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III. SCATTERING ON A BOUND STATE

Berkeley, California

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ABSTRACT

The results of two previous papers are carried on to a formulation of the Fadeev equations due to C. Lovelace. This allows showing that the nonrelativistic scattering amplitude for an elementary particle on a bound state is a meromorphic function of the total angular momentum. An extension of the helicity amplitudes to complex values of the total angular momentum and of the spin of the bound state is indicated. One is thus able to define the scattering amplitude for a particle on a Regge pole when the energy of the Regge pole is in the physical sheet.

In two previous papers[1] we have shown how the scattering amplitude for a three-particle collision can be extended to complex values of the total angular momentum J. We found that this extended amplitude satisfies an integral equation whose kernel has elements that are analytic functions of J. Obviously, this is not enough to prove that the amplitude is a meromorphic function of J and this problem needs further investigation. However, if we reduce the kernel to values of the magnetic quantum numbers M and M' smaller than, say, an integer N, this truncated kernel is operator-analytic as a function of J. On the other hand, the scattering amplitude deduced from the truncated kernel is a meromorphic function of J which coincides with the physical amplitude for all integral values of J smaller than M. This is, in our opinion, a very strong hint that the true amplitude itself is meromorphic although one must recognize that the convergence of these truncated amplitudes has to be considered carefully. Anyway, we shall in the present paper accept as an ansätz that the three-body scattering amplitude is meromorphic.

This discussion in papers I and II has been restricted to the case in which no pair of particles can give rise to a bound state. This leaves out of the discussion the fundamental problem of the scattering of a particle on a bound state. Arguments have been advanced by Udgaonkar and Gell-Mann,[2] on the basis of a model, and by Newton[3] in a work which we have already discussed, that the corresponding amplitude has cuts. On the other hand, J. B. Hartle,[4] in a work which closely parallels that of Newton, did not find any evidence for these cuts.[5] Applying the method of the two preceding papers, we shall show that, actually, the amplitude is a meromorphic function J.

In order to formulate the equations for this problem, we use an extension of the Fadeev equations[6] which have been given by C. Lovelace.[7] The necessary background is provided in Sec. 2. In Sec. 3, one shows how to extend the

scattering amplitude of a particle on a bound state to a complex value of the angular momentum. In Sec. 4, one considers the extension to complex values of J of the Born approximation to which the results of Sec. 3 do not apply directly. In Sec. 5, we show how the helicity amplitude for physical or complex values of J can be extended to complex values of the spins of the bound states, i.e., we define the scattering amplitude for a particle on a Regge pole in nonrelativistic theory.

2. THE LOVELACE EQUATIONS

In the two preceding papers, we have used the Fadeev equations as they were originally written by Fadeev himself[6] and we have indicated how the results compare to another set of equations found by Weinberg.[8] When one wants to consider explicitly the possibility of bound states, it is convenient to work with a third formulation of the problem, due to Lovelace, which is very close to the original Fadeev equations. Needless to say, all these formulations are actually equivalent.

Whereas we considered only a scattering amplitude for the reaction 3 particles → 3 particles which we called $T(z)$, we now consider also states made up by one particle and a bound state of the two others. We label by an index 0 a state of three particles and by an index 1, for instance, the state made up by particle 1 and a bound state of the pair (23). Accordingly, we introduce a set of sixteen amplitudes $T_{\alpha\beta}(z)$, where $\alpha, \beta = 0, 1, 2, 3$; it is convenient to call V_1 the potential of particles 2 and 3 and V_0 a three-body potential which we shall suppose to be identically zero. Accordingly, we call $T_1(z)$ the two-body scattering amplitude of particles 2 and 3 and $T_0 = 0$. Lovelace has shown that one can write the following set of equations,

$$T_{\alpha\beta}(z) = \sum_{\delta\neq\beta} V_\delta - \sum_{\gamma\neq\alpha} T_\gamma(z)\, G_0(z)\, T_{\gamma\beta}(z), \tag{1}$$

where T_{00} is the three-body scattering amplitude. The amplitude for the scattering of particle 1 on a bound state of particles 2 and 3 with wave function $\phi(\underset{\sim}{q}_{23})$ is equal to

