UCRL-11380

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INTEGRATION OF THE SUTTON POINT SOURCE ATMOSPHERIC DIFFUSION KERNEL

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Submitted for publication to Nuclear Chemistry UCRL-11380

UNIVERSITY OF CALIFORNIA
Lawrence Radiation Laboratory
Berkeley, California

AEC Contract No. W-7405-eng-48

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Atmospheric Diffusion

Most analyses of the atmospheric transport of radioactive material following a nuclear accident employ the well-known formula obtained by Sutton by an approximate integration of the basic instantaneous point source expression which he proposed (1,2). The calculations described here were designed to examine the accuracy of the steady state Sutton formula and to investigate the effect of a time varying source on the validity of the approximate integral evaluation.

The concentration at a time t and a point x meters downwind, y meters crosswind, and z meters above the ground, from an instantaneous point source of unit strength at x = y = Q, z = h at zero time is:

$$\chi_{pt}(x,y,z,t) = \frac{1}{\pi^{3/2} C_x C_y C_z (\bar{u}t)^{3m/2}} \left\{ \exp - \left[\frac{\frac{(x-\bar{u}t)^2}{C_x^2} + \frac{y^2}{C_y^2} + \frac{(z+h)^2}{C_z^2}}{(\bar{u}t)^m} \right] + \exp - \left[\frac{\frac{(x-\bar{u}t)^2}{C_x^2} + \frac{y^2}{C_y^2} + \frac{(z-h)^2}{C_z^2}}{(\bar{u}t)^m} \right] \right\} \quad (1)$$

$\bar{u}$ is the mean wind speed in the x direction and C_x, C_y, C_z and m = 2 - n are parameters representing the prevailing meteorological dispersion conditions.

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(1) AECU - 3066

(2) O. G. Sutton, Proc. Royal Soc. (London) 135A, 143 (1932).

Consider a continuous, time varying source of strength $Q_o f_r(t) f_s(t)$. The function $f_r(t)$ denotes radioactive decay, and $f_s(t)$ describes any non-radioactive alteration in the emission rate. $f_s(t)$ might represent the decay of the pressure in a containment vessel following a loss-of-coolant accident. The rate of release of fission products depends upon the volumetric flow rate from the vessel (which is governed by the internal pressure) as well as the rate of radioactive decay. The two functions must be separated because the radioactive decay continues to reduce the concentration while the puff is in transit from the source to a downwind point, while $f_s(t)$ applies only to the source rate itself.

The concentration in the puff emitted at t' is decreased by decay in transit by the ratio $f_r(t' + x/\bar{u})/f_r(t')$. The downwind concentration at time t is obtained by integration of (1):

$$\chi(x,y,z,t) = Q_o \int_0^t f_r(t' + x/\bar{u})\, f_s(t')\, \chi_{pt}(x,y,z\ t-t')dt' \quad (2)$$

In order to evaluate the integral, set $w = (x/\bar{u})/(t-t')$. Substituting Eq. (1) in Eq. (2) yields:

$$\chi(x,y,z,t) = \left[\frac{Q_o}{\pi c_y c_z \bar{u} x^m}\right] \cdot \frac{1}{\sqrt{\pi A}} \int_{\frac{x}{\bar{u}t}}^{\infty} f_r\left[t - \frac{x}{\bar{u}}\left(\frac{1}{w}-1\right)\right] f_s\left[t - \frac{x}{\bar{u}}\frac{1}{w}\right] \quad (3)$$

$$w^{m-2}\left\{\exp - w^m\left[\frac{(1-1/w)^2}{A} + S_+\right] + \exp - w^m\left[\frac{(1-1/w)^2}{A} + S_-\right]\right\}$$

where

$$A = \frac{C_x^{\,2}}{x^{2-m}} \tag{4}$$

$$\xi_\pm = \frac{1}{x^m}\left[\left(\frac{y}{c_y}\right)^2 + \left(\frac{z \pm h}{c_z}\right)^2\right] \tag{5}$$

