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THE INELASTIC SCATTERING OF PROTONS FROM CARBON

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INTRODUCTION

The purpose of this paper is to study the spectrum of high energy protons that have been inelastically scattered from carbon. In particular, it will be shown that this spectrum depends in a rather direct way upon the momentum distribution of particles within the target nucleus. Thus, by comparison with experimental data on scattering, it is possible to test various assumptions for this distribution and thereby gain some knowledge of the ground state of the carbon nucleus.

To illustrate the method a simple problem, that of scattering from a harmonically bound particle, is solved first. This calculation embodies many of the important features of scattering from the carbon nucleus and, in addition, possesses the advantage of being soluble directly (at least in Born approximation) since the harmonic oscillator wave functions and matrix elements are known. Comparison of results obtained in the two ways then gives some idea of the validity of the approximations used and also provides further insight into the mechanism of the scattering process.

The second part of the paper deals with the more complicated problem of scattering from the nucleus. In this section the method illustrated before is used to obtain formulae for the energy spectra of protons scattered at various angles from carbon. Finally, by making various assumptions about the momentum distribution in the

nucleus, we attempt to fit what experimental data is available thereby testing the theory and also providing some information as to which momentum distribution is most acceptable.

I. METHOD AND ILLUSTRATIVE PROBLEM

Consider a harmonically bound particle (coordinate $\vec{r}_1$, Hamiltonian H_o), initially in the ground state, which is being bombarded by another of the same mass whose coordinate is labelled $\vec{r}_o$. Furthermore, assume that a potential V acts between the two and let the incoming momentum, $\vec{p}$, be large enough that scatterings induced by this potential can be described in the Born approximation. Finally, assume that the incoming particle is unaffected by the oscillator potential which binds the other. This system, which will serve as an illustration of the method, is about the simplest imaginable which represents, even crudely, the interaction of a fast nucleon with a neutron or proton within the nucleus.

Our problem, then, is to obtain (in so far as possible) a formula, involving only the ground state of the harmonic oscillator, which describes the energy distribution of the scattered particles. To do this consider the total current of particles scattered in the direction $\vec{e}$ ($\vec{e}$ is a unit vector in the direction of observation). It is

$$J = (M/2\pi\hbar^2)^2 \sum_{n}' \left\{ \left| \iint \psi_o^*(r_1)\, e^{-i\vec{p}\cdot\vec{r}_o}\, V(r_o - r_1)\, e^{i\vec{k}_n\cdot\vec{r}_o} \psi_n(r_1)\, d\vec{r}_1\, d\vec{r}_o \right|^2 (\hbar k_n/M) \right\} \tag{1}$$

or, in a self-evident matrix notation,

$$J = (M/2\pi\hbar^2)^2 \sum_{n}' \iint \left[e^{-i\vec{p}\cdot\overrightarrow{r_o - r_o'}}\, e^{i\vec{k}_n\cdot\overrightarrow{r_o - r_o'}}\, (\hbar k_n/M) V_{on}(r_o) V_{no}(r_o') \right] d\vec{r}_o\, d\vec{r}_o' \tag{2}$$

Here

$$\hbar^2 k_n^2/2M = \hbar^2 p^2/2M - (E_n - E_o)\ ,\ \vec{k}_n = k_n \vec{e}\ , \tag{3}$$

the ψ_n's are wave functions of the particle bound in the oscillator potential, and $E_n = (n+\frac{1}{2})h\omega$ where ω is the frequency. In what follows it will be shown that it is possible to transform J, which is a sum of the form

$$(M/2\pi\hbar^2)^2 \sum_{n}' |M_{on}|^2 (\hbar k_n/M)\ ,$$

into an integral

$$J = (M/2\pi\hbar^2)^2 \int \left|F(q)\right|^2 (\hbar q/M)\, dq \tag{4}$$

where q has replaced k_n and $\left|F(q)\right|^2$ may be interpreted as the probability of scattering into the differential momentum interval dq. Furthermore, the original requirement will be satisfied because $\left|F(q)\right|^2$ depends only upon the ground state wave function of the bombarded system and thus a formula for the energy distribution will have been obtained that does not involve any of the excited states.

In order to perform the transformation which eliminates these excited states the first step will be to carry out the sum over n. For instance, if there were no n dependence in the term $e^{i\vec{k}_n\cdot\vec{r}_o - \vec{r}'_o}(\hbar k_n/M)$ this could be done immediately to give

$$\sum_n V_{no}(r_o)\, V_{on}(r'_o) = \left\langle V(r_o)\, V(r'_o)\right\rangle_{00} \tag{5}$$

Of course, in the present case such a simple procedure is not possible. What will be done instead is to expand

$$e^{i\vec{k}_n\cdot\vec{r}_o - \vec{r}'_o}(\hbar k_n/M)$$

into a power series in $(E_n - E_o)$. This gives rise to a sequence of sums of the form

$$\sum_n \left[(E_n - E_o)^n\, V_{on}(r_o)\, V_{no}(r'_o)\right]$$

each of which can be summed to give zero-zero matrix elements of products and/or commutators of H_o and V. For example, if we let $\vec{p}' = p\,\vec{e}$, the first three terms in the expansion are

$$e^{i\vec{k}_n\cdot\overrightarrow{r_o-r_o'}}\,(\hbar k_n/M) = e^{i\vec{p}'\cdot\overrightarrow{r_o-r_o'}}\,(\hbar p/M)$$

$$\Big\{ 1 - \tfrac{1}{2}(1 + 2\,i\,\vec{p}'\cdot\overrightarrow{r_o-r_o'})(E_n-E_o)\;2\,M/\hbar^2\;p^2 -$$

$$\tfrac{1}{8}\Big[1 - i\,\vec{p}'\cdot\overrightarrow{r_o-r_o'} + (\vec{p}'\cdot\overrightarrow{r_o-r_o'})^2\Big](E_n-E_o)^2(2M/\hbar^2\;p^2)^2 \;\ldots \Big\} \tag{6}$$

In the first term there is no n dependence and the sum can be evaluated as was done in equation (5). The second term gives

$$\sum_n{}' \Big[V_{on}(r_o)(E_n-E_o)\;V_{no}(r_o')\Big]$$

$$= \sum_n{}' \Big\{ V_{on}(r_o)\,\Big\langle H_o V(r_o')\Big\rangle_{no} - V_{on}(r_o)\,\Big\langle V(r_o')\;H_o\Big\rangle_{no} \Big\}$$

which, by the usual rules of matrix multiplication, is

$$\Big\langle V(r_o)\;(H_o,\;V(r_o'))\Big\rangle_{00} \tag{7}$$

By the same method the higher sums are easily found to be

$$\sum_n{}' V_{on}(r_o)(E_n-E_o)^2\;V_{no}(r_o') = \Big\langle (V(r_o),\;H_o)(V(r_o'),\;H_o)\Big\rangle_{00} \tag{8}$$

$$\sum_{n}' V_{on}(r_o)(E_n-E_o)^3 V_{no}(r_o') = \left\langle (V(r_o), H_o)(H_o, (H_o, V(r_o'))) \right\rangle_{00} \tag{9}$$

and so forth. This expansion is now substituted back, term-by-term, into equation (2). The first term is merely

$$(M/2\pi\hbar^2)^2 \int \psi_o^*(r_o) e^{-i(\overrightarrow{p-p'})\cdot(\overrightarrow{r_o-r_o'})} (\hbar p/M)\, V(r_o-r_1)\, V(r_o'-r_1)$$

$$\psi_o(r_1)\, d\vec{r}_1\, d\vec{r}_1'\, d\vec{r}_o \tag{10}$$

The second, which is more complicated, is

$$-(M/2\pi\hbar^2)^2 \int \psi_o^*(r_1)\, e^{-i\,\overrightarrow{p-p'}\cdot\overrightarrow{r_o-r_o'}}\, V(r_o-r_1)(\hbar p/M)(2m/\hbar^2\, p^2)$$

$$(H_o, V(r_o'-r_1))\tfrac{1}{2}(1+i\,\vec{p}'\cdot\overrightarrow{r_o-r_o'})\, d\vec{r}_1\, d\vec{r}_1'\, d\vec{r}_o \tag{11}$$