$$\int d^3\underset{\sim}{q}_{23}\, d^3\underset{\sim}{q}'_{23}\, \phi^*(\underset{\sim}{q}'_{23})\, \langle \underset{\sim}{p}'_1\, \underset{\sim}{p}'_2\, \underset{\sim}{p}'_3 | \mathcal{T}_{11}(z) | \underset{\sim}{p}_1\, \underset{\sim}{p}_2\, \underset{\sim}{p}_3 \rangle\, \phi(\underset{\sim}{q}_{23}). \qquad (2)$$

(all notations unexplained here are to be found in the two preceding papers.)

3. EXTENSION TO COMPLEX VALUES OF J

The kernel of the Lovelace Eq. (1) is in fact identical to the kernel of the Fadeev equations, so that our preceding analysis applies to this equation. It is therefore possible to state a system of integral equations for the amplitudes $T_{\alpha\beta}(z) = \sum_{\delta\neq\beta} V_\delta$ projected on well-defined values J of the total angular momentum. The inhomogeneous terms of these equations as well as the terms of the kernel are holomorphic functions of J. Although this is not enough to make sure that the solution is a meromorphic function of J, we shall accept this result as an ansatz. We shall therefore define reduced matrix elements $\mathcal{T}_{M'M}{}^{J}(\omega',\omega)$, which we shall split into the first Born approximation $\mathcal{B}$ and the rest $\mathcal{U}$, as follows:

$$\mathcal{T}_{\alpha\beta\ M'M}{}^{J}(\omega';\omega) = \mathcal{B}_{\alpha\beta\ M'M}{}^{J}(\omega';\omega) + \mathcal{U}_{\alpha\beta\ M'M}{}^{J}(\omega';\omega) \ . \tag{3}$$

Let us recall that ω stands for the set $(\omega_1, \omega_2, \omega_3)$ of the kinetic energies of the three particles in the total center-of-mass system, and M and M' are respectively the projections of the total angular momentum J on two axes invariantly related to the momenta $(\underset{\sim}{p}_1, \underset{\sim}{p}_2, \underset{\sim}{p}_3)$ and $(\underset{\sim}{p}'_1, \underset{\sim}{p}'_2, \underset{\sim}{p}'_3)$. In practice, in the following, these axes will be chosen along $\underset{\sim}{p}_1$ and $\underset{\sim}{p}'_1$ so that, when we consider the scattering of particle 1 on a bound state of particles 2 and 3, M and M' will be the initial and final helicities of the bound states. (For simplicity, we always suppose that the three particles 1, 2, 3 are spinless.)

As shown in II, the momentum-space matrix elements are related to the reduced matrix elements by

$$\langle \underset{\sim}{p}'_1 , \underset{\sim}{p}'_2 , \underset{\sim}{p}'_3 | \mathcal{T}_{\alpha\beta}(z) | \underset{\sim}{p}_1 , \underset{\sim}{p}_2 , \underset{\sim}{p}_3 \rangle$$

$$= \text{constant} \sum_{JMM'} (2J + 1) \mathcal{T}^J_{\alpha\beta\, M'M}(\omega', \omega) \mathcal{D}_{M'M}^{\;J} (\psi, \theta, \phi) , \qquad (4)$$

where (ψ, θ, ϕ) are the Euler angles of the rotation, which carries a reference system linked to $(\underset{\sim}{p}_1 , \underset{\sim}{p}_2 , \underset{\sim}{p}_3)$ to a system linked to $(\underset{\sim}{p}'_1 , \underset{\sim}{p}'_2 , \underset{\sim}{p}'_3)$.