For times greater than the cloud transit time ($t > x/\bar{u}$), a more convenient form of Eq. (3) is:

$$\chi(x,y,z,t) = \left[\frac{Q_0}{\pi c_y c_z \bar{u} x^m}\right] f_r(t) f_s\left(t - \frac{x}{\bar{u}}\right)\left[F_+ e^{-\xi_+} + F_- e^{-\xi_-}\right] \tag{6}$$

where

$$F = \frac{e^{\xi}}{\sqrt{\pi A}} \int_{1/T}^{\infty} g_r(w,T)\, g_s(w,T)\, w^{m-2} \exp - w^m\left[\frac{(1-1/w)^2}{A} + \xi\right] dw \tag{7}$$

T is the time in units of the transit time:

$$T = t/(x/\bar{u}) \tag{8}$$

g_s is the function $f_s[t - (x/\bar{u})/w]\,/f_s(t - x/\bar{u})$, and g_r is the function $f_r[t - \frac{x}{\bar{u}}(\frac{1}{w} - 1)]/f_r(t)$. For $f_r(t) = 1$, t^{-b}, and $e^{-\lambda_r t}$, g_r is:

$$f_r(t) = 1, \qquad g_r(w,T) = 1 \tag{9a}$$

$$f_r(t) = t^{-b}, \qquad g_r(w,T) = [1 - \frac{1}{T}(\frac{1}{w} - 1)]^{-b} \tag{9b}$$

$$f_r(t) = e^{-\lambda_r t}, \quad g_r(w,T) = \exp[\Gamma_r\,(1/w - 1)] \tag{9c}$$

where $\Gamma_r = \lambda_r(x/\bar{u})$.

If the source decay function can be approximated by a simple exponential, $f_s(r) = e^{-\lambda_s t}$ and

$$g_s(w,T) = \exp\ [\Gamma_s\ (1/w - 1)] \tag{10}$$

If the source is steady and only decay in transit is considered, the function g_r is $f_r[t - (x/\bar{u})(1/w - 1)]/f_r[t - (x/\bar{u})/w]$ and no explicit time functions appear in Eq. (6).

For $f_r = e^{-\lambda_r t}$, g_r reduces to $e^{-\lambda_r x/\bar{u}}$, independent of time.

If $f_r(t) = f_s(t) = 1$ and $F_+ = F_- = 1$, Eq. (6) reduces to the familiar Sutton formula. A case of particular interest is that of the ground concentration along the cloud centerline, for which $y = z = 0$, $\xi_+ = \xi_-$, and $F_+ = F_-$. Equation (7) then reduces to:

$$F = \frac{1}{\sqrt{\pi}}\int_{1/r}^{\infty} g(w,T)\, w^{\frac{3m-2}{2}} \exp\left[\zeta(1-w^m)\right] \exp\left[-w^m \frac{(1-1/w)^2}{A}\right] dw \tag{11}$$

where $g(w,T)$ denotes the product of 9a, b or c, and a source function of the type of Eq. (10).

An analytic approximation to F can be obtained because the value of the parameter A is, in general, quite small. Since A appears in the denominator of the last exponential in Eq. (11), the integrand will be very small unless $w \approx 1$. Thus w can be set equal to unity except where it appears subtracted from unity (i.e., the second term in the integrand and w^m in the last exponential are set equal to unity). The integral is

divided into two ranges, $1/T \leqslant w \leqslant 1$ and $1 \leqslant w \leqslant \infty$. The first exponential term is expanded in a power series about $w = 1$, and the time function is written as:

$$g(w,T) = g(1,T) + \alpha(w - 1) + \ldots$$

where

$$\alpha = \left(\frac{dg}{dw}\right)_{w=1} \tag{12}$$

Equation (11) becomes:

$$F = \sum_{\ell} \frac{s^{\ell}}{\ell!}\left[a_{1,\ell} + (-1)^{\ell} a_{2,\ell}\right] + \alpha \sum_{\ell} \frac{s^{\ell}}{\ell!}\left[b_{1,\ell} - (-1)^{\ell} b_{2,\ell}\right] \tag{13}$$

where

$$a_{1,\ell} = \frac{1}{\sqrt{\pi A}} \int_{1/T}^{1} (1-w^{-1})^{\ell} \exp\left[-\frac{(1/w-1)^2}{A}\right] dw \tag{14}$$

$$a_{2,\ell} = \frac{1}{\sqrt{\pi A}} \int_{1}^{\infty} (w^{-1}-1)^{\ell} \exp\left[-\frac{(1-1/w)^2}{A}\right] dw \tag{15}$$

