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EFFECTS OF COLLECTIVE EXCITED STATES ON ALPHA PARTICLE BARRIER PENETRABILITY

Berkeley, California

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Elizabeth A. Rauscher
(Thesis)

January 5, 1965

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ABSTRACT

For the alpha decay of Po^{212} to the ground state and various excited states of the Pb^{208} daughter, there are collective surface vibrational modes, which couple these low-lying excited states to the ground state of Pb^{208}. In the present description of the transitions between these excited states of Pb^{208} and the ground state of Pb^{208} only single phonon excitations will be considered.

Interaction of the electric multipole nature occurs between the excited state levels and the ground state of the recoil lead daughter induced by its interaction with the emitted alpha particle, affecting the latter's penetration of the nuclear barrier.

The formation of the excitation in the Lead nucleus can either be produced by Po^{212} alpha decay or by (α,α') inelastic scattering on Pb^{208}, but in either case we consider that an alpha cluster exists in the nuclear surface region.

It is the purpose of the present paper to investigate the manner in which the multipole interaction affects the alpha particle penetration of the nuclear barrier.

I. INTRODUCTION

The emitted alpha particle, as it leaves the recoil nucleus and penetrates the nuclear barrier, can be described by a partial differential eigenvalue equation of the form

$$H\Psi_n = E_n\Psi_n \tag{I.1}$$

as expressed by the time independent Schrödinger picture.

Here we have a Hamiltonian, H, as a second order differential operator acting on space dependent wave functions, Ψ_n describing states of the system in "ordinary space" and the constant energy eigenvalue, E_n.

Speaking in terms of matrix operators acting upon eigenkets which are our dynamical states in Hilbert space, we can express the alpha particle decay in the Heisenberg picture as

$$(H)_{op}|n\rangle = \epsilon_n|n\rangle. \tag{I.2}$$

For the most part, we shall formulate our problem in the latter notation.

For the alpha decay of Po^{212} to Pb^{208} we shall describe the action of the nuclear barrier, represented as a non-diagonal $N \times N$ matrix, on the alpha particle as a wave vector where the effect of the multipole nature of the surface deformation of the daughter nucleus is included.

In regard to this particular problem there are four important open channels for the multipole transition from the excited Lead states to the ground state of Lead. Thus, the general barrier penetration matrix will be a 4×4 non-diagonal matrix which will describe the effect of the barrier by its action on the alpha eigenvector, a four component vector in Hilbert space.

The most important collective levels in Lead (see Fig. 1) are the 0+, ground, 3-, 4+ and 5-. We shall derive an expression for the coupling matrix elements for these levels in the region from R_0 to R_1 (see Fig. 2, page 3), where channel mixing occurs.

The first excited state is the low-lying 3-. It appears that just considering the two channel problem, where only the 0+ and 3- states are considered, is a reasonable approximation. Only single phonon exchange is

considered where $\ell = 3$ is the spin of the phonon for the octupole case. The multipolarity of the state is determined by the usual $2^{\ell} = 2^3 = 8$ or octupole state.

In order to obtain an expression for the coupling matrix element in our two channel case, we will set forth the expression for the total Hamiltonian, H, and the total wave vectors, $|n\rangle$.

The expansion of $|n\rangle$ in terms of a complete orthogonal set is performed and then substitution into $(H)_{op} |n\rangle = \epsilon_n |n\rangle$ is made. Upon multiplication by $\langle m|$ from the left, and by use of the constructed orthogonal eigenstates, we obtain a set of ordinary radial dependent differential equations describing the emitted alpha particle as it penetrates the nuclear barrier.

In order to obtain an expression for the total Hamiltonian, a nuclear model must be adopted. In the present formulation, we are considering purely collective properties of the nuclear sufrace and shall describe the nucleus as an incompressible, classical liquid drop that undergoes pure surface oscillations which can be represented by harmonic oscillators.

This hydrodynamic model, adopted by A. Bohr and B. Mottelson[1] in their early work, assumes an equilibrium spherical shape and constant volume, adiabatic conditions. Because the equilibrium shape of the lead daughter is spherically symmetrical, since it is doubly major in Z and N containing closed nuclear shells only, we need consider only vibrational excitation and not rotational excitation, as there is no permanent deformation present.

Utilizing this model we have a Hamiltonian consisting of five terms:

$$H = T + V^C + V^N + H_{coll.} + H_{int.} \qquad (I.3)$$

T is the kinetic energy operator for the system consisting of the emitted alpha and the recoil nucleus given as

$$-\frac{\hbar^2}{2\mu} \nabla_{r,\Omega}^2$$

where $\hbar$ is Planck's constant divided by 2π and μ is the reduced mass of the alpha-recoil nucleus system. The Laplacion in spherical-polar coordinates, $V = \frac{2Ze^2}{r}$ is the point density source monopole or central electro-

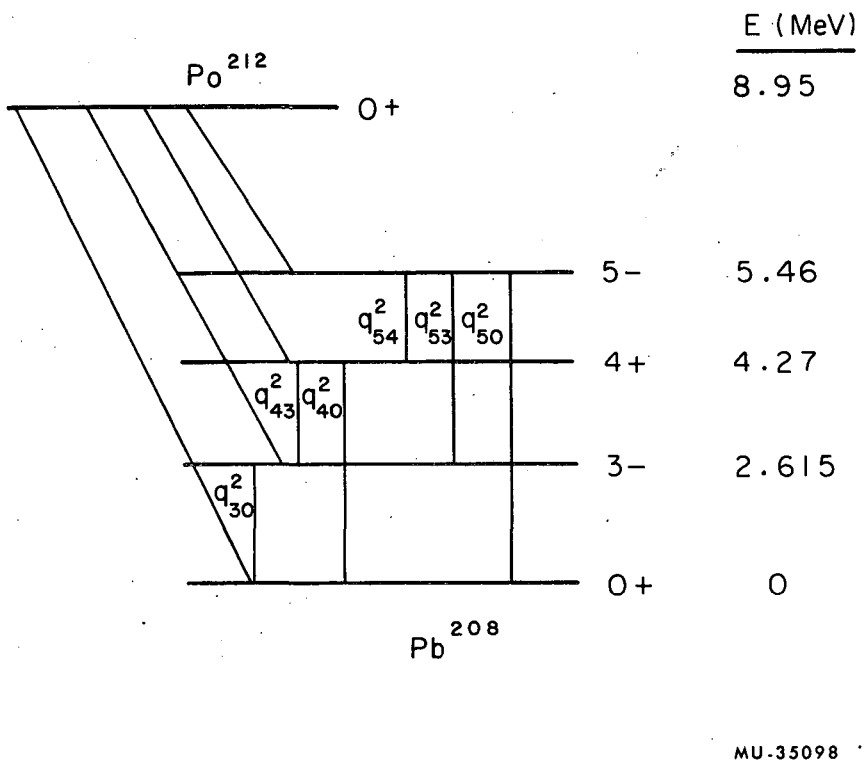

Fig. 1. The most important nuclear energy levels in Pb^{208} have been observed in inelastic-electron scattering experiments by W.C. Barber.[2] Similar values for the spacing of the lead states are also given by T. Stovel and N. Hintz.[3] The states quoted here are taken from H. Faraggi's data.[4]

static force term, where Z is the charge on the daughter and r is the radial separation of the alpha and the recoil daughter. The symbol Ω represents the angular coordinates (θ,ϕ) of the alpha particle in a space fixed spherical polar coordinate system.

The central part of the attractive nuclear potential, V^N, will be assumed to be an exponential potential of the Woods-Saxon form.

$H_{coll.}$ or H^N is the Hamiltonian operator for the internal energy of the recoil nucleus. By adoption of the before-mentioned hydrodynamic model, $H_{coll.}$ becomes

$$H_{coll.} = \sum_m [(\tfrac{1}{2})C_3\, \alpha_{3m}^2 + (\tfrac{1}{2})B_3\, \dot{\alpha}_{3m}^2]$$

where C_3 is the "surface tension" parameter and B_3 is the "Mass" parameter for the octupole case.

The expansion coefficient, α_{3m}, is the collective coordinate by which the distortion in the nuclear potential caused by the surface vibrational deformation, may be described.

The interaction Hamiltonian operator, H_{int}, is that part of the Hamiltonian which deals with the interaction involved in the alpha daughter system, due to the vibrational channel coupling.

The nuclear potential, V^N, and interaction Hamiltonian, H_{int}, are the first and second terms of a Taylor expansion of the coupling potential, $V_{coup.}(r,\Omega)$. Only terms linear in α_{3m} were kept.

H_{int} is dependent upon the nuclear deformation of the nucleus and is thus dependent of α_{3m}. It is also functionally dependent on the total barrier potential slope in the coupling region.

$$\begin{aligned} H_{int} &= \sum_m V_m(\alpha_{3m},\Omega) \\ &= -\frac{dV}{dr}\,[R(\alpha,\Omega) - R_0] \end{aligned} \qquad (I.5)$$

The position in space, $R(\alpha,\Omega)$ of the oscillating nuclear surface is defined by

$$R(\alpha,\Omega) = R_0[1 + \sum_m \alpha_{3m}\, Y_3^m(\Omega)] \qquad (I.6)$$

for the octupole case.

R_0 is the equilibrium radius. See Fig. 3. Upon substitution of $R(\alpha,\Omega)$ in the above expression for H_{int} we obtain

$$H_{int} = -\frac{dV}{dr}\left[R_0 \sum_m \alpha_{3m} Y_3^m(\Omega)\right] \tag{I.7}$$

The five components making up our eigenvalue equation can be written as

$$(T + V^C + V^N + H_{coll.} + H_{int} - E_\alpha)\ |n\rangle = 0 \tag{I.8}$$

where E_α is the total energy eigenvalue of the system.

The total wave function, ψ, is written, in the two level approximation, as

$$r\psi = u_0(r)\ Y_0^0(\Omega)\chi_{00}(\alpha_3) + \frac{u_3(r)}{\sqrt{7}} \sum_m (-)^m\ Y_3^m(\Omega)\chi_{3,-m}(\alpha_3) \tag{I.9}$$

where the separation of variables into functions of r, Ω and α_3, is applicable.

The radial part of the total wave function is given by $U_\ell(r)$, the angular part by $Y_\ell^m(\Omega)$, the spherical harmonics and $\chi_{\ell,-m}(\alpha_3)$ the part of the wave function depending on the distortion in the nuclear potential of the daughter nucleus.

$r\psi$ can also be expressed as

$$r\psi = u_0(r)\ Y_0^0(\Omega)\chi_{00}(\alpha_3) + u_3(r)\{Y_3^m(\Omega),\ \chi_{3,-m}(\alpha_3)\}_{0+} \tag{I.10}$$

The bracket notation is introduced to indicate the sum over m, and, that the total angular momentum is coupled to zero the angular momentum of the initial state.

Before the sum over angular momentum, ℓ, has been performed, as in the above expression, we have the following general expression

$$r\psi(\vec{r},\alpha_3) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{(-)^m u_\ell(r)}{\sqrt{2\ell+1}}\ Y_\ell^m(\Omega)\ \chi_{\ell,-m}(\alpha_3)$$

or (I.11)

$$r\psi = \sum_{\ell=0}^{\infty} \frac{u_\ell(r)}{\sqrt{2\ell+1}}\{Y^m(\Omega),\ \chi_{\ell,-m}(\alpha_3)\}_{\ell\pi}$$

where $Y_\ell^m(\Omega)$ and $X_{\ell,-m}^{(\alpha_3)}$ form complete orthonormal sets.

