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Berkeley, California

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March 15, 1965

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ABSTRACT

We investigate the hypothesis that the ultraviolet singularities of quantum field theory are connected with conformal invariance at very high energies, and we demonstrate how they can be removed in an elementary way.

The main point is that a whole class of new and independent solutions of the field equation is connected with the conformal group. These solutions cannot be described within the framework of the usual test function class S of the distribution calculus. But it is possible to define new test functions which are appropriate for dealing with these new solutions and their essential singularity on the light cone.

In the quantum field theory of the Klein-Gordon equation one now obtains two sets of creation and annihilation operators, the second of which belongs to the new solution. The metric of the Hilbert space concerned is semidefinite and degenerate. It also turns out that the vacuum is infinitely degenerate. But the secondary vacua have a vanishing norm. As an immediate consequence of the semidefinite metric the mass-independent singularities on the light cone of the generalized Green's function cancel each other.

The results obtained bear a striking resemblance to the ideas already discussed by Heisenberg in his attempt to remove the fundamental difficulties of current quantum field theory.

An unknown world of new physical quantities, which is isomorphic to that of the linear momenta but nevertheless quite different, seems to be connected with the additional conformal symmetry at very high energies.

I. INTRODUCTION

It has been conjectured previously[1,2] that the fundamental difficulties in quantum field theory caused by the ultraviolet singularities of the Green's functions,[3] are closely connected with conformal invariance at very high energies. The reasons are the following: we obtain the nonrelativistic limit of the relativistic kinematics by expanding the basic relation[4]

$$E = (\underset{\sim}{p}^2 + m^2)^{\frac{1}{2}}$$

between the energy E, the momentum $\underset{\sim}{p}$ and the rest mass m of a particle in a power series of $\underset{\sim}{p}/m$ and considering only the first term in $\underset{\sim}{p}$:

$$E = m(1 + \underset{\sim}{p}^2/m^2) \approx m + \tfrac{1}{2}\underset{\sim}{p}^2/m \quad .$$

This approximation changes the group theoretical structure of the relativistic framework, and instead of the Lorentz group we have the Galilei group in the nonrelativistic limit. If we now consider the extreme relativistic limit, we can expand the above energy-momentum relation in a power series of $m/|\underset{\sim}{p}|$ and again consider only the first term in $\underset{\sim}{p}$:

$$E = p(1 + m^2/\underset{\sim}{p}^2) \approx p, \quad \text{with} \quad p = (\underset{\sim}{p}^2)^{\frac{1}{2}} \quad .$$

If we confine ourselves to this extreme relativistic approximation, we again have a different group theoretical framework: in addition to Lorentz invariance we also have invariance under dilatations and the special four-parameter conformal group. The physical background and some aspects of the mathematical structure of this larger symmetry group are discussed in references.[1,2,5,6] Several authors[7-14] also have analyzed the dilatation invariance in connection with special field theoretical models.

There are two basic difficulties which make it very hard to study the physical significance of conformal invariance at very high energies. The first one is connected with the experimental situation: the energies of the present accelerators are not high enough to test the consequences of the conformal invariance with sufficient accuracy.[5] But this problem is mainly an economical one and there is hope that in the near future larger accelerators will be built.

The second difficulty is by far more fundamental and is connected with the ultraviolet singularities of quantum field theory. Because of this lack of consistency one cannot rigorously examine a particular field theory mathematically and ask whether it really has the properties at very high energies, which are suggested by conformal invariance.

But perhaps one should reverse the question: is it possible that the ultraviolet singularities have their origin in the conformal invariance at very high energies? There are several reasons which favor this conjecture, and though many questions remain to be answered, we present strong evidence in this paper that those singularities are

indeed connected with the conformal symmetry and that they can be removed by taking into account the whole fifteen-parameter conformal group.

The crucial point is that the eigenfunctions of the energy-momentum operators no longer form a complete set if one considers the whole conformal group. This was proved in reference 1 for all finite dimensional representations. The situation is, of course, far more complicated for representations of infinite degree. In the latter case there are unitary representations,[15,16] but if one considers special physical examples, it seems as if these unitary representations are not the physically interesting ones.[1]

We here consider the Klein-Gordon equation, first without rest mass and later with rest mass, and obtain the following results: the eigenfunctions of the four-parameter special conformal group have an essential singularity on the light cone and for that reason they cannot be expanded in terms of plane waves. In order to regularize these eigenfunctions we define in Section II a new class of test functions, which can be constructed by a simple mapping of the test function class S in conventional distribution theory.[17,18]

In Section III we consider the Klein-Gordon equation without rest mass and its two different types of classical solutions: the first ones are the usual superpositions of plane waves; the second ones are the superpositions of the eigenfunctions of the special conformal group. It is possible to define a metric for a joint subclass of these two classes by means of the usual scalar product of the classical Klein-Gordon equation. The metric is semidefinite and degenerate.

This subclass of solutions serves as the class of test functions in Section IV, where we consider the quantum field theory of the Klein-Gordon equation in the framework of operator-valued distributions. We get two independent sets of creation and annihilation operators, and the metric of the Fock space of the two combined sets is semidefinite, but remains positive definite for each set alone.

Because of the commutation relations between the two sets of creation and annihilation operators, we can define a conformal invariant generalization of, for instance, the usual Green's function $\Delta^{(+)}(x-y)$, which now vanishes in the case of rest mass zero and which does not contain the mass independent singularities on the light cone for a particle with finite rest mass.

