (V.1) and (V.5). We prove (V.12) by showing that the relationship (V.13) is valid on the whole positive axis of the physical sheet except for the points mentioned above:

$$\det \underset{\approx}{W} = \frac{D^*(k_1)}{D(k_1)} \quad \text{for all } k_1 > 0 \text{ except for isolated points.} \tag{V.14}$$

By inserting the definitions (II.19) and (IV.1b), one sees the proof will be completed if we can show

$$\det{}_{\text{open}}\left\{ \sum_i E^{-1}_{mi}(k_1, \cdots, k_N) E_{i\ell}(-k_1, \cdots, -k_N) \right\}$$

$$= \frac{\det\{E_{\ell m}(-k_1, \cdots, -k_n, +k_{n+1}, \cdots, +k_N)\}}{\det\{E_{\ell m}(k_1, \cdots, k_N)\}} \quad . \tag{V.15}$$

Here, n is thenumber of open channels, and $\det_{\text{open}}$ means the determinant of the submatrix the indices of which refer to open channels only. The proof of (V.15), although not difficult, is cumbersome because it uses some of the concepts of the theory of determinants (18). Since it does not seem to give new physical insight and because of its length, we defer it to the Appendix.

Equation (V.12) then holds on the whole positive axis of the physical sheet except for some points. Among these, let us first consider the points k_1^a where $D(k_1^a) \neq 0$, $k_i \neq 0$ for $i = 2, \cdots, N$, but where $\sin(\xi^a + \lambda_m)^{1/2} a = 0$ for at least one m. Both sides of (V.14) are well defined at k_1^a and are continuous functions of k_1 for real k_1 in a neighborhood of k_1^a. Therefore, since (V.12) holds in a neighborhood of k_1^a, it also holds at k_1^a itself. Among the branch points $k_i = 0$, $i = 2, \cdots, N$, we first consider those for which $D(k_i = 0) \neq 0$. For $k_1 < (e_i)^{1/2}$, the matrix $\underset{\approx}{W}$ has $i - 1$ rows and columns, and for $k_1 > (e_i)^{1/2}$, it has i rows and columns. However, the continuity argument still applies to the right-hand side of Eq. (V.14). This shows that we can choose the sum of the eigenphase shifts $\delta_s(k_1)$ so that it is continuous across threshold, i.e., so that $\lim_{k_i \to +0} \delta_s(k_i) = \lim_{ik_i \to i0} \delta_s(k_i)$,

where the two limits have to be taken along the positive real and positive imaginary axis. In order to be able later to define our choice of phases in simple terms, let us formulate this choice of $\delta_s(k_1)$ as the

Condition 1. Unless there is a bound state at threshold, $\delta_s(k_1)$ is continuous at threshold.

If we fulfill Condition 1, Eq. (V.12) is valid on sections of the positive real axis of the physical sheet that have zeros of D as their end points. Let now $k_1^a > 0$ be a σ-fold zero of D not coinciding with a branch point. Then, from our definition of the path C it follows that for $\epsilon > 0$ and in analogy to (V.8),

$$\lim_{\epsilon \to 0} \delta(k_1^a - \epsilon) = \lim_{\epsilon \to 0} \delta(k_1^a + \epsilon) - \pi\sigma. \tag{V.16}$$

Therefore, we can fulfill (V.12) for real values of k_1 both in the right and in the left neighborhood of k_1^a if we require $\delta_s(k_1)$ to jump discontinuously by $\pi\sigma$ at k_1^a, according to (V.16). A similar argument applies to σ-fold zeros of D at the branch points. We can thus formulate

Condition 2. Let a bound state occur at k_1^a with k_1^a real, $k_1^a > 0$, so that D has a σ-fold zero in the parameter of local uniformalization. At $k_1 = k_1^a$, $\delta_s(k_1)$ is a discontinuous function of k_1. It fulfills, for $\epsilon > 0$, the equation

$$\lim_{\epsilon \to 0} \delta_s(k_1^a - \epsilon) = \lim_{\epsilon \to 0} \delta_s(k_1^a + \epsilon) - \pi\sigma \tag{V.17a}$$

if the bound state does not occur at threshold, and the equation

$$\lim_{\epsilon \to 0} \delta_s(k_1^a - \epsilon) = \lim_{\epsilon \to 0} \delta_s(k_1^a + \epsilon) - \frac{\pi\sigma}{2} \tag{V.17b}$$

if the bound state does occur at threshold.

If we fulfill both Conditions 1 and 2, Eq. (V.12) holds on the whole positive real axis of the physical sheet, and we have, from Eq. (V.9),

Levinson's Theorem: The sum $\delta_s(k_1)$ of the eigenphase shifts of the scattering matrix is determined only modulo π. Let it be chosen so that Conditions 1 and 2 are fulfilled. Then,

$$\delta_s(0) - \delta_s(\infty) = \pi(m + \frac{1}{2}\sigma), \qquad (V.18)$$

where m is the number of bound states not embedded in the continuum, each counted t-fold if D has a t-fold zero, and where $\sigma = 0$ for $D(0) \neq 0$. Otherwise, if D has a τ-fold zero at $k_1 = 0$, $\sigma = \tau$.

The above form of the theorem is a satisfying generalization of Levinson's theorem for single-channel potential scattering. It allows us to interpret the poles of $\underset{\approx}{S}$ close to the real axis in another way. Let $\underset{\approx}{S}$ have a simple pole at k_1^a, where k_1^a is positive real and $k_i(k_1^a) \neq 0$, $i = 2, \cdots, N$. At $k_1 = k_1^a$, we have a nondegenerate bound state embedded in the continuum. If we change the potential parameters $\lambda_1, \cdots, \lambda_N$ and $\underset{\approx}{0}$ slightly, we can make the pole of $\underset{\approx}{S}$ move off the real axis. It now gives rise to a sharp resonance, and as a consequence of Condition 2, we obtain the

Corollary. Let a sharp isolated resonance in the cross section be caused by a simple pole of the S matrix at k_1^a close to the real axis so that $|\mathrm{Im}\, k_1^a| << \mathrm{Re}\, k_1^a$, $\mathrm{Im}\, k_1^a < 0$. In the limit of vanishing $\mathrm{Im}\, k_1^a$, we have

$$\delta_s(\mathrm{Re}\, k_1^a + \mathrm{Im}\, k_1^a) + \pi = \delta_s(\mathrm{Re}\, k_1^a - \mathrm{Im}\, k_1^a). \qquad (V.19)$$

This relation holds approximately for $|\mathrm{Im}\, k_1^a| \lesssim \mathrm{Re}\, k_1^a$.

We believe that this corollary completes in a very satisfactory manner the description of resonant states given in Section IV. Unfortunately, we cannot prove the reverse, namely that if δ_s increases sharply by π, we must have a pole of the S matrix close to the real axis. Similarly, we can understand better what happens if there is a bound state at threshold, $k_i = 0$, $i = 1, \cdots, N$. If $D(k_i)$ has a simple zero at $k_i = 0$, then the sum of the

eigenphase shifts jumps by $\frac{\pi}{2}$ at $k_i = 0$. This is in perfect agreement with Eq. (IV.37). If $D(k_i)$ has a zero of higher order than the first, this zero does not affect the cross section, and the sum of the eigenphase shifts increases by a multiple of π. (It follows from our discussion in Section IV that if $D(k_i)$ has a zero of higher order than the first at $k_i = 0$, then the order of this zero must necessarily be even.) One possible shortcoming of our generalized version of Levinson's theorem is that we cannot show that a zero of D of higher order σ than the first in the parameter of local uniformalization always implies the existence of a τ-fold degenerate bound state with $\sigma = \tau$. We can show only $\sigma \geq \tau \geq 1$. However, one may trust that in most cases all zeros of D are simple.

VI. DISTRIBUTION OF THE POLES OF $\underset{\approx}{S}$ ON $\hat{U}(N)$ AND ITS DEPENDENCE ON THE POTENTIAL PARAMETERS

We want to discuss the distribution of the poles of $\underset{\approx}{S}$ on $\hat{U}(N)$ as an illustration to the foregoing sections. We are of course, not able to describe the position of the poles completely for arbitrarily strong coupling between the channels. In Section VI.A, the reader is reminded of the distribution for single-channel square-well potential scattering; in Section VI.B the more general case of N channels is dealt with. Some detailed properties of the two-channel case are investigated in Section VII.

A. Reminder of the Single-Channel Case (8)

We remind the reader of Fig. 1 from Nussenzveig's paper (8). It shows the distribution of poles of the S function in the k_1 plane for scattering of an S-wave particle by an attractive square-well potential. This corresponds exactly to our single-channel case.[4] The position of the poles is given as a function of the depth of the potential well, $|\lambda_1|$, in units of $\hbar^2/2Ma^2$. If we put $a \approx 5$ f, then $\hbar^2/2Ma^2 \cong 1$ MeV, where M is the nucleon mass. This means that k^2 gives the energy in units of MeV.