It will prove to be convenient to define a new vector, angular momentum, coupled representation $\mathcal{Y}_\ell^m(\Omega,\alpha_3)$ which is related to the uncoupled representation, $Y_\ell^m(\Omega)X_{\ell,-m}^{(\alpha_3)}$, by a unitary transformation

$$\mathcal{Y}_\ell^m(\Omega,\alpha_3) = \sum_{m=-\ell}^{\ell} C(\ell\ell 0;m-m0)\, Y_\ell^m(\Omega)X_{\ell,-m}^{(\alpha_3)} \tag{I.12}$$

where the C's are the Condon and Shortley vector addition, or Clebsch-Gordon coefficients.[5]

Returning now to the Heisenburg picture, we express $\mathcal{Y}_\ell^m(\Omega,\alpha_3)$ as $|\ell\ m\rangle$. We will denote $\mathcal{Y}_{00}^{00}$ by $|00\rangle$ and $\mathcal{Y}_{33}^{00}$ by $|33\rangle$, indicating the coupling to zero previously indicated by the bracket notation.

ψ or $|n\rangle$ can now be written as

$$|n\rangle = \frac{u_0}{r}\,|00\rangle + \frac{u_3(r)}{r}\,|33\rangle \tag{I.13}$$

and upon substitution of $|n\rangle$ in our eigenvalue equation and by the use of the harmonic oscillator eigenvalue property of H_{coll} (here, a seven-dimensional harmonic oscillator), we obtain

$$(H-E_\alpha)\,|n\rangle = -\frac{\hbar^2}{2u}\frac{d^2}{dr^2}\,u_0\,|00\rangle - \frac{\hbar^2}{2u}\frac{d^2u_3}{dr^2}\,|33\rangle + (V^C+V^N-E)\left[\frac{u_0}{r}|00\rangle + \frac{u_3}{r}\,|33\rangle\right]$$

$$+\frac{7\hbar\omega}{2}\frac{u_0}{r}|00\rangle + \frac{9\hbar\omega}{2}\frac{u_3}{r}|33\rangle + R_0\frac{dV^N}{dr}\left[\sum_m \alpha_{3m}Y_3^m(\Omega)\right]\left(\frac{u_0}{r}\,|00\rangle + \frac{u_3}{r}|33\rangle\right) = 0 \tag{I.14}$$

Multiplying from the left by $\langle 00|$ and by the use of the orthogonality condition,

$$\langle \ell'm'|\ell m\rangle = \delta_{\ell\ell'}\,\delta_{mm'} \tag{I.15}$$

we have our ordinary differential equation or radial wave equation where the eigenfunction is dependent upon radial separation only.

$$-\frac{\hbar^2}{2u}\frac{d^2u_0}{dr^2} + \left(V^C + V^N + \frac{7\hbar\omega}{2} - E_\alpha\right)u_0 + R_0\frac{dV^N}{dr}\frac{u_3}{r}\sum_m \langle 00|\alpha_{3-m}Y_{3m}\,|33\rangle = 0 \tag{I.16}$$

The $7\hbar\omega/2$ term is the zero-point energy of the harmonic oscillator but we may redefine the zero of energy so that we can drop this term.

By virtue of the properties of spherical harmonics, the above equation becomes

$$-\frac{\hbar^2}{2u}\frac{d^2u_0}{dr^2} + (V^C + V^N - E_\alpha)u_0 + R_0\frac{dV^N}{dr}\sqrt{\frac{7}{4\pi}}\frac{u_3}{r}\langle\chi_0(\alpha_3)|\alpha_{3m}|\chi_1(\alpha_{3m})\rangle = 0 \qquad (I.17)$$

for single phonon transitions.

By use of an operator property of the harmonic-oscillator (cf., Schiff[6]), the coupling matrix element,

$$R_0\frac{dV^N}{dr}\sqrt{\frac{7}{4\pi}}\langle\chi_0|\alpha_{3m}|\chi_1\rangle$$

becomes

$$R_0\frac{dV^N}{dr}\sqrt{\frac{7}{8\pi}}\frac{\hbar^{1/2}}{(B_3C_3)^{1/4}}$$

Simlarly, to obtain the second ordinary couple differential equation for the two-channel case, we multiply on the left by $\langle 33|$ and obtain

$$-\frac{\hbar^2 d^2u_3}{2udr^2} + (V^C + V^N + \frac{9\hbar\omega}{2} + \frac{\hbar^2\, 12}{2ur^2} - E_\alpha)u_3 + R_0\frac{dV^N}{dr}u_0\sqrt{\frac{7}{8\pi}}\frac{\hbar^{1/2}}{(B_3C_3)^{1/4}} = 0 \qquad (I.18)$$

Again, we redefine the energy zero by subtracting the zero-point energy $7/2\hbar\omega$ leaving $\hbar\omega$, the energy of the 3- state in Pb^{208}.

Instead of expressing our energy quantities in terms of MeV we shall, for simplicity, multiply through by $\frac{2u}{\hbar^2}$ and express these quantities in $1/f^2$ units of reciprocal fermis squared. In the present paper, the coupling term will be defined as $q^2(r)$, so that

$$q^2(r) \equiv \frac{2u}{\hbar^2}R_0\frac{dV^N}{dr}\sqrt{\frac{7}{8\pi}}\frac{\hbar^{1/2}}{(B_3C_3)^{1/4}} \qquad (I.19)$$

Our radial wave equations then reduce to

$$\frac{d^2u_0}{dr^2} + \frac{2u}{\hbar^2}(E_\alpha - V^\alpha)\, u_0 + q^2 u_3 = 0 \tag{I.20}$$

and

$$\frac{d^2u_3}{dr^2} + \frac{2u}{\hbar^2}\left(E_\alpha - \hbar\omega - V^\alpha - \frac{12\,\hbar^2}{2ur^2}\right) u_3 + q^2 u_0 = 0$$

The V^N term has been absorbed into the V^C Coulomb term, and is designated V^α.

It will prove to be convenient to define the following quantities.

$$\begin{aligned} k_0^{\,2}(r) &= \frac{2u}{\hbar^2}\,(E_\alpha - V^\alpha) \\ k_3^{\,2}(r) &= \frac{2u}{\hbar^2}\left[E_\alpha - \hbar\omega - V^\alpha - \frac{\hbar^2}{2u}\,\frac{12}{r^2}\right] \end{aligned} \tag{I.21}$$

where $k_0^{\,2}$, $k_3^{\,2}$ and q^2 are expressed in units of $1/f^2$.

The coupled equations for the octupole case thus reduce to the simple form

$$\begin{aligned} u_0'' + k_0^{\,2} u_0 + q^2 u_3 &= 0 \\ u_3'' + k_3^{\,2} u_3 + q^2 u_0 &= 0 \end{aligned} \tag{I.22}$$

where the indicated derivative is taken with respect to the radial distance, r.

In the following section we shall derive an expression for the matrix elements of a 2×2 matrix, for the octupole case, which describes the effect of the nuclear barrier upon an emitted alpha wave function. This matrix will describe our coupling barrier penetration problem for the two channel case.

II. THE BARRIER PENETRATION MATRIX, IN THE CASE WHERE COUPLING OCCURS

In this section we shall present a method for "decoupling" our coupled differential equations. We shall begin by considering the two channel case, as is done in H. A. Weidenmüller's[7] work on many channel scattering.

The two channel case is, of course, only relevant to the approximation where octupole excitation is the only type of excitation that occurs and the open channels involve only the final states of the daughter that are the 0+ and 3-. No assumption as to the relative magnitudes of the coupling matrix element q^2 and the energy matrix element, $k_{30}^2 \equiv k_3^2 - k_0^2$, will be made.

As will be seen, when q^2 and k_{30}^2 are numerically evaluated, the assumption that $k_{30}^2 \ll q^2$ is not valid, and $k_{30}^2 \simeq 0$ is not a reasonable approximation for this problem.

We shall calculate the transformation matrix, denoted by B, which transforms A, the matrix Hamiltonian from a non-diagonal form to a diagonal representation. The fact that the off-diagonal elements of A are not zero can be ascribed to the coupling effect.

We shall find a new set of basis vectors $|V\rangle$ in the representation where A is diagonal. These vectors may be obtained from the basis vector, $|U\rangle$, of the "old" representation, where A is not in diagonal form, by finding the eigenvalue $\lambda_{\mp}$ that belongs to the eigenvectors $|V\rangle$ and the diagonal form of A. $|V\rangle$ can be constructed from $\lambda_{\mp}$, as we shall demonstrate.

We make use of the characteristic equation in conjunction with the secular determinant in order to obtain $\lambda_{\mp}$.

Let us return to our two coupled partial wave equations which we derived in Section I. They are:

$$\begin{aligned} u_0'' + k_0^2 u_0 + q^2 u_3 &= 0 \\ u_3'' + k_3^2 u_3 + q^2 u_0 &= 0 \end{aligned} \qquad \text{(II.1)}$$

which can be expressed, in matrix notation, as:

$$\begin{pmatrix} k_0^2 & q^2 \\ q^2 & k_3^2 \end{pmatrix}\begin{pmatrix} u_0 \\ u_3 \end{pmatrix} = -\begin{pmatrix} u_0 \\ u_3 \end{pmatrix}'' \qquad \text{(II.2)}$$

Let A be defined by

$$A \equiv \begin{pmatrix} k_0^2 & q^2 \\ q^2 & k_3^2 \end{pmatrix}$$

and the eigenvectors of A as:

$$|U\rangle \equiv \begin{pmatrix} u_0 \\ u_3 \end{pmatrix}$$

Then the above equation becomes:

$$A|U\rangle = -\ |U\rangle''$$

this is our matrix representation of our two coupled, differential equations with eigenvalues E_0 and E_3 respectively.

As stated before, the eigenvalues $\lambda_{\mp}$ can be obtained from the characteristic equation which is $\bar{A}|X\rangle = 0$, where

$$\bar{A} \equiv \begin{pmatrix} k_0^2 - \lambda_{\mp} & q^2 \\ q^2 & k_3^2 - \lambda_{\mp} \end{pmatrix}$$

and $|X\rangle$ is some vector.

Setting det $\bar{A}$, where det $\bar{A}$ is the secular determinant of $\bar{A}$, equal to zero, we obtain the following equation in terms of $\lambda_{\mp}$.

$$(k_0^2 - \lambda_{\mp})(k_3^2 - \lambda_{\mp}) - q^4 = 0 \qquad \text{(II.3)}$$

giving us the two eigenvalues, $\lambda_{\mp}$, as:

$$\lambda_{\mp} = \frac{(k_0^2 + k_3^2) \mp \sqrt{(k_0^2 + k_3^2)^2 - 4(k_0^2 k_3^2 - q^4)}}{2} \qquad \text{(II.4)}$$

We are free to specify a zero of energy quite arbitrarily and can thus simplify the above expression.

