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ABSTRACT

Velocity moments of the Vlasov equation are expanded systematically in m/e and equations obtained giving the fluid velocity transverse to $\underset{\sim}{B}$ through second order. These equations include all "finite gyro-radius" effects. No a priori form is imposed on the velocity distribution. The transverse electric field $E_\perp$ is allowed to be large. For straight field lines the equations for purely transverse motion are closed. For weak instabilities ($E_\perp$ small) closed equations are obtained only for low plasma pressure.

I. INTRODUCTION

In spite of the interest in finite gyro-radius corrections to the hydromagnetic equations, no complete derivation of these effects has been given for a Vlasov plasma. In particular, the corrections to the pressure equations of Chew, Goldberger, and Low (CGL)[1] have not been discussed in general. Previous work has been either incomplete[2,3] or complicated by the inclusion of collisions and restricted to a local Maxwellian velocity distribution in lowest order (e.g., references 4 and and 5). Concurrent work by Rosenbluth and Simon[6] discusses the complete set of moment equations describing weak instabilities of a low-pressure plasma in a uniform magnetic field. The present work includes an alternate derivation of their basic equations.

We outline here a systematic expansion of the moment equations for a Vlasov plasma in a strong magnetic field. This expansion is the same as the usual B^{-1} expansion of the Vlasov equation, and is equivalent to treating the particle motion as adiabatic. No a priori assumptions are made about form of the velocity distribution. The electric field transverse to $\underset{\sim}{B}$ is allowed to be large so that the "$\underset{\sim}{E} \times \underset{\sim}{B}$" velocity may be of the same order as the thermal velocity.

In lowest order the resulting equations are essentially those of CGL.[1] These equations describe the transverse motion of the plasma through first order and the parallel motion in zero order.

Through the next order the equations resemble the hydrodynamics of a nonideal fluid. Previous calculations of certain off-diagonal components of the pressure tensor $\underset{\sim}{P}$ [2,3] are found to be incorrect if $\underset{\sim}{P}$ is anisotropic in zero order. The CGL pressure equations are modified

by the appearance of transverse heat flows and other effects. The transverse heat flows involve the fourth moments of the zero-order velocity distribution $f^{(0)}$. In the absence of collisions $f^{(0)}$ is not necessarily Maxwellian, and at the initial time the fourth moments must be specified independently of the density and kinetic temperatures. Their time dependence is given by relations similar to the CGL pressure equations.

This system of equations is not closed because quantities such as the parallel heat flows remain undetermined. For generality these undetermined quantities are retained. In special cases, such as a plasma independent of position along a magnetic field of constant direction, the moment equations for purely transverse motion together with Maxwell's equations form a closed set, provided that the frequency seen by a moving fluid element is not too low. For weak instabilities, however, the moment equations close only if the plasma pressure is low or other special conditions are satisfied. In more general cases the equations may form a useful guide to the processes that may be expected, even if they are not closed.

The method followed here consists of expanding the moment equations directly.[2,7] This is algebraically simpler than finding these equations from an expansion of f, as was done in reference 3, and gives results in a form convenient for iteration to higher order.

The moment equations and expansion procedure are discussed in Section II, and the finite gyro-radius corrections presented in Section III. In Section IV these results are modified to apply to weak instabilities. The relation of the equations obtained to those used by Roberts and Taylor[8] to discuss the stability of a plasma in a gravitational field is indicated, and the Rosenbluth-Simon equations[6] are derived.

II. THE MOMENT EQUATIONS

The first few moments of the Vlasov equation may be written

$$\partial\rho/\partial t + \nabla\cdot\rho\mathbf{u} = 0 \quad , \tag{1}$$

$$(d\mathbf{u}/dt)_{\parallel} + \rho^{-1}(\nabla\cdot\mathbf{P})_{\parallel} - (e/m)E_{\parallel} = 0 \quad , \tag{2}$$

$$\mathbf{u}_{\perp} - \mathbf{u}_E = \Omega^{-1}\mathbf{e}_1 \times [d\mathbf{u}/dt + \rho^{-1}\nabla\cdot\mathbf{P}] \quad , \tag{3}$$

$$\mathbf{G} \equiv d\mathbf{P}/dt + \mathbf{P}\nabla\cdot\mathbf{u} + \nabla\cdot\mathbf{Q} + [\mathbf{P}\cdot\nabla\mathbf{u}]^s = \Omega[\mathbf{P}\times\mathbf{e}_1]^s \quad , \tag{4}$$

$$d\mathbf{Q}/dt + \mathbf{Q}\nabla\cdot\mathbf{u} + \nabla\cdot\mathbf{R} + [\mathbf{Q}\cdot\nabla\mathbf{u}]^s - [\rho^{-1}\mathbf{P}\nabla\cdot\mathbf{P}]^s = \Omega[\mathbf{Q}\times\mathbf{e}_1]^s \quad , \tag{5}$$

$$d\mathbf{R}/dt + \mathbf{R}\nabla\cdot\mathbf{u} + \nabla\cdot\mathbf{S} + [\mathbf{R}\cdot\nabla\mathbf{u}]^s - [\rho^{-1}\mathbf{Q}\nabla\cdot\mathbf{P}]^s = \Omega[\mathbf{R}\times\mathbf{e}_1]^s \quad , \tag{6}$$