The commutator appearing here is easily found to be

$$-(\hbar^2/2M)\,(\nabla_1^2\, V(r_o'-r_1) + 2\,\vec{\nabla}_1\, V(r_o'-r_1)\cdot\vec{\nabla}_1) \tag{12}$$

To evaluate these derivatives fourier analyze $V(r_o'-r_1)$ and $\psi_o(r_1)$, i.e., write

$$V(r_o'-r_1) = \int V_{\vec{k}}\, e^{i\,\vec{k}\cdot\overrightarrow{r_o'-r_1}}\, d\vec{k}\ ; \quad \psi_o(r_1) = \int N_{\vec{\ell}}\, e^{i\vec{\ell}\cdot\vec{r}_1}\, d\vec{\ell} \tag{13}$$

This is not an unreasonable way to calculate the integral since the square of the fourier transform of ψ_o determines the momentum distribution which we wish to appear in the final formula. Substituting the transforms back into equation (11) gives

$$-(M/2\pi\hbar^2)^2 \int \Big\{ \psi_o^*(r_1)\, e^{-i\,\overrightarrow{p-p'}\cdot\overrightarrow{r_o-r_o'}}\, (\hbar p/M)\, V(r_o-r_1)\, \tfrac{1}{2}(1+i\vec{p}'\cdot\overrightarrow{r_o-r_o'})$$

$$1/p^2\, (k^2+2\vec{k}\cdot\vec{\ell})\, N_{\vec{\ell}}\, V_{\vec{k}}\, e^{i\,\vec{k}\cdot\overrightarrow{r_o'-r_1}}\, e^{i\vec{\ell}\cdot\vec{r}_1} \Big\}\, d\vec{r}_1\, d\vec{r}_o'\, d\vec{r}_o\, d\vec{\ell}\, d\vec{k} \tag{14}$$

Now, by making the change of variables $\vec{R} = \overrightarrow{r_o'-r_o}$, all the integrations except that over $\vec{R}$ can be performed easily to give

$$-(M/2\pi\hbar^2)^2 \int \left|N_{\ell}\right|^2 \left|V_{\vec{k}}\right|^2 \frac{(k^2+2\vec{k}\cdot\vec{\ell})}{p^2} \left(\frac{\hbar p}{M}\right) e^{i\,(\vec{p}-\vec{p}'+\vec{k})\cdot\vec{R}}$$

$$\tfrac{1}{2}(1+i\,\vec{p}'\cdot\vec{R})\, d\vec{R}\, d\vec{\ell}\, d\vec{k} \tag{15}$$

In exactly the same way the contribution to the current from the third term in equation (6) can be calculated. It is

$$- \left(\frac{M}{2\pi\hbar^2}\right)^2 \int\left\{\left|N_\ell\right|^2 \left|V_{\vec{k}}\right|^2 e^{i(\vec{p}-\vec{p}'+\vec{k})\circ\vec{R}} \left(\frac{\hbar p}{M}\right)\left(\frac{k^2 + 2\vec{k}\circ\vec{\ell}}{p^2}\right)^2 \right.$$

$$\left. \frac{1}{8}\left[(\vec{p}'\circ\vec{R})^2 - i(\vec{p}'\circ\vec{R}) + 1\right]\right\} d\vec{R}\, d\vec{\ell}\, d\vec{k} \tag{16}$$

Unfortunately, the higher terms in the expansion (eq. (6)) <u>cannot</u> be evaluated in this simple manner since they always invalue commutators of the form $(H_o, (V(r_o), H_o))$ which will contain derivatives of the Hamiltonian, H_o. However, if the series converges well (convergence will be discussed later), the first three terms should be adequate to determine J. They are

$$J = \left(\frac{M}{2\pi\hbar^2}\right)^2 \int\left\{\left|N_\ell\right|^2 \left|V_{\vec{k}}\right|^2 e^{i\overrightarrow{p-p'+k}\circ\vec{R}} \left(\frac{\hbar p}{M}\right)\right.$$

$$\left.\left[1 - \left(\frac{1+ i\, p'\circ R}{2}\right)\left(\frac{k^2 + 2\vec{\ell}\circ\vec{k}}{p^2}\right) - \left(\frac{(\vec{p}'\circ\vec{R})^2 - i\,\vec{p}'\circ\vec{R} + 1}{8}\right)\left(\frac{k^2 + 2\vec{\ell}\circ\vec{k}}{p^2}\right)^2\right]\right\}$$

$$\vec{dR}\, \vec{dk}\, d\vec{\ell} \tag{17}$$

Looking back to the expansion (equation (6)) we see that in J, $(E_n - E_o)2M/\hbar^2\, p^2$ has been replaced by $\frac{k^2 + 2\,\vec{k}\circ\vec{\ell}}{p^2}$ so, to the approximation to which we are working, the series can be summed

again to give the original $e^{i\vec{k}_n\cdot\overrightarrow{r_o-r_o'}}(\hbar k_n/M)$ except that in it $(E_n-E_o)(2M/\hbar^2)$ is everywhere replaced by $(k^2+2\vec{k}\cdot\vec{\ell})$. This simplifies the formula for the current considerably and by using the abbreviations

$$q = (p^2 - k^2 - 2\vec{k}\cdot\vec{\ell})^{\frac{1}{2}} \qquad \vec{q} = q\vec{e}$$

it can be written in the following compact form.

$$J = (M/2\pi\hbar^2)^2 \int \left|N_\ell\right|^2 \left|V_{\vec{k}}\right|^2 e^{i(\vec{q}-\vec{p}-\vec{k})\cdot\vec{R}} (\hbar q/M)\, d\vec{\ell}\, d\vec{k}\, d\vec{R} \tag{18}$$

The $\vec{R}$ integral gives a delta function of $\vec{q}-\vec{p}-\vec{k}$, so that when we carry out the $\vec{k}$ integration $\vec{k}$ must be replaced throughout by $\vec{p}-\vec{q}$. Furthermore, since q is a function of $\vec{k}$ a jacobian is introduced into the formula for the current. Thus J becomes

$$J = (M/2\pi\hbar^2)^2 \int \left|N_\ell\right|^2 \left|V_{\overrightarrow{p-q}}\right|^2 (\hbar q/M) \left(\frac{d\vec{\ell}}{1 - \frac{\vec{p}\cdot\vec{\ell}}{q^2}}\right) \tag{19}$$

where the term in the denominator is the jacobian and q is now calculated from the equation

$$q^2 = p^2 - (\overrightarrow{p-q})^2 - 2\,\overrightarrow{p-q}\cdot\vec{\ell} \tag{20}$$

which is obtained by eliminating $\vec{k}$ from the two relations $\vec{q} = \vec{p} + \vec{k}$ and $q = (p^2 - k^2 - 2\vec{k}\cdot\vec{\ell})^{\frac{1}{2}}$. Equation (20) is particularly interesting because it is just the one that would be obtained by eliminating the recoil momentum, $\vec{r}$, from the two classical conservation laws

$$\vec{p} + \vec{\ell} = \vec{q} + \vec{r} \quad \text{and} \quad p^2 + \ell^2 = q^2 + r^2 .$$

Thus $\vec{q}$ is the momentum a particle would have after scattering in the direction $\vec{e}$ from another particle moving with momentum $\vec{\ell}$. When viewed with this fact in mind the formula for the current is quite reasonable because $|N_\ell|^2$ represents the probability of finding a momentum $\vec{\ell}$ and $\vec{q}$ is just the momentum the bombarding particle would have after hitting another moving with $\vec{\ell}$.

