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ABSTRACT

We investigate a special model to see if it is possible to extend the three-body scattering amplitude to complex values of the total angular momentum J by an integral equation with completely continuous kernel, or by extending the Fredholm solution of the Fadeev equations from integral J to complex J. This model is a helium atom with infinitely heavy nucleus, neglecting the interaction between electrons and replacing the Coulomb potentials by a superposition of finite-range Yukawa potentials. One finds poles and cuts in J which depend not only upon the total energy but also upon the subenergies of the electrons. Accordingly, the problems stated above have no solution.

I. INTRODUCTION

There have been recently several attempts to elucidate the analytic properties of the nonrelativistic three-body scattering amplitude as a function of the total angular momentum J.[1-4] They were motivated by the importance assigned in high-energy physics to the Regge poles that are known to be present in the two-body case; either nonrelativistic,[5] for solutions of Bethe-Salpeter equations,[6,7] or in the strip approximation to the Mandelstam representation.[8]

This problem presents certain difficulties which are absent in the two-body case. Notwithstanding the facts that the kinematics is more involved and the collision matrix is disconnected, two essential new difficulties appear when J is made complex:

(a) The triangular inequalities for the coupling of relative angular momenta, or inequalities of the form $|M| \leq J$ for the projection of the angular momentum on some axis, are no longer true.

(b) The full three-body scattering amplitude has complex singularities as a function of some angle cosines.

In the previous papers, OI and OII,[4] this problem has been approached through the Fadeev equations, which provide a good mathematical formulation of nonrelativistic three-body systems.[9,10] In OI, the Fadeev equations for a given J were written using as a complete set of commuting observables the three energies ω_1, ω_2, and ω_3 of the particles in their total center-of-mass system, the total angular momentum J and its projections M_z on a space-fixed axis, and M on a "body fixed" axis belonging to a reference system linked once for all in a well-defined way to the triangle formed by the three linear momenta. In OII, the

continuation of these equations to complex J was discussed, with the result that it is not possible to perform such a continuation because of difficulty (a).

When J is complex, the Fadeev equations look like

$$T^{J(i)}_{MM'}(\omega,\omega';z) = B^{J(i)}_{MM'}(\omega,\omega';z) + \sum_{j\neq i} \sum_{M''=-\infty}^{+\infty} \int d\omega'' K^{J(i,j)}_{MM''}(\omega,\omega'';z) T^{J(j)}_{M''M'}(\omega'',\omega';z)$$

$$\text{for} \quad i,j = 1,2,3, \qquad \text{(I.1)}$$

where z is the complex energy, which is put on the energy shell $z = \sum_i \omega_i = \omega_i'$ at the end of the calculation, and ω represents the set $\{\omega_1\omega_2\omega_3\}$. An infinite summation over M'' replaces the finite summation over $|M''| \leq J$ when J is no longer an integer. It was shown that there are two facts that forbid the continuation: the kernel of (I.1) is unbounded because of its exponential behavior in the helicity variables M when $M \to \infty$; and when J reaches a physical value these equations couple the $T_{MM'}{}^J$ matrix for "sense" and "nonsense" channels, i.e., couple $T_{MM'}{}^J$ with $|M|$, $|M'| < J$ and $T_{MM'}{}^J$ with $|M|$, $|M'| > J$; so we do not get back the physical set of equations used as a starting point.

It is convenient to point out here that these kinds of situations are present in all the hitherto considered approaches to the problem. Either using the Schrödinger equation in configuration[1,2] or in momentum spaces,[11] or using the Fadeev equations with another set of quantum numbers which completely determine the eigenstates of the total

angular momentum,[3] one always ends up somewhere with certain unbounded operators which spoil the mathematical analysis, because of the presence of difficulty (a).

There exists in principle a way of avoiding the previously mentioned difficulty, and it consists in extending to complex J the Fredholm solution $N_{MM'}{}^{J}(\omega,\omega';z)/D^{J}(z)$ of the Fadeev equations for physical J, instead of extending the equations themselves. With this method summations over M'' are computed before going to complex J, and it was shown in OII that when this continuation is carried out every term in the series of the numerator and the denominator is an analytic function of J in a certain right-half plane. However, in order to be able to say something rigorous about the scattering amplitude $T_{MM'}{}^{J}(\omega,\omega';z)$, one has to solve the formidable problem of finding whether or not the Fredholm series are uniformly convergent for complex J.

The aim of this paper is to present a particular three-body model as a counter example which shows that the Fredholm denominator--or Jost function--$D^{J}(z)$ does not exist for nonintegral values of J.