The position of the poles is a function of $|\lambda_1|$, and for $|\lambda_1| \to 0$ there are infinitely many simple poles located at $k_1 = n\pi - i\infty$, $n = 0, \pm 1, \pm 2, \cdots$. With increasing $|\lambda_1|$, they move on curves that are almost straight lines parallel to the imaginary axis. The pole on the imaginary axis moves faster than the others; it stays on the imaginary axis for all values of $|\lambda_1|$ and, after having crossed the real axis, corresponds to a bound state. The other poles, which must occur in pairs since they must be situated symmetrically with respect to the imaginary axis, move along the lines shown in Fig. 1 of (8). Eventually, they merge into a double pole at $k_1 = -i$, then separate again.

They move along the imaginary axis, one towards positive imaginary values of k_1, the other towards negative imaginary values of k_1. The first of them, after having crossed the real axis, becomes equivalent to a bound state. This way, more and more bound states appear with increasing $|\lambda_1|$.

We have calculated the phase shift $\delta(k_1)$ as a function of k_1 for $|\lambda_1|a^2 = 50$, which corresponds to a potential depth of 50 MeV. We have two bound states, at $k_1 = 4.55128\ i$ and at $k_1 = 6.51728\ i$. There is one pole of $S(k_1)$ on the negative imaginary axis, at $k_1 = -6.03146\ i$. In Table II, we give the positions of the first six poles of $S(k_1)$ in the lower half of the k_1 plane.

Since we have two bound states, we know from Levinson's theorem that $\delta(0) = 2\pi$ if we require $\delta(+\infty) = 0$. The function $\delta(k_1)$ for $k_1 \leq 30$ is shown in Fig. 7. It equals $(3/2)\pi$ and $(1/2)\pi$ at $k \approx 2$ and $k \approx 14.6$, respectively. Obviously, $\delta(k_1)$ is not a very smooth function of k_1, and the arrows on the real axis coincide more or less with those values of k_1 for which δ and hence the cross section $\sin^2\delta$ display little bumps. The arrows indicate the real parts of the numbers given in Table II. It is clear that the bumps have something to do with the poles of $S(k_1)$. It is gratifying to see that the bumps are more easily discernible if Im k_1 is small than if it is large according to Table II.

For potential scattering, Humblet (4,20) has shown that $S(k_1)$ is a meromorphic function of k_1 and can be written as an infinite sum over pole terms plus an entire function [see Eq. (1.54) of the first of references 4]. In order to see which influence the pole terms have, we have calculated the sum over the pole terms excluding the two bound states and the state on the negative imaginary axis. Unfortunately, Humblet's formula contains an arbitrary parameter M. We have chosen M = 1. The sum converges essentially like $1/n^2$, and we have calculated the first 100 terms to obtain an accuracy better than one part in 10^{-4}.

The result is shown in Fig. 8. We plot 1 - Re $S(k_1)$-- this is proportional to the cross section--as a function of k_1 (curve C). Curve A shows the contribution to + $S(k_1)$ of the first resonance at around $k_1 = 3.2$ (see Table II). We have added the contributions from the poles at k_1 and at $-k_1^*$. Curve B shows the total contribution of all resonance terms; it practically coincides with A for $k_1 < 3.5$, and differs from it for larger values of k_1, mainly because of the influence of the second resonant state at $k_1 = 8.3$. Curve D is the difference between curves C and B, i.e., the contribution from the entire function and from the three states on the imaginary axis.

We see that most of the cross section is due to the entire function and not to the pole contributions. (The contributions from the bound states and the state at $k_1 = 6.03\,i$ are very large, but monotonic functions of k_1.) We also see that curve D still displays the bumps, although less pronouncedly than curve C does. This has to do both with the fact that M is arbitrary and with the fact that the resonances are far from the real axis ($\Gamma \geq 1$ MeV), and not isolated. In such a situation, it is simply not possible to decompose the scattering function $S(k_1)$ uniquely into a resonance part and a nonresonance part. This can also be seen from Fig. 7, where the poles do not cause $\delta(k_1)$ to jump by π but produce only barely discernible little bumps.

From this example, we learn that maxima of the cross section need not have anything to do with poles of $S(k_1)$. The maxima at $k_1 \approx 2$ and $k_1 \approx 14.6$ are consequences of Levinson's theorem, and are not caused by nearby poles of $S(k_1)$. Poles too far away from the real axis cannot be discerned very well, and we shall henceforth concentrate on narrow resonances in the cross sections, i.e., resonances for which $\mathrm{Im}\, k_1 < \frac{1}{10}$ or $\Gamma < 10$ keV. We may hope that these are indeed caused by nearby poles of the S matrix.

B. The Case of N Channels

We study this case by first depicting the position of the poles for vanishing coupling between the channels and by later adding a few remarks about the more general case. In the limit of vanishing coupling between the channels, the S matrix is diagonal, and the ith diagonal element of $\underset{\approx}{S}$ contains the expression for elastic scattering in channel i. Therefore, it would suffice to study the problem on N different complex planes--one for each k_i. There, the poles of $\underset{\approx}{S}$ would be situated just as described in Section VI.A. The distribution of the poles on $\hat{U}(N)$ can thus be obtained by mapping the complex planes for the N wave numbers k_i onto the Riemann surface $\hat{U}(N)$.

Since each sheet--except for its cuts and the point $k_1 = \infty$--is an exact copy of the k_1 plane, the distribution of poles of the S_{11} element on each sheet of $\hat{U}(N)$ is the same as described in Section VI.A. To see how the poles of the element S_{ii}, $i \neq 1$, are distributed, we divide the 2^{N-1} sheets of the surface into two classes. All sheets for which the (i-1)st element, τ_{i-1}, of the labeling vector $\underset{\approx}{\tau}$ is positive belong to one class, the other ones to the other class. The way the k_i plane is mapped onto all sheets belonging to the first class was indicated in a special case in Fig. 3. From this mapping, one sees that poles on the positive imaginary axis of the k_i plane are mapped onto the positive imaginary or the real axes of all the sheets in the first class. If they are on the real axis, they appear symmetrically with respect to the origin. A simple pole at $k_i = i(e_i)^{1/2}$ appears at $k_1 = 0$, where it will be a twofold pole in k_1, the parameter of local uniformalization. This can be understood from the arguments presented in Section IV.B. A pole on the negative imaginary axis of the k_i plane will, for $\mathrm{Im}\, k_i < -(e_i)^{1/2}$, be mapped onto the negative imaginary axis of all sheets of the first class. The image of the piece of the imaginary k_i axis with $-(e_i)^{1/2} \leq \mathrm{Im}\, k_i \leq 0$ lies, by definition of the sheets, on the part of the real axis of the sheets of the second class

for which $-(e_i)^{1/2} \leq \mathrm{Re}\, k_1 \leq (e_i)^{1/2}$. Again, each pole has two images, which lie symmetric with respect to the origin. Finally, resonant states at k_i^a, so that neither $\mathrm{Re}\, k_i^a$ nor $\mathrm{Im}\, k_i^a$ vanish, are mapped into the lower half plane of the sheets of the first class, where they lie symmetric with respect to the imaginary k_1 axis. The mapping of the poles onto the sheets of the second class is obtained by taking a sheet of the first class, reflecting it around its real axis, and exchanging the poles on the real axis with k_1 between $-(e_i)^{1/2}$ and $+(e_i)^{1/2}$, as described above. Since all sheets differ in at least one element τ_i of their labeling vectors from the physical sheet, we see that the physical sheet is indeed the only sheet in the upper half plane of which there are no poles of any element of $\underset{\approx}{S}$ (except on the imaginary axis).

It is helpful to investigate the motion of the poles as functions of the λ_i, still with no coupling between the channels, and for $\lambda_i \geq -e_i$, so that according to (II.9) and (II.10) we deal with attractive potentials. We shall be interested only in the physical sheet and the sheets adjacent to it (see Fig. 4). We begin with channel 1. Let us assume we have two bound states. They correspond to two poles on the positive imaginary axis of the physical sheet. As λ_1 decreases, they move towards the real axis. Simultaneously, the one pole that must exist on the negative imaginary axis of all sheets does the same. For a particular value of λ_1, the first pole crosses the point $k_1 = 0$ and then enters the sheet $\{-1, \cdots, -1\}$, moving down on the negative imaginary k_1 axis. Simultaneously, a pole enters the negative imaginary axis of the physical sheet from the adjacent sheet $\{-1, \cdots, -1\}$, and moves down. At $k_1 = -i$, two poles coming from larger and smaller values of $\mathrm{Im}\, k_1$ merge on each sheet, and from here on the motion of the pair is confined to one sheet, and is as described in Section VI. A. For sufficiently small values of λ_1, the second pole leaves the positive imaginary k_1 axis, too, and moves toward

smaller values of Im k_1, staying on the imaginary axis and moving on whichever sheet is adjacent to the one on which it was before it crossed the point $k_1 = 0$.