To obtain the differential cross section for scattering into an increment dq of outgoing momentum the integral in the current must be carried out over values of $\vec{\ell}$ consistent with a constant q. To do this rewrite equation (20) as

$$q^2 - qp\cos\theta + p\ell\cos\xi - q\ell\left[\cos\xi\cos\theta + \sin\xi\sin\theta\cos(\phi - \eta)\right] = 0$$

where θ, ϕ; ξ, η are the polar angles of $\vec{q}$ and $\vec{\ell}$, both measured from $\vec{p}$. Now let us make a change of variables from $d\eta$ to dq, i.e., $d\eta = \frac{d\eta}{dq}\,dq$. A short calculation gives

$$\frac{d\eta}{dq} = \frac{\left(1 - \frac{p \cdot \vec{\ell}}{q^2}\right) q}{\left[\gamma^2 (1 - \cos^2 \xi) - (\alpha + \beta \cos \xi)^2\right]^{\frac{1}{2}}} \tag{21}$$

where

$$\alpha = q(p \cos \theta - q)$$
$$\beta = \ell(p - q \cos \theta)$$
$$\gamma = q \ell \sin \theta$$

With this change of variables the current is

$$J = \left(\frac{M}{2\pi \hbar^2}\right)^2 \int \frac{\left|N_\ell\right|^2 \left|V_{\overrightarrow{p-q}}\right|^2 \left(\frac{\hbar q}{M}\right) d(\cos \xi)\, \ell^2\, d\ell\, q\, dq}{\left[\gamma^2 (1 - \cos^2 \xi) - (\alpha + \beta \cos \xi)^2\right]^{\frac{1}{2}}} \tag{22}$$

The limits of integration on the $\cos \xi$ integration are the branch points of the denominator so it can be evaluated by contour integration provided we assume $\left|N_\ell\right|^2$ independent of $\cos \xi$. The result is

$$J = \frac{1}{2\pi} \left(\frac{M}{\hbar^2}\right)^2 \int \frac{\left|N_\ell\right|^2 \left|V_{\overrightarrow{p-q}}\right|^2 \ell\, d\ell\, q \left(\frac{\hbar q}{M}\right) dq}{\left|\overrightarrow{p - q}\right|} \tag{23}$$

The limits on the final integration over ℓ are found by requiring that $\frac{d\eta}{dq}$ be real for some value of $\cos \xi$ lying between -1 and 1.

This condition turns out to be

$$\ell > \frac{|q - p\cos\theta|\, q}{|\vec{p - q}|}$$

which, incidentally, is just the restriction that would be obtained classically. For example, if $q = p\cos\theta$ the formula gives, as it should, the minimum value of zero for ℓ. The upper limit on ℓ, of course, is infinity.

Before carrying out the ℓ integration an expression must be obtained for $|N_\ell|^2$. In the case of the harmonic oscillator this quantity is

$$\left(\frac{1}{\pi^{3/2}}\right)\left(\frac{\hbar}{M\omega}\right)^{3/2} e^{-\ell^2 \hbar/M\omega} .$$

So the integral can be performed immediately to give

$$J = \frac{1}{2\pi^{5/2}}\left(\frac{M}{\hbar^2}\right)^2 \sqrt{\hbar/M\omega}$$

$$\int \frac{e^{-\frac{q^2(q-p\cos\theta)^2}{|\vec{p-q}|^2}(\hbar/M\omega)}}{|\vec{p-q}|} \left|V_{\vec{p-q}}\right|^2 (\hbar q/M)\, q\, dq \tag{24}$$

This is the final result in which, as was mentioned previously, q replaces k_n and

$$\frac{d^2\sigma}{d\Omega\, dq} = \frac{1}{2\pi^{5/2}} (M/\hbar^2)^2 \sqrt{h/M\omega}\, \frac{e^{-\frac{q^2(q - p\cos\theta)^2}{|\overrightarrow{p-q}|^2}(\hbar/M\omega)}}{|\overrightarrow{p-q}|}$$

$$\left|V_{\overrightarrow{p-q}}\right|^2 q/p \quad q \tag{25}$$

is the differential cross section. Actually, this formula is valid only when $q - p\cos\theta$ is not too large (this will become clear later on) so in the slowly varying terms it is within the limits of accuracy to replace q by $p\cos\theta$. This gives the somewhat simpler formula for the current

$$J = \frac{1}{2\pi^{5/2}} (M/\hbar^2)^2 \sqrt{h/M\omega}$$

$$\int e^{-\operatorname{ctn}^2\theta(q - p\cos\theta)^2(\hbar/M\omega)} \left|V_{\overrightarrow{p-q}}\right|^2 (\hbar p/M)(\cos^2\theta/\sin\theta)\, dq \tag{26}$$

and a corresponding simple expression for the differential cross section.

To gain some idea of the validity of equation (24) we will now solve the same problem using the known oscillator functions and then compare the results. Consider a transition to a final state of

energy $(n+\frac{1}{2})h\omega$ and with the scattered particle moving away with momentum $\vec{q}$ ($\vec{q}$ is connected with n by the energy conservation law

$$\hbar^2 p^2/2M = \hbar^2 q^2/2M + n\hbar\omega).$$

The wave functions of the oscillator are products of Hermite functions, but if we choose the z_1 axis along $\overrightarrow{p-q}$ the only one of the Hermite functions that changes its quantum number during the scattering is that having z_1 as argument. The others integrate out immediately to give unity. Therefore, the matrix element, M, is given by

$$M = \frac{1}{(n!\ 2^n)^{\frac{1}{2}}} \left(\frac{M\omega}{\hbar}\right)^{\frac{1}{2}} \iint H_n(\xi) e^{-\xi^2} e^{i\,\overrightarrow{p-q}\cdot\vec{r}_o} V(r_o - r_1)\, d\vec{r}_1\, d\vec{r}_o \tag{27}$$

where $\xi = \sqrt{\frac{M\omega}{\hbar}}\, z_1$ ($\sqrt{\frac{\hbar}{M\omega}}$ is the size of the ground state wave function for the harmonic oscillator). By making the change of variables $\overrightarrow{r_o - r_1} = \vec{r}$ and using the generating function for Hermite polynomials the matrix element is easily evaluated. Its square, which determines the transition probability, is

$$|M|^2 = \frac{1}{n!\ 2^n} \left(\frac{\hbar}{M\omega} |\overrightarrow{p-q}|^2\right)^n e^{-\frac{\hbar}{M\omega}\frac{|\overrightarrow{p-q}|^2}{2}} \left|V_{\overrightarrow{p-q}}\right|^2 \tag{28}$$

For what follows we are interested in the case $\hbar\omega \ll \frac{\hbar^2 p^2}{2M}$ which means that n, in the above matrix element, can be looked upon as a continuous variable although, actually, it is discrete. Using this idea the current of scattered particles is found to be

$$J = \int \frac{(\hbar/M\omega)(M/2\pi\hbar^2)^2}{n!\ 2^n} \left(\frac{\hbar}{M\omega} \left|\overrightarrow{p-q}\right|^2 \right)^n e^{-\frac{\hbar}{M\omega}\frac{\left|\overrightarrow{p-q}\right|^2}{2}} \left|V_{\overrightarrow{p-q}}\right|^2 (\hbar q/M) q\, dq \tag{29}$$

where n is everywhere to be replaced by

$$n = \frac{\hbar^2}{2M}\left(\frac{p^2-q^2}{\hbar\omega}\right) = \frac{p^2-q^2}{k_o^2} \quad \text{with} \quad k_o^2 = \frac{2M\omega}{\hbar^2}$$