In order to investigate the existence of $D^{J}(z)$, we examine the limit where one of the particles becomes infinitely heavy while the interaction between the two other particles vanishes. Since the Fadeev equations remain valid in that case, we presume that, if $D^{J}(z)$ exists in the general case, it would also exist in this limiting model, which is introduced in Section II. It is analogous to a simplified helium atom in which a superposition of Yukawa potentials replaces the Coulomb potentials, the interaction between the two "electrons" being

neglected. In Section III, the singularities of $T_{MM'}^{J}(\omega,\omega';z)$ in the J plane for fixed $M = M' = 0$ are analyzed. The information we can get about the full three-body amplitude, using the Neumann series expansion of the Fadeev equations is enough for obtaining such singularities using the generalization of the Froissart-Gribov formula[12,13] proposed in OII. It is found that there are no Regge poles--that is, singularities of the form $J = J(z)$--but there appear for a given value of z an infinite number of singularities which depend upon the subenergies, and which are restricted to a finite region near the origin.

Finally, these results are used in Section IV to prove that in this model the Jost function $D^{J}(z)$ does not exist for complex J, and its relevance with respect to the whole three-body Regge poles problem is also discussed.

II. THE MODEL

Let us consider a system of three particles, with the following assumptions:

(i) The masses are $m_1 = m_2 = m$; $m_3 = M = \infty$.

(ii) If V_{ij} represents the interaction between particles i and j, we take $V_{13} \neq 0$, $V_{23} \neq 0$, but $V_{12} \equiv 0$. Moreover, the nonvanishing interactions are taken to be superpositions of Yukawa potentials.

Let us first discuss the kinematics. We denote by $\vec{k}_\alpha$ (where $\alpha = 1,2,3$) the momenta of the three particles, which satisfy

$$\vec{k}_1 + \vec{k}_2 + \vec{k}_3 = 0 \tag{II.1}$$

in the total center-of-mass system. For the two independent momenta we can introduce, for example, the center-of-mass momentum $\vec{p}_\alpha$ of the (β,γ) subsystem (here $\alpha \neq \beta \neq \gamma$; $\alpha,\beta,\gamma = 1,2,3$ or cyclic permutations) and the momentum $\vec{q}_\alpha$ of the particle α relative to the (β,γ) subsystem:

$$\vec{p}_\alpha = \frac{1}{\sqrt{(m_\beta + m_\gamma)2m_\beta m_\gamma}} [m_\gamma \vec{k}_\beta - m_\beta \vec{k}_\gamma] , \tag{II.2}$$

$$\vec{q}_\alpha = \frac{1}{\sqrt{2m_\alpha(m_\beta + m_\gamma)(m_\alpha + m_\beta + m_\gamma)}} [m_\alpha(\vec{k}_\beta + \vec{k}_\gamma) - (m_\beta + m_\gamma)\vec{k}_\alpha] . \tag{II.3}$$

In our model,

$$\begin{aligned} \vec{p}_2 &= \vec{q}_1 = \vec{k}_1 , \\ \vec{p}_1 &= -\vec{q}_2 = \vec{k}_2 . \end{aligned} \tag{II.4}$$

The third particle can have any linear momentum, because of its infinite mass; its energy $\omega_3 = p_3^{\,2}/2M$ being always zero. In the initial state, we define its momentum by

$$\vec{k}_3 = -(\vec{k}_1 + \vec{k}_2) = -(\vec{p}_2 + \vec{p}_1) \tag{II.5}$$

in order to stay within the total center-of-mass system. Because there is no interaction between particles 1 and 2, the angle between $\vec{k}_1$ and $\vec{k}_2$--or $\vec{p}_1$ and $\vec{p}_2$--is an irrelevant variable, so without loss of generality we can choose both $\vec{p}_1$ and $\vec{p}_2$ collinear in the initial state. The same is true when applied to the final state, but of course not when applied to both of them simultaneously.

We must now define the angular variables in the three-body scattering amplitude. The initial configuration is given by a triangle of sides $(\vec{p}_1, \vec{p}_2, \vec{k}_3)$ and the final one by another triangle of sides $(\vec{p}_1', \vec{p}_2', \vec{k}_3')$. Then we choose the body-fixed axis z along $\vec{p}_1$, and the z' axis along $\vec{p}_1'$. The initial triangle lies in the xz plane, and the final one in the x'z' plane. The three-body scattering amplitude is then a function of the subenergies $\omega_1\omega_2\omega_1'\omega_2'$ and of the Euler angles $(\alpha\beta\gamma)$ which determine the rotation necessary for taking the x y z system of axis into the x'y'z' system. One has

$$\hat{p}_1 \cdot \hat{p}_1' = \cos\beta \quad , \tag{II.6}$$

where the notation $\hat{p} = \vec{p}/|\vec{p}|$ has been used. When the vectors $\vec{p}_1'$ and $\vec{p}_2'$ are collinear, one also has

$$\hat{p}_2 \cdot \hat{p}_2' = \cos\beta = \cos\theta \cos\beta - \sin\theta \sin\beta \cos\infty \quad , \tag{II.7}$$

where $\cos\theta$, the angle between $\vec{p}_1$ and $\vec{p}_2$, is given by

$$\cos\theta = \frac{(\vec{p}_1 + \vec{p}_2)^2 - p_1^2 - p_2^2}{2p_1p_2} \quad . \tag{II.8}$$

