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OPTICAL-MODEL AND MANDELSTAM REPRESENTATION

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ABSTRACT

It is shown that it is always possible to find explicitly an optical potential which fits a scattering amplitude satisfying the Mandelstam assumptions. The potential can be constructed by making a series of integral transformations on the momentum-transfer discontinuity of the amplitude. In the case in which the asymptotic behavior in energy of the amplitude is given by the Regge pole hypothesis, it is shown that, although the range of the potential (behavior $e^{-\mu_0 r}$) is constant, the size of the interaction region increases with energy owing to the nonuniformity in energy of the asymptotic behavior of the potential as a function of distance.

I. INTRODUCTION

It has often been stated, in a loose sense, that the predictions of the optical model do not fit with the Regge-pole asymptotic behavior of the scattering amplitude, except if one chooses a potential which decreases logarithmically with energy and has a range which increases logarithmically.[1,2,3] As the range of a potential that fits the analytic properties of the amplitude must be given by the smallest exchanged mass, it seems that one should reject either the optical potential or the Regge pole hypothesis.

It is the purpose of this paper to show that one can always find an optical potential that fits any amplitude satisfying the Mandelstam analyticity assumptions.[4] This is based on a very remarkable identity that relates the scattering amplitude to a certain function $\psi(y)$ in a way that is closely analogous to the relation between the scattering amplitude and the eikonal (Eq. 26). Moreover, the function $\psi(y)$ can be given a simple expression in terms of partial-wave amplitudes that allows to relate it to the momentum-transfer discontinuity of the total amplitude.

These relations are presented in Section II, where we study the relation between a scattering amplitude and a function $\phi(z)$ whose Taylor-series coefficients are the partial-wave amplitudes. The function $\psi(y)$ appears as the Borel transform of $\phi(z)$. We obtain an integral representation of the scattering amplitude in terms of $\psi(y)$, converging in a region that always contains the diffraction peak.

In Section III we recall the essential formulas of the optical model treated in the eikonal approximation. We also show that the potential can be written explicitly in terms of the eikonal.

In Section IV, we derive the optical potential that fits a given scattering amplitude, and we give its analyticity properties as well as its behavior near the origin.

In Section V we investigate the behavior of the potential for large values of r, particularly when the amplitude behaves asymptotically according to the Regge pole hypothesis. It is shown that the range of this potential, i.e., the coefficient of r in the exponential decrease for large values of r, is a constant. However, this asymptotic behavior is not uniform in energy, and we must introduce the notion of size of the potential, i.e., the value of r for which it is different from zero. It is found that this size increases with energy so that, finally, the apparent disagreement between the optical model and the Regge pole hypothesis is too narrow a choice of the potentials.

Incidentally, this work provides a rather general justification of the optical model at high energies.

II. INTEGRAL REPRESENTATIONS OF A SCATTERING AMPLITUDE

We are going to indicate two integral representations of a scattering amplitude $F(s, \cos\theta)$, which are suggested by the theory of functions of a complex variable. For our purposes the scattering amplitude is defined by its partial-wave expansion:

$$F(s, \cos\theta) = \sum_{\ell} (2\ell + 1)\, a_{\ell}(s)\, P_{\ell}(\cos\theta) . \qquad (1)$$

Along with $F(\cos\theta)$, it is convenient to introduce another function of another variable $\phi(z)$ which has the partial-wave amplitudes for Taylor coefficients:

$$\phi(z) = \sum_{\ell} (2\ell + 1) a_{\ell}\, z^{\ell} . \qquad (2)$$

It is easy to show that if the series (2) converges inside a circle of radius R, the series (1) converges inside an ellipse with foci at ± 1 and with semi-axes $\frac{1}{2}\left|R \pm \frac{1}{R}\right|$.

It is possible to relate these two functions by an integral transformation, using the Laplace representation of Legendre polynomials,

$$P_{\ell}(\cos\theta) = \frac{1}{\pi} \int_0^{\pi} [\cos\theta + i \sin\theta \cos\alpha]^{\ell}\, d\alpha , \qquad (3)$$

which gives, after being introduced into Eq. (1) and performing a change in the integration variable,[5]

$$F(\cos\theta) = \frac{1}{\pi} \int \frac{\phi(z)dz}{[2z \cos\theta - z^2 - 1]^{\frac{1}{2}}} , \qquad (4)$$

where the range of integration is between the two roots $\cos\theta \pm i \sin\theta$ of the denominator.