Since n was assumed to be fairly large Stirling's formula is now applicable, giving for J

$$J = \int \left(\frac{\hbar}{M\omega}\right)^{\frac{1}{2}} \left(\frac{M}{2\pi\hbar^2}\right)^2 \frac{1}{(p^2-q^2)^{\frac{1}{2}}}$$

$$\exp\left\{\left(\frac{p^2-q^2}{k_o^2}\right)\log\left(\frac{\left|\overrightarrow{p-q}\right|^2}{k_o^2}\right) - \frac{\left|\overrightarrow{p-q}\right|^2}{k_o^2} - \left(\frac{p^2-q^2}{k_o^2}\right)\left[\log\left(\frac{p^2-q^2}{k_o^2}\right) - 1\right]\right\}$$

$$\left|V_{\overrightarrow{p-q}}\right|^2 \frac{hq}{M}\, q\, dq \ . \tag{30}$$

This rather forbidding formula can be understood somewhat better by expanding the term in the exponential into a power series in $q - p \cos\theta$. If we keep only terms up to and including $(q - p\cos\theta)^2$ it is easy to verify that J is transformed into the following considerably simpler expression.

$$J = \frac{1}{(2\pi)^{5/2}} (M/\hbar^2)^2 \sqrt{\frac{\hbar}{M\omega}} \frac{e^{-\operatorname{ctn}^2\theta(q-p\cos\theta)^2 \frac{\hbar}{M\omega}}}{(p^2-q^2)^{\frac{1}{2}}} \left|V_{\overrightarrow{p-q}}\right|^2 (\hbar q/M)\, q\, dq \tag{31}$$

In a similar way the formula for the differential cross section simplifies; namely

$$\frac{d^2\sigma}{dq\, d\Omega} = \frac{1}{(2\pi)^{5/2}} (M/\hbar^2)^2 \left(\frac{\hbar}{M\omega}\right)^{\frac{1}{2}} \frac{e^{-\operatorname{ctn}^2\theta(q-p\cos\theta)^2 \frac{\hbar}{M\omega}}}{(p^2-q^2)^{\frac{1}{2}}} \left|V_{\overrightarrow{p-q}}\right|^2 \frac{q^2}{p}\, dq \tag{32}$$

These formulas are the ones to be compared with equations (24) and (25). Actually, the two pairs of formulas are quite similar. In particular, if we make the approximation $q = p\cos\theta$ in the slowly varying parts of equation (31) it reduces exactly to equation (26) (which is the approximate form of (24)). A better test, which gives some idea of the errors involved in equation (24), is to check the difference in position of the maximum of the differential cross section in the two cases. This difference turns out to be

$k_o^2/4\ p \cos\theta$ which means a percentage error of $k_o^2/4\ p^2 \cos^2\theta$. For the case of scattering from carbon (as will be seen in the next section) k_o^2/p^2 is about one-tenth, so that we should expect errors of around five percent in our calculations of cross sections for nuclear scattering. This five percent error in equation (24) arises, of course, because higher terms in the series (eq. (6)) were neglected. Indeed, by examining the convergence of this series we can see that this is just the sort of accuracy to be expected. In this expansion (eq. (6)) we wrote

$$k_n = \sqrt{p^2 - \frac{2M}{\hbar^2}(E_n - E_o)}$$

and expanded into powers of $\frac{2M(E_n - E_o)}{\hbar^2}$. Instead, we could also have written

$$k_n = \sqrt{p^2 - \frac{2M}{\hbar^2}(\overline{E} - E_o) - \frac{2M}{\hbar^2}(E_n - \overline{E})} \tag{33}$$

(where $\overline{E}$ is the average energy of excitation) and then expanded into powers of $\frac{2M}{\hbar^2}(E_n - \overline{E})$. Actually, this last is the natural expansion parameter; our only reason for using the former method was that it is much less cumbersome and gives the same answer as the second (at least for the terms we kept in the expansion). Using this new parameter it is easy to estimate the size of the error by examining the fourth term in the expansion of $e^{i\vec{k}_n \cdot \overrightarrow{r_o - r'_o}}$ (the expansion of k_n itself converges well). If we choose $\overrightarrow{r_o - r'_o}$

of the order of nuclear forces and use for $E_n - \overline{E}$ the experimentally observed half-width of the spectrum, the fourth term turns out to be a few percent of the first three, thus giving an error of the same size as that estimated by comparing the two solutions of the harmonic oscillator problem.

Earlier we mentioned that the connection between p, q, and $\mathcal{L}$ (eq. (20)) is just that obtained from the classical conservation laws of energy and momentum by eliminating the recoil. One might think to improve this approximation by using somewhat different laws, namely $\vec{p} + \vec{\mathcal{L}} = \vec{q} + \vec{r}$ and $p^2 = q^2 + r^2 + \alpha^2$ where α^2 takes account of some departure from the classical equations. Such an approach is suggested, in the case of scattering from carbon, by the Born approximation where the two outgoing particles are treated as plane waves and closure is used to eliminate the final states of the residual nucleus. Unfortunately, this procedure seems to give no better results (at least for the oscillator problem) than the other method. For instance, the position of the maximum of the differential cross section is given in this case by

$$q_{max} = p \cos\theta \left[1 - \frac{\alpha^2}{p^2 \cos^2\theta} + \frac{3\, k_o^2 \sin^2\theta}{4\, p^2 \cos^4\theta} \right]$$

which cannot, by any choice of α^2, be brought into agreement with with the maximum obtained from equation (32) which is at

$$q_{max} = p \cos\theta \left[1 + \frac{k_o^2}{4 p^2 \cos^2\theta} + \frac{k_o^2 \sin^2\theta}{2 p^2 \cos^4\theta} \right]$$

The inference to be drawn from this fact is that equation (25) represents the best obtainable approximation to the differential cross section which does not involve the excited states of the bombarded system in an essential way. This conclusion is reinforced by considering the expansion (eq. (6)) in which the first three terms were easily calculable, and differed markedly in form from the fourth which could not be evaluated at all.

It is interesting to note that formula (25) is just the one that would be obtained from the impulse approximation. Thus, the method we have used to derive this equation is useful mainly because it gives some idea of the errors involved and at the same time indicates that the impulse approximation probably cannot be improved.

This completes the discussion of the harmonic oscillator problem. Having illustrated the method, we will now apply it to the more interesting problem of the scattering of protons from carbon.

II. SCATTERING FROM CARBON

For scattering from carbon the analogue of equation (1) is

$$J = \left(\frac{M}{2\pi\hbar^2}\right)^2 \sum_n \left\{ \int\left[\psi_o^* \, e^{i\overrightarrow{k_n - p}\cdot\vec{r}_o} \sum_{i=1}^{12} V(r_o - r_i)\psi_n\right] d\tau \quad x \right.$$

$$\left. \int\left[\psi_n^* \, e^{i\,\overrightarrow{p - k_n}\cdot\vec{r}_o} \sum_{j=1}^{12} V(r_o - r_j)\psi_o\right] d\tau \left(\frac{\hbar k_n}{M}\right)\right\} \tag{34}$$

where now V is the nucleon-nucleon interaction which depends upon the coordinates, spins, and isotopic spins of the nucleons involved and the ψ_n's are wave functions for the twelve particles which originally formed the carbon nucleus. Actually, equation (34) is not the complete expression for the current in this case for we have neglected terms that arise from 'exchange collisions' in which the bombarding particle is stopped and another particle driven out of the nucleus. However, these terms can be handled in exactly the same way as the one appearing in equation (34) so in order to save writing we will omit them until we get to the final formulas and treat equation (34) as if it represented the whole current. Furthermore, if we consider scattering at fairly large

angles (say $\theta \geqslant 20^{\circ}$) the interference terms ($i \neq j$) which appear in J only contribute appreciably for final states in which two of the particles that were in the carbon are moving very fast (compared to nuclear velocities). However, the probability of transferring large amounts of momentum to two particles in a single collision is small so in the following we can neglect the interference terms. With this simplification, equation (34) can be rewritten as

$$J = \left(\frac{M}{2\pi\hbar^2}\right)^2 \sum_{n,i} \left| \int \psi_o^* \, e^{i\,\vec{k}_n - \vec{p}\cdot\vec{r}_o} \, V(r_o - r_i) \psi_n \, d\tau \right|^2 \left(\frac{\hbar k_n}{M}\right) \tag{35}$$

Incidentally, it should not be inferred from what was said above that the interference terms are always unimportant for they give the exclusion effects which modify the forward scattering. For example, using the Fermi model of the nucleus it is easy to verify that they cut the scattering cross section to zero in the forward direction.

