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ON THE THREE-BODY SCATTERING AMPLITUDE
II. EXTENSION TO COMPLEX ANGULAR MOMENTUM

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March 17, 1964

ERRATUM

TO: All recipients of UCRL-11186

FROM: Technical Information Division

Subject: UCRL-11186, "On the Three-Body Scattering Amplitude. II. Extension to Complex Angular Momentum," Roland L. Omnes, December 18, 1963.

Please make the following correction on subject report.

The derivation of Eq. (25) from Eq. (24) is wrong. The considerations of parity made at that occasion make no sense. However, it is easy, using the integration on u and v to derive a correct proof. One can build a proof which makes no use of the poorly converging formula (17). This proof will be given in the published version of the paper as well as a more detailed analysis of the two-body scattering amplitude.

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ON THE THREE-BODY SCATTERING AMPLITUDE. II. EXTENSION TO COMPLEX ANGULAR MOMENTUM

Roland L. Omnes

December 18, 1963

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ABSTRACT

The Fadeev equations for the three-body scattering amplitudes of a given total angular momentum J--as derived in a preceding article--are extended to complex values of the angular momentum. The main steps are provided by an analysis of the properties of rotation matrices for complex angular momentum. Of particular importance are rotation matrices of the second kind, which are natural generalizations of Legendre functions of the second kind. It is shown that the terms of the kernel are analytic functions of the total angular momentum J for any value bigger than -1/2 . This is a generalization of the cancellation of Amati-Fubini-Stanghellini cuts that was discovered by Mandelstam. Although a complete mathematical analysis of the analytic properties of the solution is not given, there is no evidence for any singularity except poles. Therefore the essential results of Regge for the two-body scattering amplitude may certainly be extended to three-particle nonrelativistic channels. A Sommerfeld-Watson transformation allows us to derive from these results the asymptotic properties of the three-body scattering amplitude when one reaction angle becomes infinite.

1. INTRODUCTION

This article is devoted to an extension of the reduced Fadeev equations--as derived in a preceding paper (later on called I)--to complex values of the total angular momentum.

The reason for this study is the current interest in the possible limitations of the Chew-Frautschi hypothesis,[1] according to which the two-body scattering amplitude is a meromorphic function of the total angular momentum, and all elementary particles are members of a Regge pole trajectory.[2] If this hypothesis be true, it would follow that it is possible to build up a theory of strong interactions without arbitrary parameters[3] and to analyze the high-energy diffraction peak in terms of the Regge pole trajectories and residues.[4] The existence of singularities other than poles would complicate this last analysis but it would not necessarily invalidate the first statement. Because of the importance of such problems, it seems worthwhile to take up a systematic analysis of the properties of scattering amplitudes in terms of a complex angular momentum.

When one wants to extend the results of Regge[2] to the relativistic scattering amplitude, two essentially new features enter in, namely: (a) the existence of processes described in the Mandelstam representation[5] by the so-called third double-spectral function that are ultimately due to crossing, and (b) the nonconservation of the number of particles that necessitates a study of the many-particle states as possible intermediate or final states.

It was first realized by Gribov and Pomeranchuk[6] that crossing leads to essential singularities of the scattering amplitude for two spinless particles at negative integral values of the angular momentum.

On the other hand, Amati, Fubini and Stanghellini[7] pointed out the possible existence of cuts in the total angular momentum due to many-particle intermediate systems, as is shown in Fig. 1 . It was further shown by Mandelstam[8] and by Polkinghorne[9] that, in fact, these cuts are not in the physical sheet and do not have to be considered. However, Mandelstam[10] also showed that analogous graphs, like those in Fig. 2, lead indeed to cuts, and that the Gribov-Pomeranchuk singularities are on a well-defined sheet linked to these cuts.

An important feature of the graphs in Fig. 1 is that they have a nonrelativistic interpretation, and it would be very interesting to know if, for all such nonrelativistically interpretable processes, there are no cuts; or--stated otherwise--if the nonrelativistic amplitudes for two particles going into many particles or for many-particle scattering, are meromorphic functions of the total angular momentum. Clearly, the first problem is then to analyze the three-particle scattering amplitude.

This problem has already been considered by Newton[11] and Hartle,[12] who, using the same method, arrive at opposite conclusions. While Newton claims that there are cuts of the three-body scattering amplitude extending up to infinite values of the total angular momentum J , Hartle finds it to be a meromorphic function. As we have indicated in I , these two authors have used a formulation of the problem that turns it into a study of matrices with continuous indices. This is clearly unsuitable for any correct mathematical analysis. Furthermore, they solve their reduced Schrodinger equation by a method essentially equivalent to a use of the Lippmann-Schwinger equation,[13] which is well known to be an insufficient formulation of the three-body problem.[14]

We therefore think that the results of Newton and Hartle cannot be accepted and that it is necessary to find a formulation of the problem free of their drawbacks. The solution we have found is set forth as follows.

In Section 2, the reduced Fadeev equations found in I are written explicitly after an iteration. The reason for that iteration is that only the square of the Fadeev kernel is completely continuous,[15,16] and we need at least to start from such a kernel for a sound mathematical analysis. This iterated kernel is an integral upon two-body scattering amplitudes that involves rotation matrices, one of the rotation angles being integrated from 0 to π.

In Section 3, some properties of the two-body scattering amplitudes that enter into the Fadeev kernel are analyzed. As they are off-the-energy-shell amplitudes, they are defined by Lippmann-Schwinger equations (which are reliable for a two-body process). The method used here is due to Lee and Sawyer.[17] Of particular importance is the asymptotic behavior of these amplitudes when a parameter that enters into the Fadeev kernel tends to infinity. A cancellation between the Regge asymptotic behavior as the scattering angle tends to infinity and the behavior of the Regge pole residues when the energy tends to infinity is found. This effect leads ultimately to the nonappearance of cuts found in Section 5.

In Section 4, we introduce what we call "rotation matrices of the second kind," which are related to the customary rotation matrices in the same way as the Legendre functions of the second kind are related to Legendre polynomials. It is shown that it is possible to replace in the kernel the rotation matrices by matrices of the second

kind, the essential integration now being made in the complex plane and enclosing the singularities of the two-body scattering amplitudes and of the propagation denominators. This procedure is a generalization of the Gribov-Froissart formula.[18]

In Section 5, we extend the second form of the kernel to complex angular momenta. This extension is unique, according to a theorem by Carlson.[19] It is found that, owing to the results in Section 2, the kernel elements are analytic functions of the angular momentum, which generalizes the cancellation of the Amati-Fubini-Stanghellini cut found by Mandelstam. Although we do not provide an analysis of the properties of the solution, this seems to be a good hint that the three-body scattering amplitude is a meromorphic function of the total angular momentum.

Section 6 shows how these results can be used, through a Watson-Sommerfeld transformation, to investigate the asymptotic properties of the three-body scattering amplitude when an angle tends to infinity.[20]

In Appendix A some useful properties of the Jacobi functions are recalled. These are found mostly in the book by Szego.[21] Appendix B states the definitions and properties of the rotation matrices of the first and second kind for complex values of the angular momentum. The results to be found here are mostly new, so far as we know.

This paper is restricted to the case in which the particles are spinless and do not have bound states. We intend to investigate the problem of the scattering of a particle on a bound state in a further paper.[22]

Although the analysis is specifically made on the iterated Fadeev equations, it can be applied as well to the Weinberg equations, which are another formulation of the three-body scattering problem.[23] We have

checked that--as it must be--the results are the same. The Weinberg equations can be extended to the many-particle problem, and--at first sight--it seems that most of the present work can be carried on to this more general case.