As before, let us define $k_{30}^2 \equiv k_3^2 - k_0^2$ or the energy spacing of the 3- level which is seen to be 2.61 MeV (see Fig. 1).

Calculating k_0^2 and k_3^2 directly as defined by $k_i^2 = \frac{2\mu}{\hbar^2}(E_i - V)$ where the index $i = 0$ or 3.* As we shall see, in view of the greater detail given in Sec. IV, $k_0^2 = 3.12\ 1/f^2$ and $k_3^2 = 3.96\ 1/f^2$. The zero on the $\lambda_{\mp}$ energy axis can be shifted down so that $k_0^2 = 0$ MeV and $k_3^2 =$ 2.61 MeV, and the same results are obtained. The change of form from

$$A = \begin{pmatrix} k_0^2 & q^2 \\ q^2 & k_3^2 \end{pmatrix},$$

to

$$A = \begin{pmatrix} 0 & q^2 \\ q^2 & k_{30}^2 \end{pmatrix}$$

is obtained by this shift in the energy scale and for $k_0^2 = 0$ scale $\lambda_{\mp}$ becomes:

$$\lambda_{\mp} = \frac{k_{30}^2 + \sqrt{k_{30}^4 + 4q^4}}{2} \qquad \text{(II.5)}$$

For the second form, the shift is made so that the zero point of the energy axis coincides with $k_0^2 = 0$. It is evident that $\lambda_{\mp}$ for $k_0^2 = 0$ MeV can be obtained from $\lambda_{\mp}$ for $k_0^2 = 16.0$ MeV or $3.12\ 1/f^2$ by subtracting 16.0 MeV from the latter.

---

*The quantities, k^2 and q^2 are treated as $|k_i^2|$ and $|q^2|$, that is to say, they are treated as positive quantities.

Now let us proceed to construct the eigenvectors $|V\rangle$, belonging to the eigenvalues $\lambda_{\mp}$, such that $|V\rangle$, defined as $|V\rangle = \binom{v+}{v-}$, satisfies the eigenvalue equation: $A|V\rangle = \lambda_{\pm}|V\rangle$.

For the purpose stated above, let us return again to our characteristic equation: $\bar{A}|X\rangle = 0$ where $\bar{A}$ acts on some two-dimensional vector ket in Hilbert space with compnents $\binom{x}{y}$ such that $\bar{A}|X\rangle = 0$ is true.

From the characteristic equation we obtain two linearly dependent equations:

$$(k_0^2 - \lambda_{\mp})\, x + q^2\, y = 0$$

$$\text{and} \qquad q^2\, x + (k_3^2 - \lambda_{\mp})\, y = 0 \qquad \text{(II.6)}$$

Either of these two equations may be used to construct $v_{\mp}$, the components of $|V\rangle$, so let us arbitrarily choose the first equation in the form:

$$x/y = \frac{-q^2}{(k_0^2 - \lambda_{\mp})} \qquad \text{(II.7)}$$

The same ratio x/y obtained from the second equation results in a redundant expression, since the above two equations are not linearly independent. This situation is a consequence of the type of vector space with which we are dealing.

For simplicity we shall return to the energy scale where $k_0^2 = 0$ and then the ratio x/y becomes $x/y = -q^2/\lambda_{\mp}$ and $y/x = -\lambda_{\mp}/q^2$.

We can now construct two normalized vector component wave functions $v+$ and $v-$ from the vector components x and y of $|X\rangle$ as

$$v_+ = \frac{1}{b_+}\begin{pmatrix} 1 \\ \lambda_+/q^2 \end{pmatrix}$$

$$v_- = \frac{1}{b_-}\begin{pmatrix} 1 \\ \lambda_-/q^2 \end{pmatrix} \qquad \text{(II.8)}$$

where v_+ and v_- are eigenkets or column matrices and v_+^+ and v_-^+ are row matrices.

$1/b_+$ and $1/b_-$ are obtained from the normalization conditions on v_+ and v_-.

The condition on (x,y) or $\langle x|y\rangle = 1$ or $x^2 + y^2$ gives $b_\mp = 1+\lambda_\mp/q^2$ where $v_\mp$ are mutually orthonormal eigenvectors.

So we have $v_\mp$ as:

$$v_- = \frac{1}{\sqrt{1+\lambda^2-/q^4}} \begin{pmatrix} 1 \\ \lambda_-/q^2 \end{pmatrix} \qquad \text{(II.9)}$$

$$v_+ = \frac{1}{\sqrt{1+\lambda^2+/q^4}} \begin{pmatrix} 1 \\ \lambda_+/q^2 \end{pmatrix}$$

We will construct a transformation matrix that will transform the matrix A, in its non-diagonal representation with basis vectors, $|U\rangle = \binom{u_0}{u_3}$ to a representation where A is a doagonal matrix, denoted as D, with basis vectors $|V\rangle = \binom{v_-}{v_+}$.

In matrix notation we say $D = \tilde{B}\,AB$ where $\tilde{B}$ denotes the transpose of B. That is to say, the representation of A as $\langle u_3|A|u_0\rangle$ is not diagonal and $\langle v_-|D|v_+\rangle$ is diagonal.

Certain group properties of B can be obtained by considering B_i as an element of a general transformation group, G, i.e., $B_i \ni G$. By virtue of the group properties of G it is true that we have the following commutator relation $[A,B_i] = 0$.

This follows from the postulate that the Hamiltonian is invariant with respect to the transformations of a group, G. Thus $\langle u_3|A|u_0\rangle$ and $\langle v_-|D|v_+\rangle$ are equivalent.

Here we are dealing with the dynamical states $|u_0\rangle$ and $|u_3\rangle$ and $|v_+\rangle$ and $|v_-\rangle$ in Hilbert space. These vector components, $|u_0\rangle$ and $|u_3\rangle$ are the components of $|U\rangle$, a vector in a two dimensional Hilbert space. These two components comprise a complete set of basis vectors in this space.

We can obtain the new set of basis vectors $|V\rangle$ from the old set $|U\rangle$ by the operation: B as $|V\rangle = B|U\rangle$. It is seen that the Hamiltonian A, as a matrix operator, remains unchanged under the transformation, B.

In actuality, we are defining B in this manner to be that specific transformation which carries $|U\rangle$ into $|V\rangle$ as $|V\rangle \equiv B|U\rangle$ or written as

$$\begin{pmatrix} v_- \\ v_+ \end{pmatrix} = \left(\begin{pmatrix} \xi & -\eta \\ \eta & \xi \end{pmatrix} \right) \begin{pmatrix} u_0 \\ u_3 \end{pmatrix} \tag{II.10}$$

where

$$B \equiv \left(\begin{pmatrix} \xi & -\eta \\ \eta & \xi \end{pmatrix} \right)$$

The above form of B has been chosen for a specific reason, as will become apparent later when the expressions for the matrix elements of B are determined.

B is the proper transformation matrix first: if it commutes with A, as mentioned above, and second: if it is unitary.

The unitarity of the B matrix is again dependent upon the properties of the transformation group G, of which B is a member. The unitarity condition for a transformation is indicative of the fact that any transformation made by that group element will preserve the orthonormality of the wave function upon which it acts.

The conservation of particle number is indeed a consequence of unitarity of the transformation where normalization remains undisturbed by the action of a transformation matrix such as B.

B is real as all its elements are real and therefore $B = B^*$. From the unitarity condition we have $B\tilde{B} = I$ and from this condition we have the implied condition that $B^{-1} = \tilde{B}$, i.e. B inverse is equal to B transpose. This fact will be used in subsequent calculations.

Let us look at the resulting form of the separated equations. We started with the following coupled equation in matrix notation

$$|U\rangle'' + A|U\rangle = 0 \quad \text{or} \quad A|U\rangle = -|U\rangle'' \tag{II.11}$$

But $A = \tilde{B}\,D\,B$ since $D = B\,A\,\tilde{B}$
So that $|U\rangle'' + A|U\rangle = 0$ becomes

$$|U\rangle'' + B\,D\,\tilde{B}|U\rangle = 0 \tag{II.12}$$

Using our definition of B as the matrix which transforms $|U\rangle$ into $|V\rangle$ as $|V\rangle \equiv B|U\rangle$ in conjunction with the eigenvalue $\lambda\mp$, we have

$$D|V\rangle = \lambda \pm |V\rangle \tag{II.13}$$

where D is the diagonal form of A, so that

$$|V\rangle'' + D|V\rangle = 0 \tag{II.14}$$

we obtain the following two separated second order differential equations:

$$\begin{aligned} v_+''(r) + \lambda_+(r)v_+(r) &= 0 \\ v_-''(r) + \lambda_-(r)v_-(r) &= 0 \end{aligned} \tag{II.15}$$

The exact form of the matrix B can easily be expressed in terms of $v_\mp$ by the following parametrization method. When numerical evaluations are made later on, it will be seen that this parametrization is useful in this regard. As is well known, any 2×2 real, unitary matrix can be expressed as:

$$R = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

Since B is real and unitary, it qualifies as a value which R can take on so we will express B in this manner.

In order to obtain the elements of B from the elements of R, we can picture a right triangle of sides λ_- and q^2 making an angle of ϕ opposite to the side of length λ_- and with hypotenuse $\sqrt{q^4 + \lambda_-^2}$. From this triangle we see that

$$\sin\phi = \frac{\lambda_-}{\sqrt{q^4 + \lambda_-^2}} \tag{II.16}$$

and $\cos\phi = \dfrac{q^2}{\sqrt{q^4 + \lambda_-^2}}$ so that B can be written as

$$B = \begin{pmatrix} \dfrac{q^2}{\sqrt{q^4 + \lambda_-^2}} & -\dfrac{\lambda_-}{\sqrt{q^4 + \lambda_-^2}} \\ \dfrac{\lambda_-}{\sqrt{q^4 + \lambda_-^2}} & \dfrac{q^2}{\sqrt{q^4 + \lambda_-^2}} \end{pmatrix} \tag{II.17}$$

At this point in our discussion we introduce the coupling barrier penetration matrix, Q. Q is a real symmetric matrix. We shall see how Q is expressed in terms of the transformation matrix, B, and the WKB integral penetration matrix, P, which describes the manner in which the barrier affects the partial wave amplitudes u_0 and u_3 when no coupling is present. For this purpose let us consider the step by step process by which an alpha particle escapes the coupling region of the barrier.

We shall consider an eigenvector $\begin{pmatrix} u_0 \\ u_3 \end{pmatrix}$ as describing an alpha cluster which exists at some raduis, R_0. We shall regard this alpha cluster as a purely out-going wave at this radius, and that its nature is purely independent of the statistical manner in which it was formed. In this regard we shall consider the nucleus as a black box and not look at the manner in which shell model wave functions, not derivable from a potential, are calculated.

Let us for now assume that $\begin{pmatrix} u_0 \\ u_3 \end{pmatrix}$ exists, and let us "carry" this alpha particle wave vector through the coupling region of the barrier and find out what effect the coupling has on this emitted alpha particle.