$$b_{1,\ell} = \frac{1}{\sqrt{\pi A}} \int_{1/T}^{1} (1-w)(1-w^{-1})^{\ell} \exp\left[-\frac{(1/w-1)^2}{A}\right] dw \tag{16}$$

$$b_{2,\ell} = \frac{1}{\sqrt{\pi A}} \int_{1}^{\infty} (w-1)(w^{-1}-1)^{\ell} \exp\left[-\frac{(1-1/w)^2}{A}\right] dw \tag{17}$$

with the transformation $s = (1/w - 1)/\sqrt{A}$, Eq. (14) becomes:

$$a_{1,\ell} = \frac{1}{\sqrt{\pi}} \int_{0}^{\frac{T-1}{\sqrt{A}}} \left[1 - \frac{1}{(1+\sqrt{A}s)^{2}}\right]^{\ell} \frac{1}{(1+\sqrt{A}s)^{2}} e^{-s^2} ds \tag{14a}$$

Since the parameter A is small, the first two terms in the integrand of (14a) can be expanded and only first order terms in s retained. In addition, since the leading and most important term in $a_{1,o}$ is $\frac{1}{2}$ erf $[(T-1)/\sqrt{A}]$, T-1 need only be greater than $\sim 2\sqrt{A}$ to permit replacement of the upper limit with infinity.

The concentration at a fixed downwind position on the plume centerline as a function of time after the start of steady emission is shown in Fig. 1. The concentration is nearly zero until the center of the first puff has reached the downwind point. At T = 1, the concentration is approximately one-half of the steady state value, and by $T \approx 1 + 2\sqrt{A}$, the steady concentration has been attained. The profiles become sharper as the dispersion coefficient C_x decreases. If diffusion in the direction of wind travel were neglected, the time response would be a step function at T = 1. The succeeding analysis (and the usual Sutton equation) is valid only for $T > 1 + 2\sqrt{A}$, for which Eq. (14a) becomes:

$$a_{1,\ell} = m^{\ell} A^{\ell/2} \int_0^{\infty} \lambda^{\ell} e^{-\lambda^2} d\lambda - 2 m^{\ell} A^{\frac{\ell+1}{2}} \int_0^{\infty} \lambda^{\ell+1} e^{-\lambda^2} d\lambda \qquad (14b)$$

The remaining integrals can be treated similarly. Carrying terms up to $\ell = 4$ in $a_{1,\ell}$ and $a_{2,\ell}$ and $\ell = 3$ in the b integrals:

$$F = 1 + (ms)\left(\frac{ms}{4} - 1\right) A + \frac{(ms)^3}{4}\left(\frac{ms}{8} - 1\right) A^2 + \alpha\left[\left(\frac{ms}{2} - 1\right) A + \frac{3}{4}(ms)^2\left(\frac{ms}{6} - 1\right) A^2\right] \qquad (15)$$

Steady Sources

For steady sources ($\alpha = 0$), Eq. (15) gives values of F which agree to within 3% with those computed numerically from Eq. (11), provided that $A \leqslant 0.01$ and $\xi \leqslant 10$. There is little interest in values of ξ greater than ~ 10, since the $e^{-\xi}$ term in Eq. (6) will have reduced the concentration so far below the centerline ($\xi = 0$) value that even a substantial correction to such a small number is not significant. Along the plume centerline, the Sutton equation is good to better than 1 part in 1000. (i.e., F = 1). Carrying additional terms in the expansion does not improve the accuracy for large ξ or A, since the assumption that the second term and w^m in the last exponential of the integrand of Eq. (11) fails.

The parameter A can range from zero (several kilometers downwind) to a maximum of ~ 0.1 (for $C_x = 0.5$, $n = 0.2$, and $x = 100$ meters). The region $0.01 \leqslant A \leqslant 0.1$ represents the region encountered under lapse conditions between 50 m and 1 km from the source. The approximate correction, Eq. (15) does not hold in this range, but the solid lines of Fig. 2 show the F values calculated by direct numerical integration of Eq. (11) for $n = 0.2$. Correction factors for other values of n can be estimated by using the scaling suggested by Eq. (15), in which F is a function of $(m\xi)$. For a given ξ, A, and $n \neq 0.2$, F is obtained from Fig. 2 by interpolating to the given A and a value of ξ equal to $(2 - n)/1.8$ times the actual value. Thus for $A = 0.075$, $\xi = 10$, $n = 0.5$, interpolation in Fig. 2 at $\xi = 8.35$ gives $F = 7.9$; the value

computed directly from Eq. (11) with $n = 0.5$ is 8.2

The correction factor F can be as large as several thousand, but only at distances so far from the centerline that the concentration is still negligible.[5] The magnitude of the correction factor for realistic situations is illustrated by the following:

The factor F at a position 100 m downwind of a 50 m stack under typical lapse conditions ($n = 0.2$, $C_x = C_z = 0.4$) is 1.45, or the ground concentration is $\sim$ 45% greater than predicted by the usual Sutton equation. The ground concentration 1 km downwind of a 50 m stack under typical inversion conditions ($n = 0.5$, $C_x = 0.4$, $C_z = 0.07$) is 86% greater than that predicted by the Sutton equation.

Unsteady State Sources

The parameter α corresponding to the source variations of Eqs. (9a), (9b) and (9c) and (10) is 0, b/T, and Γ, respectively. The correction factors are given by Eq. (15) to within 5% for $A \leqslant 0.01$ and $\xi \leqslant 10$; for larger A values, F is shown as the dashed line in Fig. 2 for $\Gamma = 1$. ($\Gamma = 1$ could represent, for example, a source with a half life of 15 min. at a point 1300 m downwind with a wind speed of 1 m/sec.) In all cases, a decaying source increases F beyond that for steady sources. This reflects the contribution of the more concentrated puffs which have just passed the receptor point. The effect becomes quite pronounced for $A \gtrsim 0.08$ and $\xi > 7$.

Although the characterization of the source decay by the single term α is not valid for large A, the magnitude of the deviation of F from unity should depend primarily on this parameter. A source decaying as $t^{-1.2}$, for example, behaves at $T \approx 2.4$ as an exponentially decreasing source for which $\Gamma = 0.5$. (The α values are the same.) For $0 \leqslant \Gamma \leqslant 1$, linear interpolation from the values in Fig. 2 gives satisfactory results.

For a source with a half life considerably larger than the transit time, the correction factor F is very nearly equal to the steady state value, and the time variation of the downwind concentration is governed almost entirely by the factor between the bracketed terms of Eq. (6). The source function $f_s(t - x/\bar{u})$ implies that the downwind concentration at time t is the same as that from a steady source equal to the actual source strength at $t - x/\bar{u}$.

Figures

1. Concentration Change with Time at a Downwind Point on Cloud Centerline, $n = 0.2$.

2. Correction Factor to Sutton Formula, $n = 0.2$, $T = \infty$; Solid Line, $g(w,T) = 1$; Dashed Line, $g(w,T) = \exp[\ (1/w-1)]$