Since these mass independent singularities cause the main difficulties in conventional quantum field theory,[3] it should now be possible to construct consistent quantum field theories without ultraviolet divergencies. Whether this is indeed the case has to be determined by analyzing special examples, for instance, quantum electrodynamics or the pseudoscalar coupling $\bar{\psi}(x)\gamma_5\,\psi(x)\,\varphi(x)$, etc. The above discussed method should at least be applicable for all couplings with a dimensionless coupling constant, for in these theories the interaction terms become conformal invariant in the limit of negligible rest mass, too. The special significance of this class of field theories in connection with the high energy limit has recently been emphasized by A. Bastai et al.[19]

In Section V we compare our results with the qualitatively similiar approach of Heisenberg and co-workers to the problem of ultraviolet divergencies.[20-22] We also discuss briefly the possible physical significance of the eigenvalues, which belong to the generators of the special conformal group and which seem to constitute an isomorphic, yet unknown, world to the world of the linear momenta. Finally, we mention how the field theoretical concept of causality is concerned by the conformal group.

II. THE GENERATORS OF THE SPECIAL CONFORMAL GROUP AND THEIR EIGENFUNCTIONS

The special conformal group consists of the transformations[1]

$$x^{\mu'} = \sigma^{-1}(x)(x^{\mu} - c^{\mu}x^2), \quad \text{with} \quad \mu = 0, 1, 2, 3,$$

$$\sigma(x) = 1 - 2cx + c^2x^2 \quad ,$$

$$x^2 = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 \quad , \tag{1}$$

where c^{μ} are the four real group parameters. These transformations have the basic property

$$ds'^2 = \sigma^{-2}(x)ds^2 \ , \quad ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \quad ,$$

$$ds'^2 = (dx'^0)^2 - (dx'^1)^2 - (dx'^2)^2 - (dx'^3)^2 \ , \tag{2}$$

and therefore induce a local change in the units of length. If we denote by R the transformation

$$x^{\mu} \rightarrow Rx^{\mu} = -x^{\mu}/x^2 \ , \quad \text{with} \quad \mu = 0, 1, 2, 3, \tag{3}$$

then the special conformal transformations (1) can be written as[1]

$$x^{\mu'} = RT(c)\, Rx^{\mu} \quad , \quad \text{with} \quad \mu = 0, 1, 2, 3 \ ,$$

where $T(c)$ means a translation $x^\mu \to x^\mu + c^\mu$. We get therefore the eigenfunctions of the generators of the infinitesimal conformal transformations by applying R to the plane waves, the corresponding eigenfunctions of the translation group,

$$e^{ipx} \to e^{-ih\cdot x/x^2} ,$$

where h is a four-fector, the analog to the momentum four-fector p.

The appropriate mathematical framework to deal with the eigenfunctions e^{ipx} is the theory of distributions on the test function class S.[17,18] Since the functions e^{-ihx/x^2} have an essential singularity on the light cone, they do not fit into this framework. But we can define a new type of distributions by considering the class S_R of test functions $\psi(x)$, where, for instance,

$$\psi(x) = \frac{1}{(x^2)^4} \varphi(Rx), \quad \text{with} \quad \varphi(x) \in S .$$

Since $\varphi(x)$ vanishes faster than any power of x if x goes to infinity, $\psi(x)$ vanishes faster than any power of x^2 on the light cone.

The analog to the Fourier transformation is the H transformation,

$$g(x) = \frac{1}{(2\pi)^4} \int d^4h \, \tilde{g}(h) \, e^{-ihx/x^2} ,$$

of a certain function $g(x)$. It follows that

$$g(Rx) = \frac{1}{(2\pi)^4} \int d^4h \, \tilde{g}(h) \, e^{ihx} ,$$

and we therefore have the inversion

$$\tilde{g}(h) = \int d^4x \, g(Rx) \, e^{-ihx} ,$$

or, by again making the transformation R under the integral,

$$\tilde{g}(h) = \int \frac{d^4x}{(x^2)^4} \, g(x) \, e^{ihx/x^2} .$$

For the test functions $\psi(x)$ it follows that

$$\tilde{\psi}(h) = \Box_h^{\,4} \, \tilde{\varphi}(h) , \quad \text{with} \quad \Box_x = \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x_\mu} ,$$

where $\tilde{\varphi}(h)$ is the Fourier transform of $\varphi(x) \in S$. This means that $\tilde{\psi}(h) \in S$!

If $g(x)$ is a distribution on S_R, we have

$$\begin{aligned} g(x) \, \langle \psi(x) \rangle \equiv \int d^4x \, g(x) \, \psi(x) &= \frac{1}{(2\pi)^4} \int d^4h \, \tilde{g}(h) \, \tilde{\varphi}(-h) \\ &\equiv \frac{1}{(2\pi)^4} \, \tilde{g}(h) \, \langle \tilde{\varphi}(-h) \rangle . \end{aligned} \tag{4}$$

For $\tilde{g}(h) = 1$ we get the analog of the usual δ function in x space

$$\delta_R(x) = \frac{1}{(2\pi)^4} \int d^4h \; e^{-ihx/x^2} \quad , \tag{5}$$

with

$$\delta_R(x) \langle \psi(x) \rangle = \left[(x^2)^4 \; \psi(x) \right]_{\frac{x^\mu}{x^2} = 0 \; ; \text{ where } \mu = 0,1,2,3.}$$

The δ function in h space has the representation

$$\delta(h) = \frac{1}{(2\pi)^4} \int \frac{d^4x}{(x^2)^4} \; e^{ih\cdot x/x^2} \quad . \tag{6}$$