This simple pattern of motion is considerably more complicated for $i \neq 1$. For sufficiently large λ_i, two poles lie on the positive imaginary k_1 axis of all sheets with $\tau_{i-1} = +1$, and on the negative imaginary axis of all sheets with $\tau_{i-1} = -1$. One of the two causes one pole to lie on the negative imaginary axis of all sheets with $\tau_{i-1} = +1$, or positive imaginary axis of all sheets with $\tau_{i-1} = -1$. As λ_i decreases, the pair of poles on the positive imaginary axes of the sheets with $\tau_{i-1} = +1$ moves toward the real axis until one of them reaches the point $k_1 = 0$ on the real axis. There, it is joined by the pole belonging to the pair on the negative imaginary axes of the sheets with $\tau_{i-1} = -1$. We have a double pole in the 2^{N-2} points $k_1 = 0$, with $\tau_{i-1} = +1$. When we decrease λ_i further, the two poles separate and move in opposite directions on the real axis until they reach the points $k_1 = \pm(e_i)^{1/2}$, $\tau_{i-1} = +1$.[5] The pair of poles belonging to the physical sheet leaves this sheet and enters the real axis of the sheet with $\tau_k = +1$ for $k < i-1$, or -1 for $k \geq i-1$. We now have pairs of poles on the real axis of all sheets with $\tau_{i-1} = -1$. In the meantime, the pole on the negative imaginary axis of the sheets with $\tau_{i-1} = +1$ has moved towards the real axis, too. At $k_1 = 0$, $\tau_{i-1} = -1$, it was joined by the pole on the positive imaginary axis of the sheet with $\tau_{i-1} = -1$. Again, the two poles have separated and moved in opposite directions along the real axis of the sheets with $\tau_{i-1} = -1$. At $k_1 = \pm(e_i - 1)^{1/2}$, $\tau_{i-1} = -1$, they merge with the poles discussed above and form pairs of double poles. As λ_i decreases further, they separate again, one pole of each pair leaving the real axis towards positive values of Im k_1, the other one towards negative values of Im k_1. This way, finally two poles situated symmetrically with respect to the imaginary axis emerge below the points

$k_1 = \pm(e_i - 1)^{1/2}$ onto the lower half of the physical sheet and move on this half plane toward ∞. For very small potential strength their paths become asymptotically equal to the ones described in Section VI.A.

We have seen that the only poles ever to be on the positive real axis of the physical sheet are the ones corresponding to a bound state, as it should be. This is not true for any other sheet. Furthermore, if all λ_{ij} are sufficiently large, there will be $\frac{N-i}{\pi}[(e_{i+1})^{1/2} - (e_i)^{1/2}]$ bound states between any two neighboring thresholds for zero coupling between the channels. (This is because the asymptotic distance between two poles is π.) Thus, with increasing numbers of open channels the density of bound states becomes more and more reduced. We expect the same statement to apply to the density of narrow resonances, unless the interaction between the channels changes things drastically. This is supported by the fact that for arbitrarily strong coupling between the channels, the conditions for the existence of a bound state embedded in the continuum become more numerous with increasing numbers of open channels, according to Section IV.E.

With vanishing interaction between the channels, we could treat all sheets on the same footing. This is no longer true as we turn on the coupling. In particular, a pole (bound state) at k_1^a on the positive real axis of the physical sheet in general moves into the sheet adjacent to the real axis at this particular point, when the coupling is turned on. Since we know that poles always occur symmetrically on each sheet with respect to the imaginary k_1 axis, the same holds for its mirror pole at $-k_1^a$. This way, two poles leave the physical sheet simultaneously and become resonant states, and the symmetry among the sheets is destroyed.

VII. NUMERICAL EXAMPLES: THE TWO-CHANNEL CASE

Several authors (e.g., references (7, 9, 12, 13) have investigated various aspects of this case, and we restrict ourselves to those aspects that seem to be of interest in the framework of the present investigation. These are the motion of the poles on the Riemann surface as a function of the interaction strength between the channels (with particular emphasis upon poles close to the real axis of the physical sheet), the demonstration of Levinson's theorem, and properties of a single resonance as mentioned in Section IV. We begin with a suitable parametrization of the two-channel case.

A. Parametrization

Let the symmetric matrix A_{mn} of Eq. (II.9) with m, n ≤ 2 be given by

$$A_{mn} = \begin{pmatrix} b_1 & c \\ c & b_2 \end{pmatrix}, \tag{VII.1}$$

where b_1, b_2, and c are real numbers. Let the eigenvalues of this matrix be λ_1, λ_2 where

$$\lambda_{1,2} = \frac{1}{2}(b_1 + b_2) \pm \frac{1}{2}\left[(b_1 - b_2)^2 + 4c^2\right]^{1/2}. \tag{VII.2}$$

The elements of the orthogonal matrix 0_{mn} defined in Eq. (II.10) are then given by

$$0_{mn} = \frac{1}{A^{1/2}} \begin{pmatrix} \lambda_1 - b_2 & c \\ -c & \lambda_1 - b_2 \end{pmatrix}, \tag{VII.3}$$

where

$$A = (\lambda_1 - b_2)^2 + c^2. \tag{VII.4}$$

We introduce the abbreviation

$$z_i = (\xi + \lambda_i)^{1/2} \cot(\xi + \lambda_i)^{1/2} a, \quad \text{for } i = 1, 2, \tag{VII.5}$$

and have, for the matrix $E_{\ell m}$ of Eq. (II.16),

$$E_{\ell m} = \begin{matrix} z_1 - ik_1 + \frac{c^2}{A}(z_2 - z_1) & \frac{(\lambda_1 - b_2)c}{A}(z_1 - z_2) \\ \frac{(\lambda_1 - b_2)c}{A}(z_1 - z_2) & z_2 - ik_2 + \frac{c^2}{A}(z_1 - z_2) \end{matrix} \tag{VII.6}$$

so that

$$\det E_{\ell m} = (z_1 - ik_1)(z_2 - ik_2) - \frac{c^2}{A}(z_1 - z_2)(ik_1 - ik_2). \tag{VII.7}$$

The elements of the S matrix are given by

$$S_{11} = e^{-2ik_1 a} (\det E_{\ell m})^{-1} \left[(z_1 + ik_1)(z_2 - ik_2) + \frac{c^2}{A}(z_1 - z_2)(ik_1 + ik_2)\right],$$

$$S_{22} = e^{-2ik_2 a} (\det E_{\ell m})^{-1} \left[(z_1 - ik_1)(z_2 + ik_2) - \frac{c^2}{A}(z_1 - z_2)(ik_1 + ik_2)\right], \tag{VII.8}$$

$$S_{12} = S_{21} = -2i(k_1 k_2)^{1/2} \left[\exp(-ik_1 a - ik_2 a)\right] \frac{(\lambda_1 - b_2)c}{A} \frac{z_1 - z_2}{\det E_{\ell m}}.$$

From Eqs. (VII.7) and (VII.8) it is obvious that it is useful to introduce λ_1, λ_2 and $\alpha = \frac{c^2}{A}$ with $0 \leq \alpha \leq 1$ as the independent parameters by means of which we characterize our problem. In terms of these parameters (and the threshold parameter $e_2 \equiv e$), the physically relevant parameters are given by

$$b_1 = (1-a)\lambda_1 + a\lambda_2,$$

$$b_2 = a\lambda_1 + (1-a)\lambda_2,$$

$$c = \pm [a(1-a)]^{1/2}(\lambda_1 - \lambda_2),$$

$$\frac{c(\lambda_1 - b_2)}{A} = \pm [a(1-a)]^{1/2},$$

$$V_{11} = -b_1,$$

$$V_{22} = -b_2 - e,$$

$$V_{12} = -c. \qquad \text{(VII.9)}$$

It is also useful to realize that Eqs. (VII.7) and (VII.8) are invariant under the substitution

$$\lambda_1 \longrightarrow \lambda_2,$$

$$\lambda_2 \longrightarrow \lambda_1,$$

$$a \longrightarrow 1-a. \qquad \text{(VII.10)}$$

so that in many cases it suffices to have a vary in the range $0 \leq a \leq \frac{1}{2}$. In the following, we shall put $a = 1$ throughout.

