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THE METHOD OF BOOLEAN SIGNATURES:
AN APPLICATION OF BOOLEAN ALGEBRA TO THE
STATISTICAL ANALYSIS OF EXPERIMENTS

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May 5, 1964

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ABSTRACT

A method is presented that simplifies the physical content of an experimental result by removing a maximal number of redundant variables. A set of N primary binary variables is postulated to characterize each "event" of the sample, and an N-bit binary signature is constructed that summarizes the behavior of each event relative to these N variables. For well-chosen variables and an unbiased experimental sample, the sets of signatures occurring frequently can be uniquely decomposed into physical subsets if an appropriate kind of physical relation holds among the variables. This gives a global picture of the correlations among the basic variables, facilitates the separation of real from background events and gives a physical picture of the physical channels contributing to the experimental distribution. The Boolean minimization approach initiated by Quine is used and the calculations are performed, for large problems, by a computer. A detailed example from high-energy physics is given.

Separation of fundamental from redundant variables, that often tedious necessity, can be facilitated by a Boolean Algebraic technique that decomposes an experimental sample as economically as possible into physically significant parts. With certain physical assumptions, the decomposition is unique. The calculation, which has been reduced to a computer code, is based on an analysis initiated by Quine.[1] The major variables of the problem, however, must be expressible as binary variables. We discuss the method by using an example from high-energy physics.

An experiment[2] was performed in the Lawrence Radiation Laboratory's 72-in. Berkeley hydrogen bubble chamber to study the reaction $\pi^- + p \rightarrow \pi^0 + n$, as discussed in the preceding paper.[3] There were two major experimental difficulties, broadly speaking—detectability and validity. There were large variations in the detectability of experimental events and many events were of questionable validity. Because validity and detectability may be correlated it is important to study all the variables together and achieve a global picture so that we may compute correctly the simultaneous influence of detection biases and background contamination.

The seven variables chosen to characterize each event are listed in Table I. Variables X_2, X_3, and X_4 refer to the proton track and its situation relative to the film plane (which is parallel to the plane of the glass top of the chamber). Variables X_5 and X_6 refer to the missing mass calculated for the particles assumed to be emitted in the original reaction. Like X_6, X_1 is also a measure of the validity of the event. The parameter is less than 2.706 for 90% of valid events (one constraint). The variable X_0 is included because we are interested only in events having forward neutral pions, π^0's and we wish to examine any correlations that may arise. Variables X_2 and X_5 measure the stringency imposed on the corresponding validity requirements X_1 and X_6 and, consequently, their reliability.

Thus each event we analyze is characterized by a series of seven binary variables. Consider, for example, an event for which $\cos\theta^{c.m.} = 0.62$, $\chi^2 = 1.45$; $\delta\lambda = 1.52$ deg, $L_p = 0.91$ cm, $\lambda = -41.1$ deg, $\delta MM = 20.0$ MeV/c^2, RD = 1.50 (i.e., MM = 165.0 MeV/c^2 instead of the ideal 135.00). The values of the variables (X_6, X_5, X_4, X_3, X_2, X_1, and X_0) are (0, 1, 0, 0, 0, 0, and 1). We could write them in a more compact string 0100001, read it as a binary number $(0100001)_2 = (33)_{10}$ and consider its decimal equivalent, 33, the signature of the event. In other words, for the set of variables X_i, we construct $\sum_{i=0}^{6} X_i \cdot 2^i$ and call that the signature of the event. Because the signature is based on binary variables X_i, we may call it a Boolean signature. This signature expresses unambiguously in a single number, the character of the event with respect to the basic binary variables.

Having computed the signature for each event, we plot the distribution of signatures. In general we may expect that not all possible signatures will be represented in the statistical sample. Rather, if the basic variables are well chosen and are physically significant, we may expect that certain combinations of variables will occur more frequently than others. In particular, there might be correlations between variables that will exclude the occurrence of certain combinations. One must remember that nearly all the physics lies in the choice of variables. Indeed, the method does no more than summarize economically the physics thus fed in. To understand the effect of choice, we take the two extreme cases.