III. THE SOLUTIONS OF THE KLEIN-GORDON EQUATION

As to the physical applications the situation is as follows: since at low energies the discontinuous energy and mass spectra play an important part in atomic physics, conformal invariance cannot always be valid; for the conformal group transforms a given value of a physical quantity continuously into other values of the same quantity. But it seems as though at extremely high energies, where the rest masses and the discontinuous part of the energy spectra are negligible, the dilatations and the special conformal group become essential.[2,5,6]

If, for example, we neglect the rest mass in the Klein-Gordon equation, the equation

$$\Box F(x) = 0 \tag{7}$$

is invariant under the special conformal group (1), which has been well known since 1910 (a comprehensive list of references is given in reference 1) ! The essential new point here is that a whole class of new and independent solutions is connected with this additional invariance. This class is perhaps not as important for classical field theory as it obviously is for quantum field theory, and this may be the reason why no one yet has considered these solutions in connection with modern quantum field theory--as far as I know. It also turns out that the mathematical calculus of operator-valued distributions[23] is extremely appropriate for dealing with these new functions. Taking

into account this additional physical and mathematical framework, it is quite easy to remove the troublesome singularities on the light cone in current field theory, at least those which are mass-independent !

If the plane waves

$$e_p(x) = \frac{1}{(2\pi)^{3/2}} e^{ipx} , \text{ with } p^2 = 0 , \tag{8}$$

and their superpositions

$$f(x) = \frac{1}{(2\pi)^{3/2}} \int d^4p \, \delta(p^2) \, \tilde{f}(p) \, e^{ipx} \tag{9}$$

are solutions of Eq. (7), then the functions

$$g_h(x) = \frac{1}{x^2} e_h(Rx) = \frac{1}{(2\pi)^{3/2}} \frac{1}{x^2} e^{-ihx/x^2} , \text{ with } h^2 = 0 , \tag{10}$$

and their superpositions

$$g(x) = \frac{1}{(2\pi)^{3/2}} \frac{1}{x^2} \int d^4h \, \delta(h^2) \, \tilde{g}(h) \, e^{-ihx/x^2} \tag{11}$$

are solutions, too.

For any two solutions $F_1(x)$, $F_2(x)$ of the Klein-Gordon equation, (7), we have the scalar product

$$(F_1 , F_2) = -i \int d^3 \underset{\sim}{x} \left(F_1^* \partial_0 F_2 - \partial_0 F_1^* F_2 \right)_{x^0 = 0} .$$

For $p_0 = (\underset{\sim}{p}^2)^{1/2} \geqslant 0$ we get

$$(e_{p_1}, e_{p_2}) = 2p_0\, \delta(\underset{\sim}{p}_1 - \underset{\sim}{p}_2) \quad , \tag{12}$$

or

$$(f_1, f_2) = \int \frac{d^3\underset{\sim}{p}}{2p_0}\, \tilde{f}_1^{\,*}(p)\, \tilde{f}_2(p) \; .$$

In a similar way we obtain, for $h_0 = (\underset{\sim}{h}^2)^{1/2} \geqslant 0$,

$$(g_{h_1}, g_{h_2}) = 2h_0\, \delta(\underset{\sim}{h}_1 - \underset{\sim}{h}_2) \tag{13}$$

or

$$(g_1, g_2) = \int \frac{d^3\underset{\sim}{h}}{2h_0}\, \tilde{g}_1^{\,*}(h)\, \tilde{g}_2(h) \quad ,$$

where $\delta(\underset{\sim}{h})$ is the three-dimensional analog to the four-dimensional δ function (6).

The most important quantity is

$$(e_p, g_h) = \gamma(p, h) \equiv \left[p_0 \Delta_h - h_0\, \Delta_h^2 \right] t(\underset{\sim}{p}, \underset{\sim}{h});\; h_0 , p_0 \geqslant 0 ,$$

$$\text{where } \Delta = \text{Laplace operator,} \tag{14}$$

$$t(\underset{\sim}{p}, \underset{\sim}{h}) = \frac{1}{(2\pi)^3} \int d^3\underset{\sim}{x}\; e^{i\underset{\sim}{p}\underset{\sim}{x}}\; e^{-i\underset{\sim}{h}\cdot\underset{\sim}{x}/\underset{\sim}{x}^2} \quad .$$

Here $t(\underset{\sim}{p}, \underset{\sim}{h})$ is the mixed Fourier-H- transform of the distribution

$$\delta(\underset{\sim}{x}) * \delta_R(\underset{\sim}{x}) = \int d^3\underset{\sim}{y}\, \delta(\underset{\sim}{x} - \underset{\sim}{y})\, \delta(R\underset{\sim}{x} - \underset{\sim}{y}) .$$

This is the convolution of the distributions $\delta(\underset{\sim}{x})$ and $\delta_R(\underset{\sim}{x})$ (see Appendix); $\gamma(p, h)$ has the essential properties

$$\int \frac{d^3\underset{\sim}{p}}{2p_0}\, \gamma(p, h_1)\, \gamma(p, h_2) = 2h_0\, \delta(\underset{\sim}{h}_1 - \underset{\sim}{h}_2) , \tag{15}$$

$$\int \frac{d^3\underset{\sim}{h}}{2h_0}\, \gamma(p_1, h)\, \gamma(p_2, h) = 2p_0\, \delta(\underset{\sim}{p}_1 - \underset{\sim}{p}_2) .$$