A truly complete solution of the problem would include an examination of the analytic properties of the solution and not only of the kernel. This is far from being a matter-of-fact extension. For instance, if one wants to investigate whether the kernel K is completely continuous, it is necessary to examine trace KK^+ to see if it is bounded. However, the properties of this trace, which is a doubly infinite sum, require knowledge about the asymptotic properties of the rotation matrices of the second kind when two indices tend to infinity. This means knowing the asymptotic properties of the hypergeometric function $F(a, b, c, z)$ when two independent linear combinations of a, b, c tend to infinity. Apparently this problem has not been considered by mathematicians.[24] With the problem cast in a well-defined form, we hope that it can be useful.

2. REDUCED FADEEV EQUATIONS

Let us start from the Fadeev equations (I.44) which we shall write in an abbreviated from as

$$\mathcal{T}^{(i)} = \mathcal{T}_{k\ell} - \sum_{(j)} K^{(i,j)} \mathcal{T}^{(j)}$$

where

$$i, j, k, \ell = 1, 2, 3; \quad i \neq j, \; i \neq k, \; i \neq \ell \quad \text{and} \quad k \neq \ell . \tag{1}$$

Equation (1) cannot be used directly to define an extension of the collision matrix to complex values of a parameter (namely, here, the total angular momentum) because the kernel $K^{(i,j)}$ is not completely continuous.[15] We therefore iterate Eq.(1), which gives

$$\mathcal{T}^{(i)} = \mathcal{T}_{k\ell} - \sum_{(j)} K^{(i,j)} \mathcal{T}_{\ell m} + \sum_{j} K^{2(i,j)} \mathcal{T}^{(j)}, \tag{2}$$

where $\ell \neq j$, $m \neq j$. According to the expression (I.47) for the kernel K, its square K^2 has the form

$$K^2 = \begin{pmatrix} H_{12}+H_{13} & H_{13} & H_{12} \\ H_{23} & H_{21}+H_{23} & H_{21} \\ H_{32} & H_{31} & H_{31}+H_{32} \end{pmatrix} , \tag{3}$$

where a typical term is, for instance, H_{12}:[25]

$$H_{12M'M}(\omega',\omega, z) = \int \frac{1}{p_1' \; p_2} F_{23}(\omega', \omega'', u, z - \omega_1')$$

$$\times F_{13}(\omega'', \omega, v, z - \omega_2) \, d_{M'M_1}^{J} (- \alpha_1') \, e^{iM_1 u}$$

$$\times d_{M_1M_2}^{J}(\theta_{21}'') \, e^{iM_2 v} \, d_{M_2M}^{J}(\alpha_2) \left[(\omega_1 + \omega_2 + \omega_3'' - z)\right.$$

$$\left.\times (\omega_1 + \omega_2 + \omega_3 - z)\right]^{-1} du \, dv \, d\omega_3'' . \tag{4}$$

When we wrote Eq.(4), we took into account the equality $\theta_{21}'' = \alpha_2'' - \alpha_1''$, and we have integrated ω_1'' and ω_2'', which--according to the delta functions in Eq.(I.47)--are given by

$$\omega_1'' = \omega_1' \text{ and } \omega_2'' = \omega_2 . \tag{5}$$

The integrations over u and v in Eq.(4) go from 0 to 2π. The integration over ω_3'' is over a limited range such that, according to Eq.(5),

$$|p_1' - p_2| \leq p_3'' < p_1' + p_2 . \tag{6}$$

According to the relation (I.16) between ω_3'' and θ_{21}'', one can also write

$$d\omega_3'' = -2(m_1 m_2 \omega_1' \omega_2)/m_3 \, d \cos \theta_{12}'' , \tag{7}$$

the integration upon $\cos \theta_{12}''$ going from -1 to $+1$.

3. PROPERTIES OF THE TWO-BODY SCATTERING AMPLITUDE

We need to know the analytic properties of $F_{23}(\omega_3', \omega_3'', z - \omega_1')$ as a function of ω_3'', as well as its asymptotic behavior when ω_3'' tends to infinity.

As was shown in Eq.(I.40), F^{23} is equal to the two-body scattering amplitude $f_{23}(\vec{q}_{23}, \vec{q}_{23}, \zeta)$, which is itself defined as the solution of the Lippmann-Schwinger equation. We shall write that equation in the case for which the potential is a pure Yukawa potential $ge^{-\mu r} r^{-1}$:

$$f(\vec{q}, \vec{q}, \zeta) = \frac{g}{(\vec{q} - \vec{q}')^2 + \mu^2} + \int \frac{d^3 \vec{k}}{(2\pi)^3} \frac{1}{k^2 - \zeta} \times \frac{g}{(\vec{q} - \vec{k})^2 + \mu^2} f(\vec{k}, \vec{q}', \zeta) . \tag{8}$$

If one analyzes the singularities of the solution of Eq.(8) according to the method of Blankenbeckler, Goldberger, Khuri, and Treiman,[26] it is found that $f(\vec{q}, \vec{q}', \zeta)$, as an analytic function of ζ, has only poles at the bound-state energies and a cut extending from 0 to ∞. As a function of q^2 (and q'^2), it has a cut starting from $q^2 = 0$ and a cut going along $(\zeta)^{1/2} + i\mu_0$ (where $\mu \leqslant \mu_0 < \infty$). As a function of the cosine, $\hat{q} \cdot \hat{q}'$, it has the customary Landau singularities.

More can be said if we apply the Lee and Sawyer method to Eq.(8). Projecting out the ℓth partial wave $a_\ell(q^2, q'^2, \zeta)$ from Eq.(8), we get the equation

$$a_\ell(q^2, q'^2, \zeta) = \frac{g}{2q} Q_\ell\left(\frac{q^2 + q'^2 + \mu^2}{2qq'}\right) + \frac{g}{\pi q}$$

$$\int_0^\infty Q_\ell\left(\frac{q^2 + k^2 + \mu^2}{2qk}\right) \frac{dk}{k^2 - \zeta} \; a_\ell(k^2, q'^2, \zeta), \qquad (9)$$

where Q_ℓ is the Legendre function of the second kind. One sees in Eq.(9) that $a_\ell(q^2, q'^2, \zeta)$ has a right-hand cut going from $q'^2 = 0$ to $+\infty$ and two superposed left-hand cuts going from $q'^2 = -\infty$ to $(i\mu \pm q)^2$. The Fredholm determinant of Eq.(9) depends only upon ζ and is in fact identical to the Lee and Sawyer determinant. Its zeros are the Regge poles of the two-body scattering amplitude with a complex energy ζ.

The asymptotic behavior of $f_{23}(\vec{q}_{23}, \vec{q}'_{23}, \zeta)$, when ω_{23}'' tends to infinity can be deduced from the Sommerfeld-Watson formula:

$$f(\vec{q}, \vec{q}', \zeta) = \frac{1}{2} \int \frac{a_\ell(q^2, q'^2, \zeta)}{\sin \pi\ell} P_\ell(-\hat{q} \cdot \hat{q}') \, d\ell \; . \qquad (10)$$

Taking into account only the leading Regge pole $a(\zeta)$, one has

$$f(\vec{q}, \vec{q}', \zeta) = \frac{G(q^2, q'^2, \zeta)}{\sin \pi \alpha(\zeta)} (\hat{q} \cdot \hat{q}')^{\alpha(\zeta)} \; . \qquad (11)$$

However, we are in fact interested only in the limit ω_3'' tending to infinity, where not only $\hat{q} \cdot \hat{q}'$, but also q'^2 tends to infinity, so that a precise knowledge of the asymptotic behavior of the residue $\mathcal{G}(q^2, q'^2, \zeta)$ is also needed.