Let us now consider in more detail the scattering of particle 1 on a bound state of particles 2 and 3 . We choose the initial reference system as having its z axis along $\underset{\sim}{p}_1$ and its x axis in the plane determined by $(\underset{\sim}{p}_1 , \underset{\sim}{p}_2 , \underset{\sim}{p}_3)$. If one calls σ the spin of the bound state, then the wave function is given by

$$\psi(\underset{\sim}{q}_{23}) = \phi_\sigma(q_{23}) \underset{\sim}{P}^M_\sigma (\cos \gamma_1) , \qquad (5)$$

where $\underset{\sim}{q}_{23}$ is the relative momentum of particles 2 and 3 in the center-of-mass system of the bound state and γ_1 is the angle between $\underset{\sim}{p}_1$ and $\underset{\sim}{q}_{23}$. The formulas that more precisely define these kinematical variables can be found in I.

If one introduces a fixed reference system with its z axis along $\underset{\sim}{p}_1$, calling χ and χ' the azimuthal angles of $\underset{\sim}{q}_{23}$ and $\underset{\sim}{q}'_{23}$, noticing that θ and ϕ are the polar angles of $\underset{\sim}{p}'_1$, and that everything is invariant with respect to a rotation along $\underset{\sim}{p}_1$, then, introducing Eqs. (2) and (5) into Eq. (4), one gets

$$\langle \underset{\sim}{p}'_1 , J' , M' \mid \mathcal{T}_{11}(z) \mid \underset{\sim}{p}_1 , J, M \rangle$$

$$= \text{constant} \sum_J (2J+1)\, e^{i(M-M')\phi}\, d_{M'M}^{\;J}(\theta)\, A_{MM'\sigma\sigma'}^{\;J}(\omega_1' , \omega_1 , z) \qquad (6a)$$

where

$$A_{MM'\sigma\sigma'}^{\;J}(\omega_1 , \omega_1' , z) = \int \phi_\sigma(q_{23}')\, P_\sigma^{M'}(\cos \gamma'_1)$$

$$\mathcal{T}_{11M'M}^{\;J}(\omega', \omega)\, \phi_\sigma(q_{23})\, \underline{P}_\sigma^{M}(\cos \gamma_1)\, d^2 q_{23}\, d^2 q_{23}' , \qquad (6b)$$

where, for instance, $d^2 q_{23} = q_{23}^2\, d\, q_{23}\, d \cos \gamma_1$. Comparing this result with the Jacob-Wick formula shows that the integral in Eq. (6b) is the helicity amplitude for the scattering of particle 1 on the bound state.

Now, the matrix $\mathcal{T}_{11}$ that appears in Eq. (6b) consists of the two parts $\mathcal{B}_{11}$ and $\mathcal{U}_{11}$. We have agreed that $\mathcal{U}_{11}$ is a meromorphic function of J. Moreover, the position of its poles depends only on z. Therefore, the contribution of $\mathcal{U}$ to the helicity amplitude is a meromorphic function of J as long as the integral in Eq. (6b) converges uniformly. The domain of convergence will depend on the asymptotic properties of $\mathcal{U}^J$ as well as the value of J. However, we don't know enough yet about these asymptotic properties to find out the convergence domain.

4. ANALYTIC PROPERTIES OF THE BORN APPROXIMATION

The foregoing discussion does not apply to $\mathcal{B}_{11}^{J}$, namely to the matrix element of the potentials V_2 and V_3. We shall now show directly that the contribution of V_2, for instance, is a meromorphic function of J. Let us use the same fixed reference system where the polar angles of $\underset{\sim}{p}_1$, $\underset{\sim}{q}_{23}$, $\underset{\sim}{q}'_{23}$, $\underset{\sim}{p}'_1$ are $(0,0)$ (α, χ), (α', χ'), (θ, ϕ), with $\gamma_1 = \alpha$ and

$$\cos \gamma'_1 = \cos \alpha' \cos \theta + \sin \alpha' \sin \theta \cos (\phi - \chi') . \tag{7}$$