In this model the two-body T matrix for particles 1 and 2 is identically zero, so $T_3(z) \equiv 0$ in the Fadeev equations, and they look like

$$\begin{pmatrix} T^{(1)}(z) \\ \\ T^{(2)}(z) \end{pmatrix} = \begin{pmatrix} T_1(z) \\ \\ T_2(z) \end{pmatrix} - \begin{pmatrix} 0 & T_1(z) \\ \\ T_2(z) & 0 \end{pmatrix} G_0(z) \begin{pmatrix} T^{(1)}(z) \\ \\ T^{(2)}(z) \end{pmatrix} . \tag{II.9}$$

The three-body collision matrix is given by the matrix element $\langle \bar{p}_1' \bar{p}_2' | T(z) | \bar{p}_1 \bar{p}_2 \rangle$ where

$$T(z) = T^{(1)}(z) + T^{(2)}(z) \quad . \tag{II.10}$$

It is convenient to recall here what the matrix elements of $T_i(z)$, $i = 1,2$, and $G_0(z)$ are:

$$\langle \bar{p}_1' \bar{p}_2' | T_i(z) | \vec{p}_1 \vec{p}_2 \rangle = \langle \bar{p}_1' | T_i(z - \omega_i) | p_i \rangle \delta^{(3)}(\bar{p}_j' - \bar{p}_j) \quad ; \qquad i \neq j \quad , \tag{II.11}$$

where the matrix element in the right-hand side is the off-the-energy-shell two-body scattering amplitude of the pair jk, and $\omega_i = p_j^2/2m$. Also, in this particular model,

$$\langle \bar{p}_1' \bar{p}_2' | G_0(z) | \bar{p}_1 \bar{p}_2 \rangle = \frac{1}{\omega_1 + \omega_2 - z} \delta^{(3)}(\bar{p}_1' - \bar{p}_1) \delta^{(3)}(\bar{p}_2' - \bar{p}_2) \tag{II.12}$$

We will consider only the connected part $U(z) = U^{(1)}(z) + U^{(2)}(z)$ of the amplitude, where

$$\begin{pmatrix} U^{(1)}(z) \\ \\ U^{(2)}(z) \end{pmatrix} = \begin{pmatrix} T^{(1)}(z) \\ \\ T^{(2)}(z) \end{pmatrix} - \begin{pmatrix} T_1(z) \\ \\ T_2(z) \end{pmatrix} . \tag{II.13}$$

The Fadeev equations then become

$$U^{(1)}(z) = -T_1(z)G_0(z)T_2(z) + T_1(z)G_0(z)T_2(z)G_0(z)T_1(z) + T_1(z)G_0(z)T_2(z)U^{(1)}(z) , \tag{II.14}$$

with a similar equation for $U^{(2)}(z)$.

III. SINGULARITIES IN THE J PLANE

In this particular model, we analyze the singularities in the J plane of the three-body scattering amplitude $T_{MM'}{}^{J}(\omega,\omega';z)$. This will be performed in two steps:

(a) Using the Fadeev equation (II.14), we compute in a certain approximation the connected part of the full three-body scattering amplitude, $\langle \bar{q}_1'\bar{q}_2'|U^{(1)}(z)|\bar{q}_1\bar{q}_2\rangle$.

(b) Once we have computed different approximations for the $U^{(1)}(z)$ matrix, we look at the singularities in J of $T_{MM'}{}^{J}(\omega,\omega';z)$, using the extension to complex J proposed in OII, which is essentially a generalization of the Froissart-Gribov formula for the two-body case:

$$T_{MM'}{}^{J}(\omega,\omega';z) = \int_0^{2\pi} d\alpha d\gamma \oint d\cos\beta \langle \bar{p}_1'\bar{p}_2'|U^{(1)}(z)|\bar{p}_1\bar{p}_2\rangle \mathcal{E}_{MM'}{}^{J}(\alpha,\beta,\gamma) . \tag{III.1}$$

Here $\mathcal{E}_{MM'}{}^{J}(\alpha,\beta,\gamma)$ are the rotation matrices of the second kind introduced in OII, and the contour of integration in the $\cos\beta$

plane runs along the singularities of $U^{(1)}(\bar{p}_1'\bar{p}_2';\bar{p}_1\bar{p}_2;z)$, which, by introducing adequately the notion of signature, can always be mapped into the right half $\cos\beta$ plane. It must be stressed here that these singularities are in general complex. For simplicity, we consider only the case $M = M' = 0$, in which the function $\mathcal{G}_{MM'}^{J}(\alpha,\beta,\gamma)$ reduces to the more familiar Legendre functions of the second kind, $Q_J(\cos\beta)$, and the α and γ integrations are not relevant. The generalization of the results to values of M and M' different from zero is only a technical problem and does not present any new difficulty.