where ρ is the density $mn = m\langle f\rangle$, $\mathbf{u}$ the flow velocity $n^{-1}\langle\mathbf{v}f\rangle$, $\mathbf{P}$ the pressure tensor $m\langle\boldsymbol{\nu}\boldsymbol{\nu}f\rangle$ with $\boldsymbol{\nu} = \mathbf{v} - \mathbf{u}$, $\mathbf{Q}$ the heat flux tensor $m\langle\boldsymbol{\nu}\boldsymbol{\nu}\boldsymbol{\nu}f\rangle$, $\mathbf{R} = m\langle\boldsymbol{\nu}\boldsymbol{\nu}\boldsymbol{\nu}\boldsymbol{\nu}f\rangle$, $\mathbf{S} = m\langle\boldsymbol{\nu}\boldsymbol{\nu}\boldsymbol{\nu}\boldsymbol{\nu}\boldsymbol{\nu}f\rangle$, $d/dt = \partial/\partial t + \mathbf{u}\cdot\nabla$, $\mathbf{e}_1 = B^{-1}\mathbf{B}$, $\Omega = eB/mc$, e and m are the particle charge and mass, and $\mathbf{u}_E = cB^{-2}\mathbf{E}\times\mathbf{B}$. Projections perpendicular and parallel to $\mathbf{B}$ are indicated by the subscripts $\perp$ and $\parallel$. The superscript s indicates that the quantity is to be symmetrized by adding to it all cyclic permutations of its vector factors (or tensor indices). Thus

$$[\mathbf{P}\times\mathbf{e}_1]^s = \mathbf{P}\times\mathbf{e}_1 - \mathbf{e}_1\times\mathbf{P} \quad ,$$

and

$$\{[\mathbf{Q}\cdot\nabla\mathbf{u}]^s\}_{ijk} = Q_{ijn}\partial u_k/\partial x_n + Q_{kin}\partial u_j/\partial x_n + Q_{jkn}\partial u_i/\partial x_n \quad .$$

Gravitational forces are included by letting $\mathbf{E}$ be the equivalent electric field, equal to the actual field plus m/e times the gravitational acceleration.

We consider these equations in the large-B (adiabatic) limit corresponding to assuming the quantity m/e (i.e., Ω^{-1}) to be small. To make the ordering of small quantities clear, Eqs. (1) through (6) must be written in dimensionless form. This may be accomplished by expressing $\underset{\sim}{r}$, t, $\underset{\sim}{B}$, $\underset{\sim}{E}$, and n in the units L, τ, B_0, E_0, and n_0 characteristic of the fluid motion and $\underset{\sim}{u}$ in terms of a representative thermal velocity v_{th}. If the dimensionless quantities are indicated by primes and $\underset{\sim}{P}$ written $n_0 m v_{th}^2 \underset{\sim}{P}'$, Eq. (3) becomes

$$\underset{\sim}{u}'_{\perp} - \frac{cE_0}{v_{th}B_0}\underset{\sim}{u}'_E = \varepsilon \frac{\underset{\sim}{e}_1}{B'} \times \left[\frac{L}{v_{th}\tau}\frac{\partial \underset{\sim}{u}'}{\partial t'} + \underset{\sim}{u}'\cdot\nabla'\underset{\sim}{u}' + \frac{\nabla'\cdot\underset{\sim}{P}'}{\rho'}\right] . \qquad (7)$$

The adiabatic expansion parameter $\varepsilon = mcv_{th}(eB_0L)^{-1} << 1$ enters this equation in the same way as Ω^{-1} enters Eq. (3). Equation (7) involves two other parameters, $cE_0(v_{th}B_0)^{-1}$ and $L(\tau v_{th})^{-1}$, which we assume independent of ε in this and the following section. The modifications necessary when both these parameters are of order ε are discussed in Section IV.

The expansions of $\underset{\sim}{E}$ and $\underset{\sim}{B}$ are found from Maxwell's equations in dimensionless form. These equations involve the parameters $\beta = n_0 m v_{th}^2 B_0^{-2}$ and $v_{th}^2 c^{-2}$ in addition to those appearing in Eq. (7). The appropriate expansions at $\underset{\sim}{E}$ and $\underset{\sim}{B}$ depend on the relative order of these parameters.

We now expand in ε all quantities appearing in Eqs. (1) through (6), writing them in the form $\rho = \rho^{(0)} + \rho^{(1)} + \ldots$, etc., and regarding m/e and Ω^{-1} to be proportional to ε. To simplify notation, the expansion of $\underset{\sim}{E}$ and $\underset{\sim}{B}$ will not be carried out explicitly. The appearance of $(e/m)E_{\parallel}$ in Eq. (2) indicates that $E_{\parallel}$ must be of first order, since $(e/m)E_{\parallel}$ is then of zero order.

It will be noticed that Eq. (3) expresses $\mathbf{u}_\perp^{(n)}$ in terms of $\mathbf{E}$, $\mathbf{B}$, and quantities of lower order, but that (1) and (2) do not give $\rho^{(n)}$ and $u_{\parallel}^{(n)}$ directly in terms of lower-order quantities. This is a general feature of the moment equations: moments of the form $\langle \alpha(\phi) f \rangle$, where ϕ is the azimuthal angle of $\mathbf{v}$ about the direction $\mathbf{e}_1$ and $\alpha(\phi)$ is a function of $\mathbf{v}$ which averages to zero over ϕ, are easily expressed directly in terms of moments of lower order. However, moments of the form $\langle \beta(v_\perp^2, v_{\parallel}) f \rangle$ are determined by differential equations which do not involve Ω^{-1}.