When q' tends to infinity, the inhomogeneous term in Eq.(9) behaves as $q'^{-\ell-2}$ times a function that does not depend on q'. As the kernel in Eq.(3) does not depend on q' itself, the full solution will behave as $q'^{-\ell-2}$, so long as it is defined. The solution will not be defined when the values of ζ and ℓ correspond to a Regge pole, but by letting ζ tend continuously to such a value, it is clear that the residue behaves as $q'^{-\alpha(\zeta)-2}$ when q' tends to infinity.

When ω_3'' tends to infinity, both q_{23}'' and $\hat{q}' \cdot \hat{q}''$ behave as $\omega_3''^{1/2}$, so that the two exponents $\alpha(\zeta)$ in the Regge factor and its residue in Eq.(11) cancel. Therefore,

$$F_{23}(\omega', \omega'', z - \omega_1') \sim \omega_3''^{-1} \quad \text{when} \quad \omega_3'' \to \infty . \tag{12}$$

This is equivalent to the statement that the full solution of Eq.(8) has the same kind of asymptotic behavior as the Born approximation when ω_3'' tends to infinity.

Finally, let us notice that, due to Eqs.(20a and 20b) of I, the singularities of F_{23}, which are all outside the physical region $|\hat{q} \cdot \hat{q}| < 1$, are also outside the domain $-1 < \cos\theta_{12}'' < 1$.

4. AN ALTERNATIVE FORM OF THE KERNEL

In order to extend Eq.(4) to complex values of J, it is necessary to give it such a form that the extension is uniquely defined. We shall see in Section 5 that this is not the case for Eq.(4). This problem is already well known in the case of two-body scattering, where the partial-wave scattering amplitudes for real values of ℓ can be indifferently written in terms of the total scattering amplitude $f(\cos\theta)$, as

$$\int_{-1}^{+1} P_\ell(\cos\theta)\, f(\cos\theta)\, d\cos\theta \tag{13a}$$

or

$$\int_C Q_\ell(\cos\theta)\, f(\cos\theta)\, d\cos\theta \tag{13b}$$

where the contour C encloses the segment $(-1, +1)$ in the plane of complex $\cos\theta$. However, when ℓ is made complex, the expression (13a) blows up exponentially when $\mathrm{Im}\,\ell \to \infty$ as $e^{\pi|\mathrm{Im}\ell|}$ whereas (13b) stays bounded. According to Carlson's theorem,[19] Eq.(13b) is the unique interpolation, which has this property and which makes a Sommerfeld-Watson transformation possible.

Equation (4) is very analogous to (13a) because, when $M_1 = M_2 = 0$, $d^J_{M_1M_2}(\theta_{21}'')$ is equal to $P_J(\cos\theta_{12}'')$, whereas the integration over ω_3'' is equivalent, according to Eq.(7), to an integration over $\cos\theta_{12}''$ from -1 to $+1$.

When M_1 and M_2 are not both equal to zero, it is possible to

relate $d_{M_1M_2}{}^J(\theta_{21}'')$ to the Jacobi polynomials $P_n^{(\alpha,\beta)}(\cos\theta_{21}'')$ according to[25]

$$d_{M_1M_2}{}^J(\theta_{21}'') = \left[\frac{(J+M_1)!(J-M_1)!}{(J+M_2)!(J-M_2)!}\right]^{1/2} \left(\cos\theta_{21/2}''\right)^{M_1+M_2}$$

$$\times \left(\sin\theta_{21/2}''\right)^{M_1-M_2} P_{J-M_1}^{(M_1-M_2,\ M_1+M_2)}\left(\cos\theta_{21}''\right) . \qquad (14)$$

This equation is valid only for $M_1 - M_2$ and $M_1 + M_2$ both positive or zero, and other expressions are needed in the other cases. However, Eq. (14) will be enough for our purposes.

We indicate in Appendix A the properties of the Jacobi polynomials that will be needed in the following. Together with the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$, one can introduce the so-called Jacobi function of the second kind $Q_n^{(\alpha,\beta)}(x)$.[21] For n an integer, it is an analytic function of x in the plane cut from -1 to $+1$, whereas for n not an integer it is analytic in the plane cut from $-\infty$ to $+1$. We introduce the new matrices

$$e_{M_1M_2}{}^J(\theta_{21}'') = \left[\frac{(J+M_1)!(J-M_1)!}{(J+M_2)!(J-M_2)!}\right]^{1/2} \left(\cos\theta_{21/2}''\right)\left(\sin\theta_{21/2}''\right)^{M_1-M_2}$$

$$\times\ Q_{J-M}^{(M_1-M_2,M_1+M_2)}\left(\cos\theta_{21}''\right) \qquad (15)$$

together with the matrices analogous to the full rotation matrices

$$\mathcal{E}_{M_1M_2}{}^{J}(\psi, \theta, \varphi) = e^{-iM_1\psi}\, e_{M_1M_2}{}^{J}(\theta)\, e^{-iM_2\varphi} . \tag{16}$$

It is shown in Appendix B that they enjoy the remarkable property

$$\mathcal{E}_{M_1M_2}{}^{J}(R'R) = \sum_{M=-\infty}^{M=+\infty} \mathcal{E}_{M_1M}{}^{J}(R')\, \mathcal{D}_{MM_2}{}^{J}(R) . \tag{17}$$

Equation (17) means that, if we replace $\mathcal{D}_{M_1M_2}{}^{J}(\theta_{21}'')$ by $\mathcal{E}_{M_1M_2}{}^{J}(\theta_{21}'')$ in Eq. (4), this is equivalent to replacing the full expression

$$d_{M'M_1}{}^{J}(-\alpha_1')\, e^{iM_1u}\, d_{M_1M_2}{}^{J}(\theta_{21}'')\, e^{iM_2v}\, d_{M_2M}{}^{J}(\alpha_2) , \tag{18}$$

which is nothing but

$$\mathcal{D}_{M'M}{}^{J}\left[R(-\alpha_1', u, \theta_{21}'', v, \alpha_2)\right] , \tag{19}$$

(where R is the product of a rotation α_2 around the y axis, v around the z axis, θ_{21}'' around y, u around z, and lastly $-\alpha_1'$ around y) by

$$\mathcal{E}_{M'M}^{J}\left[R(-\alpha_1', u, \theta_{21}'', v, \alpha_2)\right], \tag{20}$$

as we shall show are the end of this section.

The analogy of Eqs. (13a) and (13b) is that, when $f(x)$ is an analytic function of x in a neighborhood of the segment $[-1, +1]$, the two quantities

$$\int_{-1}^{+1} (1 - x)^{\alpha}(1 + x)^{\beta} f(x)\, P_n^{(\alpha,\beta)}(x)dx \tag{21a}$$

and

$$\frac{1}{i\pi} \int_C (y - 1)^{\alpha}(y + 1)^{\beta} f(y)\, Q_n^{(\alpha,\beta)}(y)\, dy \tag{21b}$$

are equal.[21] Here C has the same meaning as in Eq. (13b).