First, assume that all possible combinations of variables (signatures) are represented approximately equally. The result of the analysis is that all the variables should be considered redundant; no combinations of variables characterize the experimental set. The experiment does not distinguish any physically significant combinations from among the possible combinations presented.

Second, consider the case in which only one combination of variables (signature) occurs. In that case there are no redundant variables. The experiment distinguishes exactly one physical combination, and each variable is needed to specify it. (If for example, one variable were redundant, then two combinations differing only in the one redundant variable, would occur.)

In each of these extreme cases, the method gives no information that is not immediately apparent on inspection. The cases do illustrate, though, the domain of the method; the compact expression of the results of an experiment relative to a pre-chosen set of binary variables.

One exceedingly important assumption should be stated explicitly. We assume any two events to be similar to each other if they differ in the value of precisely one variable. If the kind of similarity being treated in a problem is not of that kind, the analysis we follow is useless in its simple form.

We have specified, then, a set of basic binary variables each of which has a definite value for each experimental event. For each event, we compute a Boolean signature relative to the set of binary variables, and we examine the distribution of the experimental signatures. Some signatures are relatively popular, and some, unpopular. Here, a second and last insertion of physical information occurs. One must decide, for each signature, whether it is interesting. What is called "interesting" depends on the problem. A signature may be interesting because it occurs often, because it fails to occur, or for any reason. The signatures we choose, for a given application, to call interesting are expressed by this procedure in their most compact form.

Is there, for an arbitrary set S, of interesting Boolean signatures, a unique most compact expression? No, in general there are several. Each of these, however, has a certain set of terms in common with all the others. Each such term is called an "essential cell".

We can illustrate the terminology with a simple example. Suppose we have three binary variables and that the interesting signatures are $S = (0) + (1) + (3) + (4) + (6)$. We remind ourselves that the signature (0) means $X_0 = 0$, and $X_1 = 0$, and $X_2 = 0$; the signature (6) means $X_0 = 0$, and $X_1 = 1$, and $X_2 = 1$. It is usual to indicate $X_i = 1$ by $\overline{X}_i$, and $X_i = 0$ simply by X_i. Thus the signature (0) is written $X_2X_1X_0$, and the signature (6), $\overline{X}_2\overline{X}_1X_0$. The set S, then represents the union of interesting cases which is written, in Boolean Algebra, as a sum of products

$$S = (0) + (1) + (3) + (4) + (6)$$

$$S = X_2X_1X_0 + X_2X_1\overline{X}_0 + X_2\overline{X}_1\overline{X}_0 + \overline{X}_2X_1X_0 + \overline{X}_2\overline{X}_1X_0 = S_0.$$

Now the following relations, among others, hold in Boolean Algebra:

$X + X = X$; $X + \overline{X} = 1$; $X\overline{X} = 0$; $0 + X = X$; $0 \cdot X = 0$; $1 + X = 1$; $1 \cdot X = X$.

With these, we seek to simplify the expression for S.

For example the second and third terms in S, $X_2X_1\overline{X}_0 + X_2\overline{X}_1\overline{X}_0$ have the common factor $X_2\overline{X}_0$. We remove it and express the terms as $X_2\overline{X}_0(X_1 + \overline{X}_1) = X_2\overline{X}_0$ (since we have $X + \overline{X} = 1$ for all X in this algebra). Similarly, with the fourth and fifth terms, we have $\overline{X}_2X_1X_0 + \overline{X}_2\overline{X}_1X_0 = \overline{X}_2X_0(X_1 + \overline{X}_1) = \overline{X}_2X_0$. Thus we can express S in a more succinct form $S = X_2X_1X_0 + X_2\overline{X}_0 + \overline{X}_2X_0 = S_1$, which has 3 terms with 7 variables in all, where the original expression, S_0 had 5 terms and 15 variables. The expression S_1, however, is not yet as succinct as possible. We can simplify it by using the theorem

$$X_2\overline{X}_0 = X_2\overline{X}_0 + X_2X_1\overline{X}_0.$$

We prove this by writing

$$X_2\overline{X}_0 = X_2\overline{X}_0(X_1 + \overline{X}_1) = X_2X_0X_1 + X_2\overline{X}_0\overline{X}_1.$$