Let us consider, for simplicity, the case in which the potential V_2 is a pure Yukawa potnetial with range μ^{-1}. Its contribution to the helicity amplitude is given, according to Jacob and Wick, by

$$\int \phi_\sigma(q'_{23})\, \underset{\sim}{P}_\sigma^{M'}(\gamma'_1)\, e^{-iM'\chi'}\, e^{-i(M-M')\phi}\, d_{M'M}^{J}(\theta)$$

$$\times \frac{1}{|\underset{\sim}{q}'_{13} - \underset{\sim}{q}_{13}|^2 + \mu^2}\, \phi_\sigma(q_{23})\, P_\sigma^{M}(\gamma_1)\, e^{iM\chi}$$

$$\times d^3 q_{23}\, d^3 q'_{23}\, d\cos\theta\, d\phi . \tag{8}$$

Noticing that the quantity to be integrated depends only upon $u = \chi - \phi$ and $u' = \chi' - \phi'$. one gets

$$\int \phi_\sigma(q'_{23})\, P_\sigma^{M'}(\gamma'_1)\, e^{-iM'u'}\, d_{MM'}^{J}(\theta)\, e^{iMu}\, \phi_\sigma(q_{23})\, \underline{P}_\sigma^{M}(\gamma_1)$$

$$\times \left[|q_{13} - q'_{13}|^2 + \mu^2 \right]^{-1} (q_{23}\, q'_{23})^2\, dq_{23}\, dq'_{23}$$

$$\times d\cos\alpha\, d\cos\alpha'\, d\cos\theta\, du\, du', \tag{9}$$

where, in the case of equal masses,

$$|\underset{\sim}{q}_{13} - \underset{\sim}{q}'_{13}|^2 = \left[\tfrac{3}{2}(\underset{\sim}{p}_1 - \underset{\sim}{p}'_1) - (\underset{\sim}{q}_{23} - \underset{\sim}{q}'_{23}) \right]^2$$

$$= \tfrac{9}{4}(p_1^{\,2} + p_1^{\,'2}) + q_{23}^{\;2} + q_{23}^{\;'2} - \tfrac{9}{2}\, p_1 p'_1 \cos\theta$$

$$- \tfrac{3}{2}\, p_1\, q_{23} \cos\alpha + \tfrac{3}{2}\, p_1\, q'_{23} \cos\alpha'$$

$$+ 2\, q_{23}\, q'_{23}\, [\cos\alpha \cos\alpha' + \sin\alpha \sin\alpha' \cos(u - u')]$$

$$+ 3\, p'_1\, q_{23}\, [\cos\alpha \cos\theta + \sin\alpha \sin\theta \cos u]$$

$$- 3\, p'_1\, q'_{23}\, [\cos\alpha' \cos\theta + \sin\alpha' \sin\theta \cos u'] \,. \tag{10}$$

Let us consider Eq. (9). The integrations upon u and u' can be split into an integration upon $\frac{1}{2}(u + u)$ and upon $\frac{1}{2}(u - u')$. The integration upon $\frac{1}{2}(u + u')$, taking Eqs. (7) and (10) into account will give $(\sin\theta)^{|M-M'|}$

times a function analytic in cos θ in a neighborhood of the segment (-1, +11). This is precisely the factor that allows one to apply the reciprocity formula between Jacobi functions of the first and second kinds (Eqs. A-10 and A-12 of II). Therefore, when J is an integer, Eq. (9) can be replaced by

$$\int \phi_\sigma (q'_{23})\, \underline{P}_\sigma^{M}(\gamma'_1)\, e^{-iM'u'}\, e_{MM'}^{J}(\theta)\, e^{iMu}\, \phi_\sigma(q_{23}) \underline{P}_\sigma^{M}(\alpha)$$

$$\times \left[|\underset{\sim}{q}_{13} - \underset{\sim}{q}'_{13}|^2 + \mu^2 \right]^{-1} (q_{23}\, q'_{23})^2\, dq_{23}\, dq'_{23} \tag{11}$$

$$\times d\cos\alpha\, d\cos\alpha'\, d\cos\theta\, du\, du',$$

where the integration upon cos θ is now made along a contour Γ which encloses the singularity of $\left[|\underset{\sim}{q}_{13}{}^2 - \underset{\sim}{q}'_{13}|^2 + \mu^2 \right]^{-1}$. The integral (11) converges for any value of J, since the singularities enclosed by Γ are only simple poles.