The inverse of Eq. (4) can be written as

$$\phi(z) = \int_{-1}^{+1} N(x,z)\, F(x)dx, \qquad (5)$$

where the kernel $N(x,z)$ is defined by its Legendre expansion,

$$N(x,z) = \sum_{\ell} (\ell + \tfrac{1}{2}) z^{\ell} P_{\ell}(x) . \tag{6}$$

It is easy to evaluate this series by derivation of the generating function for Legendre polynomials,

$$[1 - 2xz + z^2]^{-\frac{1}{2}} = \sum_{\ell} z^{\ell} P_{\ell}(x) , \tag{7}$$

to get

$$\phi(z) = \frac{1}{2} \int_{-1}^{+1} \frac{(1 - z^2) F(x) dx}{(1 - 2xz + z^2)^{\frac{1}{2}}} . \tag{8}$$

The main use of Eqs. (4) and (8) is to relate the holomorphy domains D of $F(x)$ and Δ of $\phi(z)$. Let us mention without proof that

$$D = \left\{ x \,|\, x = \tfrac{1}{2}(z + z^{-1}); \quad z \in \Delta \right\} , \tag{9}$$

$$\Delta = \left\{ z \,|\, z = x + (x^2 - 1)^{\frac{1}{2}}; \quad x \in D \right\} . \tag{10}$$

In Eq. (10), the sign of the square root is positive fox x larger than 1.

It has been shown by Emile Borel that, given a function $\phi(z)$ through its Taylor expansion (2), one can get an integral representation of $\phi(z)$ in terms of the entire function[6]

$$\psi(y) = \sum_{\ell} \frac{(2\ell + 1)}{\ell!} a_{\ell} y^{\ell} \tag{11}$$

as

$$\phi(z) = \int_0^{\infty} e^{-\eta} \psi(z\eta) d\eta . \tag{12}$$

Generally, the domain of convergence of the integral (12) is different from the circle of convergence of the series (11) and contains it.

Let us introduce Eq. (12) into Eq. (4). After a few elementary transformations, we get

$$F(\cos\theta) = \int_0^\infty e^{-y\cos\theta} J_0(y\sin\theta)\,\psi(y)dy\,, \qquad (13)$$

which is the main tool that will be used in this paper.[7] Let us note that, rather than by this somewhat lengthy sequence of integral transforms, Eq. (13) can be derived directly from Eq. (1) by using the Hobson formula,[8]

$$P_\ell(\cos\theta) = \frac{1}{\ell!}\int_0^\infty e^{-y\cos\theta} J_0(y\sin\theta)y^\ell\,dy, \qquad (14)$$

where J_0 is the conventional Bessel function.

We shall need in the following the domain of uniform convergence of the integral in Eq. (13). Since, according to Eq. (11) and the boundedness of the coefficients a_ℓ, $\psi(y)$ is an entire function, the convergence of Eq. (12) will be completely determined by the asymptotic behavior of $\psi(y)$ when y tends to $+\infty$. In order to get this behavior, we shall make the supplementary assumption that $F(s, \cos\theta)$ satisfies a dispersion relation in $\cos\theta$ such as given by the Mandelstam hypothesis,[4]

$$F(\cos\theta) = \frac{1}{\pi}\int_{x_0}^\infty \frac{A(x)dx}{x - \cos\theta}\,. \qquad (15)$$

As is well known, the partial-wave amplitude a_ℓ can be expressed in terms of the discontinuity $A(x)$ by[9]

$$a_\ell = \frac{1}{\pi}\int_{x_0}^{\infty} A(x)\, Q_\ell(x)dx \,. \tag{16}$$

It will be more convenient to introduce another function related to A(x),[10]

$$R(u) = \frac{1}{\pi}\int_{x_0}^{u+(u^2-1)^{\frac{1}{2}}} A(x)\, \frac{dx}{(1-2ux+x^2)^{\frac{1}{2}}}\,, \tag{17}$$

which has essentially the same properties, as a distribution, as A(x). The partial-wave amplitude can then be expressed as the Mellin transform of this function R(u):

$$a_\ell = \int_{\rho^{-1}}^{\infty} R(u)u^{-\ell-1}\, du\,, \tag{18}$$

where

$$\rho = x_0 - (x_0^{\,2} - 1)^{\frac{1}{2}}\,. \tag{19}$$

Introducing this representation into the series expansion (11), we get

$$\psi(y) = \int_0^{\rho} \frac{d\lambda}{\lambda}\, R\left(\frac{1}{\lambda}\right)(1+2\lambda y)e^{\lambda y}\,, \tag{20}$$

which is essentially a Laplace transform integral. This representation shows that $\psi(y)$ behaves like $e^{\rho y}$ when y tends to infinity with a positive real part. Incidentally, its asymptotic behavior when y tends to infinity with a negative real part is determined by the behavior of R(u) when u tends to infinity, i.e., by the Regge singularities of $a_\ell(s)$.[11]