First a transformation to diagonal form is made in order that the set of basis vectors belonging to the diagonal form of A will be carried through the barrier in the "de-coupled" form.

$$\begin{pmatrix} v_-(R_0) \\ v_+(R_0) \end{pmatrix} = B \begin{pmatrix} u_0(R_0) \\ u_3(R_0) \end{pmatrix} \qquad \text{(II.18)}$$

by definition of B.

Then the new wave vector $\begin{pmatrix} v_-(r) \\ v_+(r) \end{pmatrix}$ may propagate through the barrier under the influence of the barrier as prescribed by the diagonal barrier penetration matrix, denoted by P.

Therefore,

$$\begin{pmatrix} v_-(R_1) \\ v_+(R_1) \end{pmatrix} = P \begin{pmatrix} v_-(R_0) \\ v_+(R_0) \end{pmatrix} \qquad \text{(II.19)}$$

where P represents that part of the barrier between R_0 and R_1. (Details of the structure of P will be elucidated later.) See Fig. 2, Sec. III for an interpretation of R_0 and R_1.

We obtain our original wave vector by the operation of B again as:

$$\begin{pmatrix} u_0(R_1) \\ u_3(R_1) \end{pmatrix} = \tilde{B} \begin{pmatrix} v_-(R_1) \\ v_+(R_1) \end{pmatrix} \qquad \text{(II.20)}$$

We have made use of the fact that $B^{-1} = \tilde{B}$, since B is a real unitary matrix, as is evidenced by the relation det B = 1, as was elaborated upon earlier.

Briefly stated we then have, in vector notation,
$|V(R_0)\rangle \equiv B|U(R_0)\rangle$ by definition of the transformation B, $|V(R_1)\rangle = P|V(R_0)\rangle$ by the action of the nuclear barrier on the outgoing alpha particle wave function.

$|U(R_1)\rangle = \tilde{B}|V(R_1)\rangle$ by transformation back to our original basis vectors.

The total transformation of these matrix operations carry the alpha particle partial wave amplitudes through the coupling region of the barrier and thus comprise the operations of the matrix we denote as Q. Q becomes $Q = \tilde{B}\,P\,B$.

The matrix elements of Q will be denoted as:

$$Q = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \qquad \text{(II.21)}$$

since Q is real and symmetric, that is to say, Q is Hermitean.

$Q^+ = Q$ where Q^+ is the Hermitean conjugate of Q and is a combination of the complex conjugate and transpose operations: $\tilde{Q}^* \equiv Q^+$.

Returning again to the parametrization method previously described, expressions for the matrix elements of Q may be obtained.

First, however, let us depart from the above discussion for a brief consideration of the barrier penetration matrix P. Further discussion of P will appear in the next section.

P is a diagonal matrix of the form

$$P = \begin{pmatrix} K_- & 0 \\ 0 & K_+ \end{pmatrix} \tag{II.22}$$

where the diagonal elements of P, $K_\mp$ will be approximated by WKB exponential penetrability expressions, the arguments of which are dimensionless integrals.

A diagonal matrix of the form of P would describe the barrier adequately if coupling of the partial wave equations did not occur.

It is the off-diagonal elements of Q which describe the coupling effect in the inner part of the barrier region.

Returning to the evaluation of α, β, and γ by the right triangle method, we have the expression for Q as

$$Q = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} K_- & 0 \\ 0 & K_+ \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \tag{II.23}$$

so that α, β, and γ become:

$$\begin{aligned} \alpha &= K_+ \sin^2\phi + K_- \cos^2\phi \\ \beta &= (K_+ - K_-) \sin\phi \cos\phi \\ \gamma &= K_- \sin^2\phi + K_+ \cos^2\phi \\ &\text{where } \tan\phi = \lambda \mp/q^2 \end{aligned} \tag{II.24}$$

This method of expressing the elements of Q makes numerical evaluation of Q easier.

The next section contains a detailed discussion of the structure of the barrier as it is made up of nuclear, coulomb, and centrifugal barrier componenets. We also consider the manner in which the coupling affects the usual barrier penetration calculation. We shall also perform the numerical calculation of Q.

Let us first, before numerically evaluating Q, look at the barrier regions in greater detail.

III. NUMERICAL EVALUATION OF THE BARRIER PENETRATION MATRIX FOR THE TWO CHANNEL CASE

We have seen how Q is obtained as a barrier penetration matrix in a mathematical formalism. Now let us investigate the physical interpretation of Q.

In regard to Fig. 2 it is apparent that there are essentially four regions in the potential energy diagram to be considered in the solution of the barrier penetration problem.

The first, of course, is the nuclear interior, that is, the region inside R_0 (the inner classical turning point).

The second is the coupling region from R_0 to R_1 (where R_1 is the radius at maximum barrier height) where the barrier penetration matrix is not diagonal and was defined as our Q.

The third is the region from R_1 to R_2 (where R_2 is the outer c.t.p.). We assume that coupling does not occur here and that the barrier penetration matrix is diagonal and that the barrier can be described by a simple WKB (Wentzel, Kramers, Brillouin) calculation.

The fourth is the region, outside of R_2, where the problem can be taken out to infinity, where laboratory observations are made.

Microscopic models have been developed to set the boundary conditions on alpha wave functions at the outer surface of region one. In essence, they take shell model product wave functions for individual nucleons and project out cluster probabilities. We shall not be concerned here with the microscopic models, but shall consider the general effects of surface coupling of outgoing alpha channels under a wide range of inner boundary conditions.

We consider our alpha waves as purely outgoing waves at infinity and as of the same phase at the nuclear surface, which, for our problem, coincides quite closely to R_0.* The latter requirement is forced by the nature of the problem of decay of a quasi-stationary state, where there is a standing wave inside the potential barrier.

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*The nuclear radius, R_0, is given by $R_0 = (1.2A^{1/3} + 2)f$. For the case of P_0^{212}, R_0 is found to be $9.15^0 f$.

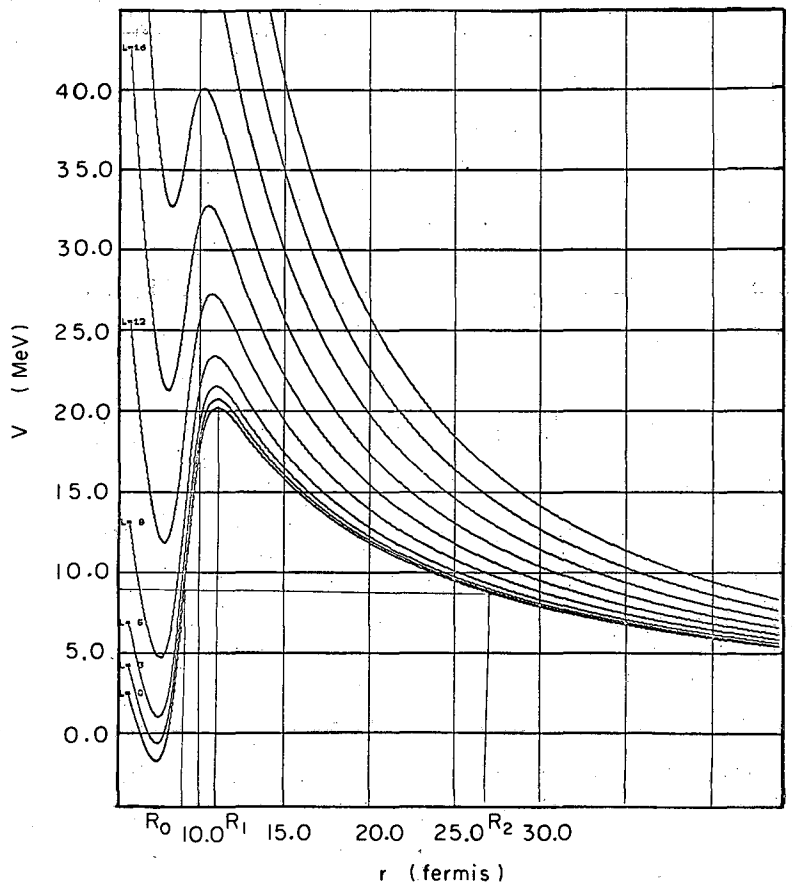

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Fig. 2. This represents the total potential barrier seen by the outgoing or scattered alpha for the Po^{212} to Pb^{208} transition. The total potential is given by $V^T(R) = V^C(R) + V^N(R) + V^L(R)$ and is plotted by a program developed by Poggenburg.[11] The nuclear potential, for the present case, was taken as a Woods-Saxon potential. V^L was taken to be the Langer form of the centrifugal barrier factor as $V^L(R) = \hbar^2/2\mu(l + 1/2)^2/R^2$, and V^C is the Coulomb term.

In the region R_0 to R_1, Q operates on the alpha particle wave vector, as previously discussed, and transforms it to a new wave vector at R_1. It will be noted that since Q involves a WKB calculation by virtue of its dependence on matrix P, we must be careful about the use of Q at or near the inner c.t.p. as WKB calculations are not adequate for this region or near R_2. (For details refer to the discussion in Morse and Feshback.)[8]

At the radius R_1 we can apply a straightforward WKB calculation on the alpha waves. We then extend the problem to infinity where the relative intensities of the alpha particle groups can be calculated. Again, discretion must be used in the region about R_2. At each radius R_0, R_1, and R_2, as a boundary condition, the alpha waves must be matched.

The solution to the barrier problem involves the last three regions.

The solution in the last two regions, as stated before, involves a simple WKB type calculation. The region of interest is the second, between R_0 and R_1 and this is the region under consideration in this paper.

We consider that the alpha particle cluster exists at R_0. We then ask "What effect does the barrier have on it?" Region two can be illustrated in another manner by looking at a more "physical" picture. In this regard, consider Fig. 3. The model represented by Fig. 3 is described further, in this section, under the discussion of the collective coordinate. We have the easily excitable "pear shape," $\ell = 3$, oscillation if only the octupole coupling is considered and the higher levels, the 4+, 5-, etc., are ignored. As we shall see, this is probably not a poor approximation.

The alpha cluster is assumed to form somewhere in the nuclear interior but octupole coupling does not occur until the cluster reaches R_C, or core radius. At the distance R_C, the alpha cluster can exchange angular momentum and energy with the daughter nucleus. (The angular momentum exchanged, as stated before, can be even or odd, as $\ell^{\pi} = 0$, 3-, 4+, 5-, where π is the parity of the phonon state or mode of oscillation.) No coupling occurs inside of R_C.