The Hamiltonian, H_o, of the carbon nucleus is the sum of two terms, T_o and V_o, which represent the kinetic and potential energy respectively. In the previous calculation for scattering from a harmonically bound particle the commutator (V, H_o), which entered equations (7), (8), (9), and (11), was equal to (V, T_o) and this fact was used to obtain equation (12). However, in the

present case (V, V_o) is no longer zero because the potentials contain spin and isotopic spin portions. Therefore, before we can apply the equations of the preceding section, the contribution of these extra commutator terms must be shown to be small. A typical one of them is

$$\sum_{\substack{i,\ell \\ i\neq\ell}} \left\langle \left((\vec{\sigma}_i\cdot\vec{\sigma}_o),\ (\vec{\sigma}_i\cdot\vec{\sigma}_\ell)\right)\ V_o^{spin}(r_o-r_i)\ V^{spin}(r_i-r_\ell) \right\rangle_{00}$$

$$= \sum_{\substack{i,\ell \\ i\neq\ell}} \left\langle (2\vec{\sigma}_i\cdot\vec{\sigma}_o \times \vec{\sigma}_\ell)\ V_o^{spin}(r_o-r_i)\ V^{spin}(r_i-r_\ell) \right\rangle_{00} \tag{36}$$

which, if we use the shell model to describe the carbon nucleus, is easily seen to be zero. Similarly, the other types of commutators (arising from tensor or isotopic spin dependent potentials) are also zero in this approximation. Thus, within the validity of this model it is justifiable to drop the commutator terms and use all the formulas that were derived in connection with the harmonic oscillator problem. This, admittedly, is a rather uncertain procedure but the agreement of the final results with experiment justifies it to some extent and, in addition, it would not be possible in our present state of knowledge to evaluate these terms even if we should decide to keep them.

Having dropped terms arising from the commutator (V, V_o), the analysis proceeds just as it did in the previous section and we may use equation (23) to get the following expression for the differential cross section

$$\frac{d^2\sigma}{d\Omega\, dq} = \left(\frac{1}{2\pi}\right)\left(\frac{M}{\hbar^2}\right)^2 \sum_{i=1}^{12} \int \chi^* \frac{N_{\ell}^{i*} \left| V^i_{\overrightarrow{p-q}} - V^i_{\overrightarrow{p-\ell}} \right|^2}{\left|\overrightarrow{p-q}\right|} N_{\ell}^i \chi\, d\tau\, \frac{q^2}{p}\, \ell\, d\ell \tag{37}$$

where

$N_{\ell}^i = \int \psi_o\, e^{-i\vec{\ell}\cdot\vec{r}_i}\, d\vec{r}_i$ (a function of eleven variables), $V^i_{\overrightarrow{p-q}}$ is the fourier transform of $V(r_o - r_i)$, the χ's are spinors, and the $d\tau$ integration is over all the variables that appear in N_{ℓ}^i. Notice that we have included the 'exchange terms' which were omitted from the equation for the current at the beginning of this section. Furthermore, if the Born approximation is valid (and we will assume that it is) the term above involving the potentials can be replaced by the nucleon-nucleon cross section. This gives

$$\frac{d^2\sigma}{d\Omega\, dq} = 2\pi \sum_{i=1}^{12} \iint \frac{\left|N_{\ell}\right|^2 \sigma_i}{\left|\overrightarrow{p-q}\right|}\, q\, \ell\, d\ell\, d\tau \tag{38}$$

In making this substitution we are extrapolating somewhat since q is not equal to $p \cos\theta$ as it should be for the potential

terms to be proportional to the cross section. However, provided q is fairly close to $p \cos \theta$ (as it will be) the error involved is small. We have also dropped the superscript i from N_ℓ in the above formula since, aside from small coulomb effects, we expect the momentum distribution for all particles in the carbon nucleus to be the same. Finally, by multiplying (38) by $\frac{dq}{dE}$ we can get a formula for the cross section for scattering into the energy interval dE. It is

$$\frac{d^2\sigma}{d\Omega\, dE} = 2\pi \int \frac{|\mathfrak{N}_\ell|^2}{|\overrightarrow{p-q}|} \sum_{i=1}^{12} \sigma_i \frac{M}{\hbar^2}\, \ell \, d\ell \tag{39}$$

where

$$|\mathfrak{N}_\ell|^2 = \int |N_\ell|^2 \, d\tau .$$

If, in this formula, we use for $|\mathfrak{N}_\ell|^2$ the gaussian momentum distribution given by Huddlestone and Henley[1] the ℓ integration can be done immediately (the limits are the same as before) to give the following formula for the differential cross section

$$\frac{d^2\sigma}{d\Omega\, dE} = \frac{1}{\alpha \pi^{\frac{1}{2}}}\, e^{-\frac{q^2(q-p\cos\theta)^2}{\alpha^2 |\overrightarrow{p+q}|^2}} \left(\sum_{i=1}^{12} \sigma_i \right) \left(\frac{M}{\hbar^2} \right) \tag{40}$$

(here α^2 is a constant such that $h^2\alpha^2/2M = 16$ Mev). Another possible momentum distribution, given by Chew and Goldberger[2], is

$$\left|\eta_\ell\right|^2 = \frac{\beta}{2\pi^2\,(\beta^2+\ell^2)^2} \qquad (h^2\beta^2/2M = 18 \text{ Mev.})$$

which leads to the formula

$$\frac{d^2\sigma}{d\Omega\, dE} = \frac{\beta}{\pi}\,\frac{1}{|\overrightarrow{p-q}|}\,\frac{\left(\sum_{i=1}^{12}\sigma_i\right)}{\left(\beta^2+\frac{q^2(q-p\cos\theta)^2}{|\overrightarrow{p-q}|^2}\right)}\left(\frac{M}{\hbar^2}\right) \tag{41}$$

for the differential cross section. In order to see whether either of these energy spectra is in agreement with experiment we have plotted, for $\theta = 41.5^\circ$ and a bombarding energy of 340 Mev, equations (40), (41), and the experimental spectrum*. From the high energy portion of this graph where multiple effects are least important it is clear that the Chew-Goldberger distribution is incorrect because it contains too many high momentum components.

On the other hand, the gaussian distribution gives a fair fit so we will do all further calculations with it. However,

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* Mr. John Cladis has kindly given me permission to use his data on the energy distribution of 340 Mev protons scattered from carbon. His curves, which are still preliminary, are for angles of scattering of 22° and 41.5°.

before we can use equation (41) to make a quantitative comparison with the experimental data, a correction must be made for multiple scatterings of the bombarding proton within the carbon nucleus. At 22° this correction is small (it is the low energy tail of the experimental curve in figure 5), but at larger angles it becomes increasingly important, affecting both the shape of the spectrum and the position of the maximum, (see, for example, fig. 6). To get some idea of this effect we will calculate the energy spectrum of doubly scattered protons and then assume that scatterings in which a greater number of collisions occur give similar distributions. Actually, this approximation will not be bad since the energy distributions of multiply scattered protons are all quite smooth and, moreover, the processes of higher multiplicity probably will not contribute very highly. What error there is will consist of underestimating the low energy portion of the spectrum and overestimating the higher part for, as the number of collisions goes up, the energy of the outgoing particles tends to be lower.