Let us now consider the first-order contribution to the connected part $U^{(1)}(z)$:

$$\frac{T_1(\vec{p}_1',\vec{p}_1,z-\omega_1')T_2(\vec{p}_2',\vec{p}_2,z-\omega_2)}{\omega_1'+\omega_2-z} \quad . \tag{III.2}$$

Restricting ourselves to the case $M = M' = 0$, and putting z on the energy shell, we find the generalized Froissart-Gribov formula reads

$$T_{00}^{J}(p_1'p_2',p_1p_2,E)$$

$$= \frac{2\pi}{\omega_1'-\omega_1}\int_0^{2\pi} d\alpha \oint d\cos\beta T_1(p_1',p_1,\cos\beta,\omega_2')T_2(p_2',p_2'\cos B,\omega_1)Q_J(\cos\beta) \quad , \tag{III.3}$$

where $E = \omega_1 + \omega_2 = \omega_1' + \omega_2'$. The contour of integration in the $\cos\beta$ plane runs along the singularities of the function T_1T_2.

The singularities in J of $T_{00}^{J}(p_1'p_2', p_1p_2, E)$ as given by Eq. (III.3) are determined by the asymptotic behavior in $\cos\beta$ of T_1 and T_2. This fact can be shown in the following way: we can rewrite (III.3) as

$$T_{00}^{J}(p_1'p_2', p_1p_2, E)$$

$$= \frac{2\pi}{\omega_1' - \omega_1}\int_0^{2\pi} d\alpha\Big[\int^{\infty} d\cos\beta\, T_1(q_1'p_1, \cos\beta, \omega_2')\Delta T_2(p_2', p_2, \cos B, \omega_1)Q_J(\cos\beta) +$$

$$+\int^{\infty} d\cos\beta\, \Delta T_1(p_1'p_1, \cos\beta, \omega_2')T_2(p_2'p_2, \cos B, \omega_1)Q_J(\cos\beta)\Big]. \tag{III.4}$$

This is of the form

$$a_J = \int^{\infty} dx f(x) Q_J(x) , \tag{III.5}$$

and if $f(x) \sim x^{\alpha}$ when $x \to \infty$, it is known that a_J has poles at $J = \alpha - n$; $n = 0, 1 \ldots$.

The Regge pole analysis can be extended to the off-the-energy-shell two-body scattering amplitude;[14] so, when $\cos\beta \to \infty$, $T(p,p'; \cos\beta, z)$ exhibits the Regge behavior.

$$T(p,p'; \cos\beta; z) \sim g(p,p';z)(\cos\beta)^{\alpha(z)} . \tag{III.6}$$

The discontinuities ΔT have the same power behavior as the amplitudes in $\cos\beta$, so both terms in Eq. (III.4) yield essentially the same singularities. The asymptotic behavior of the first integrand, for example, is given by

$$T_1(p_1', p_1 \cos\beta, \omega_2') \sim g_1(p_1', p_1; \omega_2')(\cos\beta)^{\alpha(\omega_2')} , \qquad \text{(III.7)}$$

$$\Delta T_2(p_2', p_2, \cos B, \omega_1) \sim g_2(p_2', p_2; \omega_1) f(\theta, \omega_1, \alpha)(\cos\beta)^{\alpha(\omega_1)} , \qquad \text{(III.8)}$$

where

$$f(\theta, \omega_1, \alpha) = (\cos\theta - i \sin\theta \cos\theta)^{\alpha(\omega_1)} . \qquad \text{(III.9)}$$

Then

$$T_1 \Delta T_2 \sim g_1 g_2 f(\cos\beta)^{\alpha(\omega_1)+\alpha(\omega_2')} . \qquad \text{(III.10)}$$

Therefore, we conclude that this first-order contribution to the connected part yields poles in T_{00}^{J} at

$$J = \alpha(\omega_1) + \alpha(\omega_2') - n; \quad n = 0, 1 \ldots . \qquad \text{(III.11)}$$

We consider next the contriubtion to $U^{(1)}(z)$ of the second iteration,

$$\int p_1''^2 dp_1'' \frac{T_2(\vec{p}_2', \vec{p}_2, z - \omega_2'')}{(\omega_1' + \omega_2'' - z)(\omega_1 + \omega_2'' - z)} F_1(p_1', p_1'', p_1, \cos\beta, z - \omega_1', z - \omega_1) , \qquad \text{(III.12)}$$

where the function F_1 is defined by

$$F_1(p_1',p_1'',p_1,\cos\beta;z-\omega_1',z-\omega_1) = \int d\hat{p}_1'' T_1(p_1',p_1'',\hat{p}_1'\cdot\hat{p}_1'',z-\omega_1')$$

$$\times\, T_1(p_1'',p_1,\hat{p}_1''\cdot\hat{p}_1,z-\omega_1) \quad . \tag{III.13}$$