Then, using $X + X = X$, we have

$$X_2\bar{X}_0 = X_2X_1\bar{X}_0 + X_2\bar{X}_1\bar{X}_0 + X_2X_1\bar{X}_0$$

$$= X_2\bar{X}_0 + X_2X_1\bar{X}_0$$

$$S = X_2X_1X_0 + X_2\bar{X}_0 + X_0\bar{X}_2 = S_1$$

$$= X_2X_1X_0 + X_2X_1X_0 + X_2\bar{X}_0 + X_0\bar{X}_2$$

$$= X_2X_1(X_0 + \bar{X}_0) + X_2\bar{X}_0 + X_0\bar{X}_2 = X_2X_1 + X_2\bar{X}_0 + X_0\bar{X}_2 = S_2.$$

Thus S_2 has three terms, as does S_1, but contains only six variables where S_1 has seven. The expression S_2 is therefore more economical than S_1.

By a precisely analogous argument we can show that $\bar{X}_2X_0 = \bar{X}_2X_0 + \bar{X}_2X_1X_0$ and ultimately, that

$$S = X_1X_0 + X_2\bar{X}_0 + \bar{X}_2X_0 = S_3.$$

Collecting our expressions we have

$$S_2 = X_2X_1 + X_2\bar{X}_0 + \bar{X}_2X_0; \qquad S_3 = X_1X_0 + X_2\bar{X}_0 + \bar{X}_2X_0,$$

two different minimal expressions for S that are equally succinct. Notice that two terms $X_2\bar{X}_0$ and $\bar{X}_2X_0$ appear in both minimal expressions. Each of these is what we have called an essential cell. (We have not proved that these expressions are the most succinct in the form of sums of products, nor that they are the only ones. Both of these assertions are true and are assumed in this discussion.)

We now give a geometric interpretation of the procedure we followed in the example. Each signature based on three binary variables corresponds to one of the $2^3 = 8$ vertex points of a three-dimensional cube. The set S forms a subset of these vertex points. For example, the signature $(6) = \bar{X}_2\bar{X}_1X_0$ meaning "$X_2 = 1$, and $X_1 = 1$, and $X_0 = 0$" corresponds to the vertex point $(X_2, X_1, X_0) = (1, 1, 0)$. In Fig. 1, all vertex points corresponding to interesting signatures are marked with circles containing the corresponding signature.

In our Boolean reduction above we expressed the signatures (1) + (3) = $X_2X_1X_0 + X_2X_1X_0 = X_2X_0 = (E_1)$. Geometrically this is replacing the two points (1) + (3) by the line E_1. The expression for the two points names six variables. Its cost is 1 + 2 = 3. The systematic search for common factors that leave factors of the form $X + \overline{X}$ always gives the expression having the lowest cost (cost = number of terms + number of variables named). Further, this procedure is equivalent to expressing the original set of points as a set of as few cubes of as high dimension as possible. In the example, we replaced the five points (cubes of dimension 0) by three lines (cubes of dimension 1). The economy is achieved by removing a maximum number of redundant variables (variables that are not needed to describe S).

It is useful to notice a geometric property of the cells E_1 and E_2. Why, after all, do they appear in every minimal covering of S? Let us look first at cell E_1 and vertex (3). We notice that E_1 is the largest cube composed entirely of vertices in S that contains vertex (3). Even more, every such cube is contained in E_1.