It is convenient to express $\mathbf{P}$ and $\mathbf{Q}$ in terms of moments of the above forms. We introduce unit vectors $\mathbf{e}_2$ and $\mathbf{e}_3$ such that $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ form a right-handed orthogonal system. An expansion of the $\mathbf{e}_i$, which we do not carry out explicitly, is implied by the expansion of $\mathbf{B}$. The quantities $P_\delta \equiv (1/2)(P_{22} - P_{33})$, P_{23}, P_{12}, and P_{13} are of the form $\langle \alpha(\phi) f \rangle$, and appear on the right-hand side of Eq. (4). Inverting this equation gives the relations

$$P_\delta = -(4\Omega)^{-1}\mathbf{G}:\mathbf{I}_\gamma \quad , \quad P_{23} = (4\Omega)^{-1}\mathbf{G}:\mathbf{I}_\delta \quad ,$$

and

$$(\mathbf{e}_1 \cdot \mathbf{P})_\perp = \Omega^{-1}\mathbf{e}_1 \times (\mathbf{e}_1 \cdot \mathbf{G}) \quad , \tag{8}$$

where

$$\mathbf{I}_\delta = \mathbf{e}_2\mathbf{e}_2 - \mathbf{e}_3\mathbf{e}_3 \quad \text{and} \quad \mathbf{I}_\gamma = \mathbf{e}_2\mathbf{e}_3 + \mathbf{e}_3\mathbf{e}_2 \quad .$$

These equations express P_δ, P_{23}, P_{12}, and P_{13} in any order in terms of quantities of lower order. In particular, all these components vanish in zero order so that $\mathbf{P}^{(0)}$ is of the usual anisotropic diagonal form.

The remaining components of $\underset{\sim}{P}$--$p_\perp = (1/2)(P_{22} + P_{33})$ and $p_\parallel = P_{11}$-- do not appear on the right-hand side of Eq. (4). They are determined by the energy equations obtained from the contractions of Eq. (4) with $\underset{\sim}{I}_\perp = \underset{\sim}{e}_2\underset{\sim}{e}_2 + \underset{\sim}{e}_3\underset{\sim}{e}_3$ and $\underset{\sim}{e}_1\underset{\sim}{e}_1$. The results may be written

$$dp_\perp/dt + p_\perp \nabla\cdot\underset{\sim}{u} + \nabla\cdot\underset{\sim}{q}^\perp + (\underset{\sim}{P}\cdot\nabla\underset{\sim}{u}):\underset{\sim}{I}_\perp + \underset{\sim}{P}:\underset{\sim}{e}_1 d\underset{\sim}{e}_1/dt + \underset{\sim}{e}_1\cdot\underset{\sim}{Q}:\nabla\underset{\sim}{e}_1 = 0 \qquad (9)$$

and

$$\frac{1}{2}\, dp_\parallel/dt + \frac{1}{2}\, p_\parallel \nabla\cdot\underset{\sim}{u} + \nabla\cdot\underset{\sim}{q}^\parallel + (\underset{\sim}{P}\cdot\nabla\underset{\sim}{u}):\underset{\sim}{e}_1\underset{\sim}{e}_1 - \underset{\sim}{P}:\underset{\sim}{e}_1 d\underset{\sim}{e}_1/dt - \underset{\sim}{e}_1\cdot\underset{\sim}{Q}:\nabla\underset{\sim}{e}_1 = 0 \quad , \qquad (10)$$

where $\underset{\sim}{q}^\perp$ and $\underset{\sim}{q}^\parallel$ are the fluxes of transverse and parallel thermal energy, $(1/2)\underset{\sim}{Q}:\underset{\sim}{I}_\perp$ and $(1/2)\underset{\sim}{Q}:\underset{\sim}{e}_1\underset{\sim}{e}_1$.

The usual energy equation of kinetic theory is obtained by adding Eqs. (9) and (10), which gives one-half the trace of Eq. (4). The first four terms in Eq. (9) and in Eq. (10) are similar in form to this energy equation. The remaining terms occur with opposite signs in Eqs. (9) and (10), and represent an exchange of energy between the transverse and parallel thermal motion. The first of these terms arises from the rotation of the magnetic field direction seen by a moving fluid element; the origin of the second is more difficult to visualize.

By use of the diagonal form of $\underset{\sim}{P}^{(0)}$ and the form of $\underset{\sim}{Q}^{(0)}$ derived in Section III, the zero-order terms of Eqs. (9) and (10) reduce to the CGL pressure equations (see Eqs. 21 and 22), which in the absence of heat flow can be integrated to give the double adiabatic conditions. The other equations of CGL theory are easily obtained from Eqs. (1) and (2) in zero order, (3) through first order, and Maxwell's equations. They give $\underset{\sim}{u}_\perp$ through first order and the other quantities in zero order.