Notice that in (III.12) $\omega_2'' = p_1''^2/2m$. In order to find out the singularities of the corresponding T_{00}^{J}, we need to know the asymptotic behavior in $\cos\beta$ for T_2 and F_1. The behavior of T_2 is immediate:

$$T_2(p_2',p_2;\cos B,z-\omega_2'') \sim g_2(p_2'',p_2,z-\omega_2'')f(\theta,z-\omega_2''\alpha)(\cos\beta)^{\alpha(z-\omega_2'')} \quad . \tag{III.14}$$

The behavior of F_1 can be obtained by using the unitarity integral (III.13), In fact, introducing the expansions in Legendre polynomials,

$$T_1^{(1)}(p_1',p_1'',\hat{p}_1'\cdot\hat{p}_1'',z_1) = \sum_{\ell}(2\ell+1)A_\ell^{(1)}(p_1',p_1'',z_1)P_\ell(\hat{p}_1'\cdot\hat{p}_1'') \quad , \tag{III.15}$$

$$T_1^{(2)}(p_1'',p_1,\hat{p}_1''\cdot\hat{p}_1,z_2) = \sum_{\ell}(2\ell+1)A_\ell^{(2)}(p_1'',p_1,z_2)P_\ell(\hat{p}_1''\cdot\hat{p}_1) \quad , \tag{III.16}$$

one has

$$F_1(p_1',p_1'',p_1,\hat{p}_1'\cdot\hat{p}_1,z_1,z_2) = \sum_{\ell}(2\ell+1)A_\ell^{(1)}(p_1',p_1'',z_1)A_\ell^{(2)}(p_1'',p_1,z_2)$$

$$\times\, P_\ell(\hat{p}_1'\cdot\hat{p}_1) \quad . \tag{III.17}$$

By making a standard Sommerfeld-Watson transformation, one sees that F_1 has contributions from the Regge poles of both $T_1^{(1)}$ and $T_1^{(2)}$; so the Regge formula looks like

$$F_1(p_1',p_1'',p_1,\cos\beta,z_1z_2) = \sum_i \frac{2\alpha_i(z_1)+1}{\sin\pi\alpha_i(z_1)} g_i^{(1)}(p_1'',p_1',z_1)$$

$$\times A^{(2)}_{\alpha_i(z_1)}(p_1'',p_1,z_2)P_{\alpha_i(z_1)}(\cos\beta)$$

$$+ \sum_j \frac{2\alpha_j(z_2)+1}{\sin\pi\alpha_j(z_2)} g_j^{(2)}(p_1'',p_1,z_2)$$

$$\times A^{(1)}_{\alpha_j(z_2)}(p_1',p_1'',z_1)P_{\alpha_j(z_2)}(\cos\beta)$$

$$+ \text{ background integral} \quad . \tag{III.18}$$

It is convenient to notice here that $A_{\alpha_j(z_2)}^{(1)}(p_1',p_1'',z_1)$ and $A^{(2)}_{\alpha_i(z_1)}(p_1'',p_1,z_2)$ are analytic functions of the p variables, with only the left-hand cuts in them, and the right-hand cut in the z variable. As a function of z_1, $A^{(1)}_{\alpha_j(z_2)}$ has the form

$$A^{(1)}_{\alpha_j(z_2)}(p_1',p_1'',z_1) = \sum_i \frac{g_i(p_1',p_1'',z_1)}{\alpha_i(z_1)-\alpha_j(z_2)} + \text{background} \quad , \tag{III.19}$$

so $A^{(1)}$ has also a pole at $z_1 = z_2$.

Considering only the leading Regge trajectory in (III.18), we find that as $\cos\beta \to \infty$, F_1 behaves in the following way:

$$F_1 \sim (\cos\beta)^{\alpha(z-\omega_1')} G_1(p_1', p_1'', p_1, z - \omega_1'', z - \omega_1)$$

$$+ (\cos\beta)^{\alpha(z-\omega_1)} G_2(p_1', p_1'', p_1, z - \omega_1, z - \omega_1') \quad . \qquad \text{(III.20)}$$