If one takes these results into account as well as the known asymptotic behavior of the Bessel function, the domain of convergence

of the integral representation (13) is found to be given by

$$-\mathrm{Re}\, x + \left|\mathrm{Re}(x^2 - 1)^{\frac{1}{2}}\right| + \rho < 0 . \tag{21}$$

It is shown in Fig. 1. One sees that it contains a segment of the real axis between ρ and x_0, i.e.,

$$x_0 - (x_0^2 - 1)^{\frac{1}{2}} < x < x_0 . \tag{22}$$

When the energy is large, one has

$$x_0 \sim 1 + \frac{t_0}{2q^2}, \qquad \rho \sim 1 - \frac{(t_0)^{\frac{1}{2}}}{q}, \tag{23}$$

where t_0 is the nearest singularity of the scattering amplitude in terms of the momentum transfer

$$t = -2q^2(1 - \cos\theta) . \tag{24}$$

In that case, the domain of momentum transfer corresponding to Eq. (22) is given by

$$-2qt_0^{\frac{1}{2}} < t < t_0 , \tag{25}$$

and it embodies the region where the diffraction peak is observed. It will be convenient to introduce new variables into Eq. (13) by setting $y = qb$, so that,

$$F(s, \cos\theta) = q\int_0^\infty J_0(qb \sin\theta)e^{-qb\cos\theta}\,\psi(qb)db . \tag{26}$$

For finite values of t, taking into account the asymptotic behavior of $\psi(y)$, one can replace the exponential $e^{-qb\cos\theta}$ by e^{-qb}

with an error which tends to zero when energy tends to infinity, so that finally,

$$F(s, \cos\theta) = \left[q \int_0^\infty J_0(qb \sin\theta) e^{-qb} \psi(qb) db \right] \left[1 + O\left(\frac{t}{q \; t_0^{\frac{1}{2}}} \right) \right]. \tag{27}$$

III. THE OPTICAL MODEL

The optical model consists in representing a high-energy target by an imaginary, energy-dependent potential $V(s,r)$.[12] If $V(s,r)$ is a sufficiently regular function of r, the Schrödinger equation can be solved in the eikonal approximation so that the scattering amplitude appears in the form[13]

$$F(s, \cos\theta) = -iq \int_0^\infty J_0(q \sin\theta) \left[e^{i\chi(b,s)} - 1 \right] b \, db , \tag{28}$$

where the eikonal $\chi(b,s)$ is related to the potential by

$$\chi(b,s) = -\frac{m}{q} \int_0^\infty V\left[s, (b^2 + z^2)^{\frac{1}{2}} \right] dz . \tag{29}$$

It will be convenient to invert Eq. (29) in order to express the potential as a function of the eikonal. To do so, let us introduce the new variables

$$u = b^{-2}, \qquad v = (b^2 + z^2)^{-1}, \tag{30}$$

and the new functions

$$u^{-\frac{1}{2}}\chi(b) = \gamma(u) , \qquad -\frac{\pi\ m}{2q}\ v^{-3/2}V\left[(b^2 + z^2)^{\frac{1}{2}}\right] = \phi(v) , \tag{31}$$

so that Eq. (29) becomes an Abel equation,

$$\gamma(u) = \frac{1}{\Gamma(\frac{1}{2})}\int_0^u \phi(v)\frac{dv}{(u - v)^{\frac{1}{2}}} = \left(\frac{d}{du}\right)^{-\frac{1}{2}} \phi(u) . \tag{32}$$

In writing the last equality, we have used the very definition of a derivative of nonintegral order. The solution of Eq. (32) is given by

$$\phi(v) = \left(\frac{d}{dv}\right)^{\frac{1}{2}} \gamma(v) = \left(\frac{d}{dv}\right)^{-\frac{1}{2}} \frac{d\gamma(v)}{dv} = \frac{1}{\Gamma(\frac{1}{2})}\int_0^\infty \frac{d\gamma(u)}{(v - u)^{\frac{1}{2}}} . \tag{33}$$