It is not quite obvious that Eq.(4) has the form (21a). In fact, according to Eq.(14), $d_{M_1M_2}^{J}$ has rather the form $(1 - x)^{\alpha/2}(1 + x)^{\beta/2}$ $P_n^{(\alpha,\beta)}(x)$. In order to see the form (21a) it is necessary to introduce first the Legendre series expansion of F_{23} and F_{13} according to Eq.(I.42):

$$F_{23}(\omega_1', \omega'', z - \omega_1') = \sum_{\ell=0}^{\infty} \sum_{M=0}^{\infty} (2\ell + 1)$$

$$\times\ a_\ell(q_{23}'^2, q_{23}''^2, Q_{23}^2) \frac{(\ell - M)!}{(\ell + M)!} \tag{22}$$

$$\times\ P_\ell^M(\cos \gamma_1')\ P_\ell^M(\cos \gamma_1'')\ \cos Mu\ ;$$

then, for a given value of M_1, we integrate upon u, which leaves us with a series of the form

$$\int F_{23}\ e^{iM_1 u}\ du = \sum \alpha_\ell\ P_\ell^{M_1}(\cos \gamma_1'')\ . \tag{23}$$

Now $P_\ell^{M_1}(\cos \gamma_1'')$ is equal to $(\sin \gamma_1'')^{M_1}$ times a polynomial in $\cos \gamma_1''$. On the other hand, we have seen that, according to Eqs.(20a) and (20b), $\sin \gamma_1''$ is equal to $\sin \theta_{23}''$ times an analytic function of ω_3'' in a neighborhood of the segment $[-1_1, +1]$. This provides the right power of $\sin \theta_{21}''$, which is necessary to give to Eq.(4) the form (21a). It is easy to justify the use of the expansion (22) in the domain where it is needed.

We have thus shown that it is legitimate to replace Eq. (4) by an analogous expression in which $e_{M_1M_2}^J(\theta_{21}'')$ replaces $d_{M_1M_2}(\theta_{21}'')$ and

for which the integration upon $\cos \theta_{12}''$ is now made on a contour that encloses -1 and $+1$. This contour may then be deformed in order to enclose the singularities of the integrand in ω_3''. We shall call Γ such a contour. Finally,

$$H_{12}(\omega', \omega) = \int_0^{2\pi} \int_0^{2\pi} du\, dv \frac{1}{i\pi} \int_\Gamma d\omega_3'' \; F_{23}(\omega', \omega'', u, z - \omega_1')$$

$$\times F_{13}(\omega'', \omega, v, z - \omega_2) \; d_{M'M}^{J}(-\alpha_1') \; e^{iM_1 u} \; e_{M_1 M_2}^{J}(\theta_{12}'')$$

$$\times e^{iM_2 v} \; d_{M_2 M}^{J}(\alpha_2) \; [(\omega_1' + \omega_2 + \omega_3'' - z)(\omega_1 + \omega_2 + \omega_3 - z)]^{-1}. \tag{24}$$

Now it is possible in Eq. (4) to replace the full expression

$$d_{M'M}^{J}(-\alpha_1') \, e^{iM_1 u} \; d_{M_1 M_2}^{J}(\theta_{12}'') \, e^{iM_2 v} \; d_{M_2 M}(\alpha_2)$$

by only one matrix $\mathcal{D}_{M'M}^{J}(R)$, where R is the product of the five rotations with angles $-\alpha_1'$, u, θ_{12}'', v, and α_2 respectively, around the axes Oy, Oz, Oy, Oz, and Oy. It would be very convenient for what follows to replace also the matrix that appears in Eq. (24) by only one rotation matrix of the second kind $\mathcal{E}_{M'M}^{J}(R)$, where R is the same matrix as above but for the complex character of $\cos \theta_{21}''$. That this is pos-

sible is suggested by Eq. (17). Equation (17) is proved for a complex value of J. In order to understand its meaning, for physical values of J, let us consider the case in which $M = M' = 0$. In that case, Eq. (17) becomes the familiar addition for the Legendre function of the second kind,

$$Q_J(\cos\theta\cos\theta' + \sin\theta\sin\theta\cos\varphi)$$

$$= \underline{P}_J(\cos\theta')\,Q_J(\cos\theta)$$

$$+ 2\sum_{m=1}^{\infty} (-1)^m\,\underline{P}_J^{-m}(\cos\theta')\,Q_J^{m}(\cos\theta)\cos m\varphi\,.$$

It is important that, when J tends to an integral value ℓ, the summation upon m does not stop at $m = \ell$. Rather, according to the equation

$$\underline{P}_J^{-m}(z) = \frac{\Gamma(J - m + 1)}{\Gamma(J + m + 1)}\left[\underline{P}_J^{m}(z) - \frac{2}{\pi}\sin(m\pi)\,e^{-im\pi}\,Q_J^{m}(z)\right],$$

$\underline{P}_J^{-m}(z)$ reduces to $2(-1)^{m+J+1}\,Q_J^{m}(z)$ when J and m are both positive integers, and m is larger than J, to $[\Gamma(J - m + 1)/\Gamma(J + m + 1)]\,\underline{P}_J^{m}(z)$

when m is smaller than J. Accordingly, the addition formula becomes

$$Q_\ell(\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\varphi) = \underline{P}_\ell(\cos\theta')\,Q_\ell(\cos\theta)$$
$$+ 2\sum_{m=1}^{\ell}(-1)^m\,\underline{P}_\ell^{-m}(\cos\theta')\,Q_\ell^m(\cos\theta)\cos m\varphi$$
$$+ 4\sum_{m=\ell+1}^{\infty}(-1)^{\ell+1}\,Q_\ell^m(\cos\theta')\,Q_\ell^m(\cos\theta)\cos m\varphi\,.$$

An essential property of this expansion is that, while the first two terms have parity $(-1)^\ell$ when $\cos\theta'$ is replaced by $-\cos\theta'$ and φ by $\varphi+\pi$, the last term has parity $(-1)^{\ell+1}$. This splitting of the addition formula (17) into two parts of opposite parity when J is an integer is in fact a general property not restricted to $M = M' = 0$. We shall use that property in order to rewrite Eq. (24) in another form. Let us first introduce the inverse of a state $|\omega\,J\,M\rangle$, labelled $P|\omega\,J\,M\rangle$. Then introduce the state with signature ζ as

$$|\omega\,J\,M\rangle_\zeta = |\omega\,J\,M\rangle + \zeta\,P|\omega\,J\,M\rangle\,, \qquad (\zeta = \pm 1)\,,$$

and the collison matrix of signature ζ as

$${}_\zeta\langle\omega'\,J\,M|T|\omega\,J\,M\rangle_\zeta = \langle\omega'\,J\,M|T^\zeta|\omega\,J\,M\rangle\,.$$

Define the notation

$$[G(\omega, \omega', \omega'', u, v)]_\zeta = G(\omega, \omega', \omega'', \alpha_1, \alpha_2, \alpha_3, \alpha_1', \alpha_2', \alpha_3', u, v)$$

$$+ \zeta\, G(\omega, \omega', \omega'', \pi - \alpha_1, \pi - \alpha_2, \pi - \alpha_3,$$

$$\pi - \alpha_1', \pi - \alpha_2', \pi - \alpha_3', u + \pi, v + \pi).$$

Finally, the kernel of the modified Fadeev equation for T^ζ can be written

$$H_{12}(\omega', \omega) = \iint_0^{2\pi} du\, dv \int_\Gamma d\omega_3'' \, [F_{23}(\omega', \omega'', u, z - \omega_1')$$

$$\times F_{13}(\omega'', \omega, v, z - \omega_2)]_\zeta \, e_{M'M}^{J}(R) \, [(\omega_1' + \omega_2 + \omega_3'' - z)$$

$$\times (\omega_1 + \omega_2 + \omega_3 - z)]^{-1} . \qquad (25)$$

The distortion of contour used while passing from Eq. (4) to Eq. (24) is justified by the fact that all the singularities of the two-body scattering amplitudes as well as of the propagation denominator $(\omega_1' + \omega_2 + \omega_3'' - z)$ are outside the interval $(-1, +1)$ of $\cos \theta_{12}''$ for complex values of ζ.