In particular, the two cubes (3) and (1) + (3) are contained in E_1, and there are no larger cubes containing (3) that are composed entirely of vertices in S. It is also true that any cube, C, composed entirely of vertices in S appears in every minimal expression for S if C contains any vertex V in S and all the vertices in S that are adjacent to V. In particular, E_1 is such an essential cell, since it contains (3), vertex in S, and all the vertices in S that are adjacent to (3), namely vertex (1). Similarly, E_2 is essential because it contains vertex (6) and the only vertex, (4), in S that is adjacent to (6).

On the other hand, C_1 is not an essential cell. It contains vertex (1), but it does not contain both the vertices in S, (0) and (3), that are adjacent to (1). Similarly, it contains vertex (0), but not all the vertices in S that are adjacent to it. In the same way, we find that C_2 is not an essential cell.

If a cell, C, is composed entirely of vertices in S, and contains every cell in S that covers some vertex V in S, it follows that the most economical covering of V (and we must cover V and every other vertex in S) contains C. If V were covered with some other cube, C', that cube, by our assumption, would be of lower dimension than C (since C contains every cube that covers V). Then C' would contain more variables than C, and the resulting expression (or covering) would be less economical than it would have been had we used C instead. The covering containing C' therefore is not minimal. This explains why all the essential cells appear in every minimal covering of S.

The first step in the reduction of any set, S, of Boolean signatures to its minimal expression, then, is the searching out of all essential cells. Those signatures, R, in S that are not covered by essential cells must be covered separately by cells C. There will be no unique minimal covering of R.

In the example given, essential cells E_1 and E_2 cover the vertices (1), (3), (4), and (6) but leave a residue R = (0) of one uncovered vertex. The latter may be covered minimally in two ways, by C_1 or by C_2. This concludes our general introduction to the problem of the reduction of an arbitrary set of Boolean signatures to its minimal form. The references give a more satisfactory, complete, and rigorous discussion of the general arguments indicated above.

We now put some physical content into our discussion. To this point our general conclusions have held for arbitrary sets of Boolean signatures.[2] The sets that occur in physical experiments are not arbitrary, and we wish to take advantage of their special qualities. In particular, we wish to show that, in a certain approximation for sets of physically significant signatures, only the essential cells are of interest and there is, therefore, a unique result. This is interesting, because it permits us to assign to an experimental

distribution a unique signature that exhibits nakedly the dependence of the distribution on what it (the distribution) finds to be the fundamental variables.

To understand the approximation and the appropriate conditions, let us consider an analogy. Suppose that a clan is a set of individual members such that each member has an intimate relationship with every other member. Assume too that no person outside the clan satisfies this description. It is usual though, for members of the clan to have intimate friends outside it. For a large enough clan, there may be one or more members who have no intimate relationships outside the clan. The relations of such a recluse define the boundaries of the clan. If every member of the clan (including the very young) had at least one intimate relationship outside the clan, then the boundaries of the clan would be harder to find. In the more clannish clans, the latter is not the case. Some members operate entirely within the clan, and their relationships, unalloyed with foreign substance, define it. We could pick a clan out of a crowd by looking for such members and knowing their friends.

The essential cells are analogous to the clannish clans, with the signatures playing the roles of the clan members. A friendship corresponds to membership in a cube that is completely filled with elements of the set S.

If we may mix our terminologies, our procedure seeks to break the set S into clans, leaving a residue R, of perhaps a few signatures that are unaffiliated with any of the clans. We then assume that the character of the set S is expressed in the clans it contains.

We need only justify our emphasis on clans (essential cells) to complete our heuristic description of the method. We assume first, that the Boolean signatures are computed relative to a well-chosen set of physical binary criteria. Second we assume that two signatures in S that differ in precisely one argument are similar in a physically interesting way. Last, we must

assume that the set S is an unbiased representative of the physically interesting signatures.