It is therefore possible to derive the potential from the eikonal by the relation

$$V(s,r) = -\frac{2q}{\pi m r^2}\int_r^\infty \frac{b\ d[d\chi(b,s)]}{(b^2 - r^2)^{\frac{1}{2}}} . \tag{34}$$

IV. DERIVATION OF THE OPTICAL POTENTIAL FROM THE SCATTERING AMPLITUDE

We are now ready to find an optical potential that fits a given scattering amplitude at high energy. More exactly, we can find a potential which, treated by the eikonal approximation, gives the amplitude up to multiplicative corrections of order $1/q$ for finite values of the momentum transfer. This is afforded by a comparison between Eqs. (27) and (28), which gives

$$e^{i\chi(b,s)} = 1 + \frac{1}{b} e^{-bq} \psi(qb,s) . \qquad (35)$$

The function $\psi(qb)$ is given in terms of the partial-wave amplitudes by Eq. (11). If the amplitude satisfies a dispersion relation in momentum transfer as in Eq. (15), $\psi(qb)$ is given by the integral formulas (17) and (20) in terms of the discontinuity of the amplitude. Finally the potential $V(r,s)$ is given in terms of the eikonal by Eq. (34).

We shall now derive some properties of this potential.

It is well known that according to the Mandelstam hypothesis the discontinuity $A(s,t)$ is an analytic function of s with a cut from the threshold value of s to infinity, together with a left-hand cut. Correspondingly, $R(u,s)$ is an analytic function of s. Therefore, $\psi(qb,s)$ is an analytic function of s and an entire function of qb, so that it is analytic in s as a function of s and b. Then $\chi(b,s)$ is analytic in s, up to logarithmic branch points which could appear in extracting the logarithm of Eq. (35). Finally, $V(r,s)$ is also an analytic function of s in the neighborhood of the real axis with a cut going from the threshold value of s to $+\infty$, a left-hand cut, and eventually logarithmic branch points.

An analogous reasoning shows that $\chi(b,s)$ is an analytic function of b, except for logarithmic branch points, one of them being located at the origin. It is not easy to find the corresponding analytic properties of the potential. However, the behavior of $V(r,s)$ near the origin can be found explicitly.

It is shown in Appendix I that, for $r \ll q^{-1}$, $V(r,s)$ is of the order of

$$V(r,s) \approx c_1 \frac{e^{i\delta_0} \sin \delta_0}{2imr} \log(qr) + \frac{A(s)}{r} , \qquad (36)$$

where δ_0 is the S-wave complex phase shift and c_1 is an absolute constant.

V. ASYMPTOTIC BEHAVIOR OF THE OPTICAL POTENTIAL

In order to determine the asymptotic behavior of $V(r,s)$, we shall first consider the case where the discontinuity $A(s,t)$ of the potential is of the form

$$A(s,t) = g^2 \delta(t - t_0) \qquad \text{for} \quad t_0 = \mu_0^2 . \qquad (37)$$

It is shown in Appendix II that the corresponding potential behaves for large r like

$$V(r,s) \sim g^2 \frac{q}{W} \left[\frac{2\pi\mu_0}{r} \right]^{\frac{1}{2}} e^{-\mu_0 r} . \qquad (38)$$

As all the operations performed in Appendix II to get Eq. (38) are linear, the asymptotic behavior of $V(r,s)$ in the general case is given by

$$V(r,s) \sim \int_{\mu_0^2}^{\infty} A(s,\mu^2) \frac{q}{W} \left[\frac{2\pi\mu}{r} \right]^{\frac{1}{2}} e^{-\mu r} \, d\mu^2 , \qquad (39)$$

i.e., $V(r,s)$ will decrease asymptotically like $e^{-\mu_0 r}$.