5. EXTENSION TO COMPLEX ANGULAR MOMENTUM

Our problem is now to extend Eqs. (24) or (25) to complex values of J. It is shown in Appendix A how the Jacobi functions of the first and second kinds can be extended to complex values of the index n. This is used in Appendix B to extend the definition of the rotation matrices of the first and second kind $d_{MM'}^{J}(\theta)$ and $e_{MM'}^{J}(\theta)$. The main results are:

a. $d_{MM'}^{J}(\theta)$ is an analytic function of cos θ in the plane cut from $-\infty$ to -1,

b. $e_{MM'}^{J}(\theta)$ is an analytic function of cos θ in the plane cut from $-\infty$ to +1,

c. $d_{MM'}^{J}(\cos\theta)$ increases like $e^{iJ\theta}$ when J tends to infinity,

d. $e_{MM'}^{J}(\cos\theta)$ is bounded when J tends to infinity, except when cos θ is between -1 and +1,

e. one has a property analogous to the group property,[28]

$$\mathcal{D}_{MM'}^{J}(R'R) = \sum_{M''=-\infty}^{+\infty} \mathcal{D}_{MM''}^{J}(R')\, \mathcal{D}_{M''M'}^{J}(R), \tag{26}$$

as well as Eq. (17),

$$\mathcal{E}_{MM'}^{J}(R'R) = \sum_{M''=-\infty}^{+\infty} \mathcal{E}_{MM''}^{J}(R')\, \mathcal{D}_{M''M'}^{J}(R). \tag{27}$$

Equations (26) and (27) are true only for certain values of the Euler angles

(possibly complex) of the rotations R and R' , according as the series in the right-hand members converge or not. We shall agree to define these series, even when they do not converge, as formally equal to the left-hand members.

According to Carlson's theorem and Property (d) , Eq. (24) written for complex values of J provides the unique extension of the kernel to complex J. The inhomogeneous term of the Fadeev equation can also be treated in the same way. [This is true when one writes the Fadeev equation for the connected T matrix, i.e., for instance, $T^{(1)} - T_{23}$.]

Property (f) shows that it makes no difference whether one makes J complex into the form (24) or the form (25) of the kernel.

According to the asymptotic properties of the two-body scattering amplitudes given by Eq. (12), and of the rotation matrices given by Eq.(B-7c), when ω_3'' or, equivalently, $\cos \theta_{12}''$ tends to infinity, the kernel (25) is formally defined for $J > -3$. However, as the asymptotic property (B-7c) of $\mathscr{E}_{MM'}^{J}$ is true only for J larger than -1/2, the kernel is defined only for

$$J > -1/2 . \tag{28}$$

In fact, this last statement is justified only if we can show that it is possible to displace the integration contour on ω_3'' in Eq. (25) to infinity without having to displace the contours on u and v to infinity. That means that the singularities of F_{23}, F_{13} and $(\omega_1' + \omega_2 + \omega_3'' - z)$ will not meet the singularities of $\mathscr{E}_{M'M}^{J}[R(-\alpha_1', u, \theta_{21}'', v, \alpha_2)]$. Let us call (λ, μ, v) the Euler angles of R. The singularities of $\mathscr{E}_{M'M}^{J}(R)$ are at $\cos \mu = \pm 1$. However, it is easily shown that, for $u, v, \alpha_1', \alpha_2$ real,

$\cos \mu = \pm 1$ corresponds to values of $\cos \theta_{21}''$ between -1 and $+1$, and we have already shown that the singularities never get into that physical region. Accordingly, the terms of the kernel are analytic functions of J for $J > -1/2$.

As it is discussed in the introduction, this result is not enough to prove that the $\mathcal{T}^{Ji}$ are meromorphic functions of J, but, at least, it does not contradict it.

6. WATSON-SOMMERFELD TRANSFORMATION

Let us suppose that, as suggested by the results of Section 5, $\mathcal{T}_{M'M}^{J}(\omega', \omega, z)$ is a meromorphic function of J. It is easy to relate that hypothesis to the asymptotic behavior of the three-body scattering amplitude.

To that effect, let us consider the amplitude $\langle \vec{p}_1' \vec{p}_2' \vec{p}_3' | T | \vec{p}_1 \vec{p}_2 \vec{p}_3 \rangle$, and let us associate two reference systems with the initial and final states, in a well-defined way that is not necessarily the same for both states (for instance, the z' axis coincides with $\vec{p}_1{}'$, and the z axis coincides with $\vec{p}_2{}'$ or is orthogonal to $\vec{p}_1 \vec{p}_2 \vec{p}_3 \cdots$). Let us call (ψ, θ, φ) the Euler angles that define the transition of the initial reference system to the final reference system. Then, according to Eq. (23) of I,

$$\langle \vec{p}' | T | \vec{p} \rangle = \text{constant} \sum (2J + 1) \, \mathcal{T}_{M'M}^{J}(\omega' \omega) \mathcal{D}_{M'M}^{J}(\psi, \theta, \varphi) . \tag{23}$$

Because the extension of $\mathcal{T}^J$ to complex values of J has been made according to the Carlson theorem, Eq. (29) can be cast into a Watson-Sommerfeld form,

$$\langle \vec{p}' | T | \vec{p} \rangle = \text{constant} \sum_{MM'} \int \frac{(2J + 1)}{\sin \pi J} \times (-1)^M \mathcal{T}_{MM'}^{J} \mathcal{D}_{M'-M}^{J}(\psi, \theta + \pi, \varphi) , \tag{30}$$

in which we have used the symmetry property, valid for J an integer

$$\mathcal{D}_{M'-M}^{J}(\psi, \theta, \varphi) = (-1)^{J+M}\mathcal{D}_{M'M}^{J}(\psi, \theta + \pi, \varphi) \quad . \tag{31}$$

Equation(30) shows that, if the singularity with higher real value of $\mathcal{T}_{M'M}^{J}(\omega', \omega, z)$ is a pole at $J = \alpha(\omega', \omega, z)$, then $\langle\vec{p}|T|\vec{p}\rangle$ behaves as $(\cos\theta)^{\alpha}$ when $\cos\theta$ tends to infinity.

APPENDIX A

In this appendix, we shall recall some useful properties of Jacobi functions.

The Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ are customarily defined as the set of orthogonal polynomials on the interval $[-1, +1]$ with the weight function $w(x) = (1 - x)^{\alpha}(1 + x)^{\beta}$, where α and β are larger than -1; the normalization of $P_n^{(\alpha,\beta)}(x)$ is effected by

$$P_n^{(\alpha,\beta)}(1) = \frac{\Gamma(n + \alpha + 1)}{\Gamma(\alpha+1)\Gamma(n+1)} = \binom{n + \alpha}{n} . \tag{A-1}$$

They satisfy the homogeneous differential equation of the second order,

$$(1 - x^2)y'' + [\beta - \alpha - (\alpha + \beta + 2)x]\, y' + n(n + \alpha + \beta + 1)\, y = 0. \tag{A-2}$$

Equation (A-2) can easily be reduced to the hypergeometric form, so that, using Eq. (A-1), one has

$$P_n^{(\alpha,\beta)}(x) = \binom{n + \alpha}{n} F\left(-n,\ n + \alpha + \beta + 1;\ \alpha + 1;\ \frac{1 - x}{2}\right) . \tag{A-3}$$

The Jacobi functions correspond to the case for which n is no longer an integer in Eq. (A-3) . Equation (A-2) has a second solution

regular at infinity, which is called a Jacobi function of the second kind and can be defined as

$$Q_n^{(\alpha,\beta)}(x) = 2^{n+\alpha+\beta} \frac{\Gamma(n + \alpha + 1)\,\Gamma(n + \beta + 1)}{\Gamma(2n + \alpha + \beta + 2)}$$

$$\times (x - 1)^{-n-\alpha-1} (x + 1)^{-\beta}$$

$$\times F\left(n + \alpha + 1,\ n + 1;\ 2n + \alpha + \beta + 2;\ \frac{2}{1 - x}\right). \qquad \text{(A-4)}$$

We need the behavior of this function when x tends to $1 + 0$. It is equal to

$$Q_n^{(\alpha,\beta)}(x) \approx 2^{\alpha-1} \frac{\Gamma(\alpha)\,\Gamma(n + \beta + 1)}{\Gamma(n + \alpha + \beta + 1)} (x - 1)^{-\alpha}$$

$$\text{when } \alpha > 0. \qquad \text{(A-5)}$$

When we use Carlson's theorem, we need to know the asymptotic behavior of the Jacobi function of the first kind when n tends to infinity. When x does not belong to the closed segment $[-1, +1]$, it is given by

$$P_n^{(\alpha,\beta)}(x) \approx (2\pi n)^{-1/2}(x^2 - 1)^{-1/4}\left\{x + (x^2 - 1)^{-1/2}\right\}^{n+1/2}, \qquad \text{(A-6)}$$

and when $u = \cos\theta$ belongs to the open interval $(-1, +1)$, it is given by

$$P_n^{(\alpha,\beta)}(\cos\theta) = 2^{1/2}(\pi n \sin\theta)^{-1/2} \cos\left\{\left(n + 1/2\right)\theta - \pi/4\right\} . \tag{A-7}$$

The norm of a Jacobi polynomial is given by

$$\int_{-1}^{+1} (1 - x)^{\alpha}(1 + x)^{\beta} \left[P_n^{(\alpha,\beta)}(x)\right]^2 dx$$
$$= \frac{2^{\alpha + \beta + 1}}{2n + \alpha + \beta + 1} \frac{\Gamma(n + \alpha + 1)\,\Gamma(n + \beta + 1)}{\Gamma(n + 1)\,\Gamma(n + \alpha + \beta + 1)} = h_n^{(\alpha,\beta)}, \tag{A-8}$$

so that a function $f(x)$, defined in the interval $[-1, +1]$, can be formally expanded as a series of Jacobi polynomials according to

$$f(x) = \sum_{n=0}^{\infty} a_n \; P_n^{(\alpha,\beta)}(x) , \tag{A-9}$$

where

$$h_n^{(\alpha,\beta)} a_n = \int_{-1}^{+1} (1 - x)^{\alpha}(1 + x)^{\beta} \; f(x) \; P_n^{(\alpha,\beta)}(x) \, dx . \tag{A-10}$$

When $f(x)$ is an analytic function of x in a neighborhood of the closed segment $[-1, +1]$ in the complex plane, then one can use the formula

$$\sum_{n=0}^{\infty} \left[h_n^{(\alpha,\beta)}\right]^{-1} P_n^{(\alpha,\beta)}(x) \; Q_n^{(\alpha,\beta)}(y)$$

$$= \frac{1}{2} \frac{(y-1)^{\alpha}(y+1)^{-\beta}}{y-x} \quad , \tag{A-11}$$

which is valid when y is outside the ellipse with foci at ± 1 and which passes through x. Equation (A-11) allows us to replace Eq. (A-10) by

$$a_n = \left\{\pi i \, h_n^{(\alpha,\beta)}\right\}^{-1} \int (y-1)^{\alpha}(y+1)^{\beta} \, Q_n^{(\alpha,\beta)}(\bar{y}) \, f(y) \, dy \, , \tag{A-12}$$

where the integration is performed along a contour that surrounds the points -1 and $+1$ and that does not contain any singularity of $f(x)$. The expansion (A-9) is then valid within any ellipse with foci at ± 1 that does not contain any singularity of $f(x)$.

Finally, it is clear from Eqs. (A-3) and (A-4) that $P_n^{(\alpha,\beta)}(z)$ and $Q_n^{(\alpha,\beta)}(z)$ are analytic functions of the complex variable z. When n is a positive integer, $P_n^{(\alpha,\beta)}(z)$ is an entire function (in fact a polynomial), whereas $Q_n^{(\alpha,\beta)}$ is analytic in the plane cut from -1 to $+1$. When n is not an integer, $P_n^{(\alpha,\beta)}(z)$ and $Q_n^{(\alpha,\beta)}(z)$ are analytic in the z plane cut, respectively, from $-\infty$ to $+1$.

Clearly, the particular case $\alpha = \beta = 0$ corresponds to Legendre functions.

The asymptotic behavior of $Q_n^{(\alpha,\beta)}(x)$ when n tends to infinity can be easily derived from Eq. (4) and the Watson formula

$$(1/2\ z - 1/2)^{-a-\lambda}\ F\left[a + \lambda,\ a - c + 1 + \lambda;\ a - b + 1 + 2\lambda;\right.$$

$$\left. \times \frac{2}{1 - z}\right] = 2^{a+b}\ \frac{\Gamma(a - b + 1 + 2\lambda)\ \Gamma(1/2)\ \lambda^{-1/2}}{\Gamma(a - c + 1 + \lambda)\ \ \Gamma(c - b + \lambda)}$$

$$e^{-(a+\lambda)\xi}(1 - e^{-\xi})^{-c+1/2}\ (1 + e^{-\xi})^{c-a-b-1/2}\ , \qquad \text{(A-13)}$$

where

$$z \pm (z^2 - 1)^{1/2} = e^{\pm\xi}\ . \qquad \text{(A-14)}$$

Equation (A-13) shows that $Q_n^{(\alpha,\beta)}(x)$ stays bounded when n tends to infinity, except when x is in the interval $(-1, +1)$.

APPENDIX B

We indicate in this Appendix some properties of the rotation matrices for complex values of the angular momentum.