We should comment on these assumptions before proceeding. A good set of criteria helps insure that the composition of the set S will contain the interesting physical information. If the criteria do not distinguish physically interesting from physically uninteresting cases, the compact expression for the signatures S stated to be "interesting" will have no physical content.

Because the method of Boolean reduction looks for cubes filled with interesting signatures (see the example above), it cannot analyze cases where adjacent vertices do not correspond to physically similar situations. For example, suppose we wished to analyze the similarities of a set of signatures that had the property that the binary representation of each had the same number of "1" bits. In particular, suppose that there were three arguments, and each member of this particular set of interesting signatures had one "1" bit. The interesting signatures then would be (001, 010, 100), S = (1) + (2) + (4). Since no two of these signatures are adjacent on the 3 cube, no reduction is possible. The method would state that the most economical expression for S in terms of sums of products is S = (1) + (2) + (4) (i.e. the form given it originally). In doing so, the method would ignore the regularity we know to exist. This emphasizes the point that it is essentially the adjacency relations of S that are analyzed. The method is unsuitable for analysis of any other kind of similarity. If a problem has another kind of similarity, it must be recast, or another method of analysis used.

The last assumption is that S be an unbiased representative of the physically interesting signatures. This statement, as well as the first two, is meant to indicate limits on the valid use of the method.

A representative set of physical signatures is unbiased if all physically interesting signatures are represented. For example, if a statistical sample is analyzed relative to too large a set of variables, many signatures may be physically interesting, but unrepresented because of the low statistics. We should regard this assumption as a restriction on the number of variables to be used in the analysis of a statistical sample. Other biases may be introduced by adding or discarding signatures in an unphysical way.

The method is most useful, obviously, where the number of redundant variables is large. A large clan is more likely to have members who operate entirely within it, having no intimate friendships outside the group. Such clans, we have seen, are easier to pick out of a crowd. Analogously, essential cells with many redundant variables (i.e., large cells) are more reliably ascertained by our method than are small cells. Granting that the results are weaker for cases of small essential cells, we must note that they are less interesting as well, since for such cases, the results are often derivable by inspection without the apparatus we have developed.

We now give results of the analysis of the experimental set of Boolean signatures derived from the actual physical experiment described at the start. The seven criteria were given in Table I, and Fig. 2 shows the distribution of the experimental Boolean signatures for some 150 events. We choose to consider interesting those signatures that occur more than twice. The set S then consists of the twenty signatures S = (0, 32, 33, 44, 66, 78, 97, 98, 99, 100, 101, 102, 103, 108, 109, 110, 111, 113, 115, 121), each involving seven variables. The "cost" of this unreduced expression, S_0, is therefore $20 + 7 \times 20 = 160$. When we reduce S, we find that there are seven essential cells (clans)—one composed of valid, and six of spurious, events. Further, one signature (33) is unaffiliated with any clan; that is, R = (33). This

supports our contention that the set of signatures will split into physical sets. Are there any other internal checks? Yes, if the essential cells really correspond to different physical sets, we should expect them not to overlap or else to overlap very little. The latter is the case. Three signatures (108, 110, 113) representing a very small number of experimental events (see Table II) appear in more than one essential cell. This may be regarded as consistent with our insistence on separateness for the ideal case. In this instance, since the statistics are meager for several of the signatures, and since there are relatively few redundant variables in most of the essential cells, we may regard our result as consistent with expectations in first order. Note too, that when a signature does appear in two essential cells, at least one cell has only one redundant variable.

Table II summarizes the results of the Boolean analysis of the part of S resolvable into essential cells. The remainder R = (33) composed of a single signature is ignored. Since our variables have been chosen so that $X_i = 0$ corresponds to the more desirable alternative in each case, events having the signature 0 are acceptable relative to each criterion. We are most anxious, therefore, to examine the essential cell containing the signature 0 to see if it contains any other signatures. The implication of the presence of events of such a signature is that the distribution regards them as physically similar relative to the basic variables. We might say, alternatively, that one or more variables thought originally to characterize the clan of valid events are redundant from the point of view of the distribution. We must stress that our method is phenomenological from the point of view of the physical information. It yields only the information given by the distribution, and it recognizes no gradations in fundamental importance between the variables presented to it. We may regard our procedure as an analysis of the correlations between frequency of appearance and the crude behavior of certain

variables approximated by two-valued variables. Having understood the correlations, we eliminate as many variables as possible that do not correlate with the high frequency of experimental occurrence.