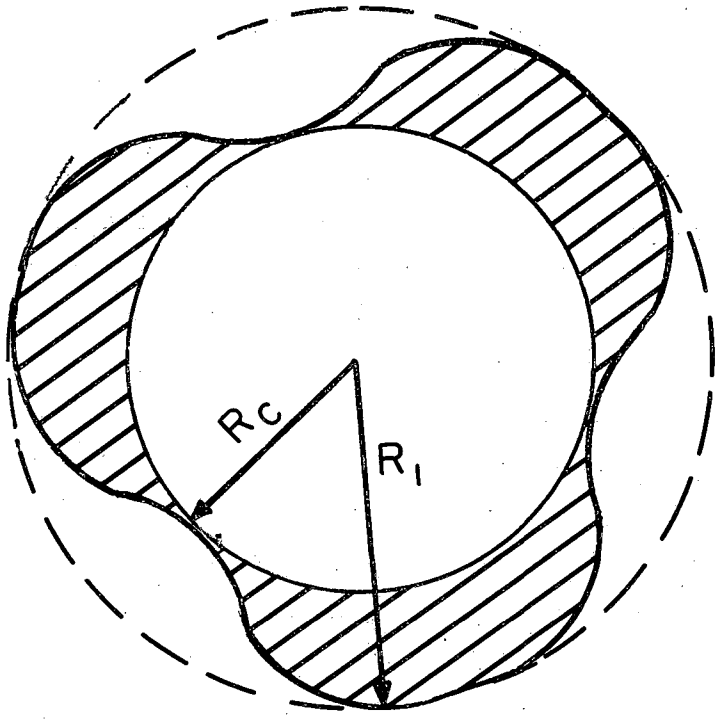

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Fig. 3. The mechanism by which the channel mixing or coupling between the recoil-excited states and the ground state, is represented by the change in momentun and energy of the emitted alpha particle through the exchange of a phonon, or a quasi-particle of integral spin, between it and the recoil nucleus "flipping" the Pb^{208} nucleus between the 0+ and 3- states. That is, the emitted alpha can induce, through phonon exchange, a change in the daughter's spatial distribution from that of a sphere to that of a three-nodal deformation of higher energy and angular momentum.

Let us return to the octupole case. In accord with our model, the core nucleons, inside of R_c, do not participate and remain undisturbed by the octupole deformation.

In the deformed region, R_c to R_1, the nucleons take part in the total collective mode of oscillation and show a new displacement at any time, t_0.

The time average of the octupole (or other) deformation is zero, as $\langle \alpha_{3u} \rangle = 0$, where α_{3u} is the collective coordinate.*

We can consider the node as ripples moving around on the surface, or, looking along a radial direction, we can imagine an undulating or oscillating surface.

There is a net effect, as $\langle \alpha_{3u}{}^2 \rangle \neq 0$. The coupling occurs in this "deformed region" between R_c and R_1.

Of course we have a similar picture for $\ell = 4$, the 4+ level where there are 4 nodes.

In any case, it is pictured that the alpha leaves the parent at the point R_0 where R_0 is taken equal to R_c and at this point the nucleus is the daughter, that is to say, that the barrier "belongs" to the daughter. The coupling of angular momentum or "sharing" occurs between the alpha and the daughter, Pb^{208}.† This is essentially the picture presented earlier.

Let us now dispense with the numerical calculation of P, the penetration matrix, in the region between R_1 and R_2. No coupling occurs here and P can be obtained, as a diagonal matrix, by a simple WKB calculation. We shall then proceed to the numerical calculation of Q.

---

*There is no prolate or oblate, permanent deformation in Pb^{208} since it is doubly magic in Z and N. Lead, Pb^{208}, has no rotational bands, but several excited states seem to have collective character as shape oscillations about an equilibrium sphere.

†Note that R_0 of Pb^{208} and R_0 of Po^{212} do not differ to any extend due to the fact that A. of Pb does not differ greatly from A of Po^{212}, where A is the atomic number.

We shall again consider octupole deformation and, therefore, we are dealing with a 2×2 matrix operator, operating on a 2 component wave vector. The potential at maximum barrier height, $V_{max} = B$ can be read from Fig. 2 directly and is $B^{\ell=0} = 20.22$ MeV at $R_1^{\ell=0} = 10.95$f and $B^{\ell=3} = 20.75$ MeV at $R_1^{\ell=3} = 10.95$f.

As we see $R_1^{\ell=0}$ and $R_1^{\ell=3}$ do not differ noticeably. From the definition of the c.t.p. $R_0^{\ell=0}$ and $R_0^{\ell=3}$ can be found approximately equal to 9.15f and $R_2^{\ell=0} = 26.5$f and $R_2^{\ell=3} = 27.5$f by means of a hand calculation. These values compare quite favorably with those obtained from Fig. 2, which are $R_2^{\ell=0} = 27.30$ f and $R_2^{\ell=3} = 27.32$f indicating that Fig. 2 is valid for our case.

The barrier penetration matrix for the uncoupled region R_1 to R_2 is given as

$$P = \begin{pmatrix} P_0 & 0 \\ 0 & P_3 \end{pmatrix} \qquad \text{(III.1)}$$

To determine the values of P_ℓ for $\ell = 0$ and 3 and also P_ℓ values for $\ell = 4$ and 5, the WKB penetrability calculation was made with the aid of a computer program developed by N. Glendenning.[12]

This computation is valid for any arbitrary barrier slope. For our lead 208 case, the value of the angular momentum state, ℓ, the value of the α decay energy $E_\alpha = 8.95$ MeV for the ground-ground transition and a value called R low are fed in.

The values of P_ℓ are computed directly.

For the purposes of this calculation, $V^N(R)$ was taken to be a Woods-Saxon potential of the general form

$$V^N(R) = \frac{-V_0}{\left[1 + \exp\left(\frac{R - r_0 A^{1/3} - r_1}{d}\right)\right]} \qquad \text{(III.2)}$$

In the present computation the parameters were chosen as, V_0, the nuclear well depth as 35, $-(r_0A^{1/3} + r_1) = 8.71$, and d, the diffuseness, as 0.61f.

As may be noted in the lead-208 case, the Woods-Saxon potential is quite similar to that chosen for K. Poggenburg's calculation where $V^N(R)$ was given as

$$V^N(R) = \frac{-35}{\left(1 + \exp\frac{(R-1.17A^{1/3}-2.17)}{0.576}\right)} \qquad \text{(III.3)}$$

where $V_0 = +35$ again and $1.17A^{1/3} = 6.96$ for $A = 208$ and $r_1 = 2.17$.

The computation involves a search for the radius at maximum barrier height. If the barrier maximum does not exist as in Fig. 4, part (a) where only one inflection point exists, then the value of R low is read in. If the barrier maximum does exist, then the program searches again for the barrier minimum as in Fig. 4, parts (b) or (c). If the minimum is below the alpha decay energy, then the program searches for the inner turning point of the barrier as in Fig, 4, part (b) as in our case.

If the minimum is above the alpha decay energy, i.e., the inner turning point does not exist as in Fig. 4, part (c), then again the R low value read in is used.

Besides determining P_ℓ, the following quantities were also determined: R_{min}, R_1, R_2, η_ℓ, the Coulomb parameter.

The values of R_2 were determined to be $R_2^{\ell=0} = 26.39$f, $R_2^{\ell=3} = 26.66$f and $R_2^{\ell=4} = 26.84$f which compare favorably with the values observed from Fig. 2 which are $R_2^{\ell=0} = 27.30$f, $R_2^{\ell=3} = 27.32$f and $R_2^{\ell=5} = 27.8$f. The $\ell=4$ wave was not plotted.

The values of P_ℓ were found to be:

$$P_0 = 3.105 \times 10^{-14}$$

$$P_3 = 6.665 \times 10^{-23}$$

$$P_4 = 5.771 \times 10^{-32}$$

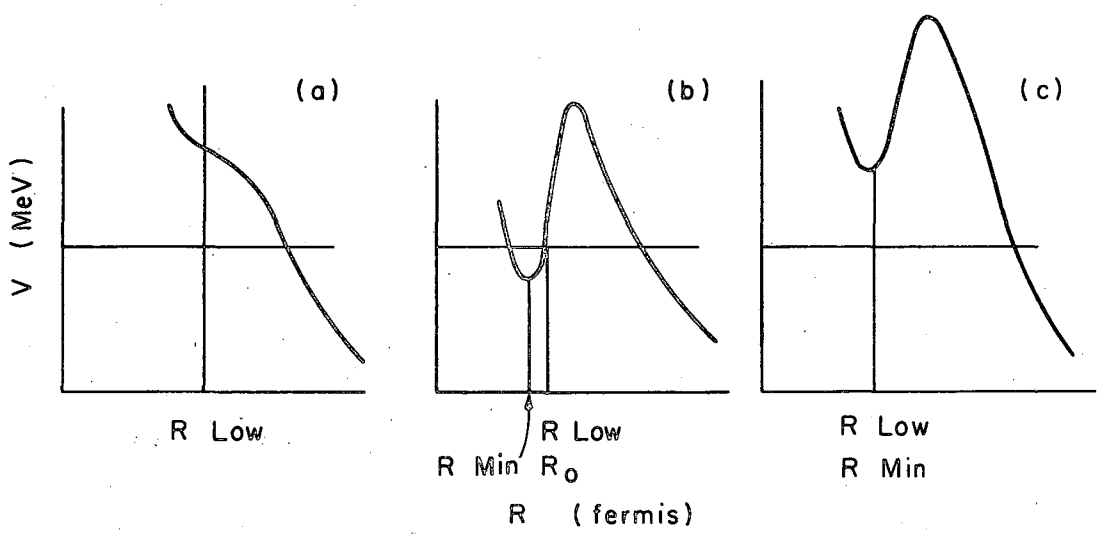

Fig. 4. The WKB barrier penetration calculation can be made for any arbitrary barrier shape by use of Glendenning's computer program in the region where coupling does not occur.

The value for the ground-ground state transition penetration factor P_0 was previously calculated by J. O. Rasmussen[9] and was found to be $P_0 = 1.32 \times 10^{-13}$. Again, as in the above calculation, the penetrability factor was calculated from R_0 out.

The potential assumed by Rasmussen for the calculation of P_0 was an Igo potential of the form:

$$V^N(R) = -V_0 \; e^{\frac{-(r-1.17A^{1/3})}{0.574}} \qquad \text{(III.4)}$$

where the well depth V_0 was taken as V_0=1100 MeV.

The Igo potential used in optical-model analysis represents the real part of the nuclear alpha-surface interaction and is determined by alpha-scattering experiments, over a wide range of target elements.[10]

The larger penetrability factor was determined when the more attractive Igo potential was used.

It was felt that for the present probelm, the Igo potential dropped off too rapidly and that the Woods-Saxon potential better represented the nuclear barrier in the surface region.

The barrier penetration matrix for the R_1 to R_2 region is given by the 2×2 diagonal matrix P, where only octupole vibrational contributions are considered in the coupling region.

$$P = \begin{pmatrix} 3.10 \times 10^{-14} & 0 \\ 0 & 6.66 \times 10^{-23} \end{pmatrix}$$

Now let us turn our attention to the coupling region once more and evaluate the non-diagonal coupling matrix, Q. For the present, let us concern ourselves with the octupole case and consider Q as a 2×2 matrix.

It may be recalled that in Sec. II, we had Q in parametric form

$$Q = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \qquad \text{(III.5)}$$

where
$$\alpha = K_+\sin^2\phi + K_-\cos^2\phi$$
$$\beta = (K_+ - K_-)\sin\psi\cos\phi$$
$$\gamma = K_-\sin^2\phi + K_+\cos^2\phi$$
and
$$\tan\phi = \lambda_{\mp}/q^2$$

In order to evaluate α, β, and γ, we must evaluate the eigenvalues $\lambda_{\mp}$ belonging to $v_{\mp}$ and also q^2, the coupling matrix element between the 0+ and 3- states of lead.