The norm of the function $F=f+g$ is

$$\begin{aligned}(F, F) = &\int \frac{d^3\underset{\sim}{p}}{2p_0}\, \tilde{f}^*(p)\, \tilde{f}(p) + \int \frac{d^3\underset{\sim}{p}}{2p_0} \frac{d^3\underset{\sim}{h}}{2h_0}\, \tilde{f}^*(p)\, \tilde{g}(h)\, \gamma(p, h) \\ &+ \int \frac{d^3\underset{\sim}{h}}{2h_0} \frac{d^3\underset{\sim}{p}}{2p_0}\, \tilde{g}^*(h)\, \tilde{f}(p)\, \gamma(p, h) + \int \frac{d^3\underset{\sim}{h}}{2h_0}\, \tilde{g}^*(h)\, \tilde{g}(h) ,\end{aligned} \tag{16}$$

where the $\tilde{f}(p)$, $\tilde{g}(h)$ are such that the norm exists (see Appendix). This norm is zero if

$$\tilde{g}(h) = -\int \frac{d^3\underset{\sim}{p}}{2p_0}\, \gamma(p, h)\, \tilde{f}(p) , \tag{17}$$

for it follows from Eqs.(17) and (15) that

$$\tilde{f}(p) = -\int \frac{d^3\underset{\sim}{h}}{2h_0}\, \tilde{g}(h)\, \gamma(p, h)$$

and

$$\int \frac{d^3 h}{2\tilde{h}_0} \tilde{g}^*(h)\, \tilde{g}(h) = \int \frac{d^3 p}{2\tilde{p}_0} \tilde{f}^*(p)\, \tilde{f}(p) \quad .$$

If we therefore have a basis $\{ \tilde{\varphi}_k(p) \}$ of the $\tilde{f}(p)$ with

$$\int \frac{d^3 p}{2\tilde{p}_0} \tilde{\varphi}_k^*(p)\, \tilde{\varphi}_\ell(p) = \delta_{k\ell} \, , \; k, \ell = 1, 2, \cdots , \tag{18}$$

we can define a basis $\{\tilde{\psi}_k(h)\}$ for the $\tilde{g}(h)$ by

$$\tilde{\psi}_k(h) = -\int \frac{d^3 p}{2\tilde{p}_0} \gamma(p,h)\, \tilde{\varphi}_k(p) \, . \tag{19}$$

It follows that the function $c_1 f_k + c_2 g_k$ has the norm

$$|c_1|^2 - c_1^* c_2 - c_2^* c_1 + |c_2|^2 \quad . \tag{20}$$

Since the determinant of this quadratic form vanishes, the metric is degenerate and semidefinite. This can easily be seen by transforming the expression (20) into diagonal form.

Further mathematical details concerning this section are given in the Appendix.

IV. QUANTUM FIELD THEORY OF THE KLEIN-GORDON EQUATION.

First we consider the Klein-Gordon equation without mass and expand the real scalar field operator $A(x)$ as usual in terms of creation and annihilation operators:

$$A(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3\underset{\sim}{p}}{2p_0} \; a(p)\, e^{ipx} + a^{\dagger}(p)\, e^{-ipx}, \quad p_0 = (\underset{\sim}{p}^2)^{1/2} .$$

Here $A(x)$, $a(p)$, and $a^{+}(p)$ are operator-valued distributions[23] on the space S of the test functions φ, and with

$$\varphi(x) = \frac{1}{(2\pi)^4} \int d^4q \; \tilde{\varphi}(q)\, e^{iqx}, \text{ with } \varphi(x) \text{ real,}$$

we have

$$A(x) < \varphi(x) > \equiv \int d^4x \; \varphi(x) \; A(x)$$

$$= \frac{1}{(2\pi)^{3/2}} \int \frac{d^3\underset{\sim}{p}}{2p_0} \; a(p)\, \tilde{\varphi}(-p) + a^{\dagger}(p)\, \tilde{\varphi}^{*}(-p)$$

$$\equiv \frac{1}{(2\pi)^{3/2}} \left[a(p) \; \langle \tilde{\varphi}(-p) \rangle + a^{\dagger}(p) \, \langle \tilde{\varphi}^{*}(-p) \rangle \right] ,$$

where $p_0 = (\underset{\sim}{p}^2)^{1/2}$

If we now put

$$\tilde{\varphi}(-p) = \tilde{\varphi}_k(p) , \quad p_0 = (\underset{\sim}{p}^2)^{1/2} \geqslant 0 ,$$

$$\left[b_k, b_\ell\right] = 0, \left[b_k^\dagger, b_\ell^\dagger\right] = 0,$$
$$\left[b_k, b_\ell^\dagger\right] = \delta_{k\ell} . \tag{25}$$

Most important are the commutation relations between the two sets

$$\{ a_k; a_k^\dagger\} , \quad \{ b_k; b_k^\dagger\} .$$

According to Eq. (14) we define

$$\left[a(p), b(h)\right] = 0, \left[a^\dagger(p), b^\dagger(h)\right] = 0,$$
$$\left[a(p), b^\dagger(h)\right] = \gamma(p,h) . \tag{26}$$

If we now make the same mapping of the basis $\{\tilde{\varphi}_k(p)\}$ on the basis $\{\tilde{\psi}_k(p)\}$ as in Eq. (19), we get

$$\left[a_k, b_\ell\right] = 0, \left[a_k^\dagger, b_\ell^\dagger\right] = 0 ,$$
$$\left[a_k, b_\ell^\dagger\right] = - \delta_{k\ell} . \tag{27}$$

The minus sign of the right-hand side of the last relation is decisive. Although this minus sign is a consequence of our special way of connecting the two sets $\{\tilde{\varphi}_k(p)\}$ and $\{\tilde{\psi}_k(h)\}$ with each other, it can always be obtained by redefinition. The reason is that the two sets

$$\{a_k;\ a_k^\dagger\}\ ,\quad \{b_k;\ b_k^\dagger\}$$

are mutually independent !