B. Resonances and Levinson's theorem

We have calculated the cross sections σ_{11}, σ_{22}, and σ_{12} defined by Eqs. (VII.8) and (II.24) for various values of the parameters. As an example, we show in Fig. 9 the function $\delta(k_1)$--i.e., the phase of the function D as defined by Eq. (V.1)--and in Fig. 10 the various cross sections for $0 \leq k_1 \leq 20$. We have chosen $e = 100$, $\lambda_1 = 40$, $\lambda_2 = -50$. For vanishing coupling between the channels--i.e., $a = 0$--we have two bound states in channel one, corresponding to two zeros of D on the positive imaginary axis of the physical sheet, and two bound states in channel two. The latter ones imply, according to Section VI.A, that D has zeros at $k_2^{(1)} = 4.55128\,i$ and $k_2^{(2)} = 6.51728\,i$. This corresponds to values of $k_1 = (k_2^2 + 100)^{1/2}$ given by $k_1^{(1)} = 8.904260$ and

$k_1^{(2)} = 7.584527$, respectively. As a consequence, Levinson's theorem requires $\delta(0) - \delta(+\infty) = 2\pi$, if δ increases by π at $k_1^{(1)}$ and at $k_1^{(2)}$. For $\alpha = 0$, δ will simply be the sum of the phase shifts in the two channels, and for a potential depth $V_{22} = -b_2 - e = -\lambda_2 - e = -50$, one of the two terms adding up to give δ was displayed in Fig. 7. For channel one, where $V_{11} = -40$, the corresponding figure will be very similar. Curve A in Fig. 9 illustrates what happens when we choose $\alpha = 0.005$, leaving λ_1, λ_2, and e unchanged. The very weak coupling between the two channels causes the poles of the S matrix originally situated at $k_1^{(1)}$ and at $k_1^{(2)}$ to move off the real axis. Thus, $\delta(k_1)$ becomes a continuous function which increases rapidly by almost π at $k_1^{(1)}$ and $k_1^{(2)}$. Except for these two points and their neighborhoods, $\delta(k_1)$ decreases monotonically with increasing k_1 and displays the little bumps that were discussed in Section VI.A. Curve B of Fig. 9 shows $\delta(k_1)$ for the same values of λ_1, λ_2 and e, but with $\alpha = 0.5$. This value of α corresponds to the maximum coupling possible between channels 1 and 2, and to $c/b_1 = c/b_2 = +9$. Still, the overall behavior of $\delta(k_1)$ is not changed appreciably. At $k_1^{(1)}$ and $k_1^{(2)}$, $\delta(k_1)$ rises by somewhat less than π, it displays a conspicuous cusp at threshold, $k_1 = 10$, and from there on it falls off monotonically until it eventually reaches the value -2π, as required by Levinson's theorem. It is surely surprising that the large change in α produces so little effect. We have also calculated $\delta(k_1)$ for values of α between 0.005 and 0.5, and have obtained a gradual transition between the values displayed in curves A and B of Fig. 9. (The same holds true, incidentally, for the cross sections shown in Fig. 10).

In the upper part of Fig. 10, we show σ_{11} as a function of k_1 for the same set of parameters. Again, curve A refers to $\alpha = 0.005$ and curve B to $\alpha = 0.5$. Curve A displays a cross section that has two broad maxima. This corresponds to Levinson's theorem, applied to channel one individually for

$\alpha = 0$: The phase δ_1 decreases from zero to -2π and thus the cross section given by $4 \sin^2 \delta_1$ must twice assume its maximum. Interruptions of this smooth pattern occur because $\alpha \neq 0$ at $k_1^{(1)}$ and $k_1^{(2)}$, where we find very narrow resonances. These are shown with a different scale for k_1 in the inserts on the lower left part of Fig. 10. [As a consequence of a theorem by Wigner (21), the cross section must vanish in the vicinity of a resonance if only one channel is open. This can easily be seen from the relation $\sigma_{11} = 4 \sin^2 \delta(k_1)$. Whenever $\delta(k_1)$ increases by π, σ_{11} has a zero.] A further interruption occurs at threshold, where $k_1 = 10.^6$ Furthermore, σ_{11} does not actually assume its maximum value 4 for $k_1 \geqslant 10$, owing to inelastic processes. For $\alpha = 0.5$, the resonances at $k_1^{(1)}$ and $k_1^{(2)}$ become broader, the behavior above threshold and for $k_1 \lesssim 14$ changes drastically owing to inelastic processes (see σ_{12} which is shown on the bottom of Fig. 10), and a little bump develops at $k_1 \approx 5$. Still, the overall behavior of σ_{11} is similar to that with $\alpha = 0$, except for the occurrence of resonances.

In the central part of Fig. 10, we show σ_{22}, which of course is defined only for $k_1 \geqslant 10$. Curve A, again showing σ_{22} for $\alpha = 0.005$, is very similar to curve C in Fig. 8 (except for a scaling factor that was omitted in Fig. 8). The similarity becomes even closer if one remembers that the scale on the abscissa is different in both figures. Whereas in Fig. 8 the abscissa shows what should now be called k_2 in a linear scale, the abscissa of Fig. 10 shows k_1 in a linear scale. Curve B again shows that choosing $\alpha = 0.5$ does not have a very great effect upon σ_{22}. The changes are, of course, most marked where σ_{12} (Curve B at the bottom of Fig. 10) is largest. For σ_{12}, we have not shown the case $\alpha = 0.005$ because $\sigma_{12} < 0.1$ for $10 \leqslant k_1 \leqslant 20$.

It is clear from Fig. 10 that we are dealing with two resonances at $k_1^{(1)}$ and $k_1^{(2)}$ that are fairly well isolated. Therefore, we give in Table III a few quantities relating to our discussion in Section IV. We have kept the

parameters λ_1, λ_2, and e fixed (their values are the same as given above) and have calculated the position of the pole of S originally at $k_1^{(2)}$ as a function of α. In the first column of Table III, we give the values of α, and in the second and third the values of Re k_1 and Im k_1 that define the position of the pole. Column 4 gives $\Gamma = -4 \text{ Re } k_1 \times \text{Im } k_1$. In columns 5 and 6, we give $\gamma_1/(\bar{\mu})^{1/2}$ and $\gamma_2/(\bar{\mu})^{1/2}$, where γ_1 and γ_2 are defined by (IV.14), and $\bar{\mu}$ by the right-hand side of (IV.16). It is these quantities that enter into the expansion of the S matrix (IV.31) and into the relations (IV.32). In order to display the second of relations (IV.32), we give in column 7 also $(\gamma_1)^2/\bar{\mu}$, which should be real and equal to $-\Gamma$ if (IV.32) were correct. The numbers in column 7 show clearly that this is not the case. In terms of our discussion in Section IV this means that a single-pole approximation for $\underset{\approx}{S}$ with no background terms is not permissible. In the last column we give the ratio $\Gamma(b)/\Gamma$ for b = 1.0. Here $\Gamma(b)$ is defined by

$$\Gamma(b) = \frac{\sum_{m \text{ open}} \text{Re } k_m^{\alpha} |h_m^{\alpha}(b)|^2}{\int_0^b |\underset{\sim}{H}^{\alpha}(x)|^2 \, dx} , \qquad \text{(VII.11)}$$

in analogy to Eq. (IV.28), but without the limit $b \to \infty$. We see that in this particular case the error caused by calculating Γ without using the sum over closed channels in Eq. (IV.4) is less than 2 per cent. With increasing b, this error decreases rapidly, and for b = 2.0 it is less than 1 part in 10^{-5} for all the cases considered in Table III.

C. Motion of the Poles Close to the Real Axis; Speculations About Nuclei

One of the most spectacular results of the numerical calculations shown in Section VII. B is that even for strong coupling between the channels, the poles that were (for $\alpha = 0$) on the real axis remain so close to the real axis that they cause resonances with widths much smaller than in the case of single-channel (potential) scattering. By integrating the differential equations for the motion of the poles as functions of λ_1, λ_2, and α numerically, we have studied this behavior further and have found in all cases that the poles have an astounding tendency to stay close to the real axis for values of k_1 below threshold. (Except in the vicinity of the threshold, we have never found poles close to the real axis above threshold.) Speaking more quantitatively, in the many cases investigated numerically $|\mathrm{Im}\, k_1|$ (indicating the position of the pole, as in the third column in Table III) never became larger than 0.3. Unfortunately, we have not been able to find an exact proof for the assertion that poles that were originally on the real axis cannot move far from it as α increases, and there may well be cases for which such an assertion is wrong. Still, it seems that in the majority of all cases the poles do have this tendency, and if one extrapolates this statement to many channels one is led to some curious speculations about nuclei. In the light of these results, the fact that nuclei have many sharp resonances need not be attributed necessarily to a weak coupling between the open and the closed channels. It also does not seem to be a particular property of the nuclear forces. It seems possible that any attractive, sufficiently strong two-body force will lead to many sharp resonances in the cross sections. In fact, sharp resonances may be produced even when the diagonal terms of the interaction (V_{11} and V_{22} in our example in Section VII. B) are too weak to lead to any bound states at all.

Examples for this are not hard to find (see reference 9). This is the reason why we preferred to discuss our results in terms of the eigenvalues λ_1, λ_2 and the coupling strength α rather than in terms of the original matrix elements V_{mn}. Whether our results permit one to make such speculations depends, of course, largely upon the generality of the case under investigation. We must leave this question to the judgment of the reader.

VIII. SUMMARY AND CONCLUSIONS

We believe to have given a fairly complete description of the properties of a particularly simple many-channel scattering problem. We can summarize our results as follows:

The Riemann surface $\hat{U}(N)$ upon which the elements of the S matrix are meromorphic functions is the surface of an algebraic function from which a finite number of points has been removed. The poles of the S matrix correspond to the zeros of D, which is holomorphic on $\hat{U}(N)$. Whenever $\underset{\approx}{S}$ has a pole there exists a solution of the Schrödinger equation which has only "outgoing" waves in all channels. In the neighborhood of such a pole, the physically interesting scattering solutions are dominated by this (homogeneous) solution, and as a consequence, the residues of $\underset{\approx}{S}$ at the poles factorize and thus yield the Breit-Wigner formula. However, the total width need not be equal to the sum of the individual widths relating to the open channels. This is borne out by the numerical calculations. If, however, the single-level approximation for the S matrix is valid it is clear that a relationship between total and channel widths must exist.