The simplest way to estimate the double collision spectrum is to neglect the effects of the momentum distribution of the scattering nucleons and consider the energy as uniquely determined by the angle of scattering through the formula $q = p \cos \theta$. This assumption leads to the following integral for the energy distribution

$$16 \int \sigma^2 \delta(E_1 - E_0 \cos^2 \theta_1) \delta(E - E_1 \cos^2 \theta_2) \sin \theta_1 \cos \theta_1 \sin \theta_2 \cos \theta_2 \, dE_1 \, d\theta_1 \, d\theta_2 \, d\phi_2$$

where we integrate over values of E_1, θ_1, θ_2, and ϕ_2 consistent with

$\cos\Theta = \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 \cos\phi_2$ (Θ is the angle of scattering.) (In obtaining this expression the differential cross sections for nucleon-nucleon scattering were taken to be constant in the center of mass system--the factors 16, $\cos\theta_1$, and $\cos\theta_2$ appearing when the element of solid angle is transformed from center of mass to laboratory system.) A short calculation gives for the spectrum

$$\frac{dN}{dE\, d\Omega} = \int \frac{16\sigma^2\, d\mu}{E_o\left[(1-\mu^2)(\mu^2 - \frac{E}{E_o}) - \mu^2 (\cos\Theta - \frac{E}{E_o})^2\right]^{\frac{1}{2}}}$$

where the integration is carried out over values of μ that make the square root in the denominator real. This is a standard elliptic integral which can be evaluated with a table of elliptic functions. The distribution, which is plotted for $\theta = 22^\circ$ and 41° in figure 2, has two important features that do not change when the effect of the nuclear momenta is included. These are the general smoothness of the cross section and the rise toward lower energy. On the other hand, the sharp cut off at the high energy end of the spectrum is completely wiped out by a treatment that takes into account the momentum distribution of particles within the nucleus. Unfortunately, such a treatment leads to an unmanageable multiple integral, so we have used the Monte Carlo method to

evaluate it and gain some idea of how the high energy portions of the curves in figure 2 are modified. Since this method has been discussed in considerable detail in the literature[3], we will present here only the results of the calculation. Figures 3 and 4 give histograms of the energy distributions obtained for 340 Mev protons doubly scattered from carbon at angles of 20° and 40°. In deriving these, equation (40) was used for the energy distribution after a single scattering and, to save time in doing the tedious numerical work, only angles of scattering (for a single scattering) of less than 60° were considered, which is the reason there are no points on the histograms for very low energies. Incidentally, each block in these represents about twenty five particles that were followed, from which it is clear that the statistical errors are of the magnitude to be expected. The curves, which have been drawn in a smooth way through the histograms, preserve the two features that were mentioned earlier. They will be used, along with equation (40) to obtain curves of the energy spectra.

The final step in getting these energy distributions is to obtain an estimate of the ratio of double to single collisions. To do this accurately is a formidable problem but, fortunately, all we need is a fairly rough answer since the high energy end of the spectrum, which is the best place to test the assumed momentum distribution, is not affected much by the admixture of double collisions. This ratio is derived by assuming the incident proton goes in a straight line through the nucleus (neglecting the change

in path length caused by scattering) and then calculating the probability of making a single collision. From this the multiple collision probability is easily obtained by a subtraction. The average distance the incoming proton travels in the nuclear matter is $\frac{4}{3}R$ where R is the radius of the carbon nucleus. Using this relation the probability of making a single scattering is

$$4R/3\lambda \; e^{-4R/3\lambda}$$

(λ is the mean free path in nuclear matter), and the multiple collision probability is, by subtraction

$$1 - e^{-4R/3\lambda} - 4R/3\lambda \; e^{-4R/3\lambda} \quad .$$

The ratio of these two probabilities is calculated using the values R and λ given by Fernbach[4]. Knowing it, the mixture of double and single collision curves is determined by the fact that the ratio of their areas must be given by the ratios of the probabilities above.

The final cross sections, which turn out to be about a one-to-one mixture of single and double collisions, are plotted in figures 5 and 6 (for 22° and 41.5°) along with the experimental spectra for these angles. The area under the experimental and theoretical curves is the same in each case. At 41.5° the peak of the graph is affected considerably by the admixture of double

collisions whereas at the smaller angle this is not true. Therefore the best place to test the momentum distribution will be at 22°. However, the 41.5° data does have a value because it serves to check the theory and gives us confidence in the deductions made from spectra at smaller angles. The reason the theoretical curve lies above the experimental in this case is that we have treated all multiple collisions as being double. Scatterings of high multiplicity should contribute less at high energies and therefore the theoretical curve should actually be somewhat lower there.

At 22°, unfortunately, the experimental data is rather poor and very little can be said about the momentum distribution beyond the fact that the one we have used is adequate. What is really needed, of course, is a series of fairly accurate experimental spectra at various angles. With these one could determine the momentum distribution from the data at one angle and then check it, and the whole theory as well, by examining the fit at other angles. However, until such data is available it is interesting to see that the spectra at two angles can be understood in terms of a simple model and that this fit, crude though it may be, already fixes the momentum distribution to a considerable extent.

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FIGURE CAPTIONS

1. Single collision spectra for various momentum distributions.

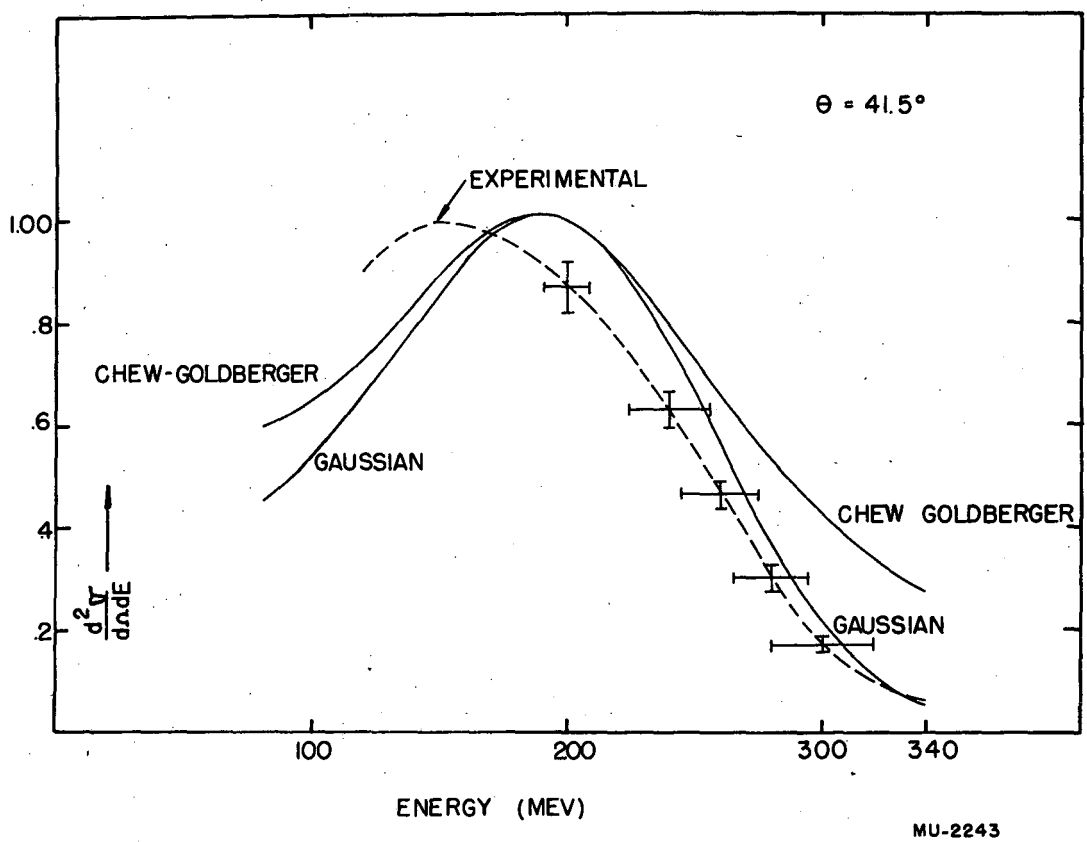
2. Double collision spectra neglecting momentum distribution of target nucleons.