The techniques used in the analysis of the first Born term cannot be applied here at once in Eq. (III.12), because of the additional integration upon ω_2''. It is necessary to make use of another trick before, which consists in changing the variable in the ω_2'' integration, by going to a new variable λ defined by

$$\lambda = \alpha(z - \omega_2'') \quad . \qquad \text{(III.21)}$$

Suppose we have a typical Regge trajectory $\alpha(E)$ for Yukawa potentials, then the contour C of integration, going in the λ plane from $\lambda = \alpha(z)$ to $\lambda = \alpha(-\infty) = -1$, is shown in Fig. 1. Next one can see what the singularities of the integrand of (III.12) are in the λ plane. The propagators yield poles at $z - \omega_2'' = \omega_1$ and $z - \omega_2'' = \omega_1'$, slightly displaced from the real axis when z is put on the energy shell, so they map into poles at $\lambda = \alpha(\omega_1) = \alpha_1$ and $\lambda = \alpha(\omega_1') = \alpha_1'$. If we now remember that the functions G_1 and G_2 are proportional to $A^{(2)}_{\alpha(z-\omega_1')}(p_1'', p_1, z - \omega_1)$ and $A^{(1)}_{\alpha(z-\omega_1)}(p_1', p_1'', z - \omega_1')$ respectively, and that these functions have left-hand cuts in the $p_1''^2$ variables, then we conclude that in the ω_2'' plane the integrand of (III.12) has cuts from $\omega_2'' = -\infty$ to some value $\omega_1'' = \xi(p_1', p_1) < 0$; so this gives a cut in the λ plane from

$\alpha(z + \xi)$ to $\alpha(+\infty)$. All these singularities are also shown in Fig. 1.

In this way, the integral in (23) transforms into an expession of the form (we suppressed irrelevant variables)

$$\oint d\cos\beta\, Q_J(\cos\beta) \int_C \frac{d\lambda}{\left[\dfrac{d\lambda}{d(z-\omega_2'')}\right]} \Psi(\lambda) \left[(\cos\beta)^{\lambda+\alpha(z-\omega_1')}\left(\frac{H_1}{\lambda-\alpha_1} + \frac{H_1'}{\lambda-\alpha_1'}\right) + (\cos\beta)^{\lambda+\alpha(z-\omega_1)}\left(\frac{H_2}{\lambda-\alpha_1} + \frac{H_2'}{\lambda-\alpha_1'}\right)\right] \tag{III.22}$$

Next the contour C can be distorted into the contour C' plus the contribution of the poles α_1 and α_1', which can be explicitly evaluated. These poles yield the following contributions to T_{00}^{J}; where z is put on the energy shell:

$$\text{a)} \quad \oint d\cos\beta\, Q_J(\cos\beta) \frac{\Psi(\alpha_1)}{\left[\dfrac{d\lambda}{d(z-\omega_2'')}\right]_{\lambda=\alpha_1}} H_1(\cos\beta)^{\alpha_1+\alpha(z-\omega_1')}$$

$$\Longrightarrow \text{pole at } J = \alpha(\omega_1) + \alpha(\omega_2') \quad ,$$

$$\text{b)} \quad \oint d\cos\beta\, Q_J(\cos\beta) \frac{\Psi(\alpha_1')}{\left[\dfrac{d\lambda}{d(z-\omega_2'')}\right]_{\lambda=\alpha_1'}} H_1'(\cos\beta)^{\alpha_1'+\alpha(z-\omega_1')}$$

$$\Longrightarrow \text{pole at } J = \alpha(\omega_1') + \alpha(\omega_2') \quad ,$$

$$\text{c)} \quad \oint d\cos\beta\, Q_J(\cos\beta) \frac{\Psi(\alpha_1)}{\left[\dfrac{d\lambda}{d(z-\omega_2'')}\right]_{\lambda=\alpha_1}} H_2(\cos\beta)^{\alpha_1+\alpha(z-\omega_1)}$$

$$\Longrightarrow \text{pole at } J = \alpha(\omega_1) + \alpha(\omega_2) \quad ,$$

$$\text{d)} \quad \oint d\cos\beta\, Q_J(\cos\beta) \frac{\Psi(\alpha_1')}{\left[\dfrac{d\lambda}{d(z-\omega_2'')}\right]_{\lambda=\alpha_1'}} H_2'(\cos\beta)^{\alpha_1'+\alpha(z-\omega_1)}$$

$$\Longrightarrow \text{pole at } J = \alpha(\omega_1') + \alpha(\omega_2) \quad .$$

In considering the remaining integral along the contour C', we can interchange the order of integration in (III.22) and get

$$\int_{C'} \frac{d\lambda}{\left[\frac{d\lambda}{d(z-\omega_2'')}\right]} \Psi(\lambda) \left[\frac{H_1}{\lambda-\alpha_1} + \frac{H_1'}{\lambda-\alpha_1'}\right] \oint d\cos\beta\, Q_J(\cos\beta)(\cos\beta)^{\lambda+\alpha(\omega_2')}$$

$$+ \int \frac{d\lambda}{\left[\frac{d\lambda}{d(z-\omega_2'')}\right]} \Psi(\lambda) \left[\frac{H_2}{\lambda-\alpha_1} + \frac{H_2'}{\lambda-\alpha_1'}\right] \oint d\cos\beta\, Q_J(\cos\beta)(\cos\beta)^{\lambda+\alpha(\omega_2)} ,$$

and the contribution to T_{00}^{J} is of the form

$$T_{00}^{J} \sim \int_{C'} d\lambda \frac{\phi^{(1)}{}_{\lambda}}{\lambda + \alpha(\omega_2') - J} + \int_{C'} d\lambda \frac{\phi^{(2)}{}_{(\lambda)}}{\lambda + \alpha(\omega_2) - J} , \qquad \text{(III.23)}$$

so this gives two cuts in the J plane;