III. FINITE GYRO-RADIUS CORRECTIONS

Extension of these equations to give the transverse velocity through second order is straightforward. Equation (3) expresses $\underset{\sim}{u}_\perp^{(2)}$ in terms of $\underset{\sim}{u}_\perp^{(1)}$, $\underset{\sim}{P}^{(1)}$, and $\rho^{(1)}$; $\underset{\sim}{P}^{(1)}$ is obtained from Eqs. (8) through (10). From (8), using the diagonal form of $\underset{\sim}{P}^{(0)}$, the form of $\underset{\sim}{Q}^{(0)}$ obtained below, and Maxwell's induction equation, we obtain

$$P_\delta^{(1)} = -(2\Omega)^{-1}\underset{\sim}{I}_\gamma : [p_\perp \nabla \underset{\sim}{u} + q_{\parallel}^{\perp} \nabla \underset{\sim}{e}_1]^{(0)} \quad , \tag{11}$$

$$P_{23}^{(1)} = (2\Omega)^{-1}\underset{\sim}{I}_\delta : [p_\perp \nabla \underset{\sim}{u} + q_{\parallel}^{\perp} \nabla \underset{\sim}{e}_1]^{(0)} \quad , \tag{12}$$

$$P_{12}^{(1)}\underset{\sim}{e}_2 + P_{13}^{(1)}\underset{\sim}{e}_3 \equiv [\underset{\sim}{e}_1 \cdot \underset{\sim}{P}^{(1)}]_\perp = \Omega^{-1}\underset{\sim}{e}_1 \times [(2p_\parallel - p_\perp)\underset{\sim}{e}_1 \cdot \nabla \underset{\sim}{u} + p_\perp (\nabla \underset{\sim}{u}) \cdot \underset{\sim}{e}_1$$

$$+ (2q_{\parallel}^{\parallel} - q_{\parallel}^{\perp})\underset{\sim}{e}_1 \cdot \nabla \underset{\sim}{e}_1 + \nabla_\perp q_{\parallel}^{\perp}]^{(0)} \quad . \tag{13}$$

The parts of these expressions proportional to $p_{\perp,\parallel}^{(0)}$ are the collisionless viscosity terms. They are well known in the isotropic case; they have been given for an anisotropic $\underset{\sim}{P}^{(0)}$ by Kaufman[2] and by Thompson.[3] Disagreement exists between Eq. (13) and the work of these authors. Thompson's results for P_{12} and P_{13} are incorrect because of an algebraic error. Kaufman's result for these components is incomplete because he assumes a constant $\underset{\sim}{B}$. This is inconsistent with a nonzero $(\underset{\sim}{e}_1 \cdot \underset{\sim}{u}^{(0)})_\perp$ and Maxwell's induction equation.

The remaining terms of Eqs. (11) through (13) represent stresses due to the zero-order heat flows. They show that $\underset{\sim}{P}^{(1)}$ need not be diagonal even if $\underset{\sim}{u}^{(0)}$ vanishes.

We consider now the determination of $\underset{\sim}{Q}$, which is needed through first order in Eq. (9) and (10). This tensor has ten independent

components which are conveniently represented in terms of $\underset{\sim}{q}^{\perp}$, $\underset{\sim}{q}^{\parallel}$, $\underset{\sim}{q}^{\delta} \equiv (1/2)\underset{\sim}{Q}:\underset{\sim}{I}_{\delta}$, and Q_{123}. All these quantities except $q_{\parallel}^{\perp}$ and $q_{\parallel}^{\parallel}$ are of the type $\langle \alpha(\phi) f \rangle$ and appear on the right-hand side of Eq. (5). Inverting this equation gives expressions for these quantities of the same form as the relations (8). These quantities therefore vanish in zero order and the only nonzero components of $\underset{\sim}{Q}^{(0)}$ are proportional to $q_{\parallel}^{\perp(0)}$ and $q_{\parallel}^{\parallel(0)}$. This fact has already been used. Inversion of Eq. (5) in first order gives

$$\underset{\sim}{q}_{\perp}^{\parallel(1)} = (2\Omega)^{-1}\underset{\sim}{e}_1 \times [\nabla R_2 - \rho^{-1}p_{\parallel}\nabla\cdot\underset{\sim}{P} + (R_1 - 3R_2)\underset{\sim}{e}_1\cdot\nabla\underset{\sim}{e}_1$$

$$+ 2(2q_{\parallel}^{\parallel} - q_{\parallel}^{\perp})\underset{\sim}{e}_1\cdot\nabla\underset{\sim}{u} + 2q_{\parallel}^{\perp}(\nabla\underset{\sim}{u})\cdot\underset{\sim}{e}_1]^{(0)} \quad , \tag{14}$$

$$\underset{\sim}{q}_{\perp}^{\perp(1)} = (2\Omega)^{-1}\underset{\sim}{e}_1 \times [\nabla R_3 - 4\rho^{-1}p_{\perp}\nabla\cdot\underset{\sim}{P} + 4\underset{\sim}{e}_1\cdot\underset{\sim}{T}]^{(0)} \quad , \tag{15}$$

$$q_{\parallel}^{\delta(1)} = -(2\Omega)^{-1}\underset{\sim}{I}_{\gamma}:\underset{\sim}{T}^{(0)} \quad , \tag{16}$$

and

$$Q_{123}^{(1)} = (2\Omega)^{-1}\underset{\sim}{I}_{\delta}:\underset{\sim}{T}^{(0)} \quad ,$$

where

$$\underset{\sim}{T} = \left(R_2 - \frac{1}{4}R_3\right)\nabla\underset{\sim}{e}_1 + q_{\parallel}^{\perp}\nabla\underset{\sim}{u} \quad ,$$

$$R_1 = m\langle v_{\parallel}^4\rangle \quad , \quad R_2 = \frac{1}{2}m\langle v_{\parallel}^2 v_{\perp}^2\rangle \quad , \quad \text{and} \quad R_3 = \frac{1}{2}m\langle v_{\perp}^4\rangle \quad .$$

The transverse components of $\underset{\sim}{q}^{\delta}$ are not required. The three scalars R_i characterize $\underset{\sim}{R}^{(0)}$, since any component of $\underset{\sim}{R}$ not proportional to one of these appears on the right-hand side of Eq. (6) and therefore vanishes in zero order.