3. Double collision spectrum at 22°.

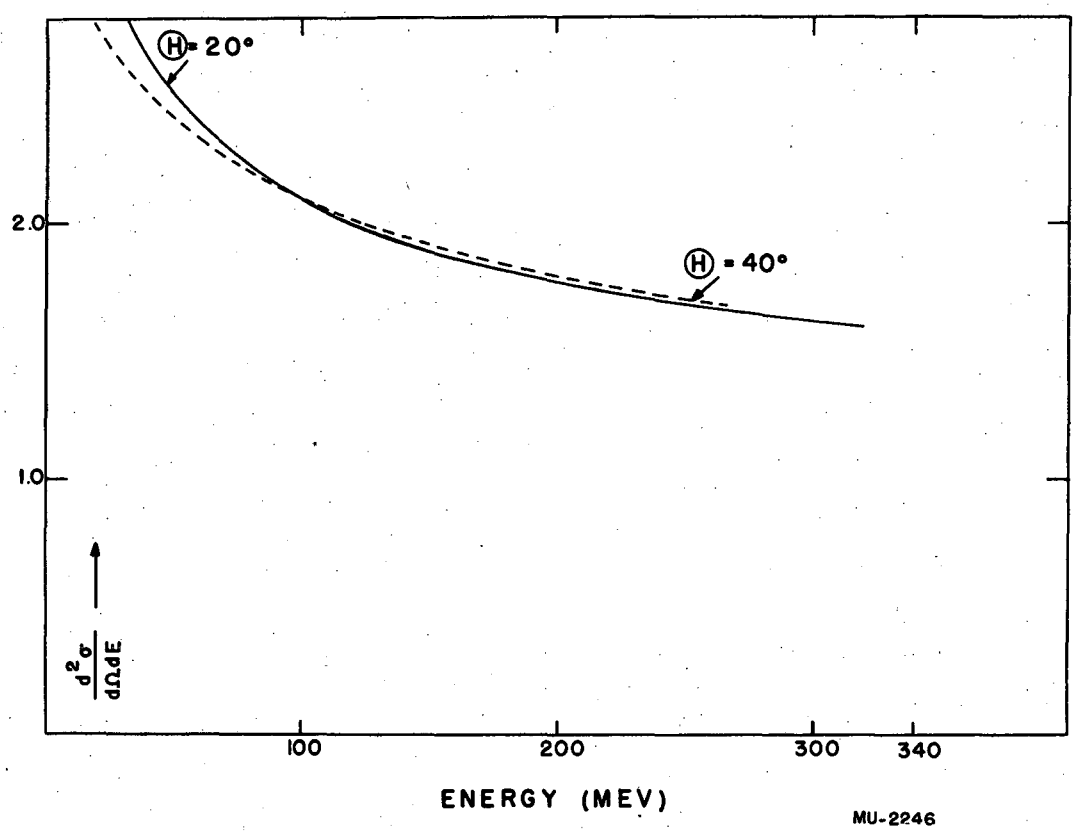
4. Double collision spectrum at 41.5°.

5. Total spectrum at 22°.

6. Total spectrum at 41.5°.


θ = 41.5°
EXPERIMENTAL
1.00
.8
.6
.4
.2
CHEW-GOLDBERGER
GAUSSIAN
CHEW GOLDBERGER
GAUSSIAN
$\frac{d^2\sigma}{d\Omega dE}$
100
200
300
340
ENERGY (MEV)

MU-2243



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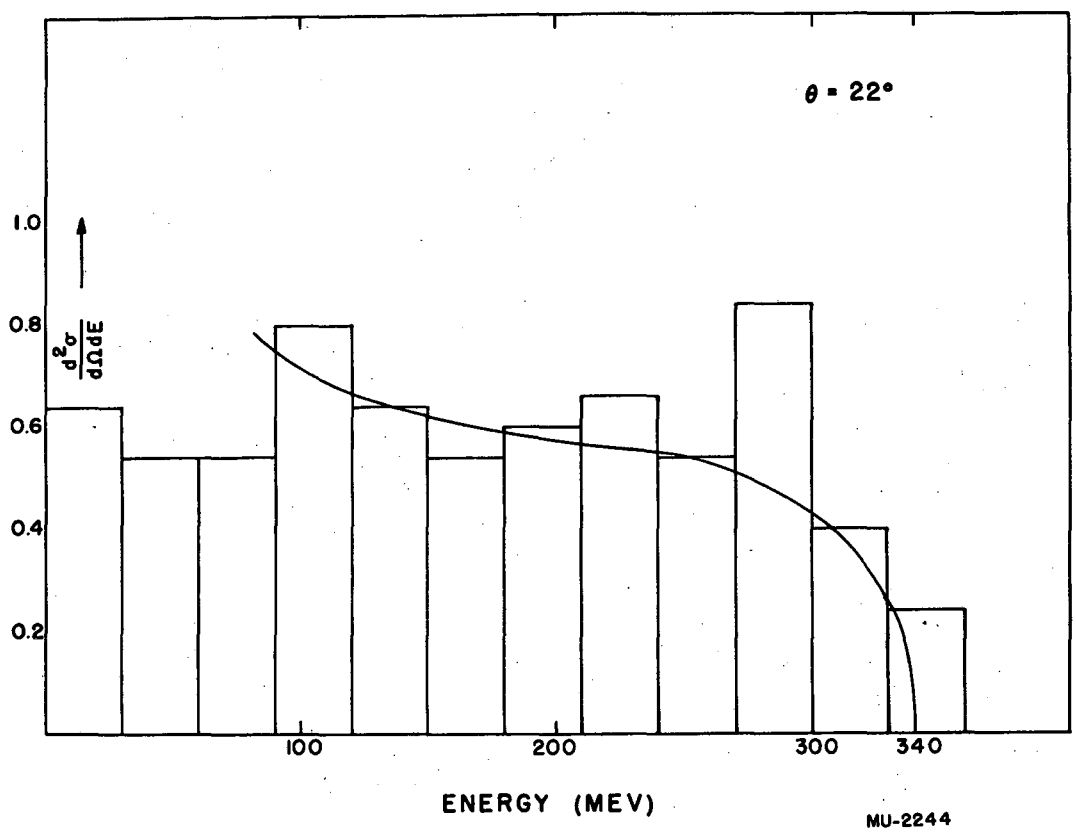
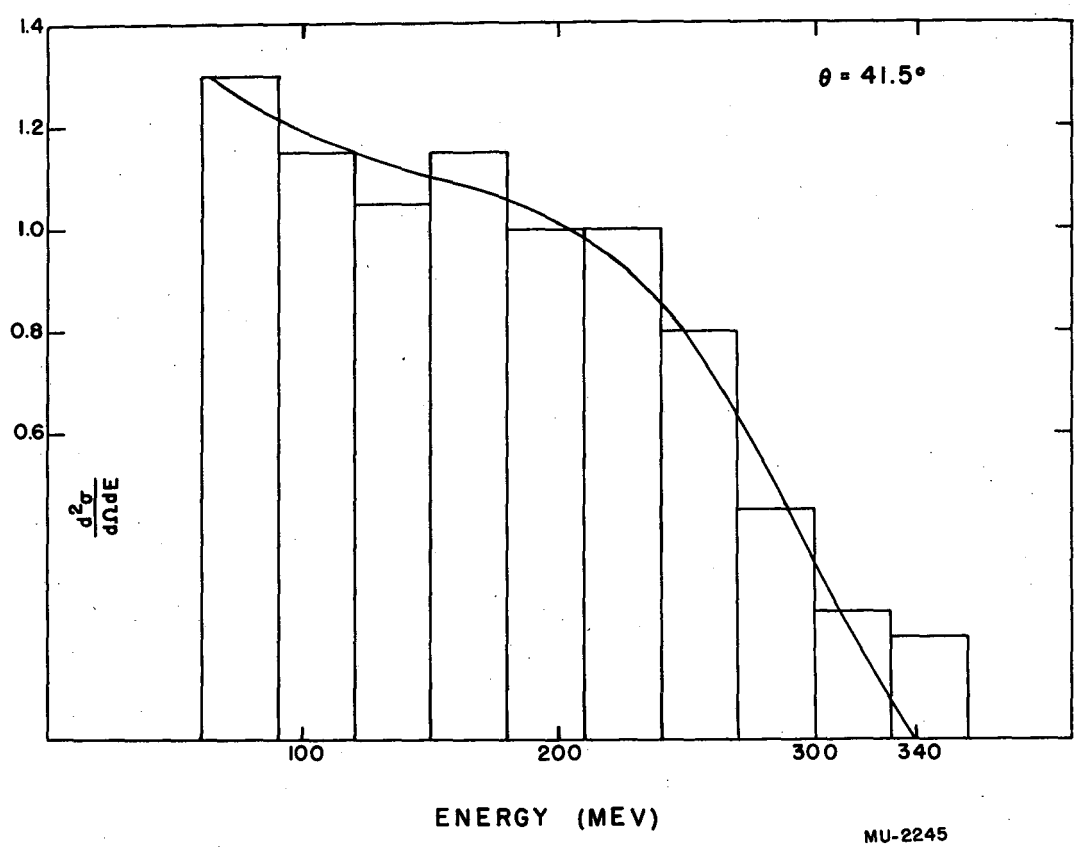
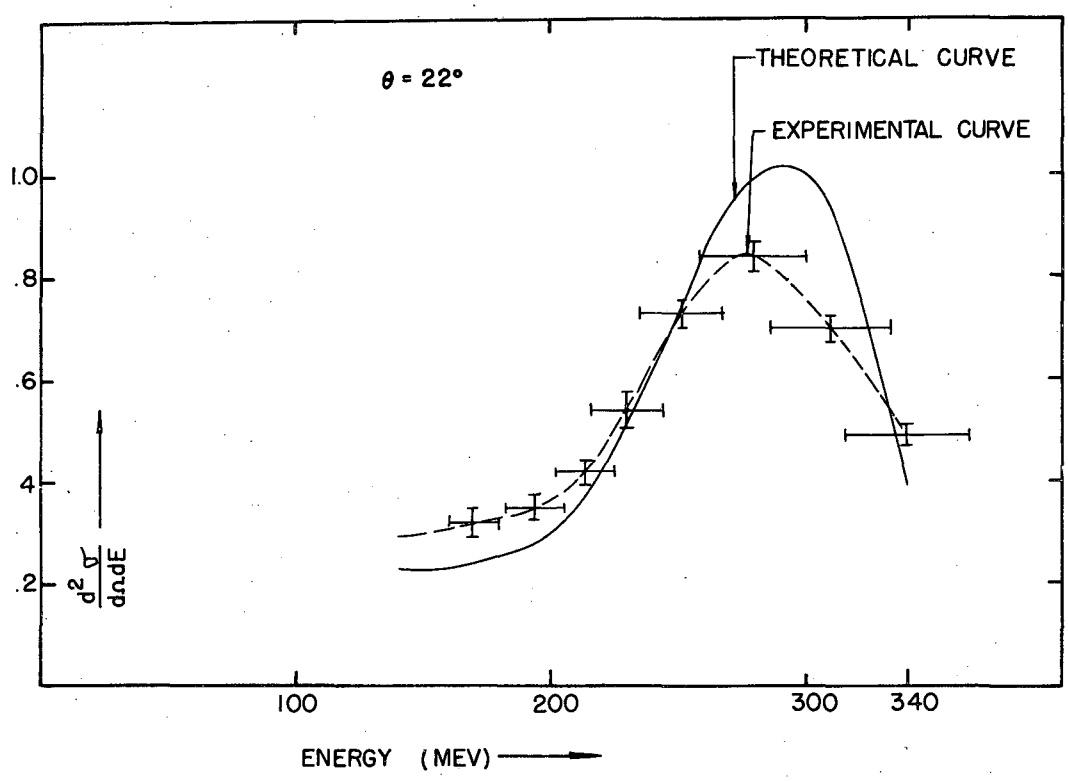