(i) a cut from $J = \alpha(z) + \alpha(\omega_2')$ to $J = \alpha(-\infty) + \alpha(\omega_2')$,

(ii) a cut from $J = \alpha(z) + \alpha(\omega_2)$ to $J = \alpha(-\infty) + \alpha(\omega_2)$.

In a completely analogous way we can analyze the contributions of the third iteration term:

$$\int \frac{p_1''^2 dp_1'' p_2''^2 dp_2''}{(\omega_1'' + \omega_2 - z)(\omega_1'' + \omega_2'' - z)(\omega_2'' + \omega_1' - z)} F_1(p_1', p_1'', p_1, \cos\beta, z-\omega_1', z-\omega_1'')$$

$$\times F_2(p_2', p_2'', p_2, \cos B, z-\omega_2'', z-\omega_2),$$

(III-24)

where

$$F_i(p_i',p_i'',p_i,\hat{p}_i\cdot p_i,z_1,z_2) = \int d\hat{p}_i'' T_i(p_i',p_i'',\hat{p}_i'\cdot\hat{p}_i'',z_1) \times T_i(p_i'',p_i,\hat{p}_i''\cdot\hat{p}_i\cdot z_2) \quad . \qquad \text{(III.25)}$$

This term gives, besides all the singularities which have been got before, the following cuts:

(i) cuts analogous to those already obtained, but in the variables ω_1,ω_1';

(ii) a cut from $J = \alpha(0) + \alpha(\omega_2)$ to $J = \alpha(\infty) + \alpha(\omega_2)$;

(iii) a cut from $J = -1 + \alpha(\infty)$ to $J = \alpha(E) + \alpha(0)$.

If we try to go to higher orders in the iterative expansions of the Fadeev equations, we can see that in a given diagram all the "blobs" connecting particles 1 and 3 can be collected in a certain function F_1 which exhibits a Regge behavior when $\cos\beta \to \infty$, and also the "blobs" connecting particles 2 and 3 in another function F_2 with the same property. For $\cos\beta \to \infty$, any arbitrary integrand shall behave as $(\cos\beta)^{\alpha_1+\alpha_2}$, and there will never be singularities in the right-half plane defined by

$$J > 2 \max_E [\mathrm{Re}\ \alpha\ (E)]$$

IV. CONCLUSIONS

We are now able to discuss the extension to complex J of the Fredholm solution $N_{MM'}^{J}(\omega,\omega',z)/D^J(z)$ of the Fadeeve equations. We

know that for a given value of z the scattering amplitude $T_{00}^{J}(\omega,\omega';z)$ has an infinite number of singularities which depend upon the subenergies. Let us take, for example, the poles at $J = \alpha(\omega_1) + \alpha(\omega_2')$. Although we have considered only a Newmann series expansion of the Fadeev equations, that type of singularity appeared in any term considered before and one intuitively expects those poles to survive after the series has been summed because they actually are interpolating singularities between bound states and resonances in our system. Then the question arises: Are they poles in $N_{00}^{J}(\omega_1',w_i'z)$? The answer is no, because for the particular values of ω_1 and ω_2' which give a bound state in each subsystem, we know that N_{00}^{J} is nonsingular, and that the singularity in T^J arises from a zero in $D^J(z)$; J being, of course, the physical value of the total angular momentum of the degenerate bound state of the three-body system. Therefore all those singularities must arise from zeros in $D^J(z)$ which for a given value of the total energy has therefore continuous lines of zeros, so that it must vanish identically.

We can conclude that this model provides a counter example in which the extension of the N^J/D^J solution of the Fadeev equations is not expected to exist. The results of this analysis also show that any attempt at reducing the three-body problem for complex J to an integral equation with completely continuous kernel must necessarily fail.

The fact that we did not find Regge poles in this example must not be taken as a proof that Regge poles do not exist in general in three-body systems, because this model is a very simplified version of an actual system, and it can be shown that in more realistic models unitarity yields Regge-type singularities in $T_{MM'}^{J}$. Nevertheless, as

real-life situations are usually more complicated than simple models, we cannot help feeling that they are not going to be the only singularities in the J plane, because the non-Regge singularities that we found depend upon so many subenergy parameters that it seems very difficult to cancel them in general and get only clean Regge singularities. If this is the case, then the Regge pole concept will not be very useful in analyzing the properties of three-body systems.