The numerical calculations show that the poles of the S matrix that correspond to bound states embedded in the continuum in the limit of vanishing coupling between the channels have a most remarkable tendency to remain in the vicinity of the real axis for almost arbitrary strength of this coupling. This and the fact that the density of such poles decreases rapidly with increasing number of open channels gives rise to some speculations about nuclei discussed in Section VII.

The generalization of Levinson's theorem to many-channel scattering yields as a consequence a further satisfactory characterization of a resonance caused by a pole of the S matrix. The sum of the eigenphase shifts increases by π over the width of the resonance as the energy increases. So far, we have not made any attempt to find out whether this sum is a measurable

quantity, since we are interested in the formal properties of resonance scattering in this paper. If, however, it should turn out that it is measurable we would have an interesting means of classifying narrow resonances in measured cross sections. It is surely regrettable at this point that we cannot show that if the sum of the eigenphase shifts increases by π we are indeed dealing with a pole of the S matrix.

Most of the above statements follow from the properties of the function D. It seems that this function plays a fundamental role in the theory of many-channel scattering, in much the same way as the Jost function does in potential-scattering theory. In fact, Newton (7) has shown that D is the determinant of what might be called the Jost matrix of many-channel scattering. In addition, by a generalization of proofs given by Peierls (10) and LeCouteur (11) he was able to show that all elements of $\underset{\approx}{S}$ can be expressed in terms of the values of D on the various sheets of the surface. This fits in very nicely with the equality between the phase of D and the sum of the eigenphase shifts of the S matrix derived in Section V. Therefore, one is tempted to believe that many of the theorems valid in potential scattering theory as reviewed recently by Faddeyev (22) can be generalized appropriately to many-channel scattering with the help of D. Also, the role played by D in the dispersion relations for many-channel scattering seems worth while investigating. On the whole, we feel that much more work should be done to completely elucidate the properties of many-channel scattering and in particular the significance of the function D.

Finally, a word about the generality of our results may be appropriate. It seems that we can extend most of our results to the case in which the potential matrix $V_{mn}(x) = 0$ for $x > a$. Whenever the potentials do not fall off sufficiently fast as $x \to \infty$ we encounter more complicated analytical structures, like logarithmic branch points. Even in this case, however,

the main results of Section IV seem to remain valid, and probably also the characterization of a resonance close to the real axis by the fact that the sum of the eigenphase shifts increases by π. Also, the introduction of angular momentum and spin should not affect the results appreciably. In the framework of many-channel scattering, then, many of our results seem quite general. As soon as we allow break-up processes of the residual system to occur, however, this is probably changed. It would certainly be most interesting if one were able to study the analytical properties introduced by a three-body breakup in a somewhat general case, and doubtless the Riemann surface is much more complicated under such circumstances. We feel that the artificial suppression of this mode of decay in this paper is the most serious handicap in trying to apply our results to a realistic scattering process.

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APPENDIX

Proof of Equation (V.15)

We introduce the following notation. Let $\underset{\approx}{L}$ be an arbitrary matrix of N rows and N columns. Let r_i and s_i, $i = 1, \cdots, n$, $n \leq N$ be two sets of positive integers, so that $1 \leq r_i \leq N$ and $1 \leq s_i \leq N$, with $r_i < r_k$ and $s_i < s_k$ for $i < k$. The minor $L(r_1, \cdots, r_n | s_1, \cdots, s_n)$ of $\underset{\approx}{L}$ is the determinant of the matrix obtained from $\underset{\approx}{L}$ by omitting all but the rows labeled $r_1, \cdots, r_n$ and all but the columns labeled $s_1, \cdots, s_n$. The algebraic complement $\bar{L}(r_1, \cdots, r_n | s_1, \cdots, s_n)$ of $L(r_1, \cdots, r_n | s_1, \cdots, s_n)$ is the determinant of the matrix obtained from $\underset{\approx}{L}$ by omitting the rows labeled $r_1, \cdots, r_n$ and the columns labeled $s_1, \cdots, s_n$, multiplied by $(-1)^{r_1 + \cdots + r_n + s_1 + \cdots + s_n}$. A special form of Laplace's theorem [see ref. (18), p. 34] can then be written as

$$\det \underset{\approx}{L} = \sum_{s_1 < s_2 < \cdots < s_n = 1}^{N} L(1, \cdots n | s_1, \cdots, s_n) \bar{L}(1, \cdots, n | s_1, \cdots, s_n). \tag{A.1}$$

We now apply these definitions to the matrix $\underset{\approx}{E}$ and its inverse. In order to show the dependence of the $k_1, \cdots, k_N$, we write the minor of $\underset{\approx}{E}$, for example, as $E\{(r_1, \cdots, r_n | s_1, \cdots, s_n), k_1, \cdots, k_N\}$, etc. The left-hand side of Eq. (V.15) can be written as

$$\det_{\text{open}} \left\{ \sum_{i=1}^{N} E^{-1}_{mi}(k_1, \cdots, k_N) E_{\ell i}(-k_1, \cdots, -k_N) \right\}$$

because of the symmetry of $\underset{\approx}{E}$. We now apply Theorem 26 on page 66 of ref. (18) and obtain, with n = number of open channels,

$$\det_{\text{open}} \left\{ \underset{\approx}{E}^{-1}(\underset{\sim}{K})\, \underset{\approx}{E}(-\underset{\sim}{K}) \right\} =$$

$$\sum_{s_1<s_2<\cdots<s_n=1}^{N} E^{-1}\left\{(1,\cdots,n|s_1,\cdots,s_n),\underset{\sim}{K}\right\} \times E\left\{(1,\cdots,n|s_1,\cdots,s_n),\ -\underset{\sim}{K}\right\}. \tag{A.2}$$

From page 90 of the same reference, it follows that

$$E^{-1}\left\{(1,\cdots,n|s_1,\cdots,s_n),\underset{\sim}{K}\right\} = \left[\det \underset{\approx}{E}(\underset{\sim}{K})\right]^{-1} \bar{E}\left\{(1,\cdots,n|s_1,\cdots,s_n),\underset{\sim}{K}\right\}. \tag{A.3}$$

Hence, we have

$$\det_{\text{open}}\left\{ \underset{\approx}{E}^{-1}(\underset{\sim}{K})\, \underset{\approx}{E}(-\underset{\sim}{K}) \right\} = \left[\det \underset{\approx}{E}(\underset{\sim}{K})\right]^{-1}$$

$$\times \sum_{s_1<s_2<\cdots<s_n=1}^{N} E\left\{(1,\cdots,n|s_1,\cdots,s_n),\ -\underset{\sim}{K}\right\} \bar{E}\left\{(1,\cdots,n|s_1,\cdots,s_n),\underset{\sim}{K}\right\}. \tag{A.4}$$

From the definitions of $\underset{\approx}{E}$ [see Eq. (II.16)], of the minor and of the algebraic complement, it follows that $E\{(1,\cdots,n|s_1,\cdots,s_n)\}$ depends explicitly only on $k_1,\cdots,k_n$ and not on $k_{n+1},\cdots,k_N$. The reverse is true for $\bar{E}\{(1,\cdots,n|s_1,\cdots,s_n)\}$. Hence, we have, using (A.1),

$$\det \underset{\approx}{E}(-k_1, \cdots, -k_n, +k_{n+1}, \cdots, +k_N) =$$

$$\sum_{s_1 < s_2 < \cdots < s_n = 1}^{N} E\left\{(1, \cdots, n | s_1, \cdots, s_n), -\underset{\sim}{K}\right\} \bar{E}\left\{(1, \cdots, n | s_1, \cdots, s_n), \underset{\sim}{K}\right\}. \tag{A.5}$$

From (A.4) and (A.5), we obtain (V.15).

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FOOTNOTES

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1. We should also exclude the points where at least one of the functions $\sin(\xi + \lambda_m)^{1/2}a$ vanishes and for which det E does not have a pole. Here, $\hat{\underset{\approx}{S}}$ may have poles, too. But these points are isolated for all finite values of k_1. They are considered further in Section IV. E.

2. It should not be necessary to point out that the terms "wave function" and "state" are not applicable for complex k_1^a. However, the word "compound state" is equally meaningless--if one is strict about it--and very helpful.

3. Davies and Baranger (13) succeed in deriving a relation similar to the second of conditions (IV.32). They do this be decomposing the S matrix into a pole term and a background matrix $\underset{\approx}{B}$ and by assuming that $\underset{\approx}{B}$ for real energies is constant over the width of the resonance. One can easily see, however, that without this rather drastic assumption about B (drastic in view of the analytically continued unitarity relationships) it is not possible to arrive at their results.

4. In the following, we shall consider only attractive potentials, since only they seem to give rise to sharp resonances in the many-channel problem.