It is interesting to see what Eq. (39) becomes when s becomes large and the corresponding asymptotic behavior of the amplitude is given by Regge poles. In that case[14]

$$A(s,\mu^2) \sim \gamma(\mu^2)\left(\frac{s}{s_0}\right)^{\alpha(\mu^2)} . \tag{40}$$

Since $\mathrm{Re}\,\alpha(\mu^2)$ is an increasing function of μ^2 for small values of μ^2, it is clear, by introducing Eq. (40) into Eq. (39), that the asymptotic behavior

$$V(r,s) \sim \int_{\mu_0^2}^{\infty} \left(\frac{s}{s_0}\right)^{\alpha(\mu^2)} \left[\frac{2\pi\mu}{r}\right]^{\frac{1}{2}} \gamma(\mu^2)\, e^{-\mu r}\, d\mu^2 \tag{41}$$

will effectively be given by $e^{-\mu_0 r}$ for r large enough, but that, when s increases, it will be necessary to go to higher and higher values of r to see this behavior. More precisely,

$$V(r,s) \sim O(e^{-\mu_0 r}) \quad \text{for} \quad r \gg \mu_0^{-1}\, \alpha'(\mu_0^2) \log\left(\frac{s}{s_0}\right) . \tag{42}$$

We propose to refine the notions used for ordinary potentials in order to encompass that case: given a potential $V(r)$ which behaves like $e^{-\mu_0 r}$ when r tends to infinity, we shall call $r_0 = \mu_0^{-1}$ the range of $V(r)$. On the other hand, if $V(r)$ becomes near to zero only for $r \sim r_1$, we shall call r_1 the <u>size</u> of the potential. In simple cases, r_0 and r_1 are of the same order and one does not distinguish between them, or r_0 does not exist as in the case of a square-well

potential. One sees from Eq. (42) that the optical potential that fits a scattering amplitude with a Regge pole behavior has a fixed range μ_0^{-1} but a size that increases logarithmically with energy.

The fact that $\alpha(\mu^2)$ is complex means that there will be strong cancellations in Eq. (41). Presumably the same kind of effects will take place for any value of r, and the potential will decrease with energy. We have not been able to determine if such is the case or if the potential itself strongly oscillates.

Finally, let us mention that it is possible that the potential we have determined behaves so wildly that the eikonal approximation cannot be applied to it. In that case, it would be possible to define other regularized potentials $V_\epsilon(r,s)$ defined, for instance, as

$$V_\epsilon(r,s) = \int \rho_\epsilon(\underset{\sim}{r} - \underset{\sim}{r}_0)\, V(\underset{\sim}{r}_0,s)\, d\underset{\sim}{r}_0 , \qquad (43)$$

where $\rho_\epsilon(\underset{\sim}{r})$ is an infinitely derivable function of support ϵ. The eikonal approximation could be applied to some $V_\epsilon(r,s)$ with a good enough approximation. Let us note, however, that the "mesh size" ϵ would have to vary with s so that the analytic properties of $V(r,s)$ as a function of s are not necessarily satisfied by the practical potentials $V_\epsilon(r,s)$. Let us note also that the singularity $\log r/r$ at the origin will be washed out by this regularization.

I want to thank V. Alessandrini for a useful discussion.

APPENDIX I

When y tends to zero,

$$\psi(y) \sim \frac{W}{q} e^{i\delta_0} \sin \delta_0 , \tag{I.1}$$

so that the eikonal behaves as

$$\psi(b) \sim \log b \quad \text{as} \quad b \to 0 . \tag{I.2}$$

Accordingly, the function $\gamma(u)$ of Eq. (31) behaves as

$$\gamma(u) \sim u^{-\frac{1}{2}} \log u \quad \text{as} \quad u \to \infty . \tag{I.3}$$

Its Fourier transform behaves like[15]

$$[F\gamma]_{(\sigma)} = c_1(\sigma + i0)^{-\frac{1}{2}} + c_2(\sigma + i0)^{-\frac{1}{2}} \log(\sigma + i0) \tag{I.4}$$

for large σ. Therefore, the Fourier transform of the function $\phi(v)$ of Eq. (31) behaves as

$$[F\phi]_{(\sigma)} \sim c'_1 + c'_2 \log(\sigma + i0) . \tag{I.5}$$

Therefore $\phi(v)$ behaves as

$$\phi(v) \sim c''_2 v^{-1} \log v \quad \text{as} \quad v \to \infty , \tag{I.6}$$

and for the potential

$$V = v^{3/2} \phi(v) \sim \frac{\log r}{r} \quad \text{as } r \to 0 . \tag{I.7}$$

