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A STUDY OF REGGE DYNAMICS IN PION NUCLEON SCATTERING

Berkeley, California

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John D. Stack

(Ph. D. Thesis)

April 27, 1965

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ABSTRACT

A theoretical study is made of the MacDowell symmetry and the properties of Fermion Regge poles. Subsequent to this, a set of Reggeized bootstrap equations for πN scattering is derived and analyzed. A careful discussion of kinematics is given, the inhomogeneous terms in the integral equation are constructed in detail, and the integral equation is transformed to one of standard Fredholm type.

INTRODUCTION

Recent years have seen a tremendous proliferation in the number and variety of known strongly interacting particles. The progress that has been made in understanding this complex of states has been mainly the result of two complementary general approaches. One of these is the study from the dynamical point of view of certain systems chosen for their simplicity. The attempts to understand the $\pi\pi$ and πN systems dynamically are perhaps the best examples of this. The other approach has been the observation of regularities in the whole spectrum of particles and their interactions and the association of these regularities with symmetries and approximate symmetries. The discovery of the conservation of isotopic spin and the broken eight fold way are results of this approach. In atomic physics the attempts to merge the dynamical model of the hydrogen atom with the regularities observed in all atomic spectra eventually led to the discovery of quantum mechanics and a complete theory of atomic phenomena. So here too it is hoped that advances in these two general directions as well as attempts to merge them together will bring about a more complete understanding of strong interactions. The present investigation is devoted to an attempt to improve the theory of πN scattering.

Historically, the attempts that have been made to understand the πN system in a dynamical way have played a large role in the development of the theory of strong interactions, and have led to a theoretical understanding of πN phenomena at low energies which is satisfying in many ways. Among the first of these attempts was the

work of Chew and Low.[1] They studied a simple static model of the πN interaction, which contained some features which are still basic ingredients of dynamical calculations. Namely, that the partial wave amplitude is an analytic function in the energy variable whose singularities are of two types; a right hand cut whose origin is the requirement of unitarity in the direct channel, and a left hand cut which represents the "force" and whose origin is the scattering in the cross channel. An important result of their analysis was a dynamical model for the N^*_{33} resonance, as due to the large attractive force from exchange of the nucleon. Further progress awaited the development of a relativistic framework for dynamical calculations. This came with the discovery of the Mandelstam representation[2] and the subsequent derivation of relativistic partial wave dispersion relations.[3,4] Frautschi and Walecka[5] used the framework thus provided to study the Chew Low model of the N^*_{33} resonance in a relativistic context. Their work qualitatively confirmed the Chew Low model in that a 3/2, 3/2 resonance was found, but at an energy rather lower than experimentally observed. Subsequent to this, Chew[6] made an important observation, again using the static model. This was that the N^*_{33} resonance in the crossed channel resulted in a strong attraction in the 1/2 1/2 state in the direct channel, in which the nucleon appears. The suggestion was then made that one could generate the nucleon as a bound state, with N^*_{33} exchange as the dominant force. The solution of the relativistic dispersion relations now required a high energy cutoff due to the high spin of the N^*_{33} resonance. Calculations

by Abers and Zemach,[7] and by Ball and Wong[8] confirmed for an appropriate choice of cutoff that the nucleon could indeed be generated as a bound state with approximately correct mass and coupling constant. Ball and Wong's work also showed that almost all the low angular momentum waves could be understood at low energies in terms of the same forces that produce the nucleon and the N^*_{33}. Thus on the basis of the above calculations and many others,[9] it is fair to say that a dynamical understanding of all low energy πN phenomena including the lowest bound state and resonance has been achieved in a way that is reasonably self consistent and in reasonable agreement with experiment. On the other hand the above calculations all contain cutoffs of one form or another to which the solutions of the bootstrap equations are quite sensitive. Nor has there been any concerted effort to understand the higher resonances in a quantitative way, particularly those at 900 and 1350 Mev which are thought to be Regge recurrences of the nucleon and N^*_{33}.

It is the purpose of the present investigation to develop a theory which while incorporating in a broad way the ideas of the calculations mentioned above, is fully Reggeized. A parameter in some ways analogous to the cutoffs of previous calculations remains in the theory, but it now has a physical significance and the theory depends on it in a much less sensitive way. Furthermore, the set of bootstrap equations studied here will make it possible to explore the conjecture that the nucleon and N^*_{33} lie on Regge trajectories, and to investigate the possibility that the two resonances mentioned above also lie on these trajectories. The work is laid out as follows:

In Section I, the basic notions of the strip approximation on which this work is based, are reviewed and discussed for the πN case. Section II is a digression on a symmetry important in πN scattering, first noted by Mac Dowell.[3] A general discussion is given, which establishes it for the general spin case and also establishes clearly the origin of the symmetry. Section III considers the πN partial wave amplitude in the complex angular momentum plane and establishes some simple analytic properties of Fermion Regge poles. Section IV contains a derivation and discussion of the basic dynamical equation as well as some further results on the behavior of Fermion poles. In Section V, the terms which play the role of forces in the dynamical equation are constructed and their qualitative behavior discussed. Section VI treats the singular behavior of the basic integral equation, leaving an integral equation of standard type. Section VII contains a brief discussion of the asymptotic behavior of the Regge parameters and some concluding remarks.