5. On their way from the origin $k_1 = 0$ to the points $k_1 = \pm(e_i)^{1/2}$, $i > 2$, these points pass through a number of branch points. At each of these branch points, they are double poles in the parameter of local uniformalization. At $k_1 = \pm(e_i)^{1/2}$, they are simple poles in the parameter of local uniformalization, k_i. From considerations of this type one can understand

how the generalized version of Levinson's theorem in Section V works for the case of vanishing coupling between the channels, and that it agrees with the same theorem applied to scattering in the N uncoupled channels individually.

6. At threshold, σ_{11} must display a cusp or a similar type of behavior which can be predicted from the knowledge of $\delta(k_1)$. This has been discussed by Newton (7).

Table I. Construction of the Riemann surface for N=3

Upper shore	{+1, +1}	{+1, -1}	{-1, +1}	{-1, -1}
Lower shore	{-1, -1}	{-1, +1}	{+1, -1}	{+1, +1}

Table II. Position of the poles in the lower half of the k_1 plane

Re k_1	3.238	8.306	12.141	15.673	19.072	22.395
Im k_1	1.066	1.316	1.519	1.689	1.836	1.963

Table III. Properties of the Residues of the Elements of $\underset{\approx}{S}$ at a Pole

a	Re k_1	Im k_1	Γ	$\gamma_1/(\bar{\mu})^{1/2}$	$\gamma_2/(\bar{\mu})^{1/2}$	$\gamma_1^2/\bar{\mu}$	$\Gamma(b = 1.0)/\Gamma$
0.005	7.5842	-0.0010	0.030	-0.1411 - 0.1026 i	10^3 (-0.8867 + 0.8780 i)	+0.009 + 0.029 i	1.020
0.100	7.5784	-0.0214	0.648	-0.6113 - 0.5150 i	10^3 (-0.9609 + 0.7782 i)	+0.108 + 0.630 i	1.019
0.200	7.5776	-0.0454	1.377	-0.7959 - 0.8120 i	10^3 (-1.0138 + 0.6358 i)	-0.026 + 1.292 i	1.018
0.300	7.5773	-0.0716	2.169	-0.9248 - 1.1077 i	10^3 (-1.0520 + 0.4835 i)	-0.372 + 2.049 i	1.015
0.400	7.5849	-0.0984	2.985	-0.9400 - 1.3904 i	10^3 (-1.0327 + 0.3045 i)	-1.050 + 2.614 i	1.013
0.500	7.5991	-0.1235	3.753	-0.8714 - 1.6444 i	10^3 (-0.9526 + 0.1314 i)	-1.945 + 2.866 i	1.011

Figure Legends

Fig. 1. The topological structure of the Riemann surface for N=3.

Fig. 2. The physical k_1 plane with its cuts.

Fig. 3. The physical k_1 plane for N=2, $e_2 = 6$, with the values of k_2 in italics.

Fig. 4. The physical k_1 plane and the sheets adjacent to the real axis.

Fig. 5. Analytic continuation of the unitarity relationships.

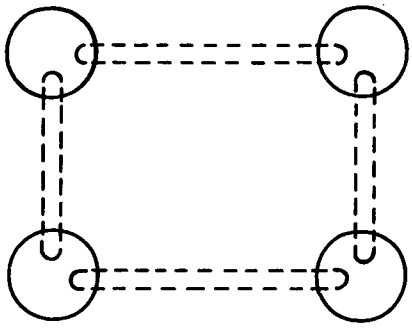
Fig. 6. The contour C for the integral in V.1.

Fig. 7. $\delta(k_1)$ as a function of k_1 for $|\lambda_1|a^2 = 50$, $k_1 \leq 30$.

Fig. 8. Cross section and pole contributions as functions of k_1.

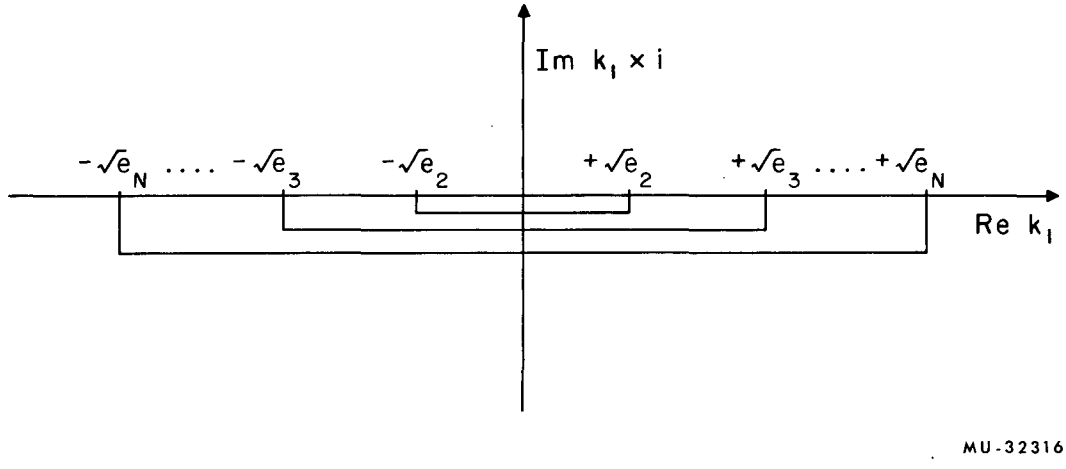
Fig. 9. The phase of D as defined by Eq. (V.1) as a function of k_1 for $\lambda_1 = 40$, $\lambda_2 = -50$, and $e = 100$. Curve A corresponds to $\alpha = 0.005$, curve B to $\alpha = 0.5$.

Fig. 10. The various cross sections as functions of k_1 for $\lambda_1 = 40$, $\lambda_2 = -50$, and $e = 100$. Curve A corresponds to $\alpha = 0.005$, curve B to $\alpha = 0.5$. The inserts show σ_{11} (curve A) with an enlarged scale for k_1.



MU-32315

Fig. 1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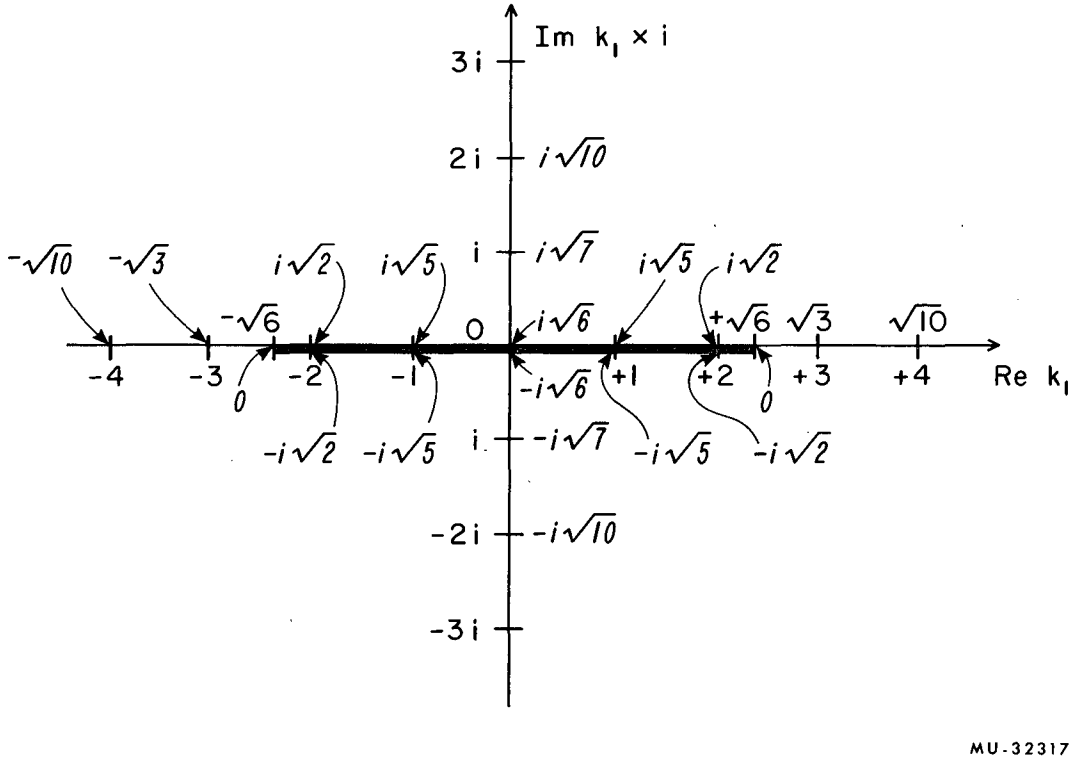
$-\sqrt{e_2}$
$+\sqrt{e_2}$
$+\sqrt{e_3}$ .... $+\sqrt{e_N}$
Re k_1
MU-32316

Fig. 2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2i
i
i
-i
-i
-2i
-3i
$i\sqrt{10}$
$i\sqrt{7}$
$i\sqrt{6}$
$-i\sqrt{6}$
$-i\sqrt{7}$
$-i\sqrt{10}$
$-\sqrt{10}$
$-\sqrt{3}$
$-\sqrt{6}$
$i\sqrt{2}$
$i\sqrt{5}$
$i\sqrt{5}$
$i\sqrt{2}$
$+\sqrt{6}$
$\sqrt{3}$
$\sqrt{10}$
0
0
0
$-i\sqrt{2}$
$-i\sqrt{5}$
$-i\sqrt{5}$
$-i\sqrt{2}$
-4
-3
-2
-1
+1
+2
+3
+4
Re k_1
MU-32317