APPENDIX II

Let us determine the behavior for large f of a potential which fits an amplitude with discontinuity

$$A(x) = g^2 \delta(x - x_0), \qquad x_0 = 1 + t_0/2q^2 . \tag{II.1}$$

One has, according to Eq. (17),

$$R(u) = \frac{g^2}{\pi} (1 - 2ux_0 + u^2)^{-\frac{1}{2}}, \tag{II.2}$$

so that

$$\psi(y) = (1 + 2y \frac{d}{dy}) \phi(y), \tag{II.3}$$

with

$$\phi(y) = \frac{g^2}{\pi} \int_0^\rho (\lambda^2 - 2\lambda x_0 + 1)^{-\frac{1}{2}} e^{\lambda y} d\lambda , \tag{II.4}$$

where we write $\rho = x_0 - (x_0^2 - 1)^{\frac{1}{2}} = x_0 - a$. For large value of y, one can replace Eq. (II.4) by

$$\phi(y) \simeq \frac{g^2}{\pi} \int_{-\infty}^\rho (\lambda^2 - 2\lambda x_0 + 1)^{-\frac{1}{2}} e^{\lambda y} d\lambda = e^{\alpha y + ay} K_0(ay). \tag{II.5}$$

Therefore, according to Eq. (35),

$$\chi(b) \simeq \frac{2g^2}{Wb} \left[\frac{\pi b q^2}{2\sqrt{t_0}}\right]^{\frac{1}{2}} e^{-\sqrt{t_0}\, b} \quad \text{as} \quad b \to \infty . \tag{II.6}$$

Putting Eq. (II.6) into Eq. (34), one gets immediately Eq. (39).

FOOTNOTES AND REFERENCES

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FIGURE CAPTION

Fig. 1. The domain of absolute convergence of the integral in Eq. (13).

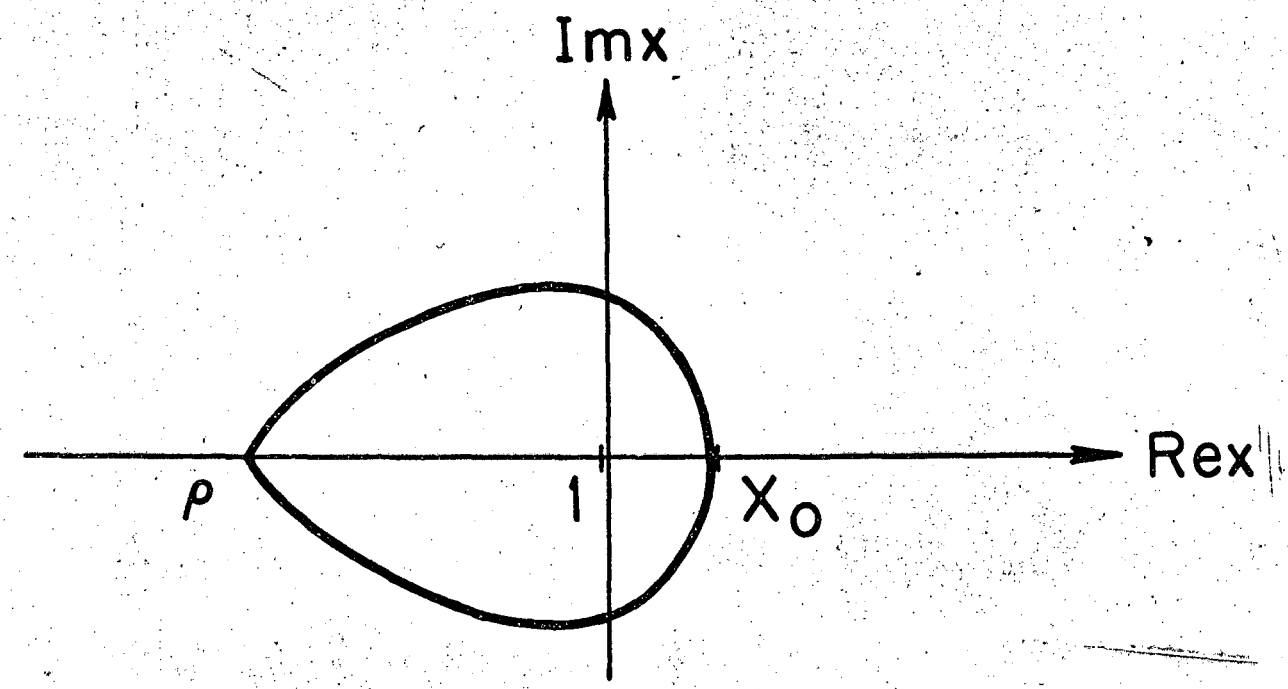

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Fig. 1.

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