With these restrictions in mind, we examine the results given in Table II. We notice that the distribution separates signatures (0) and (32) from the others, regarding them as indistinguishable from each other. This implies that we should suspect the presence of valid events [signature (0)] in the class [signature (32)] of events that are valid but weakly constrained in that the errors on the computed missing mass are large. That is, we fear that some valid events look suspicious because our information about them is not precise, just as some innocent men would be lost to society if it were the custom to convict all suspects who lacked an alibi. We apply an appropriate correction to our experimental results. That no further signature lies in the same class as our real events implies that the distribution itself distinguishes the valid events from all except perhaps those invalid events having signature 32. This may be taken to justify a confidence that no further corrections are required, and no other source of spurious, inseparable background exists.

By inspecting the larger essential cells, we can grasp a few global characteristics of our background. Examining, for example, the fifth and sixth cells in Table II (which do not overlap, are the largest cells, and summarize the behavior of well over half the rejected events), we see that both are characterized by high deviation of the missing mass from its ideal value relative to a high error in that calculated mass. Further, these events all have low dip errors. This is a global statement about the bulk of the background that is as general (economical) as the distribution permits. More detailed statements, containing equally succinct information can be had by

examining the seven essential cells in detail.[2] The cost of the final expression for S given in Table II is equal to (number of cells) + (number of variables named in the cells) = 7 + 39 = 46 as opposed to the original cost of 19 + 19 × 7 = 152 to express the 19 signatures of seven arguments each in crudest form.

We have summarized the properties of our experimental distribution, reassured ourselves somewhat that the acceptance criteria we have applied are physical, achieved a crude global characterization of the background, and subject to the assumptions of our method, found an argument to justify our estimate of contamination by inseparable background. These results were extracted with plausible assumptions, and confer, within bounds, a kind of order on our picture of the experimental results with less mechanical effort than would have been needed by conventional analyses.

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*This work was done under the auspices of the U. S. Atomic Energy Commission.

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Table I. The binary criteria used to characterize each event.

Criterion No., i	X_i relates to the phrase		X_i = (0, 1) if
0	Forward	$\cos\theta^{c.m.}$	is ($\geq$, <) 0.7
1	Chi squared (χ^2)	χ^2	is ($\leq$, >) 2.706
2	Dip error ($\delta\lambda$)	$\delta\lambda$	is ($\leq$, >) 2.5 deg
3	Projected length (L_p)	L_p	is ($\geq$, <) 0.7 cm
4	Dip [relative to top glass] (λ)	$\lvert\lambda\rvert$	is ($\leq$, >) 60 deg
5	Missing mass error (δ MM)	δ MM	is ($\leq$, >) 16.0 MeV/c^2
6	Relative deviation of the missing mass $RD = [MM - M(\pi^0)]/\delta MM$	RD	is ($\leq$, >) 1.65

Table II. Summary of Boolean analysis of fitted events.