For the sake of completeness, let us note that the same analysis could be done if, in place of fixing the values of M and M', we were to fix the relative angular momentum of one of the "electrons" with the "nucleus." (That choice has been made by Newton[1] and Choudhury.[4]) Here again, and in a much more elementary way, one could get poles dependent upon the subenergies. The same conclusion, namely that no Fredholm equations can give this result, would stand.

Finally, let us say without proof that the present model gives only Regge poles when one analyzes the scattering of one "electron" on the "hydrogen atom." Obviously, the most important problem in this field by now is to find if this is also the case in more realistic situations.

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FOOTNOTES AND REFERENCES

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FIGURE CAPTION

Fig. 1 Singularities and paths of integration in the λ plane, for a typical Regge trajectory $\alpha(E)$.

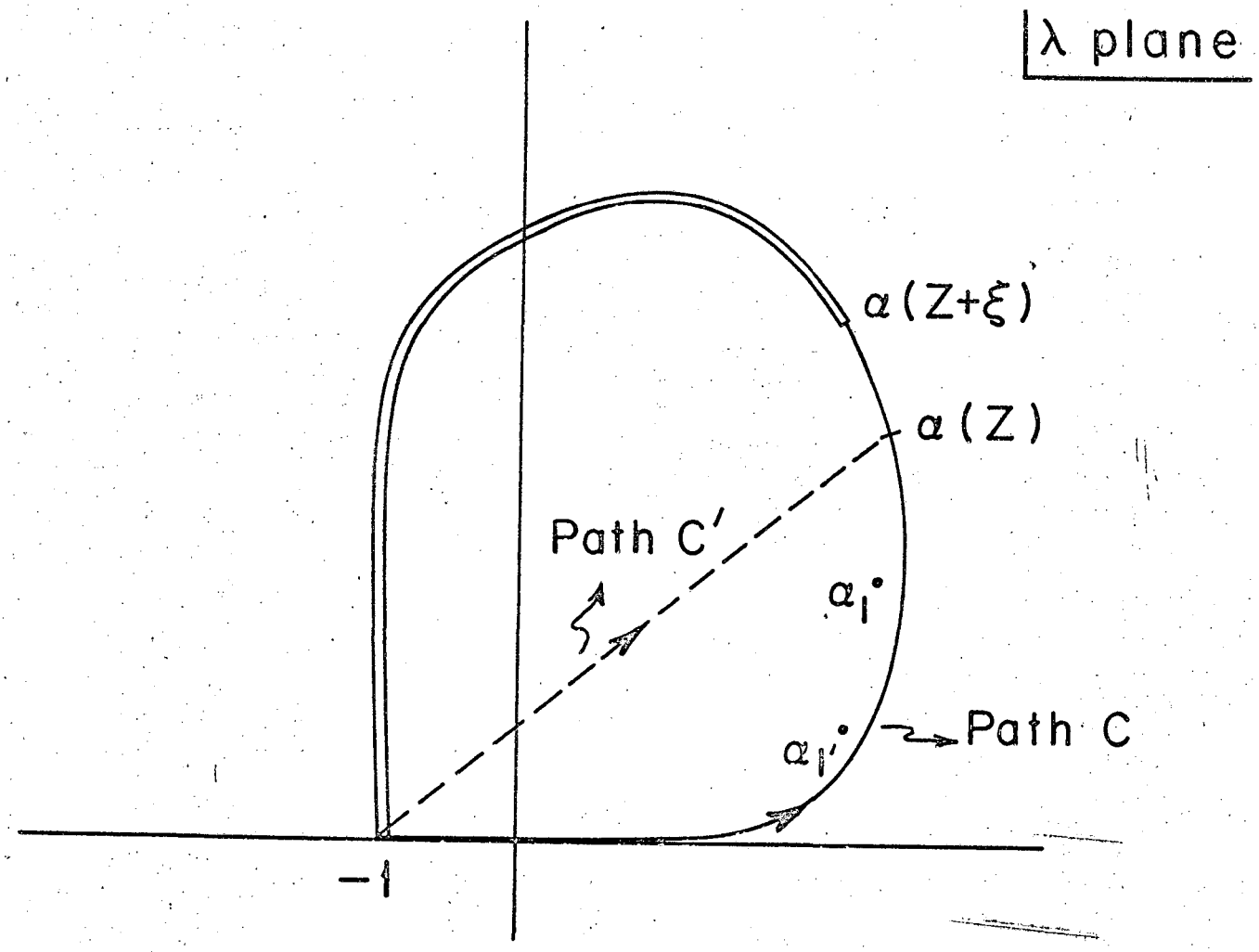

Fig. 1

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