We can construct the Hilbert space in the usual way: Let $|0\rangle$ be the unique vacuum state with the properties

$$a_k\ |0\rangle\ =\ 0,\ b_k\ |0\rangle\ =\ 0,\ \langle 0|0\rangle\ =\ 1\ ;$$

by means of the creation operators $a_k^\dagger$, $b_k^\dagger$ we then construct the state vectors

$$a_k^\dagger\ |0\rangle\ ,\ b_k^\dagger\ |0\rangle\ ,\ a_k^\dagger\ a_\ell^\dagger\ |0\rangle\ ,\ a_k^\dagger\ b_\ell^\dagger\ |0\rangle\ ,\ b_k^\dagger\ b_\ell^\dagger\ |0\rangle\ ,\ \text{etc.}$$

The metric in this Hilbert space is no longer positive definite; because of the commutation relations (27) the vector

$$c_1\ a_k^\dagger\ |0\rangle + c_2\ b_k^\dagger\ |0\rangle$$

has the norm

$$|c_1|^2 - c_1^* c_2 - c_2^* c_1 + |c_2|^2\ ,$$

which is the same as in Eq. (20). The norm of the special vectors

$$a_k^\dagger\ |0\rangle + b_k^\dagger\ |0\rangle$$

vanishes. As a consequence of the relations (27) these states also have the property

$$a_\ell \left(a_k^\dagger \, |0\rangle + b_k^\dagger \, |0\rangle \right) = 0 \ ,$$
$$b_\ell \left(a_k^\dagger \, |0\rangle + b_k^\dagger \, |0\rangle \right) = 0 \ , \text{ for all } \ell \ , \ k \ . \tag{28}$$

This means that the vacuum is degenerate. But these secondary vacua have a vanishing norm. The degeneration of the vacuum is an obvious consequence of the mass zero of the particles concerned. In this sense the above result is a kind of an inversion of the Goldstone theorem that a degenerate vacuum implies a particle without mass.[24]

There is, of course, the problem of how the probability interpretation is concerned if the metric in Hilbert space is no longer positive definite. We shall not discuss this question here because it has to be considered in connection with a detailed dynamical theory. But it is clear that the metric in the subspaces of the sets

$$\{a_k ; a_k^\dagger\} \quad \text{or} \quad \{b_k ; b_k^\dagger\}$$

respectively remains positive definite. Yet the completeness relation has to be altered, since, for example,

$$-a_k^\dagger \, |0\rangle \, \langle 0| \, b_k$$

is an additional projection operator for the state $a_k^\dagger \, |0\rangle$.

It can be seen immediately that the conformal invariant generalization of, for instance, the usual $\Delta^{(+)}$ function

$$\Delta^{(+)}(x - y) = \langle 0| A(x) A(y) |0\rangle$$

vanishes:

With $F_k(x) = A(x) \langle\varphi_k(x)\rangle + B(x) \langle\psi_k(x)\rangle$

it follows from Eqs. (22), (25), and (27) that

$$\langle 0| F_k(x) F_\ell(y) |0\rangle = 0 \text{ , for all } k \text{ , } \ell \text{ .} \tag{29}$$

The cancellation of the different parts obviously is a consequence of the semidefinite metric.

The same procedure can be used to remove the mass-independent singularities of the Green's functions for a particle with finite rest mass m. The corresponding creation and annihilation distributions are $a(p)$ and $a^\dagger(p)$ with the commutation relations

$$\left[a(p_1), a(p_2)\right] = 0, \left[a^\dagger(p_1), a^\dagger(p_2)\right] = 0 \text{ ,}$$

$$\left[a(p_1), a^\dagger(p_2)\right] = 2p_0\, \delta(\underset{\sim}{p}_1 - \underset{\sim}{p}_2), \; p_0 = \omega \equiv (\underset{\sim}{p}^2 + m^2)^{\frac{1}{2}} \text{ .}$$

We then define

$$a_k = \int \frac{d^3\underset{\sim}{p}}{2\omega} \; a(p) \; \tilde{\varphi}_k(p \; ; \; m = 0) \; ,$$

$$a_k^{\dagger} = \int \frac{d^3\underset{\sim}{p}}{2\omega} \; a^{\dagger}(p) \; \tilde{\varphi}_k^{*}(p \; ; \; m = 0) \; .$$

with the same set $\{\tilde{\varphi}_k(p \; ; \; m = 0)\}$ as in Eq. (18); $d^3\underset{\sim}{p}$ is mass-independent, too, It follows that

$$\left[a_k, \; a_\ell\right] = 0, \quad \left[a_k^{\dagger}, \; a_\ell^{\dagger}\right] = 0 \; ,$$

$$\left[a_k, \; a_\ell^{\dagger}\right] = N_{k\ell}(m) \; , \tag{30}$$

where

$$N_{k\ell}(m) = \int \frac{d^3\underset{\sim}{p}}{2\omega} \; \tilde{\varphi}_\ell^{*}(p \; ; \; m = 0) \; \tilde{\varphi}_k(p \; ; \; m = 0) = N_{\ell k}^{*}(m) \; .$$

Since $\omega^{-1}(m \neq 0) < \omega^{-1}(m = 0)$ we have

$$N_{kk}(m) < 1 \quad \text{for} \quad m \neq 0 \tag{31}$$

and the boundary conditions

$$N_{k\ell}(m = 0) = \delta_{k\ell}, \; N_{k\ell}(m = \infty) = 0 \; . \tag{32}$$