Class	Representation	Number of events	$\frac{MM - M(\pi^0)}{\delta MM}$	δMM	Dip of recoil	Projection of recoil	Dip error of recoil	χ^2	Forward π^0 ?
(0, 32	$X_6()X_4X_3X_2X_1X_0$	15	low	-	flat	long	low	low	yes
(44, 108	$()\bar{X}_5X_4X_3\bar{X}_2X_1X_0$	12	-	high	flat	long	high	low	yes
(66, 98)	$\bar{X}_6()X_4X_3X_2\bar{X}_1X_0$	15	high	-	flat	long	low	high	yes
(78, 110)	$\bar{X}_6()X_4\bar{X}_3\bar{X}_2\bar{X}_1X_0$	8	high	-	flat	short	high	high	yes
(100, 101, 102, 103, 108, 109, 110, 111)	$\bar{X}_6\bar{X}_5X_4()X_2()\ ()$	56	high	high	flat	-	low	-	-
(97, 99, 113, 115)	$\bar{X}_6\bar{X}_5()X_3X_2()\bar{X}_0$	50	high	high	-	long	low	-	no
(113, 121)	$\bar{X}_6\bar{X}_5\bar{X}_4()X_2X_1\bar{X}_0$	7	high	high	steep	-	low	low	no

FIGURE LEGENDS

Fig. 1. Geometric representation of a set S of interesting signatures

$$S = (0) + (1) + (3) + (4) + (6)$$

and its two minimal coverings

$$S_2 = C_1 + E_1 + E_2$$

and $$S'_2 = C_2 + E_1 + E_2 .$$

Fig. 2. Distribution of the experiment events among the $2^7 = 128$ possible Boolean signatures.