As is well known, the matrix that represents a rotation with Euler angles (α, β, γ) within the irreducible representation of total angular momentum J can be written

$$\mathcal{D}_{mm'}^{j}(\alpha, \beta, \gamma) = e^{-im\alpha} \, d_{m'm}^{j}(\beta) e^{-im\gamma} \quad . \tag{B-1}$$

The matrices $\mathcal{D}_{m'm}^{j}(\alpha, \beta, \gamma)$ make up an orthogonal set of functions of (α, β, γ) according to the relation

$$(8\pi^2)^{-1} \int \mathcal{D}_{mm'}^{j*}(\alpha, \beta, \gamma) \, \mathcal{D}_{m_1 m_1'}^{j_1}(\alpha, \beta, \gamma) \, d\alpha \, d \cos\beta \, d\gamma$$

$$= (2j + 1)^{-1} \, \delta_{jj_1} \, \delta_{mm_1} \, \delta_{m'm_1'} \quad . \tag{B-2}$$

An important particular case is provided by $m' = 0$, where

$$d_{m0}^{j}(\beta) = \left[\frac{(j - m)!}{(j + m)!}\right]^{1/2} (-1)^m P_j^m (\cos\beta) \quad . \tag{B-3}$$

More generally, the $d_{mm'}^{j}(\beta)$ can be related to the Jacobi polynomials according to

$$d_{mm'}^{j}(\beta) = \left[\frac{(j+m)!(j-m)!}{(j+m)!(j-m)!}\right]^{1/2} \left(\cos\frac{\beta}{2}\right)^{m+m'} \left(-\sin\frac{\beta}{2}\right)^{m-m'}$$

$$\times P_{j-m}^{(m-m',m+m')}(\cos\beta) . \qquad \text{(B-4)}$$

Equation (4) is true when

$$m - m' > 0$$

$$\text{and } m + m' > 0 . \qquad \text{(B-5)}$$

One way of proving Eq. (B-4) consists in writing the Lie equations for the rotation group in a differential form. According to Eq. (1), this leads to the differential equation for $y = d_{mm'}^{j}(x)$, where $x = \cos\beta$:

$$(x^2 - 1)y'' + 2xy' + \left[\frac{m^2 + m'^2 - 2mm'x}{1 - x^2} - j(j+1)\right] y = 0 . \qquad \text{(B-6)}$$

One then gets Eq. (4) by reducing Eq. (6) to the Jacobi form by a change of variables. The coefficients can be obtained by comparing Eq. (2) and the orthogonality relation between Jacobi polynomials (A-8).

We have shown in Appendix A how to extend the Jacobi polynomials to Jacobi functions when the index $n = j - m$ is not an integer. That provides immediately an explicit extension of the $d_{mm'}^{j}(x)$ to complex values of j. For a general value of j, the restrictions $m \leqslant j$, $m' \leqslant j$ have to be dropped, so that m and m' may now run from $-\infty$ to $+\infty$ taking any pair of integer values.

An important property of these functions is their asymptotic behavior when x tends to ± 1 or to ∞. As Eq. (6) is of the Fuchs' type, it is easily seen that its solutions behave as

$$(x - 1)^{\frac{m-m'}{2}} \quad \text{or} \quad (x - 1)^{\frac{m'-m}{2}} \quad \text{when} \quad x \to 1, \tag{B-7a}$$

$$(x + 1)^{\frac{m+m'}{2}} \quad \text{or} \quad (x + 1)^{-\frac{m+m'}{2}} \quad \text{when} \quad x \to -1 \tag{B-7b}$$

and

$$x^{+j} \quad \text{or} \quad x^{-j-1} \quad \text{when} \quad x \to \infty , \tag{B-7c}$$

so that, when $x \to \infty$, $\mathrm{Re}\, j > -1/2$, $d_{mm'}^{j}(x)$ behaves as x^{j}.

As $d_{mm'}^{j}(x)$ is related to the Jacobi functions of the first kind through Eq. (4) or (5), one can likewise define a second solution of Eq. (6) that is regular at infinity, behaving as x^{-j-1}, and is related to Jacobi functions of the second kind through

$$e_{mm'}^{j}(x) = \left[\frac{(j + m)!(j - m)!}{(j + m')!(j - m')!}\right]^{1/2} \left(\frac{1 + x}{2}\right)^{\frac{m+m'}{2}}$$

(Eq. (B-8) cont.)

$$\times\left(\frac{x-1}{2}\right)^{\frac{m-m'}{2}} Q_{j-m}^{(m-m',m+m')}(x) \ . \tag{B-8}$$

A fundamental property of the rotation matrices is their group property

$$\mathcal{D}_{mm'}^{j}(R'R) = \sum_{m''=-j}^{m''=+j} \mathcal{D}_{mm''}^{j}(R') \ \mathcal{D}_{m''m'}^{j}(R) \ , \ (j \text{ an integer}), \tag{B-9}$$

where R and R' are two rotations characterized by their Euler angles (α, β, γ) and $(\alpha', \beta', \gamma')$ and $R'R$ is the product of R' and R. It can be extended to any value of j by giving up the conditions on m'' :

$$\mathcal{D}_{mm'}^{j}(R'R) = \sum_{m''=-\infty}^{m''=+\infty} \mathcal{D}_{mm''}^{j}(R) \ \mathcal{D}_{m''m'}^{j}(R) \ . \tag{B-10}$$

A particular case of (B-10) is the well-known addition property of Legendre functions,

$$P_j(\cos\beta \cos\beta' + \sin\beta \sin\beta' \cos\varphi) = P_j(\cos\beta)P_j(\cos\beta')$$

$$+ 2 \sum_{m=1}^{\infty} \frac{\Gamma(j-m+1)}{\Gamma(j+m+1)} (-1)^m P_j^m(\cos\beta) \ P_j^m(\cos\beta') \cos m\varphi \ ,$$

which is a particular case of Eq. (11) for $m = m' = 0$, where use has been made of Eqs. (1) and (3) and where we have defined $\varphi = \gamma' + \alpha$.

One can prove Eq. (9) in several ways. For simplest proof, consider the Fourier expansion of $d_{mm'}{}^{j}(\cos\beta\cos\beta' + \sin\beta\sin\beta'\cos\varphi)$ in a series of terms proportional to $e^{im\varphi}$. Then, except for convergence considerations, Eq. (10) is equivalent to the set of equalities

$$2\pi^{-1}\int_0^{2\pi} d_{mm'}{}^{j}(\cos\beta\cos\beta' + \sin\beta\sin\beta'\cos\varphi)\, e^{-im''\varphi}\, d\varphi$$

$$= d_{mm''}{}^{j}(\cos\beta)\; d_{m''m'}{}^{j}(\cos\beta'). \qquad \text{(B-12)}$$

Equation (12) can be proved by applying the differential operator that appears in (B-6) to the left-hand side in order to prove that the right-hand side is proportional to $d_{mm''}{}^{j}(\cos\beta)$. The calculation is in fact rather tricky.

Another proof consists in considering real values of β and β', such that $|\cos\beta|$, $|\cos\beta'|$, and $|\cos\beta\cos\beta' + \sin\beta\sin\beta'\cos\varphi|$ stay less than 1 when $\cos\varphi$ runs from -1 to $+1$. Then, for m, m' and m'' fixed, Eq. (12) is true for any positive integral value of j according to Eq. (9). Furthermore, Eq. (4), together with Eq. (A-7), shows that both members do not increase as rapidly as $e^{i\pi j}$ when j tends to infinity. Both members can be then extended through Eq. (9) to complex values of j in a way that satisfies the conditions of Carlson's theorem. Therefore Eq. (B-12) is true for any value of j. The conditions on $\cos\beta$ and $\cos\beta'$ can be removed by noticing that

both members are analytic functions of $\cos\beta$ and $\cos\beta'$. Finally, Eq. (12) is true as long as $\cos\beta \neq -1$, $\cos\beta' \neq -1$, and $\cos\beta' + \sin\beta \sin\beta' \cos\varphi$ does not pass through -1 when $\cos\varphi$ varies from -1 to $+1$. In fact, even that last condition could be removed by displacing the integration contour.