In addition we define

$$a_k^{(0)} = \int d^3\underset{\sim}{p} \left[\frac{a(p)}{2\omega} \tilde{\varphi}_k(p)\right]_{m=0} ,$$

$$a_k^{(0)\dagger} = \int d^3\underset{\sim}{p} \left[\frac{a^\dagger(p)}{2\omega} \tilde{\varphi}_k(p)\right]_{m=0} ,$$

with the same commutation relations for the $a_k^{(0)}$, $a_k^{(0)\dagger}$ as for the a_k, $a_k^\dagger$ in Eqs. (22). We now set

$$a_k = a_k^{(0)} + a_k^{(m)} ,$$

$$a_k^\dagger = a_k^{(0)\dagger} + a_k^{(m)\dagger} ,$$

and assume for these operators, in accordance with the relations (22), (30), and (32), the commutation relations

$$\left[a_k^{(0)}, a_\ell^{(m)}\right] = 0, \quad \left[a_k^{(0)\dagger}, a_\ell^{(m)\dagger}\right] = 0, \quad \left[a_k^{(m)}, a_\ell^{(m)}\right] = 0, \quad \left[a_k^{(m)\dagger}, a_\ell^{(m)\dagger}\right] = 0 \tag{33}$$

$$\left[a_k^{(0)}, a_\ell^{(m)\dagger}\right] = \left[a_k^{(m)}, a_\ell^{(0)\dagger}\right] = N_{k\ell} - \delta_{k\ell}, \quad \left[a_k^{(m)}, a_\ell^{(m)\dagger}\right] = \delta_{k\ell} - N_{k\ell} .$$

In the same way as in the case without rest mass we now introduce the operators b_k, b_k^+. We merely have to substitute the quantities $a_k^{(0)}$, $a_k^{(0)+}$ for a_k, a_k^+ in Eqs. (27). The same reasons which led to the relations (33) suggest the commutation relations

$$\left[a_k^{(m)}, b_\ell\right] = 0, \left[a_k^{(m)+}, b_\ell^{+}\right] = 0 ,$$
$$\left[a_k^{(m)}, b_\ell^{+}\right] = \left[b_k, a_\ell^{(m)+}\right] = \delta_{k\ell} - N_{k\ell} . \tag{34}$$

From Eqs. (30), (33), and (34) it follows immediately that

$$\langle 0| \, (a_k + b_k) \, (a_\ell^{+} + b_\ell^{+}) \, |0\rangle = \delta_{k\ell} - N_{k\ell} . \tag{35}$$

In this generalization of the $\Delta^{(+)}$ function for a field theory with a finite rest mass the mass-independent terms cancel each other, as can be seen by setting $N_{k\ell}(0) = \delta_{k\ell}$. Since these mass-independent terms of the Green's functions cause the main troubles in quantum field theory, it now seems to be possible to construct consistent theories without divergencies. The crucial point is that, for instance, the interaction term of quantum electrodynamics is conformal invariant for negligible rest mass.[2] If one therefore makes a perturbation theory of the nonlinear equations, both types of field operators discussed above can become important at very high energies. If one discards one class one can have troubles, and this obviously has been the case in conventional quantum field theory. Whether the solution proposed here of that fundamental problem is really the right answer has, of course, to be found out by detailed comparison with experiments.

V. REMARKS

It is remarkable that many of the basic features which appear in the framework considered here have already been discussed by Heisenberg in his approach to the problem of removing the divergencies in quantum field theory:[20-22] the nondefinite metric, a degeneration of the vacuum, and even the "ghosts," which here appear to be the real quantities $h^2 = 0$. It looks as though the framework of the conformal group [inhomogeneous Lorentz group, dilatations and the group (1)] is extremely appropriate for discussing Heisenberg's ideas in a rigorous mathematical way, the origin of which is the basic structure of space and time.

The essential new physical quantities which emerge in connection with the conformal group are the four-vectors $(h_0, \underset{\sim}{h})$. Since E. Bessel-Hagen was the first who noted their existence for electrodynamics,[25] we shall call them "Bessel-Hagen momenta." These momenta seem to constitute a second world which is isomorphic to the world of the usual momenta but is nevertheless quite different ! It was shown in reference 2 by some semiclassical arguments that the Bessel-Hagen momenta may be expressed at very high energies by the four-momenta of the free particle considered, and by its distance from the origin at time $t = 0$. But the question remains whether these quantities have a physical meaning also for $h^2 \neq 0$ in a similar way to that in which the four-momenta p are essential for $p^2 \neq 0$. At present we know nothing about this second world except that it seems to merge with the world of the four-momenta at very high energies.

Finally it should be noted that the concept of local commutativity (causality) of field operators has to be altered slightly in connection with the conformal group.[1] The reason is that the transformations (1) are not linear and the coordinate lines are no longer straight. It follows from Eqs. (1) that x^2 goes over into $\sigma^{-1}(x)\, x^2$. Since the sign of $\sigma(x)$ depends on the values of the parameters c^μ, the usual definition of space-like and time-like distances is not invariant under the conformal group.[1,26] But if we define distances by the differential form (2), as is usual in general relativity, then "space-like" and "time-like" again have an invariant meaning, as can be seen immediately from Eq. (2).

Since we are dealing with local fields this should be sufficient.