If the terms in $q_{\parallel}^{\parallel}$ and $q_{\parallel}^{\perp}$ are neglected, the transverse heat flows given above agree with those given by Thompson,[3] who assumed that the $q_{\parallel}^{(0)}$'s vanish. Thompson's fluxes appear to differ from (14) and (15) because they are referred to the zero-order velocity $\underset{\sim}{u}_E + u_{\parallel}^{(0)}\underset{\sim}{e}_1$, whereas ours are referred to $\underset{\sim}{u}$. The corresponding difference in $\underset{\sim}{P}$ is of order $[\underset{\sim}{u} - \underset{\sim}{u}^{(0)}]^2$, hence does not appear in Eqs. (11) through (13).

If the distribution of $v_{\parallel}$ (the integral of f over $\underset{\sim}{v}_{\perp}$) is Gaussian, $R_1' \equiv \rho R_1/3p_{\parallel}^2 = 1$; if the distribution of $v_{\perp}$ is Gaussian, $R_3' \equiv \rho R_3/4\,p_{\perp}^2 = 1$. The quantity $R_2' \equiv \rho R_2/p_{\parallel}p_{\perp} = 1$ if $f = f_1(v_{\parallel})f_2(v_{\perp})$. For an isotropic Maxwellian $f^{(0)}$, $R_1'^{(0)} = R_2'^{(0)} = R_3'(0) = 1$, $q_{\parallel}^{\delta(1)}$ and $Q_{123}^{(1)}$ vanish, and Eqs. (14) and (15) add to give the well-known collisionless transverse heat flow[4]

$$\underset{\sim}{q}_{\perp} \equiv \underset{\sim}{q}_{\perp}^{\perp} + \underset{\sim}{q}_{\perp}^{\parallel} = (5/2)(\rho m\Omega)^{-1}p\underset{\sim}{e}_1 \times \nabla(p/n) \quad . \tag{17}$$

In general, $f^{(0)}$ need not be Maxwellian, however, and independent equations are required for the $R_i^{(0)}$. These equations are the contractions of Eq. (6) in zero order with $\underset{\sim}{e}_1\underset{\sim}{e}_1\underset{\sim}{e}_1\underset{\sim}{e}_1$, $\underset{\sim}{e}_1\underset{\sim}{e}_1\underset{\sim}{I}_{\perp}$, and $\underset{\sim}{I}_{\perp}\underset{\sim}{I}_{\perp}$. The results are

$$dR_1/dt + R_1(5\nabla\cdot\underset{\sim}{u} - 4\nabla_{\perp}\cdot\underset{\sim}{u}) = 8q_{\parallel}^{\parallel}\rho^{-1}(\nabla\cdot\underset{\sim}{P})_{\parallel} + 4S_2\nabla\cdot\underset{\sim}{e}_1 - \nabla\cdot(S_1\underset{\sim}{e}_1) \quad , \tag{18}$$

$$dR_2/dt + R_2(3\nabla\cdot\underset{\sim}{u} - \nabla_{\perp}\cdot\underset{\sim}{u}) = 2q_{\parallel}^{\perp}\rho^{-1}(\nabla\cdot\underset{\sim}{P})_{\parallel} + (S_2 - S_3)\nabla\cdot\underset{\sim}{e}_1 - \nabla\cdot(S_2\underset{\sim}{e}_1) \quad , \tag{19}$$

and

$$dR_3/dt + R_3(\nabla\cdot\underset{\sim}{u} + 2\nabla_{\perp}\cdot\underset{\sim}{u}) = -2S_3\nabla\cdot\underset{\sim}{e}_1 - 2\nabla\cdot(S_3\underset{\sim}{e}_1) \quad , \tag{20}$$

where all quantities are taken to zero order;

$$S_1 = m\langle v_{\parallel}^5\rangle \ , \quad S_2 = \frac{1}{2} m\langle v_{\parallel}^3 \, v_{\perp}^2\rangle \ , \quad \text{and} \quad S_3 = \frac{1}{2} m\langle v_{\parallel} v_{\perp}^4\rangle \ .$$

In the absence of collisions, the $S_i^{(0)}$ are not determined by the moment equations. If $f^{(0)}$ is assumed to be an anisotropic Maxwellian, Eqs. (18) through (20) yield no new information, but are consistent with the CGL pressure equations.