The domain of convergence of Eq. (11) is easily deduced from the asymptotic properties of Legendre functions when m tends to infinity. As the asymptotic properties of Jacobi functions when m or m' tends to infinity by integral values are not known--at least to our knowledge--the convergence domain of Eq. (10) cannot be fully explored. However, as obtained from the theory of Fourier series for any real or complex set of values of $\cos\beta$ and $\cos\beta'$, Eq. (10) converges inside a corona (of radii $RR' = 1$) passing through the point $\cot\beta = -\cot\beta\cot\beta'$.

In fact, when the series in Eq. (10) does not converge, we just consider the left-hand side as the correct definition of the right-hand side.

Equation (10) can be extended to the rotation matrices of the second kind as follows:

$$\mathcal{E}_{mm'}^{j}(\alpha, \beta, \gamma) = e^{-im\alpha}\, e_{mm'}^{j}(\beta)\, e^{-im\gamma} \tag{B.13}$$

and

$$\mathcal{E}_{mm'}^{j}(R'R) = \sum_{m''=-\infty}^{m''=+\infty} \mathcal{E}_{mm''}^{j}(R')\, \mathcal{D}_{m''m'}^{j}(R) \tag{B.14}$$

In order to prove Eq. (14) we first need the important identity

$$e_{mm'}^{j}(z) \sin \pi j = \frac{\pi}{2} (-1)^{m} e^{\mp i\pi j} d_{mm'}^{j}(z) - d_{-mm'}^{j}(-z) , \tag{B.15}$$

where the signs are taken according as $\mathrm{Im}\, z \gtrless 0$. Equation (15) applied to Eq. (12) shows

$$(2\pi)^{-1} \int_{0}^{2\pi} e_{mm'}^{j} (\cos \beta \cos \beta' \sin \beta \sin \beta' \cos \varphi) e^{-im''\varphi} d\varphi$$

$$= e_{mm''}^{j} (\cos \beta) d_{m''m'}^{j} (\cos \beta') (-1)^{m''} , \tag{B.16}$$

when $1 < \cos \beta' < \cos \beta$. Then, analyticity of both members of (B-16) allows us to give complex values to $\cos \beta$ and $\cos \beta'$. Then Eq. (14) follows as a consequence of the theory of Fourier series. Here again, we shall consider the left-hand member of Eq. (14) as the correct definition of the right-hand member when it diverges.

Finally, let us go back to the proof of Eq. (15). According to the symmetry properties of the $d_{mm'}^{j}$ functions, it is true for integral values of j. When j is not an integer, we notice that $d_{-mm'}^{j}(z)$ and $d_{mm'}^{j}(-z)$ both verify Eq. (6); therefore, the right-hand member of Eq. (15) also verifies Eq. (6) and is a linear combination of $d_{mm'}^{j}(z)$ and $e_{mm'}^{j}(z)$. Using the asymptotic behavior of the Jacobi functions

$$P_n^{(\alpha,\beta)}(X) \sim 2^{-n} \binom{2n + \alpha + \beta}{n} x^n, \text{ when } x \text{ tends to infinity.} \tag{B-17}$$

as well as Eq. (4), one easily verifies that the right-hand member of Eq.(15) increases less rapidly than j when z tends to infinity, so that it is proportional to $e_{mm'}^{j}(z)$. The proportionality coefficient may then be fixed by a comparison of the behavior of $d_{-mm'}^{j}(-z)$ near $z = \pm 1$ and Eq. (A-5). The behavior of $d_{-m+m}^{j}(z)$ can be related to that of $F(m' - j, j + m' + 1; m' - m + 1, \frac{1 + x}{2})$ with the help of Eqs. (4) and (A-3). By using the asymptotic behavior of the hypergeometric function

$$F(a, b, a + b - \ell, z) \sim (1 - z)^{-\ell} \frac{\Gamma(\ell)\,\Gamma(a + b - \ell)}{\Gamma(a)\,\Gamma(b)}, \tag{B-18}$$

where ℓ is a positive integer, one finds that Eq. (15) follows immediately.

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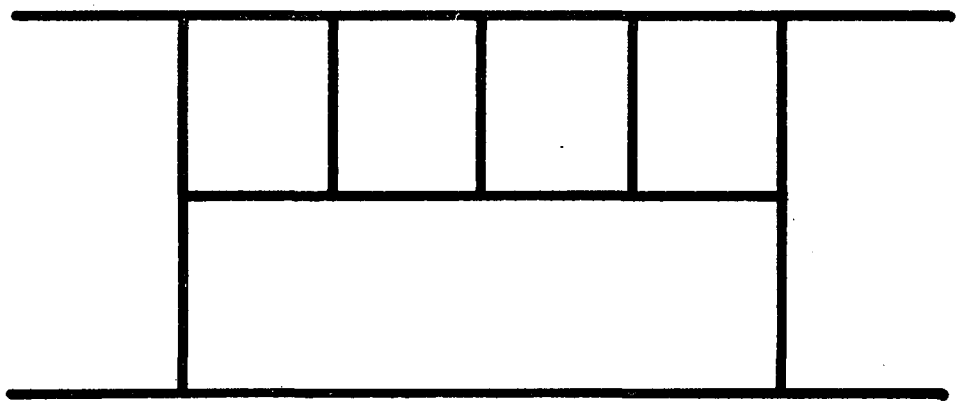

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Fig. 1.

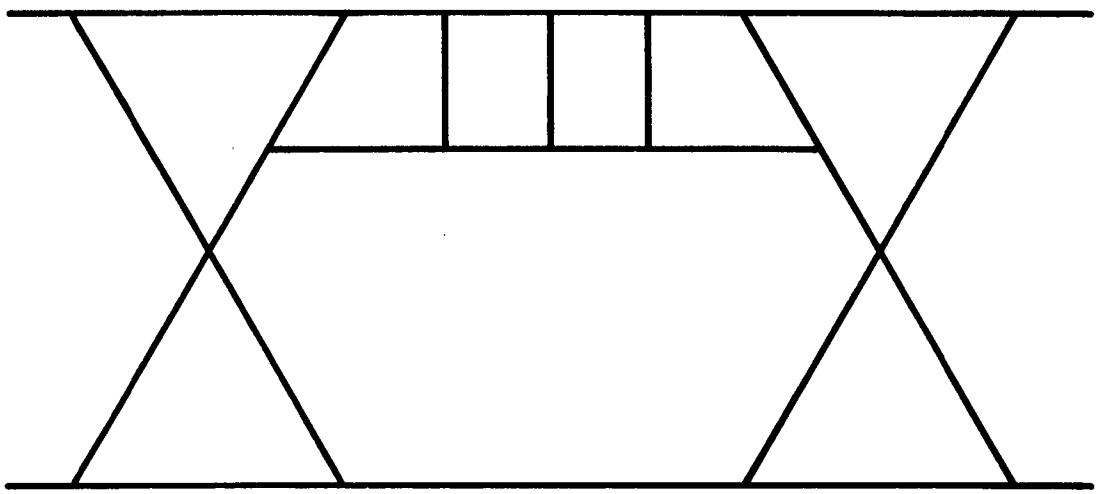

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Fig. 2.

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