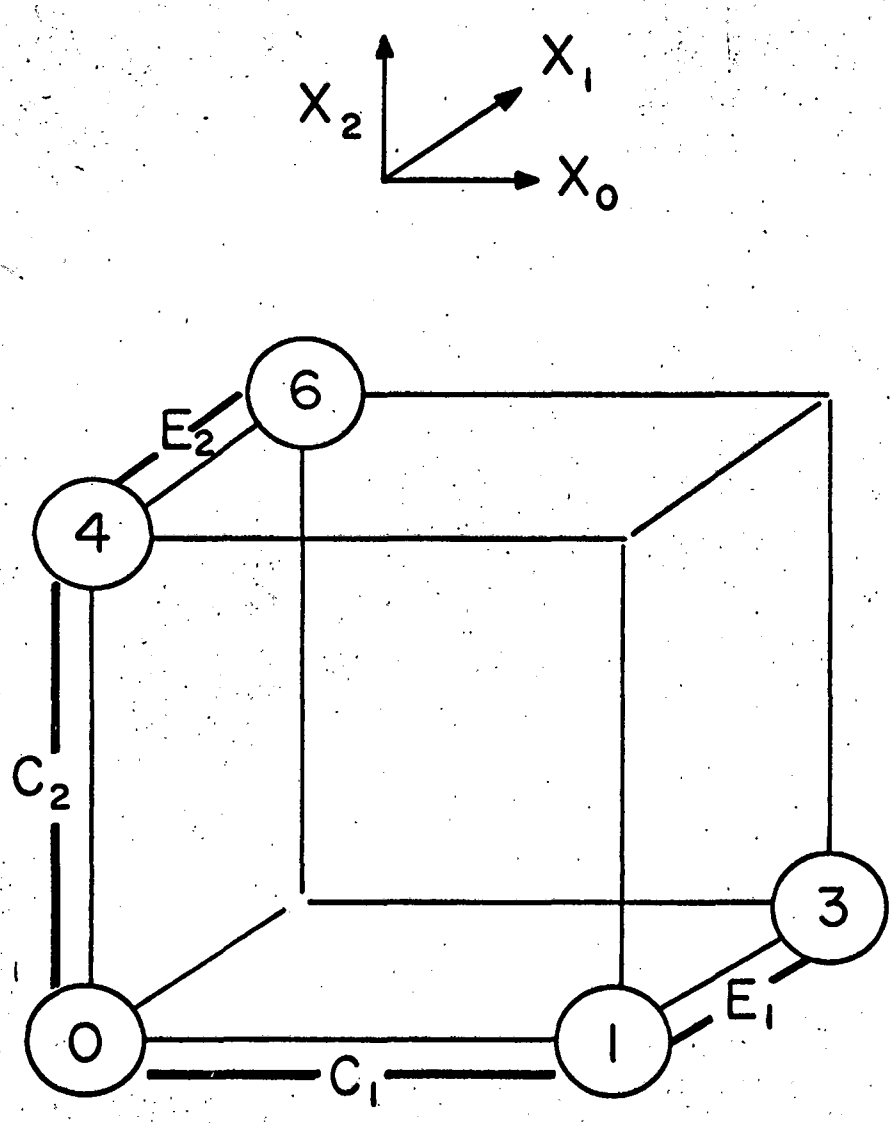

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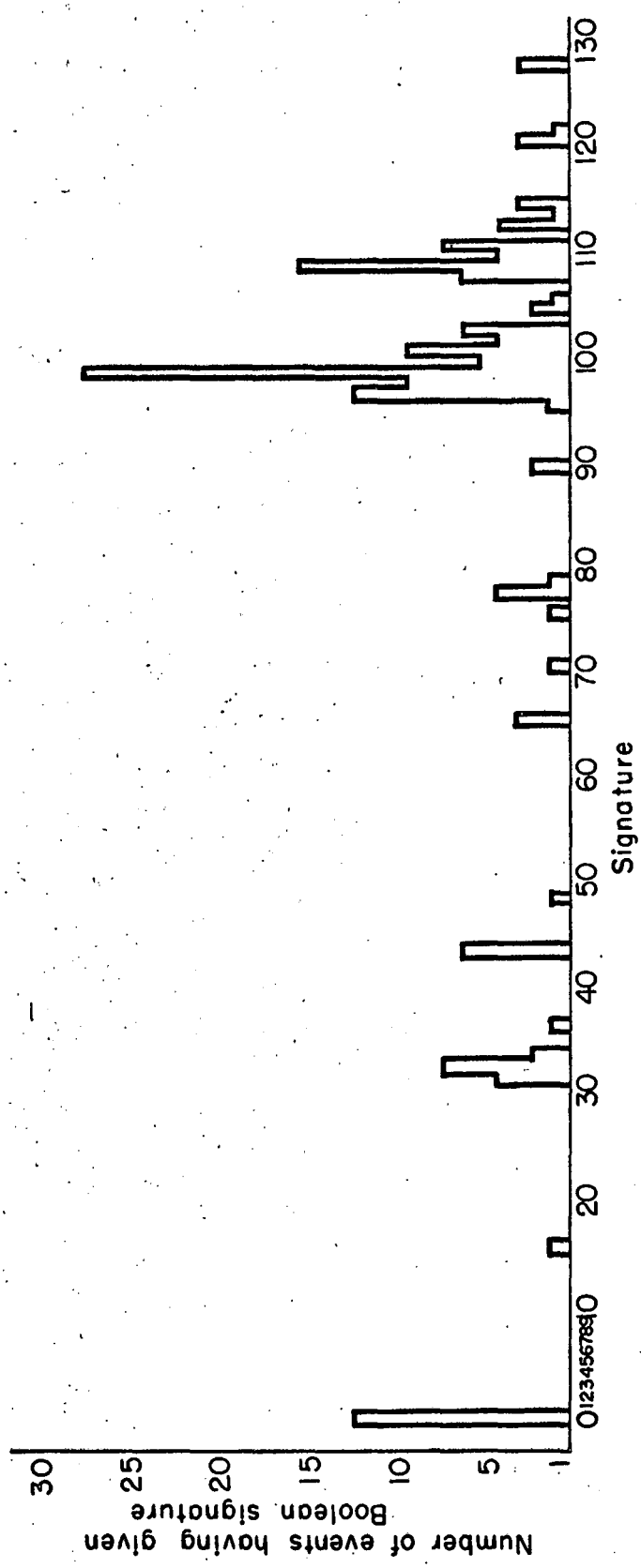
Fig. 1.



MUB-2289

Fig. 2.

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