By use of the results (11) through (16), the energy equations (9) and (10) through first order become

$$dp_{\perp}/dt + p_{\perp}(\nabla\cdot\mathbf{u} + \nabla_{\perp}\cdot\mathbf{u}) + \nabla\cdot\mathbf{q}_{\perp}^{\perp(1)} - p_{\perp}^{(0)}\alpha_1 + \alpha_2 = \Gamma_{\perp} + O(\varepsilon^2) \tag{21}$$

and

$$\frac{1}{2}\, dp_{\parallel}/dt + p_{\parallel}\left(\frac{3}{2}\nabla\cdot\mathbf{u} - \nabla_{\perp}\cdot\mathbf{u}\right) + \nabla\cdot\mathbf{q}_{\perp}^{\parallel(1)} + p_{\parallel}^{(0)}\alpha_1 - \alpha_2 = \Gamma - \Gamma_{\perp} + O(\varepsilon^2) \ , \tag{22}$$

where

$$\nabla_{\perp}\cdot\mathbf{u} = (\nabla\mathbf{u}):\mathbf{I}_{\perp} \ , \quad \alpha_1 = 2\Omega^{-1}\mathbf{e}_1\cdot[(\mathbf{e}_1\cdot\nabla\mathbf{u}) \times (\nabla\mathbf{u})\cdot\mathbf{e}_1]^{(0)} \ ,$$

$$\alpha_2 = \Omega^{-1}\mathbf{e}_1\cdot[(\nabla R_2 - \rho^{-1}p_{\parallel}\nabla p_{\perp}) \times (\mathbf{e}_1\cdot\nabla\mathbf{e}_1)]^{(0)} \ ,$$

$$\Gamma_{\perp} = -B\nabla\cdot(B^{-1}q_{\parallel}^{\perp}\mathbf{e}_1) + 2\Omega^{-1}\mathbf{e}_1\cdot[q_{\parallel}^{\perp}(\mathbf{e}_1\cdot\nabla\mathbf{e}_1) \times (\nabla\mathbf{u})\cdot\mathbf{e}_1 + (\mathbf{e}_1\cdot\nabla\mathbf{u}) \times \nabla q_{\parallel}^{\perp}]^{(0)} \ ,$$

and

$$\Gamma = \nabla\cdot(q_{\parallel}^{\perp} + q_{\parallel}^{\parallel})\mathbf{e}_1 + 2\Omega^{-1}\{q_{\parallel}^{\perp}(\nabla\mathbf{u}):[\mathbf{I}_{\delta}\mathbf{I}_{\gamma} - \mathbf{I}_{\gamma}\mathbf{I}_{\delta}]:\nabla\mathbf{e}_1\}^{(0)}$$
$$+ \Omega^{-1}\mathbf{e}_1\cdot\{[\mathbf{e}_1\cdot\nabla\mathbf{u} + (\nabla\mathbf{u})\cdot\mathbf{e}_1] \times [(2q_{\parallel}^{\parallel} - q_{\parallel}^{\perp})\mathbf{e}_1\cdot\nabla\mathbf{e}_1 + \nabla q_{\parallel}^{\perp}]\}^{(0)} \ .$$

The zero-order terms of Eqs. (21) and (22) give the CGL pressure equations. In first order these equations are modified in form by the appearance of $\mathbf{q}_{\perp}^{\perp(1)}$, $\mathbf{q}_{\perp}^{\parallel(1)}$, α_1, α_2, and the parts of Γ and $\Gamma_{\perp}$ proportional to Ω^{-1}. Adding together Eqs. (21) and (22) gives the rate of change of the total thermal energy.

The terms in α_1 of Eqs. (21) and (22) and their sum are somewhat analogous to ordinary viscous dissipation in that they arise from the contribution of the collisionless viscosity to $\underset{\sim}{P}:\nabla\underset{\sim}{u}$ and $\underset{\sim}{P}:\underset{\sim}{e}_1 d\underset{\sim}{e}_1/dt$ in Eqs. (9) and (10), are quadratic in $\underset{\sim}{u}$, and represent a transfer of energy between the fluid and thermal motions. The net transfer of energy from the flow to thermal motion produced by the collisionless viscosity does not vanish in general, as might be expected for a collisionless system, but is equal to $(p_{\parallel} - p_{\perp})\alpha_1$, and may have either sign.

The α_2 terms of Eqs. (20) and (21) represent a transfer of energy between the parallel and transverse thermal motions which comes from $\underset{\sim}{e}_1 \cdot \underset{\sim}{Q}:\nabla\underset{\sim}{e}_1$ in Eqs. (9) and (10). The parts of $\Gamma_{\perp}$ and Γ proportional to Ω^{-1} are due to the contributions of $q_{\parallel}^{\perp(0)}$ and $q_{\parallel}^{\parallel(0)}$ to $\underset{\sim}{P}^{(1)}$ and $\underset{\sim}{Q}^{(1)}$. Because zero-order heat flow can not occur in the presence of strong collisions, these terms have no analogy in ordinary hydrodynamics.

Except for $q_{\parallel}^{\perp}$ and $q_{\parallel}^{\parallel}$ through first order and the $S_i^{(0)}$, Eqs. (1) through (3), (11) through (15), and (18) through (22) for each component, together with Maxwell's equations, determine the transverse motion of the plasma through second order and the other quantities through first order.

These equations simplify considerably if $\underset{\sim}{e}_1$ is constant in space and time and if the properties of the plasma are independent of position along the field lines; the equations form a closed set if $u_{\parallel}$, $q_{\parallel}^{\perp(0)}$, and $q_{\parallel}^{\parallel(0)}$ vanish initially. These quantities then remain zero, and $p_{\parallel}$, R_1, R_2, $\underset{\sim}{q}_{\perp}^{\parallel}$, and S_i are no longer required.