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APPENDIX

We wish to discuss a few more properties of the solutions $f(x)$ and $g(x)$, introduced in Section III. We do not here intend to build up a complete and rigorous mathematical framework for the new type of distributions connected with the conformal group. Therefore, if we use the notion of the integral, this sometimes has only a symbolic meaning, in the same sense as the use of Dirac's δ function in connection with integrals. There seems to be no difficulty in dealing with these new distributions in the same way as with the known classes of distributions.[17,18]

The crucial point, which determines the structure of the Hilbert space discussed in Section IV, obviously is the fact that the two classes $\{f(x)\}$ and $\{g(x)\}$ of the classical solutions (9) and (11) overlap. If we call this joint subclass $\{u(x)\}$, the functions $u(x)$ may be expanded either in terms of plane waves e^{ipx} or in terms of the functions e^{-ihx/x^2} :

$$u(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3\underset{\sim}{p}}{2p_0} \tilde{u}_1(p) e^{ipx}$$

$$= \frac{1}{(2\pi)^{3/2}} \frac{1}{x^2} \int \frac{d^3\underset{\sim}{h}}{2h_0} \tilde{u}_2(h) e^{-ihx/x^2} \quad . \qquad \text{(A1)}$$

From this we can derive a relation between $\tilde{u}_1(p)$ and $\tilde{u}_2(h)$, which serves as condition for $u(x)$, $\tilde{u}_1(p)$, $\tilde{u}_2(h)$: the same arguments as in Section II lead to the inversion

$$\frac{\tilde{u}_2(h)}{2h_0} = -\frac{1}{(2\pi)^{3/2}}\int \frac{d^3\underset{\sim}{x}}{\underset{\sim}{x}^2}\, u\left(0, \frac{\underset{\sim}{x}}{\underset{\sim}{x}^2}\right) e^{i\underset{\sim}{h}\,\underset{\sim}{x}}$$

$$= -\frac{1}{(2\pi)^{3/2}}\int \frac{d^3\underset{\sim}{x}}{(\underset{\sim}{x}^2)^2}\, u(0,\underset{\sim}{x}) e^{i\underset{\sim}{h}\,\underset{\sim}{x}/\underset{\sim}{x}^2} \quad . \qquad \text{(A2)}$$

So that $\tilde{u}_2(h)$ and its derivatives may be reasonable functions, $u(0,\underset{\sim}{x}/\underset{\sim}{x}^2)$, $x^i u(0,\underset{\sim}{x}/\underset{\sim}{x}^2)$ etc., $i = 1,2,3$, must have an appropriate behavior if x^i goes to infinity and zero. If, for instance, $u(0,\underset{\sim}{x})$ and its derivatives vanish faster than any power of $1/x^i$ for $x^i \to \infty$, $i = 1,2,3$, and faster than any power of x^i for $x^i \to 0$, the integrals in Eqs. (A2) exist if the functions considered are continuous, too. Such functions are, for example, the products

$$\varphi(\underset{\sim}{x})\psi(\underset{\sim}{x}) \quad ,$$

where $\varphi(\underset{\sim}{x}) \epsilon S^{(3)}$ and $\psi(\underset{\sim}{x}) \epsilon S_R^{(3)}$ are the three-dimensional analogs of the four-dimensional test functions discussed in Section II.

From (A1) and (A2) it follows that

$$\frac{\tilde{u}_2(h)}{2h_0} = -\frac{1}{(2\pi)^3}\int \frac{d^3\underset{\sim}{x}}{(\underset{\sim}{x}^2)^2}\, e^{i\underset{\sim}{h}\cdot\underset{\sim}{x}/\underset{\sim}{x}^2} \int \frac{d^3\underset{\sim}{p}}{2p_0}\, \tilde{u}_1(p) e^{-i\underset{\sim}{p}\,\underset{\sim}{x}} \quad .$$

If we consider the improper limit $\tilde{u}_1(p) = 1$, we get

$$\tilde{u}_2(h) = -4\pi h_0 \delta(\underset{\sim}{h}) \quad .$$

This follows from

$$\int \frac{d^3\underset{\sim}{p}}{2p_0} \; e^{-i\underset{\sim}{p}\,\underset{\sim}{x}} = \frac{2\pi}{\underset{\sim}{x}^2}$$

and

$$\frac{1}{(2\pi)^3}\int \frac{d^3\underset{\sim}{x}}{(\underset{\sim}{x}^2)^3} \; e^{i\underset{\sim}{h}\cdot\underset{\sim}{x}/\underset{\sim}{x}^2} = \delta(\underset{\sim}{h}) \quad .$$

If, on the other hand, we have $\tilde{u}_1(p) = 2p_0\delta(\underset{\sim}{p})$, we get

$$\tilde{u}_2(h) = -\frac{1}{2\pi} \quad ,$$

since

$$\int \frac{d^3\underset{\sim}{x}}{(\underset{\sim}{x}^2)^2} \; e^{i\underset{\sim}{h}\cdot\underset{\sim}{x}/\underset{\sim}{x}^2} = \frac{2\pi^2}{h_0} \quad .$$

This last relation can be verified by making the substitution $\underset{\sim}{x} \to \underset{\sim}{x}/\underset{\sim}{x}^2$ and the consequent evaluation of the resulting integral in the complex plane.

Next we consider the subclass of solutions, whose Fourier and H transforms are connected by relation (17). If we denote these transforms by $\tilde{v}_1(p)$, $\tilde{v}_2(h)$ respectively and if we define

$$v_1^{\,a}(\underset{\sim}{x}) = \frac{1}{(2\pi)^3}\int \frac{d^3\underset{\sim}{p}}{2p_0} \; \tilde{v}_1(p)e^{i\underset{\sim}{p}\,\underset{\sim}{x}} \quad ,$$

$$v_1^{\,b}(\underset{\sim}{x}) = \frac{1}{(2\pi)^3}\int \frac{d^3\underset{\sim}{p}}{2}\,\tilde{v}_1(p)e^{i\underset{\sim}{p}\,\underset{\sim}{x}} = -i(\Delta)^{1/2}\,v_1^{\,a}(\underset{\sim}{x}) \quad ,$$

where Δ is the Laplace operator ,

we get

$$\tilde{v}_2(h) = \int d^3\underset{\sim}{x}\;\; e^{-i\underset{\sim}{h}\,\underset{\sim}{x}/\underset{\sim}{x}^2}\left[\frac{h_0 v_1^{\,a}(\underset{\sim}{x})}{(\underset{\sim}{x}^2)^2} + \frac{v_1^{\,b}(\underset{\sim}{x})}{\underset{\sim}{x}^2}\right] . \qquad \text{(A3)}$$