$K_{\mp}$, which appears in the barrier penetration matrix $P = \begin{pmatrix} K_- & 0 \\ 0 & K_+ \end{pmatrix}$ in the coupling region, can be evaluated by a WKB-type expression over the small range ΔR, $K_{\mp}$ are given as

$$K_{\mp} = e^{-\frac{1}{\hbar}\sqrt{2\mu[B-(E_\alpha-\lambda_{\mp})]}\,\Delta R} \qquad \text{(III.6)}$$

where B is the barrier height and μ is the reduced mass.

In this approximation, we have replaced the nuclear barrier by a rectangular barrier extending from R_0 to R_1 and E_α to B. It is obvious that this approximation greatly over-emphasizes the effect of the barrier. Later we discuss a triangular barrier approximation which more accurately represents the actual barrier. See Fig. 5, page 34.

In the coupling region, the interaction between the 0+ and 3- state causes a decrease in the "effective barrier height" as seen by the presence of $\lambda_{\mp}$ in the above expression.

An enhancement of the decay rate over that which would occur if coupling did not occur, is found to exist for some boundary conditions. The eigenvalues, $\lambda_{\mp}$ were previously expressed in terms of k_0^2, k_3^2, and q^2, and in order to determine the values of $\lambda_{\mp}$ we must evaluate k_0^2, k_3^2, and q^2. We shall now turn out attention to this problem.

The matrix elements k_0^2, k_3^2, and q^2 must be properly defined in energy units chosen in this case to be $1/f^2$, as stated earlier.

The equations relevant to the octupole case are

$$u''_0(r)+k_0^2(r)u_0(r)+q^2(r)u_3(r) = 0$$
$$u''_3(r)+k_3^2(r)u_3(r)+q^2(r)u_0(r) = 0 \qquad \text{(III.7)}$$
$$k_0^2(R_0) \equiv \frac{2u}{\hbar^2}(E_\alpha - V^\alpha)1/f^2$$
$$k_3^2(R_0) \equiv \frac{2u}{\hbar^2}(E_\alpha - \hbar\omega + \frac{\hbar^2}{2u}\frac{12}{R_0^2} - V^\alpha)1/f^2$$

where

$$V = (\frac{2Ze}{R} + V^N) \text{ MeV}$$

Here we have considered the evaluation of k_0^2 and k_3^2 and some "representative" radius R_0, in the coupling region R_0 to R_1. Since the coupling region extends over a relatively small interval, $R_1-R_0=\Delta R=1.8f$, compared to the total barrier extent, which is greater than 30f, the radial dependent energy variables are assumed to constant in this region, as a first approach to the solution of this problem. See Fig. 2.

The values, from the above expressions for $k_0^2(R_0)$ and $k_3^2(R_0)$, are determined to be 16.0 MeV or 3.12 $1/f^2$ and 18.6 MeV or 3.96 $1/f^2$, respectively, where the centrifugal barrier term has not been included in $k_3^2(R_0)$. The centrifugal barrier term $\left(2u/\hbar^2\right)\left(12/R_0^2\right)$ equals 0.72 MeV or 0.14 $1/f^2$.

The expression for the coupling matrix element $q^2(R_0)$ is derived in Sec. I, and is given as

$$q^2(R_0) = \frac{2u}{\hbar^2} R_0 \left.\frac{dV^N}{dr}\right]_{R_0} \sqrt{7/8\pi}\, \frac{\hbar^{1/2}}{(B_3C_3)^{1/4}} \qquad \text{(III.8)}$$

Using the relation, $\hbar\omega = \hbar\sqrt{C_3/B_3}$ we may eliminate the octupole mass parameter, B_3, from the above expression and thus obtain:

$$q^2(R_0) = \frac{2u}{\hbar^2} \left.\frac{dV^N}{dr}\right]_{R_0} \sqrt{7/8\pi}\sqrt{\frac{\hbar\omega R_0^2}{C_3}} \qquad \text{(III.9)}$$

We find $\Delta V/\Delta R=9.1$ MeV/f over the radial region $\Delta R=1.8f$ as determined from Fig. 2. The potential variation, $\Delta V=(20.22-8.95)$ MeV, so that

ΔV becomes 11.27 MeV and $\Delta R = (10.95 - 9.15)f = 1.8f$, with the result that $\Delta V/\Delta R$ becomes the above value, 9.1 MeV/f.

Once we have a value for the surface tension parameter C_3, we can uniquely determine the q^2 value. There is an uncertainty as to the precise value of C_3 and thus the value of q^2. We will set forth several methods for determining C_3 and calculate the respective values of q^2.

Bohr and Mottelson[1] derived a general expression from the hydrodynamic model considerations and arrived at the following expression for any arbitrary deformation, ℓ:

$$C\ell = (\ell-1)(\ell+2)R_0^2 w - \frac{3(\ell-1)Z^2e^2}{2\pi(\ell+1)R_0} \qquad \text{(III.10)}$$

where $w = \dfrac{15.4\ A^{2/3}}{4\pi R_0^2}$ MeV and Z is the charge on the daughter or recoil nucleus.

For the octupole case, $\ell=3$, we obtain $C_3 = 279$ MeV and therefore $B_3 = 41.5\hbar^2$/MeV. For this value of C_3, q^2 becomes 4.40 MeV or 0.86 $1/f^2$.

Lane and Pendlebury[13] quote a value of C_3 determined from electromagnetic transition probability data and found

$$C_3 = (1500 \pm 750)\text{MeV}$$

The above C_3 value was obtained by experimentally determining the lifetime for the E3 transition for the lead, Pb^{208} excited state de-excitation, into the ground state which was found to be $\tau = (4\pm2)\times10^{-11}$ sec. Then the probability per unit time for such a multipole transition is given by[14]

$$T = \frac{8\pi(\ell+1)}{C[(2\ell+1)!!]^2\hbar}\left(\frac{\omega}{C}\right)^{2\ell+1}B(\ell) \qquad \text{(III.11)}$$

for an arbitrary $E\ell$ transition, where we can convert

$$\tau \text{ to } T \text{ by } T = \frac{1}{\tau}$$

and where $B(\ell)$ is the reduced transition probability for vibrational

excitation given by

$$B(\ell) = \frac{(2\ell+1)(\frac{3}{4}\pi\ ZeR_0{}^{\ell})^2\hbar}{2\sqrt{B\ell\ C\ell}} \qquad \text{(III.12)}$$

for single phonon excitations.

Again we shall use the expression for the energy quanta in our harmonic oscillator model given before as

$$\hbar\omega_{\ell} = \hbar\sqrt{\frac{C\ell}{B\ell}}\ .$$

$C\ell$ can now be determined from the values for τ and $\hbar\omega_{\ell}$. For the Lane and Pendlebury value of C_3 we have $q^2 = 1.9$MeV.

Among the more recent methods available to us for the determination of C_3 is the DWBA (distorted wave Born approximation) analysis of inelastic scattering of charge particles. The DWBA is applicable to our coupled channel purely collective model for single phonon excitation. T. Stovel and N. Hintz[3] obtain $C_3 = 468$MeV for (P,P') scattering. R. H. Crannel[15], et al., determined the value of C_3 to be $C_3 = 793$ MeV, obtained from inelastic electron scattering. The respective values of q^2 are $q^2 = 3.41$ MeV and $q^2 = 2.82$ MeV, for these references. See Table I.

The calculation of the eigenvalues $\lambda_{\mp}$, of the system when the alpha particle is in the surface region in the octupole case, can easily be performed now that q^2 is determined. As may be recalled,

$$\lambda_{\mp} = \frac{k_{30}{}^2 \mp \sqrt{k_{30}{}^4 + 4q^4}}{2} \qquad \text{(III.13)}$$

giving $\lambda_{\mp}(R_0) = \{^{-3.74}_{6.35}\}$ MeV for C_3=210 MeV or 1.0 $1/f^2$. We shall follow though with the above example for q^2=5.07 MeV.

We may now obtain $K_{\mp}$ as a function of $\lambda_{\mp}$ by using the WKB triangle barrier approximation. See Fig. 5. This approximation is superior to the rectangular barrier approximation given previously in this section, which overestimates the effect of the barrier attenuation of the alpha wave.

Table I. Values of the surface tension parameter and respective values of the coupling constant.

C_3 [MeV]	q^2 [MeV]	q^2 [$1/f^2$]	Reference for C_3
210.	5.07	1.00	Value used for calculations in this paper
279.	4.40	0.86	(Hydrodynamic model theory)[1]
468.	3.40	0.67	((P,P') inelastic scattering)[3]
793.	2.82	0.55	((e,e') inelastic scattering)[14]
1500.	1.91	0.37	(Electromagnetic lifetime)[12]

If the coupling interaction were not present, the alpha-particle wave would be attenuated by a factor given as

$$C_0 = e^{-\sqrt{\frac{2u}{\hbar}}\int_0^{\Delta R}(V-E)^{1/2}dr} \qquad \text{(III.14)}$$

for

$$V-E = \left(\frac{\Delta E}{\Delta R}\right)r$$

so that C_0 becomes $C_0 = e^{-\frac{2u}{\hbar}(2/3)(\Delta E_0)^{1/2}\Delta R}$ where $\Delta E_0 = B-E_\alpha$.

If coupling occurs, the penetration factors, $K_{\mp}$, corresponding to the eigenvalues $\lambda_{\mp}$, can be found in terms of C_0. The upper penetration factor, K_-, can be determined from the "small triangle" in Fig. 5.

Since the "small triangle" for K_- is similar to the triangle for C_0, $\ln K_-$ is given by

$$\ln K_- = -(\ln C_0) \times \left(\frac{\Delta E_-}{\Delta E_0}\right)^{3/2} \qquad \text{(III.15)}$$

By similar triangles the lower penetration factor, K_+, is given by

$$\ln K_+ = -(\ln C_0) \times \left(\frac{\Delta E_+}{\Delta E_0}\right)^{3/2} \qquad \text{(III.16)}$$

where $\Delta E_{\mp} = E_0 + \lambda_{\mp}$.