Equations (15) and (21) then become

$$\underset{\sim}{q}_{\perp}^{\perp(1)} = (\rho\Omega)^{-1} 2 p_{\perp} \underset{\sim}{e}_1 \times [(2R_3' - 1) n \nabla (n^{-1} p_{\perp}) + (R_3' - 1) n^{-1} p_{\perp} \nabla n + p_{\perp} \nabla R_3']^{(0)} \quad , \tag{23}$$

and

$$dp_{\perp}/dt + 2p_{\perp} \nabla_{\perp} \cdot \underset{\sim}{u} + \nabla \cdot \underset{\sim}{q}_{\perp}^{\perp(1)} = O(\varepsilon^2) \quad . \tag{24}$$

The zero-order terms of Eq. (24) yield the CGL adiabatic condition for $p_{\perp}^{(0)}$. In certain cases, such as perturbations of a uniform equilibrium, a configuration depending on only one spatial coordinate, constant B and ρ, or constant B and $(p_{\perp}/n)^{(0)}$, $\cdot \underset{\sim}{q}_{\perp}^{\perp(1)}$ vanishes and the adiabatic condition holds through first order. In general, however, it is modified in first order by the transverse heat flow.

Under these conditions Eqs. (18) through (20) reduce to statements that in zero order the R_i' are conserved by the motion. Equations (1), (3), (11), (12), (23), and (24) and the conservation of $R_3'^{(0)}$ for each component and Maxwell's equations form a set which is closed for purely transverse motion.

This closed set of equations differs from those obtained on the assumption that $f^{(0)}$ is an isotropic Maxwellian[9] in that the scalar pressure is replaced by $p_{\perp}$, and that the heat flow $\underset{\sim}{q}_{\perp}$ of Eq. (17) is replaced by $\underset{\sim}{q}_{\perp}^{\perp}$ of Eq. (23) with R_3' conserved. The conservation of the R_i' implies that $f^{(0)}$ will remain Maxwellian (but not necessarily isotropic) if it has this form initially. The two energy equations then differ in form only by a factor 5/4 in the expressions for the heat flows.

IV. WEAK INSTABILITIES

For weak instabilities ($\tau^{-1} \sim \varepsilon^2\Omega$), the parameters $L(v_{th}\tau)^{-1}$ and $cE_0(v_{th}B_0)^{-1}$ are of order ε and the preceding discussion must be modified. Since $\mathbf{u}^{(0)}$ vanishes in this case, both $\mathbf{u}$ and $\mathbf{u}_E$ are now of order ε. If $q_{\parallel}^{(0)} = 0$ it follows from Eqs. (8) that P_δ, P_{23}, P_{12}, and P_{13} all vanish in first order. They are nonzero in second order, however, and make a contribution to $\rho^{-1}\nabla\cdot\mathbf{P}$ in Eq. (3) which is of the same order ($\sim\varepsilon^3 v_{th}\Omega$) as $d\mathbf{u}/dt$. These terms must be retained; ordinary hydromagnetic theory is therefore not applicable.[10]

The components P_δ, P_{23}, P_{12}, and P_{13} are found from Eqs. (8). For simplicity we restrict the discussion in this section to straight field lines and purely transverse motion. Only the transverse components of $\mathbf{P}$ are then required. They may be written in the form

$$\mathbf{P} = \left[p - \frac{\nabla\mathbf{q}_\perp^*}{2\Omega}\right]\mathbf{I}_\perp + \frac{p_\perp^{(0)}}{2\Omega}\left(\mathbf{I}_\gamma\mathbf{I}_\delta - \mathbf{I}_\delta\mathbf{I}_\gamma\right):\nabla\mathbf{u}^{(1)} + \frac{(\nabla\mathbf{q}_\perp^*)^s}{2\Omega} + O(\varepsilon^3) \quad , \tag{25}$$

where

$$\mathbf{q}^* \equiv \Omega^{-1}[\rho^{-1}p_\perp\,\nabla p_\perp - (1/4)\nabla R_3]^{(0)} = (1/2)\mathbf{e}_1 \times \mathbf{q}_\perp^{(1)} \quad .$$

The second term of Eq. (25) can be obtained from Eqs. (12) and (13) by changing $\mathbf{u}^{(0)}$ to $\mathbf{u}^{(1)}$. The terms in $\mathbf{q}^*$ arise from the contribution of $\nabla\cdot\mathbf{Q}^{(1)}$ to $\mathbf{G}$ in Eqs. (8). Equations (3) and (25) give $\mathbf{u}$ through order $\varepsilon^3 v_{th}$.

Equation (25) requires $p_\perp$ through order $\varepsilon^2\rho v_{th}^2$ and $R_3^{(0)}$. Since both $\partial/\partial t$ and $\mathbf{u}$ are small for weak instabilities the zero-order terms of Eq. (24) vanish. The first-order terms give the equation of state for $p_\perp^{(0)}$,

$$(\partial/\partial t + \underset{\sim}{u}^{(1)}\cdot\nabla)p_{\perp}^{(0)} + 2p_{\perp}^{(0)}\nabla\cdot\underset{\sim}{u}_E + \frac{\underset{\sim}{e}_1}{\Omega}\cdot\left[\frac{\nabla B}{B}\times\nabla R_3^{(0)}\right] = 0 \quad . \qquad (26)$$

Note that the calculation of $p_{\perp}^{(0)}$ now involves $R_3^{(0)}$.