Fig. 3

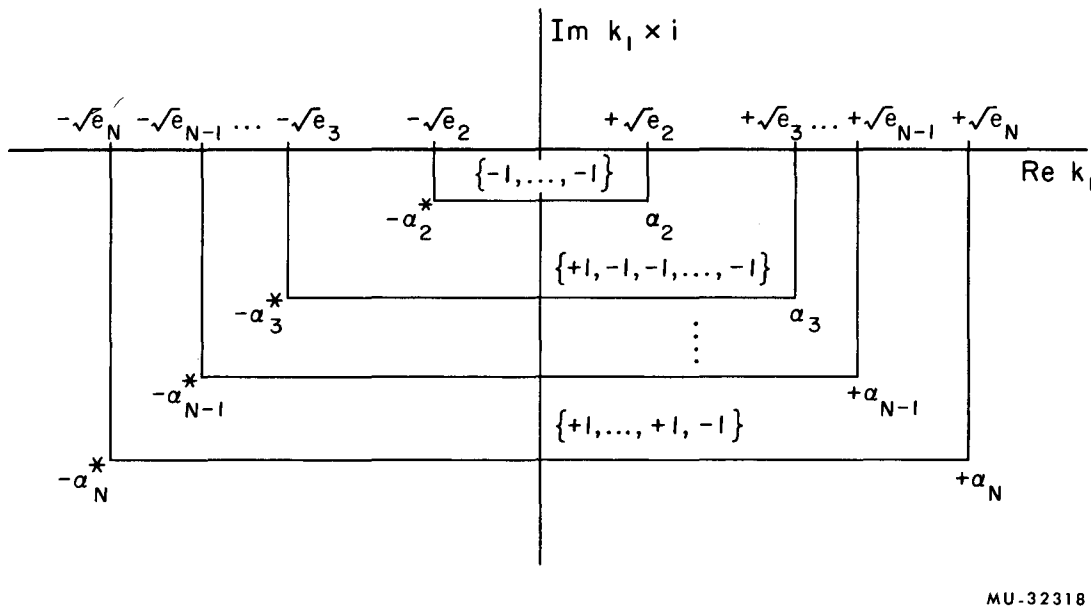

Im $k_1 \times i$
$-\sqrt{e_N}$ $-\sqrt{e_{N-1}}$ ... $-\sqrt{e_3}$ $-\sqrt{e_2}$ $+\sqrt{e_2}$ $+\sqrt{e_3}$ ... $+\sqrt{e_{N-1}}$ $+\sqrt{e_N}$
Re k_1
$\{-1, \ldots, -1\}$
$-\alpha_2^*$
α_2
$\{+1, -1, -1, \ldots, -1\}$
$-\alpha_3^*$
α_3
$-\alpha_{N-1}^*$
$+\alpha_{N-1}$
$\{+1, \ldots, +1, -1\}$
$-\alpha_N^*$
$+\alpha_N$
MU-32318

Fig. 4

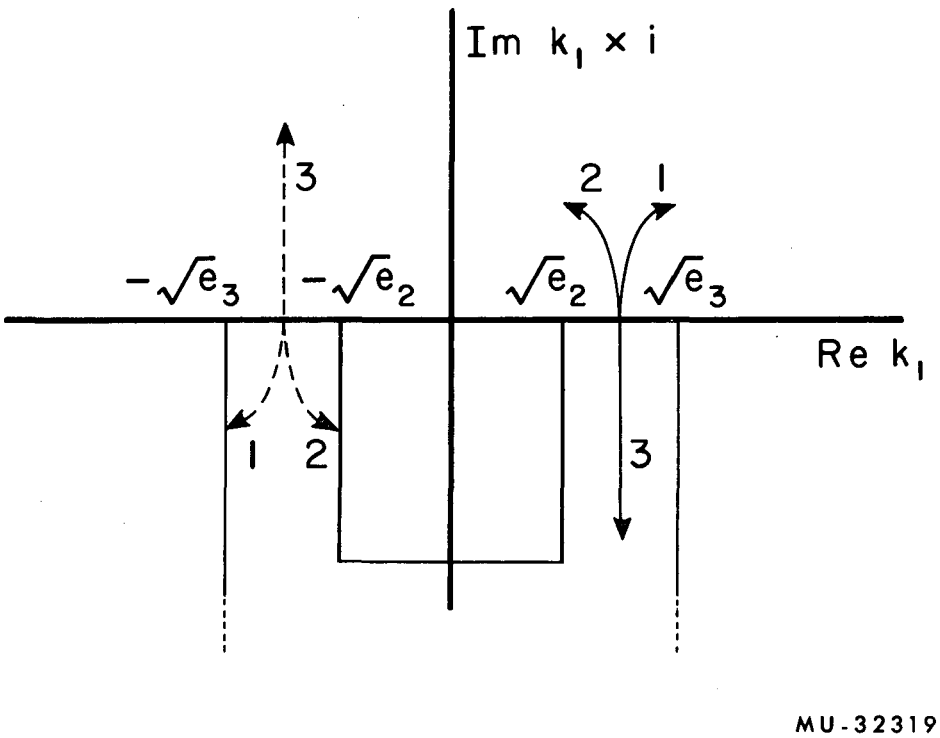
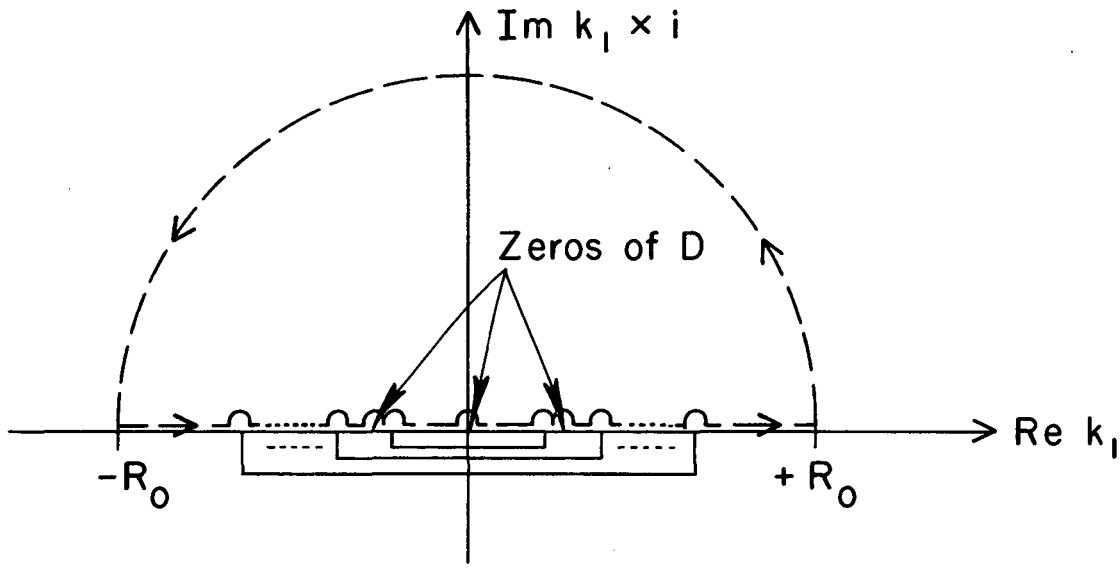

Im $k_1 \times i$
3
2
1
$-\sqrt{e_3}$
$-\sqrt{e_2}$
$\sqrt{e_2}$
$\sqrt{e_3}$
Re k_1
1
2
3
MU-32319