The values thus determined from the WKB triangle method for $K_{\mp}$ are

$$K_{\mp} = \begin{Bmatrix} 0.360 \\ 0.027 \end{Bmatrix}$$

The alpha wave vector, v_-, in the uncoupled representation, penetrates more readily when the coupling occurs because it "sees" a narrower and lower barrier, since its effective alpha decay energy is 12.69 MeV and not 8.95 MeV. See Fig. 5. The penetration factor for this wave, K_-,

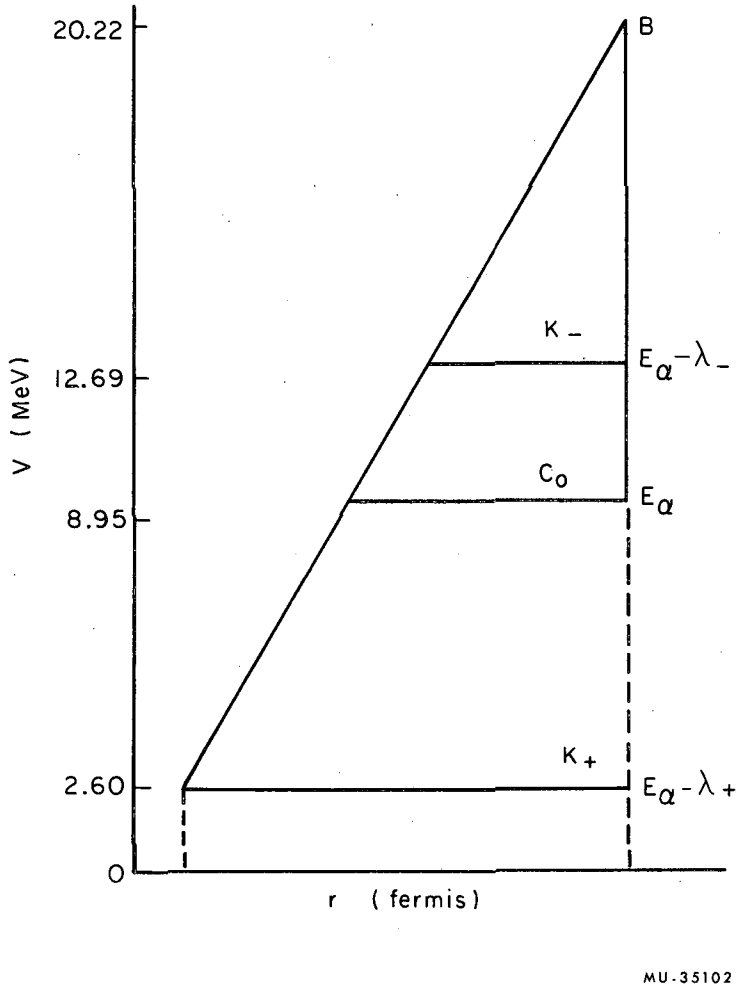

Fig. 5. The barrier penetration matrix elements $K_{\pm}$ may be obtained by application of the triangle method in the region of the barrier where coupling occurs.

is larger than the penetration factor C_0, when there is no coupling so that the alpha wave penetrates more readily.

We are now in a position to determine the matrix elements of Q and we find

$$\alpha = 0.244, \quad \beta = 0.160, \text{ and } \gamma = 0.144$$

thus Q becomes

$$Q = \begin{pmatrix} 0.244 & 0.160 \\ 0.160 & 0.144 \end{pmatrix}$$

We shall interpret the meaning of Q and its action on some two-component wave vector $\begin{pmatrix} u_0 \\ u_3 \end{pmatrix}$ by a graphical method, as will be done at the end of this section for the C_3 = 210 MeV case. See Table I. C_3 = 210 MeV is the lowest value and thus gives an estimate of the largest possible coupling effect.

The present calculation gives an order of magnitude estimate of the enhancement of alpha-particle penetrabilities due to the effects of collective excited states.

Because of the existence of a coupling in the surface region, the alpha-wave function is able to propagate more easily through the initial part of the barrier region than would be possible if the coupling were absent.

If the coupling did not exist, the ℓ=0 alpha-particle wave would be attenuated through the coupling region, from the inner turning point to the maximum barrier height, by a factor of K=0.156.

Now consider the finite coupling case with an alpha-wave vector of the proper linear combination of the ℓ=0 and ℓ=3 parts that constitute the lower uncoupled solution v_-. The v_- wave vector effectively propagates through the coupling region as an alpha group of higher energy, 3.74 MeV above E_α, where λ_-=3.74 MeV (see Fig. 5). Thus this wave vector will only be attenuated by a factor of 0.360.

As can be seen, the enhancement in the alpha-particle wave amplitude due to the coupling interaction, is about a factor of 2 and thus the enhancement in the alpha intensity is up by a factor of 4.

The hypothesis might be made that the inner boundary conditions on the alpha-particle wave will always be near the optimum to exploit the enhancement of the alpha particle's penetration through the coupling region in the decay to vibrational states.

Such is not the case in deformed daughter nuclei involving rotational states, but it could be the case for vibrational states.

Graphical methods will be used at this point to indicate the action of Q on the impinging alpha particle. Consider some input wave vector

$$\begin{pmatrix} u_0^I \\ u_3^I \end{pmatrix}$$

and by the action of Q upon it we obtain some output wave vector

$$\begin{pmatrix} u_0^0 \\ u_3^0 \end{pmatrix}$$

so that

$$\begin{pmatrix} u_0^0 \\ u_3^0 \end{pmatrix} = Q \begin{pmatrix} u_0^I \\ u_3^I \end{pmatrix} . \qquad \text{(III.17)}$$

The wave-vector components u_0^0 and u_3^0 are linearly related to the components u_0^I and u_3^I by

$$\begin{pmatrix} u_0^0 \\ u_3^0 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \begin{pmatrix} u_0^I \\ u_3^I \end{pmatrix}$$

$$u_0^0 = \alpha u_0^I + \beta u_3^I \qquad \text{(III.18)}$$

$$u_3^0 = \beta u_0^I + \gamma u_3^I$$

We assume, in writing these equations, that the super position of the u_3^I part on the u_0^I is valid.

The first case that will be considered is the one where u_0^I is taken to be unity and we shall investigate the effects on u_0^0 and u_3^0, due to the channel mixing, by letting u_3^0 deviate from zero. We now have the equations

$$u_0^0 = \alpha + \beta u_3^I$$
$$u_3^0 = \beta + \gamma u_3^I \qquad \text{(III.19)}$$

for the relation of the input to the output vector components.

In graphing u_0^0 vs u_3^I we have a linear plot where α is the intercept and β is the slope. Similarly for u_3^0 vs u_3^I, β is the intercept and γ is the slope.

The two lines intersect at $u_3^I = -4.00$ and $u_0^0 = u_3^0 = -0.40$ (see Fig. 6).

Plots of the form $|u_0^0|^2$ vs u_3^I and $|u_3^0|^2$ vs u_3^I have a minimum to the left of $u_3^I = -1.0$. The plot of $|u_0^0|^2 + |u_3^0|^2$ vs u_3^I is the sum of the above two curves (see Fig. 7).

In plotting the ratio u_0^0/u_3^0 vs u_3^I a discontinuity occurs at $u_3^0 = 0$, where $u_3^I = -1.14$. At this value of u_3^I, the ratio of u_0^0/u_3^0 goes to $+\infty$ and around and up from $-\infty$.

The discontinuity in the plot of u_3^0/u_0^0 vs u_3^I occurs at $u_3^I = -1.50$ where u_3^0/u_0^0 goes to $-\infty$. The discontinuities are avoided by plotting in the range

$$u_3^I = -1.0 \text{ to } 1.0.$$

For the plot of u_0^0/u_3^0 vs u_3^I and u_3^0/u_0^0 vs u_3^I (see Fig. 8).

The effect of the enhancement, due to the $\ell=3$ wave, can best be demonstrated in a plot of

$$\frac{|u_0^0|^2}{|u_0^0|^2 + |u_3^0|^2} \quad \text{vs} \quad u_3^I$$

and

$$\frac{|u_3^0|^2}{|u_0^0|^2 + |u_3^0|^2} \quad \text{vs} \quad u_3^I$$

The displacement of the maximum of the $\frac{|u_0^0|^2}{|u_0^0|^2 + |u_3^0|^2}$ vs u_3^I from $u_3^I = 0$ to the right of $u_3^I = 1.0$, and the displacement of the minimum of the $\frac{|u_3^0|^2}{|u_0^0|^2 + |u_3^0|^2}$ vs u_3^I from $u_3^I = 0$ is a measure of the coupling strength (see Fig. 9).

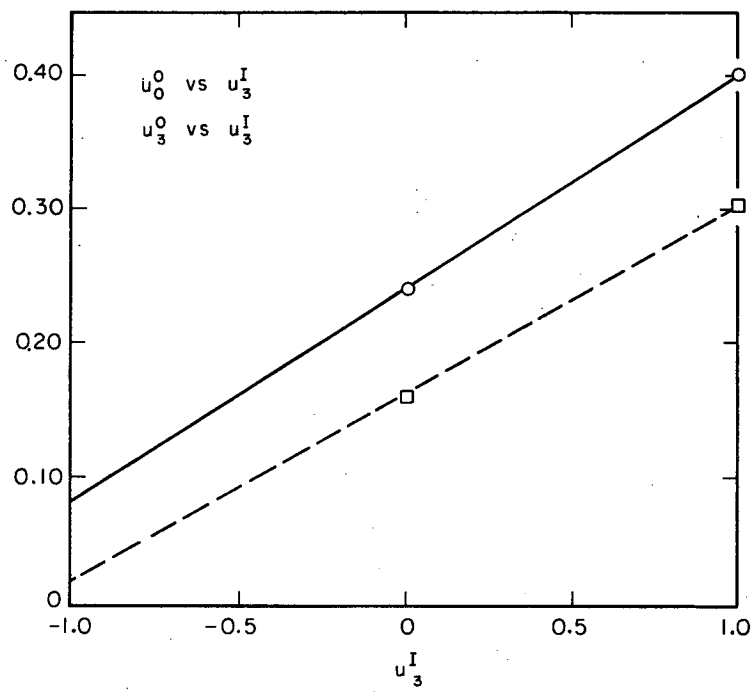

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Fig. 6. The wave vector components u_0^0 and u_3^0 are linearly related to the components u_0^I and u_3^I by the matrix elements of Q, the barrier penetration matrix in the coupling region.

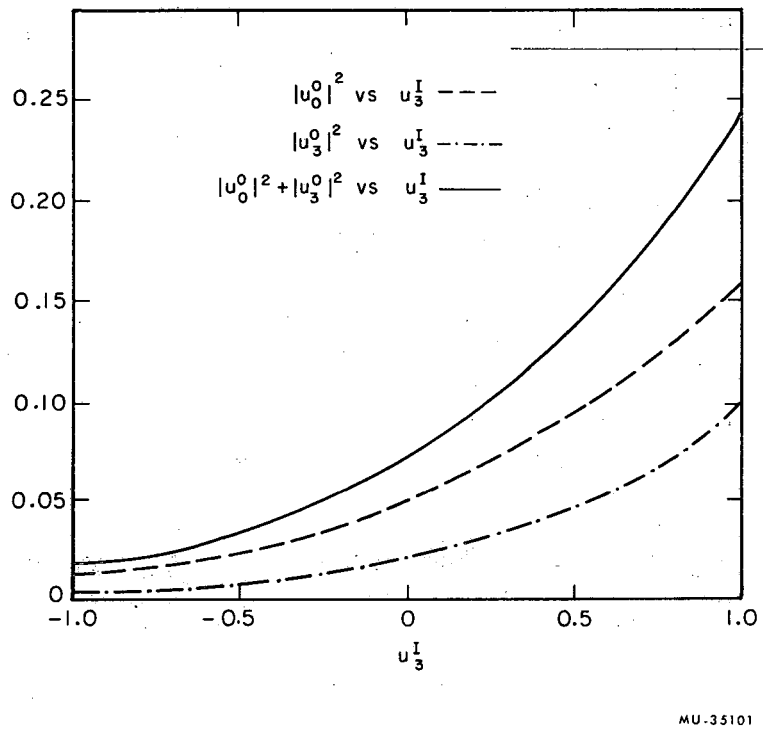

Fig. 7. The square of the wave vector components $|u_0^0|^2$ and $|u_3^0|^2$ are plotted vs. u_3^I in the quadratic representation of the effect of the coupling interaction.