First-order terms, analogous to those arising from $\nabla\cdot\underset{\sim}{q}_{\perp}^{(1)}$ in Eq. (26) and neglected in Eq. (20), must be included in the calculation of $R_3^{(0)}$. These terms introduce the sixth moment $m\langle v_{\perp}^6 f^{(0)}\rangle$. Determination of this quantity brings in still higher moments. The moment equations are not closed, even for straight field lines and purely transverse motion, unless this sequence terminates.

Termination occurs for low β. Equation (26) is then independent of R_3 and, since only $\nabla\times(\nabla\cdot\underset{\sim}{P})$ will be required,[8] the first term of Eq. (25) is not needed. Thus $R_3^{(0)}$ and the first- and second-order corrections to $p_{\perp}$ do not enter the calculation.

The situation is considerably more complicated for finite β. The first term of Eq. (25) must then be evaluated and therefore the equation for $R_3^{(0)}$ and the higher-order corrections to $p_{\perp}$ (through order ε^2) are needed. Termination of the sequence of moment equations occurs for one-dimensional geometries. Linearization about an equilibrium for which $\underset{\sim}{B}$ is uniform also produces a closed set of equations. These sets of equations include $R_3^{(0)}$ as a variable. For more general finite β configurations the moment equations are not closed.

The single fluid equations of motion and generalized Ohm's law[11] are obtained in the usual way from Eqs. (3) for ions and electrons. These equations, which now include the nonscalar contributions to $\underset{\sim}{P}$, are of the same form as the single fluid equations obtained by Rosenbluth and Simon,[6] with their unknown scalar function now determined. They differ from the

equations used by Roberts and Taylor[8] to discuss the finite gyro-radius stabilization of plasma in gravitational field, however, because the latter do not include the contribution of $\underset{\sim}{q}^*$ to $\underset{\sim}{P}$. (These terms do not affect the calculation of reference 8.) The Ohm's law of reference 8 is unnecessarily complicated because the plasma velocity enters the equation of motion only in first order (whereas the current enters through order $\varepsilon^3 nev_{th}$). The nonscalar contributions to $\underset{\sim}{P}$ and the acceleration term therefore are not required in Ohm's law.

Rosenbluth and Simon[6] have found a very simple set of moment equations describing weak transverse instabilities for low β and uniform magnetic field. Their set consists of Maxwell's equations for $\underset{\sim}{E}$, the equation of state (26), and a form of the continuity equation derived from Eqs. (1), (3), and (25). For low β, $\nabla\cdot\underset{\sim}{u}_E = 0$, and Eq. (26) reduces to $dp_\perp^{(0)}/dt = 0$, which is Eq. (A.13) of reference 6. To obtain Eq. (A.1) of reference 6 we substitute Eq. (25) into Eq. (3) and obtain, after some manipulation,

$$\rho\underset{\sim}{u} = \rho\underset{\sim}{u}_E + \frac{\underset{\sim}{e}_1}{\Omega} \times \left\{\rho^{(0)}\frac{d\underset{\sim}{u}^{(1)}}{dt} + \frac{\underset{\sim}{e}_1}{\Omega}\times[(\nabla\underset{\sim}{u})\cdot\nabla p_\perp - (\nabla p_\perp)\nabla\cdot\underset{\sim}{u}]^{(1)} + \nabla\psi\right\}, \quad (27)$$

where

$$\psi = p_\perp + (2\Omega)^{-1}[\nabla\cdot\underset{\sim}{q}^* + \underset{\sim}{P}\nabla\cdot(\underset{\sim}{e}_1\times\underset{\sim}{u})]^{(1)} ,$$

The right-hand side of Eq. (27) may be simplified by using the relation

$$\rho^{(0)}\frac{d}{dt}(\underset{\sim}{u}_E - \underset{\sim}{u})^{(1)} = \Omega^{-1}\underset{\sim}{e}_1 \times [(\nabla\underset{\sim}{u})\cdot\nabla p_\perp - (\nabla p_\perp)\nabla\cdot\underset{\sim}{u}]^{(1)} ,$$

obtained from Eq. (26) and the first-order terms of Eq. (3). The divergence of Eq. (27) then gives

$$\nabla\cdot\rho\underset{\sim}{u} = \underset{\sim}{u}_E\cdot\nabla\rho + \Omega^{-1}\nabla\cdot\left[\rho \quad \frac{d}{dt}\left(\underset{\sim}{e}_1 \times \underset{\sim}{u}_E\right)\right] + O(\varepsilon^5\Omega\rho) \quad .$$

Equation (A.1) of reference 6 is obtained by substituting this result into Eq. (3). This calculation is essentially the inverse of the derivation of the single fluid momentum equation given by Rosenbluth and Simon.

The equations found by Rosenbluth and Simon from an expansion of the Vlasov equation are thus easily obtained from an m/e expansion of the familiar conservation equations of mass, momentum, and energy. The "finite gyro radius" terms of this expansion, expressed in Eqs. (25) and (26), are easily obtained from the exact moment equations if the special assumptions of this section are made at the outset. This procedure provides an alternative derivation to that of reference 6. This alternative derivation emphasises the relation of the Rosenbluth-Simon equations to equations describing more general configurations. In particular we have shown that the moment equations do not close for weak instabilities at finite β except for one-dimensional configurations or, in the linear approximation, if the equilibrium magnetic field is uniform.

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