Fig. 5



MU-32320

Fig. 6

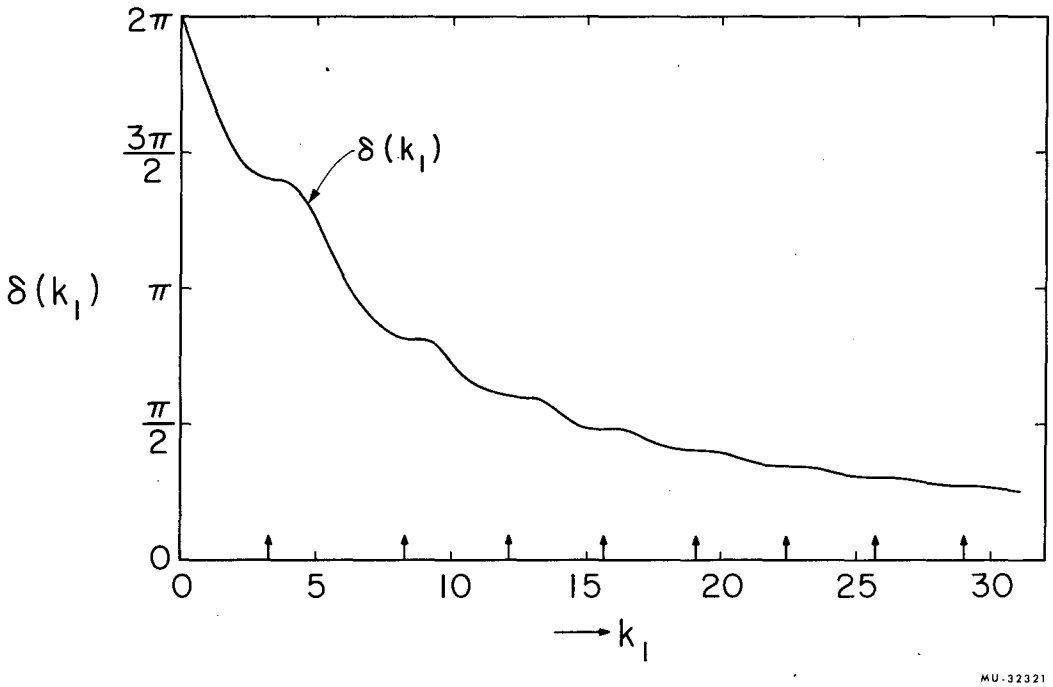


Fig. 7

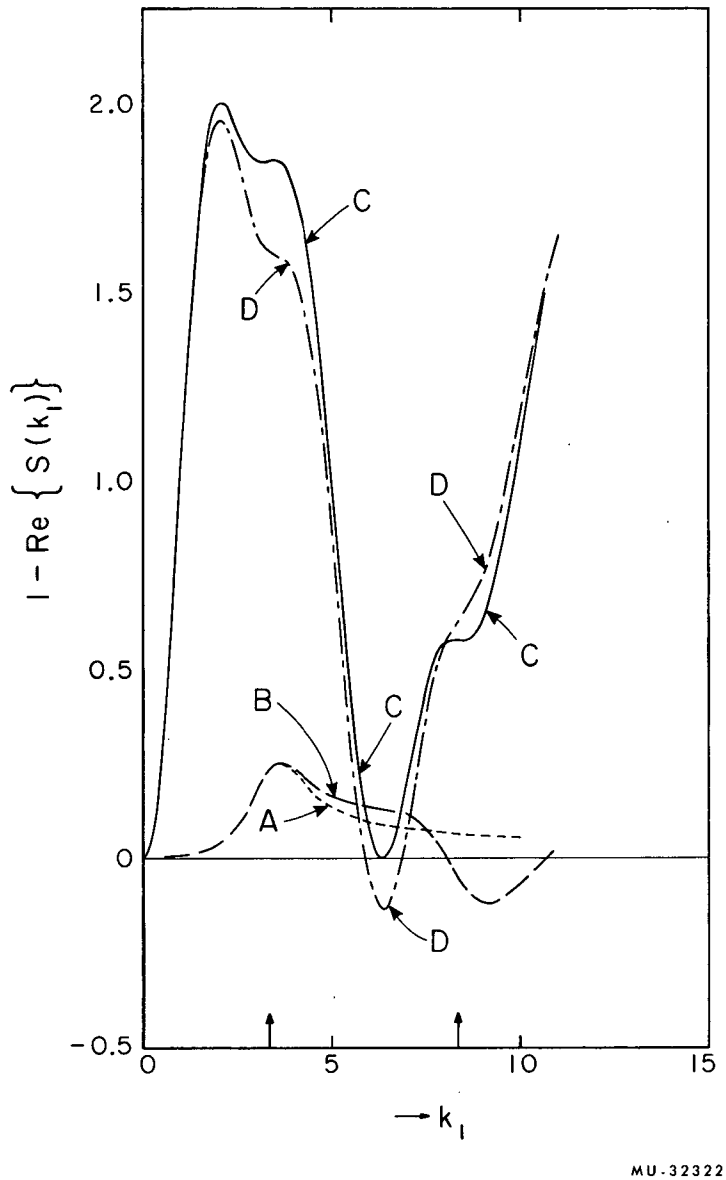

2.0
1.5
1.0
0.5
0
-0.5
0
5
10
15
$1 - \mathrm{Re}\{S(k_1)\}$
$\rightarrow k_1$
A
B
C
D
MU-32322

Fig. 8

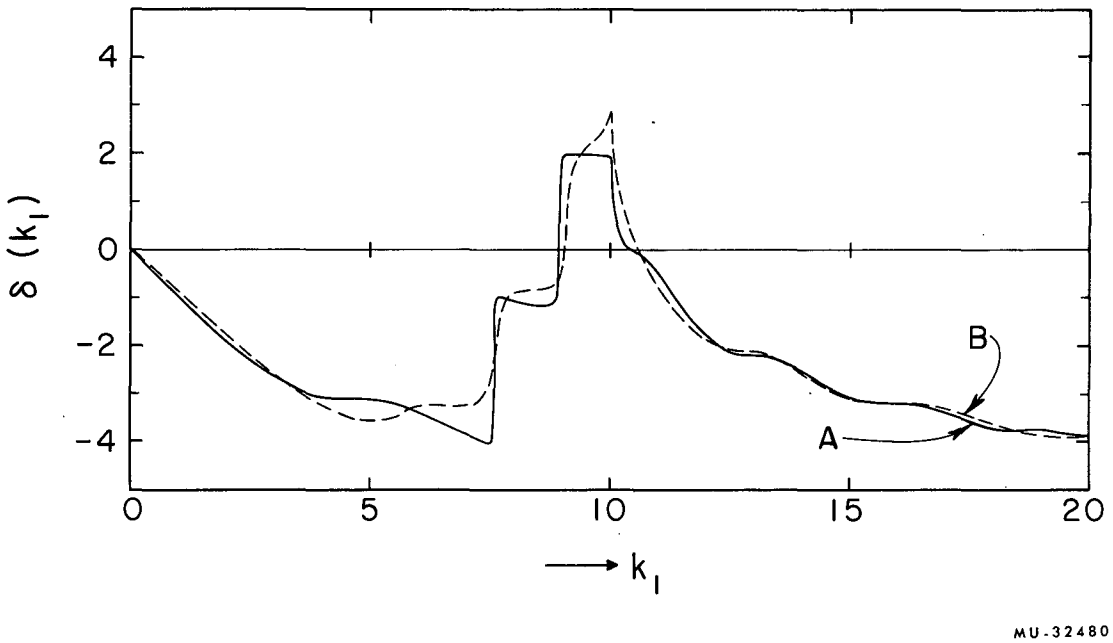


Fig. 9

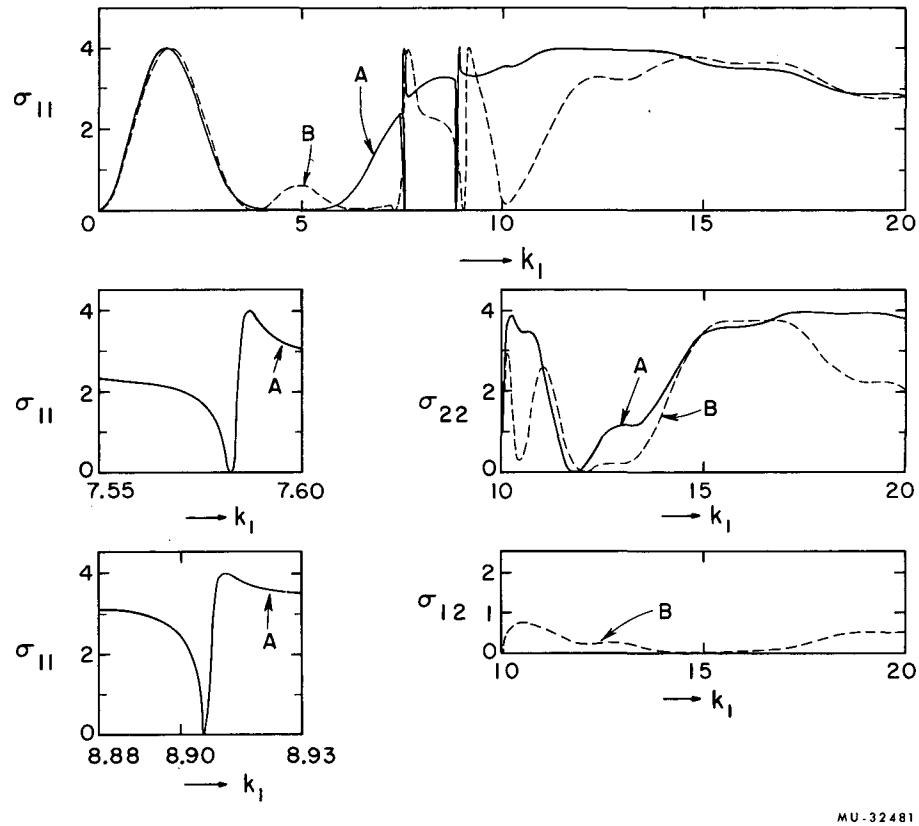

σ11
4
2
0
A
B
5
10
15
20
→ k1
σ11
7.55
7.60
σ22
σ11
8.88
8.90
8.93
σ12
1
MU-32481

Fig. 10

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