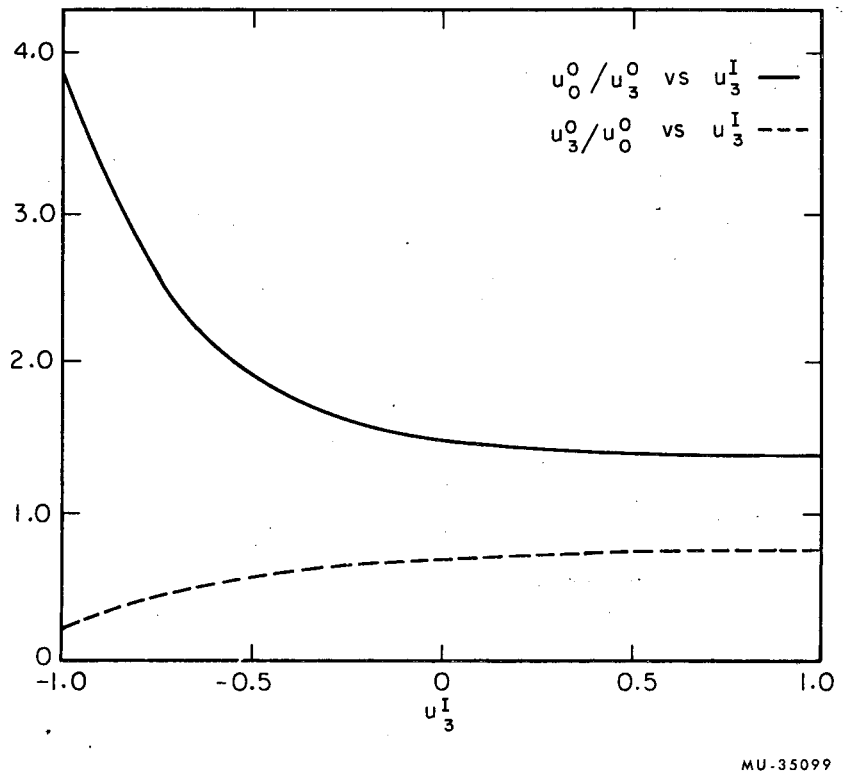

Fig. 8. The ratio of the wave vector amplitudes for the $\ell=0$ to $\ell=3$ and $\ell=3$ to $\ell=0$ waves are plotted vs. the input $\ell=3$ wave amplitude u_3^I.

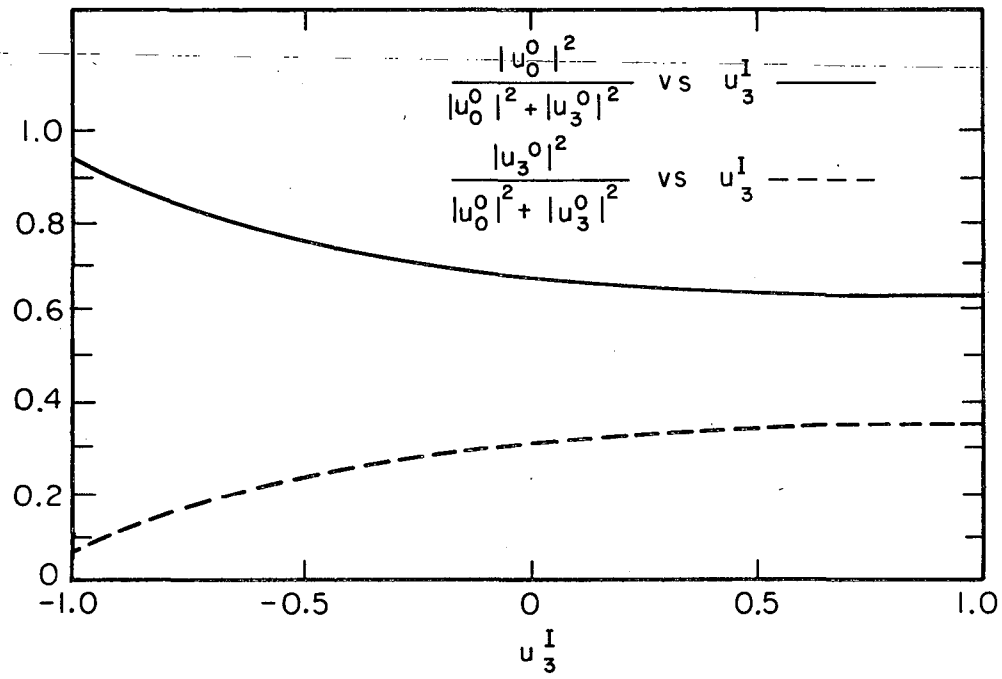

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Fig. 9. The effect of the enhancement in the ground-ground state transition, due to the $\ell=3$ wave, is seen in the plots of the square of the amplitude ratios in the input wave amplitude.

IV. THE GENERALIZED PROBLEM: THE THREE- AND FOUR-CHANNEL CASES

We have seen how the matrix Q can be obtained for the two-channel problem where only $\ell=0$ and $\ell=3$ waves are considered. The Q matrix was derived in terms of a transformation matrix, B and a diagonal barrier penetration matrix, P. The matrix B was constructed from the eigenvector $|V\rangle$ belonging to eigenvalues $\lambda_{\mp}$ which were in turn constructed out of $\lambda_{\mp}$. The eigenvalues, $\lambda_{\mp}$, satisfy the eigenvalue equation in the diagonal representation of a matrix A, a Hamiltonian 2×2 matrix which is a linear operator operating in a two dimentional Euclidean space, and are obtained by the Secular determinate method.

The matrix Q was constructed out of B and P as an operator that first transforms an eigenvector $|U\rangle$ as bases vectors in the "old" non-diagonal representation to the bases vectors $|V\rangle$ in a "new" diagonal representation and carries them through a potential barrier by P in this "new" uncoupled representation and then transforms these eigenvectors $|V\rangle$ back to the original set of bases vectors $|U\rangle$. The matrix Q found in the preceeding section was thus found to be a barrier penetration matrix, applicable for the problem where only two-channel coupling occurs. In this regard, Q will act upon an escaping alpha particle represented by $|U\rangle$ a two-component eigenket in a two-dimentional Hilbert space.

Now let us extend and generalize our problem to one with an unspecified number of open final-state channels, say N channels. The Hamiltonain matrix, A, will be an $N \times N$ matrix with associated bases vectors $U_1 \ldots, U_N$.

In general, for N open channels, that is, for N final states of the system where the alpha decay can leave the daughter in any angular momentum state, ℓ, an $N \times N$ Hamiltonian matrix Q can be constructed that would operate on an N component column vector.

There are essentially four Pb^{208} levels as determined by (α,α') and (P,P') scattering experiments that significantly contribute to the coupling. (See Fig. 1) The four significant levels in lead are seen to

to be the 0+ ground state, 3-, 4+, and 5- excited states. The relative importance of these levels decreases for higher ℓ's and thus the previous two-channel calculation is not a bad representation of the coupling effect on the alpha-particle barrier penetration.

The general problem for the lead-208 daughter thus would be completely described for a four-channel Q matrix, represented by a 4×4 matrix in a four-dimensional Hilbert space.

We will discuss briefly the three-channel case where the 4+ state of the daughter, lead, is also taken into account.

Considering the lowest three levels of lead, the ground state 0+, the excited octupole, 3- state, and the 16 pole, 4+ state, we again have our differential equations in matrix notation

$$A_{40}|U\rangle = -|U\rangle'' \qquad \text{(IV.1)}$$

where $|U\rangle$ is a three-component vector in a three-dimensional Hilbert space as

$$|U\rangle = \begin{pmatrix} U_0 \\ U_3 \\ U_4 \end{pmatrix} \qquad \text{(IV.2)}$$

and the Hamiltonian A is given as

$$A_{40} = \left(\begin{array}{cc|c} k_0^2 & q_{30}^2 & q_{40}^2 \\ q_{30}^2 & k_3^2 & q_{43}^2 \\ \hline q_{40}^2 & q_{43}^2 & k_4^2 \end{array}\right) \qquad \text{(IV.3)}$$

A_{40} will denote the 3×3 matrix. And A_{30} will denote the previously discussed 2×2 matrix.

As indicated A_{30} is a submatrix of A_{40}. The coupling terms q_{43}^2 (see Fig. 2), are assumed to be small in magnitude relative to q_{30}^2 and q_{40}^2. Hence A_{40} becomes

$$A_{40} = \begin{pmatrix} k_0^2 & q_{30}^2 & q_{40}^2 \\ q_{30}^2 & k_3^2 & 0 \\ q_{40}^2 & 0 & k_4^2 \end{pmatrix} \qquad \text{(IV.4)}$$

As is seen, the form of A_{40} is that of a real symmetric matrix which must be true for any Hamiltonian.

Again the method of secular determinants can be applied in order to determine the three eigenvalues $\lambda_{\mp,0}$ belonging to the three-component vector $|V\rangle$ in the eigenvalue expression

$$A|V\rangle = \lambda_{\mp,0}\ |V\rangle \text{ for } |V\rangle \equiv \begin{pmatrix} V_+ \\ V_- \\ V_0 \end{pmatrix} \qquad \text{(IV.5)}$$

Proceeding in a manner analogous to that used for the calculation of Q_{30} in Sec. II, for the two-channel case, we can again use the method of secular determinates in order to detemine the thre eigenvalues $\lambda_{\mp},0$ belonging to the three-component vector $|V\rangle$ in the eigenvalue expression:

$$|V\rangle = \begin{pmatrix} v_- \\ v_+ \\ v_0 \end{pmatrix} \qquad \text{(IV.6)}$$

The transformation matrices can then be deduced from $\lambda_{\mp},0$.

The values of $K_{\mp},0$ can also be determined by the use of an approximate WKB barrier calculation, so that the expression for the matrix elements of Q_{40} can be determined.

This same treatment may also be extended to the four-channel case, to determine Q_{50}.

It is felt that adequate values of C_3, C_4, and C_5 may be obtainable from a consideration of (α,α') inelastic data in order to determine the values of q_{40}^2 and q_{50}^2 as well as q_{30}^2, but for the present we shall not consider such calculations.

When such values are determined, it will be possible to calculate the matrix elements of Q_{40} and Q_{50} in a manner analogous to the calculation of Q_{30} performed in the previous section.

V. CONCLUSION

A purely collective vibrational model has been applied to the alpha decay of Po^{212} to Pb^{208}. We have calculated a non-diagonal 2×2 matrix, Q, which describes the action of the coupling part of the barrier on the emitted alpha, when the two-channel coupling approximation is considered. The effect of the barrier-penetration matrix, Q, on an outgoing two component alpha wave $|U\rangle = \begin{pmatrix} u_0 \\ u_3 \end{pmatrix}$ was examined by graphical methods and discussed.

A short discussion of the three- and four-channel cases was also included.

Further analysis of the problem, with regard to the consideration of (α,α') scattering may be in order, but for the present the determination of the form of the octupole coupling constant and its magnitude, as well as the barrier penetration matrix in the coupling region lends further insight into the alpha decay to spherical-daughter nuclei, where only vibrational excitations occur.

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