The first term apparently has the same structure as Eq. (A2). The second one is more singular than the first one, as may be seen by setting $\tilde{v}_1(p) = 1$: the resulting expression makes no sense at all in this case, contrary to the first one, which was discussed above. Equation (A3) can be analyzed in the sameway as (A2): if $v_1^{\,a}(\underset{\sim}{x})$ and $v_1^{\,b}(\underset{\sim}{x})$ vanish fast enough for x^i, $i = 1,2,3$ towards infinity and zero, the integrals exist.

Finally we wish to discuss briefly the convolutions of two distributions, the first one of which belongs to the class S of test functions, the second one to the class S_R . We shall consider only the four-dimensional case here. There is no difficulty in applying the corresponding definitions to the three-dimensional case, mentioned in connection with Eq.(14). According to the usual definition of a convolution of distributions,[17] we define the convolution of the distributions $f_1(x)$ and $g(x) = f_2(Rx)$, where the first one is defined on S , the second one on S_R , as

$$\int d^4x d^4y f_1(x) f_2(Ry - x) \chi(y) ,$$

where $\chi(y)$ is an appropriate test function. It is essential that $\chi(x)$ have an existing Fourier transform. Otherwise the definition of the convolution would be rather trivial, if it had only an H transform. With

$$f_1(x) = \frac{1}{(2\pi)^4} \int d^4p \tilde{f}_1(p) e^{ipx} ,$$

$$f_2(Ry - x) = \frac{1}{(2\pi)^4} \int d^4q \tilde{f}_2(q) e^{iq(Ry - x)} ,$$

$$\chi(y) = \frac{1}{(2\pi)^4} \int d^4\ell \tilde{\chi}(\ell) e^{i\ell y} ,$$

we have

$$\int d^4x d^4y f_1(x) f_2(Ry - x) \chi(y)$$

$$= \frac{1}{(2\pi)^8} \int d^4x d^4p d^4\ell \tilde{f}_1(p) \tilde{f}_2(p) \chi(\ell) e^{ipRx} e^{i\ell x} .$$

FOOTNOTES AND REFERENCES

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1. H.A. Kastrup, Ann. Physik (Leipzig) 9, 388 (1962). In this paper further references concerning the conformal group are given.

2. H.A. Kastrup, Physics Letters 3, 78 (1962).

3. See, for instance, N.N. Bogoliubov and D.V. Shirkov, Introduction to the Theory of Quantized Fields (Interscience Publishers, New York, 1959).

4. We use $\hbar = 1 = c$.

5. H.A. Kastrup, Nuclear Phys. 58, 561 (1964).

6. H.A. Kastrup, On the significance of the dilatations in high energy physics, in Proceedings of the International Conference on High Energy Physics, Dubna, 1964 (to be published).

7. D. Amati, S. Fubini and A. Stanghellini, Nuovo Cimento 26, 896 (1962).

8. K. Johnson, M. Baker and R.S. Willey, Phys. Rev. Letters 11, 518 (1963).

9. Th. A. J. Maris, Nuovo Cimento 30, 378 (1963).

10. R. Haag and Th. A. J. Maris, Phys. Rev. 132, 2325 (1963).

11. Th. A. J. Maris, V. E. Herscovitz and G. Jacob, Nuovo Cimento 34, 946 (1964).

12. H. Mitter, Nuovo Cimento 32, 1789 (1964).

13. M. K. Banerjee, M. Kugler, C. A. Levinson and J. E. Muzinich, Phys. Rev. 137, B 1280 (1965) .

14. H.P. Dürr, W. Heisenberg, H. Yamamoto and K. Yamazaki, Quantum electrodynamics in the nonlinear spinor theory and the value of Sommerfeld's fine structure constant (to be published).

15. Y. Murai, Progr. Theoret. Phys. (Kyoto) 9, 147 (1953) and 11, 441 (1954).

16. A. Esteve and P.G. Sona, Nuovo Cimento 32, 473 (1964).

17. I. M. Gel'fand and G. E. Shilov, Generalized Functions (Academic Press, New York and London, 1964, Vol. I).

18. W. Güttinger, Verallgemeinerte Funktionen (Institut für Theoretische Physik der Universität Munchen, 1964).

19. A. Bastai, L. Bertocchi, S. Fubini, G. Furlan and M. Tonin, Nuovo Cimento 30, 1512 (1963).

20. W. Heisenberg, Revs. Modern Physics 29, 269 (1957).

21. W. Heisenberg, in Proceedings of the International Conference on High Energy Physics, CERN 1958, p. 119.

22. H. P. Dürr, W. Heisenberg, H. Mitter, S. Schlieder and K. Yamazaki, Z. Naturforsch. 14a, 441 (1959).

23. A.S. Wightman, in Theoretical Physics (International Atomic Energy Agency, Vienna, 1963).

24. J. Goldstone, A. Salam, and S. Weinberg, Phys. Rev. 127, 965 (1962).

25. E. Bessel-Hagen, Math. Annalen 84, 258 (1921).

26. A. Gamba and G. Luzzato, Nuovo Cimento 33, 1732 (1964); we disagree, of course, with the conclusion of these authors that the conformal group has to be discarded !

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