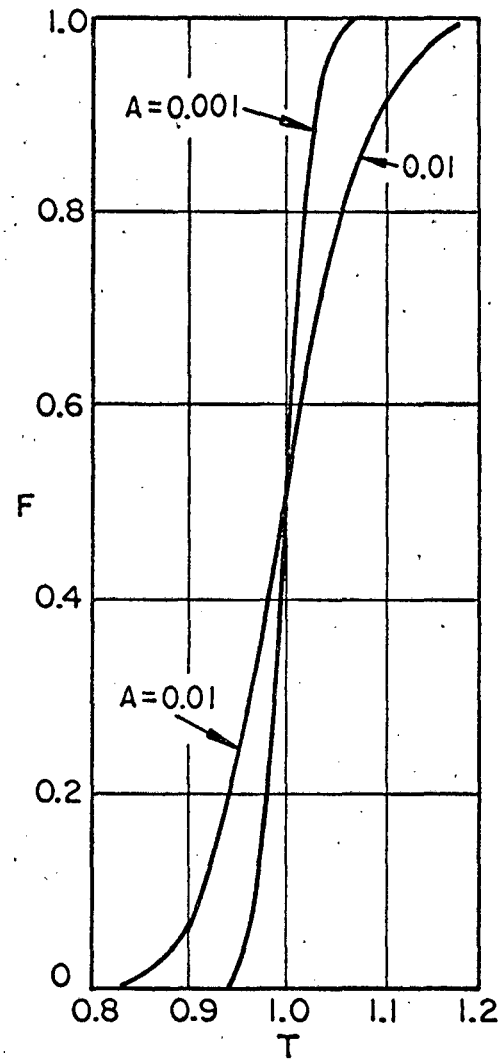
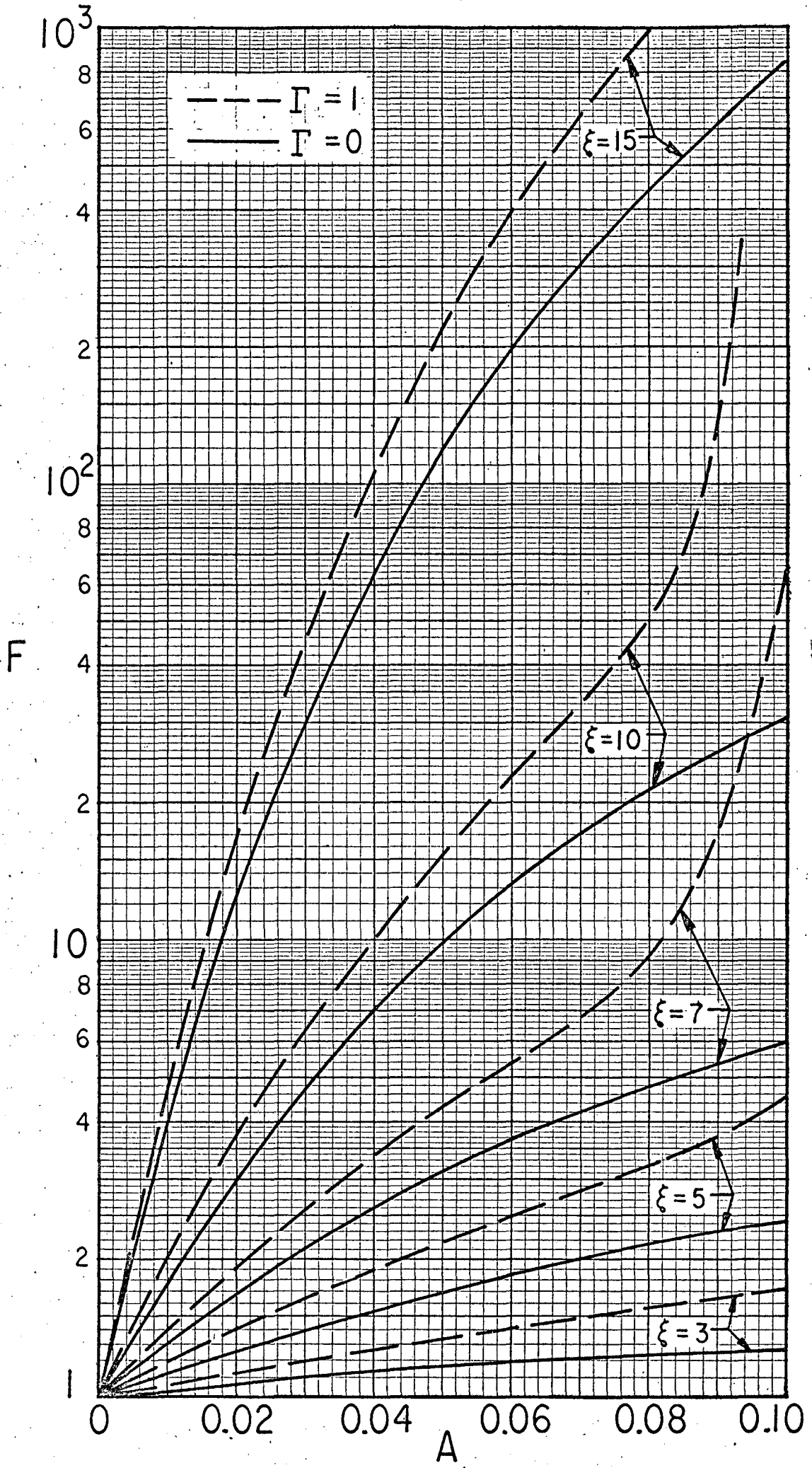


fig 1


Γ = 1
Γ = 0
ξ = 15
ξ = 10
ξ = 7
ξ = 5
ξ = 3
F
A
10^3
8
6
4
2
10^2
8
6
4
2
10
8
6
4
2
1
0
0.02
0.04
0.06
0.08
0.10