5. EXTENSION TO CONTINUOUS VALUES OF σ

Let us see if it is possible to extend Eq. (6b) to values of σ and σ' which are not the spins of physical bound states but any number, the wave-function ϕ_σ being the wave function of a Regge pole with spin σ.[9] First, in order not to spoil the convergence of the integrations upon q_{23} and q'_{23}, one will have to keep $\phi_\sigma(q_{23})$ and $\phi_{\sigma'}(q'_{23})$ square-integrable. This means that the mass m of the bound state will have to be taken in the physical sheet (namely in the complex kinetic energy plane cut from 0 to infinity), while σ will take its values on the Regge pole trajectory. Since $\sigma(m)$ is bounded in that sheet of m in which the potential is a superposition of Yukawa potentials, there does not seem to be any reason to pass from formula (6b) to an analogous formula with the Legendre functions P_σ^M replaced by functions of the second kind ϕ_σ^M, as in the Froissart-Gribov formula.[10] (Moreover, there is no evidence that such a transformation is possible here.)

A new difficulty that appears when σ takes on continuous values is that the point $\cos\gamma_1 = -1$ in $P_\sigma^M(\gamma_1)$ becomes a singular point where $\underline{P}_\sigma^M$ behaves like $(\sin\gamma_1)^{-M/2}$. We have then to show that the integral (6b) keeps a meaning. We shall give a different justification for the contributions to (6b) of $\mathcal{U}_{11}$ and of $\mathcal{B}_{11}$.

Let us first consider the contribution of $\mathcal{U}_{11}$. If Eq. (1) is written in the same form as Eq. (2) of II, where the terms of the kernel as well as the inhomogeneous terms have a form analogous to Eq. (25) of II, then one can see that the kernel contains a factor $d_{MM''}^J(-\theta'_{ij})$ and the inhomogeneous term contains a factor like $d_{M'M}^J(\theta_{ij})$, where θ_{ij}, for instance, is the angle between the reference axis i and p_j. For physical values of J, one can solve the one iterated Fadeev equations by iteration, since the kernel is completely continuous. Taking into account the foregoing remark and the

explicit form (14) of II of the rotation matrices, one sees that $\mathcal{U}_{11M'M}^{J}$ is proportional to $(\sin \theta_{ij})^{M} (\sin \theta'_{ij})^{M''}$ when the triangle $\underset{\sim}{p}_1, \underset{\sim}{p}_2, \underset{\sim}{p}_3$ collapses. Then Eqs. (20a) and (20b) of I show that $\mathcal{U}_{11M'M}^{J}$ is proportional to $(\sin \gamma_1)^{M} \sin \gamma'_1)^{M'}$ when γ_1 or γ'_1 tends to zero. This behavior can be extended by continuity to complex values of J, and it allows the integration upon γ_1 and γ'_1 in Eq. (6b) to converge.

The same result can be shown for the contribution by $\mathcal{B}_{11}$, namely: use Eq. (9) and integrate upon u, accoring to Eq. (11); it will give $(\sin \alpha)^{M}$ times an analytic function of $\cos \alpha$, which is what we need.

6. CONCLUSIONS

We have shown that if our ansätz of meromorphy for the solution of the extended Fadeev equations is correct, the scattering amplitude of a particle on a bound state is a meromorphic function of the total angular momentum J. Moreover, it is possible to define the scattering amplitude of a particle on a Regge pole as long as the energy of this Regge pole is kept within the physical sheet. This is not enough to be able to define the scattering of a particle on a resonance, as was proposed by Zwanziger.[11]

There remain two important mathematical problems to solve:

(a) to justify correctly the meromorphy of the solution,

(b) to investigate the limit process when z, the complex energy, tends to the physical energy.

These two problems will eventually be difficult but, according to the importance of the applications one can get from the notion of scattering of a particle on a Regge pole, it seems worthwhile to investigate them in order to work on a firmer basis.

A fundamental question would be why the scattering of two elementary particles and the scattering on a bound state, which are described by such different equations, have so similar analytic properties. The elucidation of the property and its connection with the basic groups (like the Galilean group) could shed new light on the problem of the formulation of good axioms for S-matrix theory.

FOOTNOTES AND REFERENCES

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