θ = 22°
1.0
0.8
0.6
0.4
0.2
$\frac{d^2\sigma}{d\Omega dE}$
100
200
300
340
ENERGY (MEV)
MU-2244

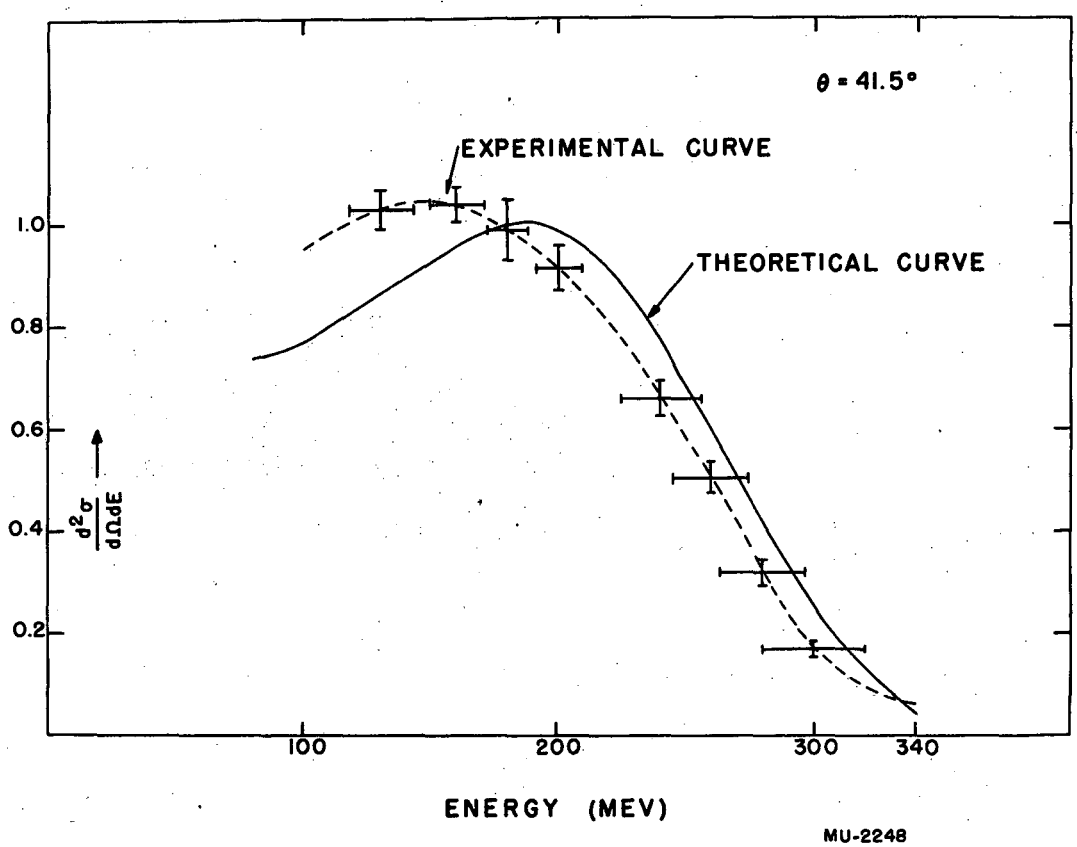
3


1.4
1.2
1.0
0.8
0.6
θ = 41.5°
$\frac{d^2\sigma}{d\Omega dE}$
100
200
300
340
ENERGY (MEV)
MU-2245

4


θ = 22°
THEORETICAL CURVE
EXPERIMENTAL CURVE
1.0
.8
.6
.4
.2
$\frac{d^2\sigma}{d\Omega dE}$
100
200
300
340
ENERGY (MEV)

MU-2247


θ = 41.5°
EXPERIMENTAL CURVE
THEORETICAL CURVE
1.0
0.8
0.6
0.4
0.2
d²σ / dΩdE
100
200
300
340
ENERGY (MEV)
MU-2248

6