SECTION I. THE STRIP APPROXIMATION

In this section, we review and discuss briefly for the πN case the basic ideas of the strip approximation developed by Chew[10] and applied by Chew and co-workers[11] to $\pi\pi$ scattering.

The basis of this approximation is an attempt to build into a single theory the general features of two body reactions of strongly interacting particles which are thought to be controlled by two body dynamics. These features fall into two ranges of energies. The first of these is the low energy region. Here scattering is concentrated in a few angular momentum states. In the most interesting cases, prominent resonances occur. While channels involving three or more particles may be open in this region, most of the scattering into such channels can be understood in terms of production of quasi-two body final states. The second energy region is very high energies. Here the scattering is almost all forward and is mainly absorptive. Essentially an infinite number of angular momentum states are involved and the variation of the amplitudes with angular momentum is extremely slow. This behavior suggests attempting to understand the situation in terms of the nearest singularities in momentum transfer. These singularities are controlled by two body dynamics in the cross channel. The intervening region of energies consists of a gradual transition between these two regimes and cannot be simply approximated by two body dynamics in either the direct or cross channels. The strip approximation attempts to join the low energy region directly

onto the high energy region. The intermediate energy region will of course be mutilated to some extent by this abrupt transition, but it is hoped that this will still allow a good approximation in the interior of the high and low energy regions.

Early attempts at a theory of this type foundered because of apparent inconsistencies between resonant behavior in one channel and reasonable high energy behavior in the cross channel.[12] The brilliant work of Regge[13] confirmed that this inconsistency was only an apparent one however, and reopened the possibility of a consistent strip approximation. In the Reggeized version of the strip approximation, the conjecture that particles and resonances are poles lying on trajectories of reasonable shape in the angular momentum plane is built in from the start. Particles and resonances are then manifestations of such poles moving through or near physical values of angular momentum in the direct channel and the high energy behavior of two body amplitudes is controlled by poles in cross channels. As has been emphasized by Chew,[10] the point at which the low energy region is joined onto the high energy region in the strip approximation also receives a rather natural interpretation in terms of Regge poles. For example the rightmost Regge pole in the s channel will dominate the t and u discontinuities as well as the st and su double spectral functions for $t, u \to \infty$ (s fixed) as long as the pole is in the right half angular momentum plane. This range of energies corresponds to the low energy resonance region mentioned previously and in this range of energies the behavior of the amplitude is controlled by the large t and u regions of the st and su double spectral functions.

Beyond the resonance region the pole curves back into the left half plane and the large t and u parts of the double spectral functions begin to fade away and nearby singularities which are controlled by cross channel poles begin to take over. This provides a natural place to join the low energy region onto the high energy region. This gives a significance to the strip width s_1, as well as a way of roughly estimating it from experiment. As in the $\pi\pi$ case treated by Chew, in the πN case studied here the above ideas will be applied in the following way: A dynamical equation based on partial wave dispersion relations will be used to generate the amplitude in the low energy region $s \in \left[(M + \mu)^2, s_1\right]$. The contributions of double spectral functions lying outside this strip will be parameterized in terms of Regge poles in the crossed u and t channels. These terms will give the dominant contributions to the double spectral functions in the low u and low t strip regions. The deep interior regions of the double spectral functions which depend in an essential way on many body dynamics will be ignored. Given the t channel $\pi\pi \to N\bar{N}$ pole parameters, a bootstrap situation then exists in which the output s channel πN poles are required to be consistent with the poles in the crossed πN channel.

II. PARTIAL WAVE AMPLITUDES AND THE MAC DOWELL SYMMETRY

πN partial wave dispersion relations differ from those for the simpler $\pi\pi$ case in essentially two respects. The first of these is the presence of unequal masses, which complicates the singularity structure. This will be discussed in more detail in later sections. The second is the necessity of working in the W plane rather than the s plane. This is the consequence of a symmetry first noted in πN scattering by Mac Dowell.[3] In this section we give this symmetry a more fundamental treatment than it has received in the past. While this will not result in any practical simplification of our treatment of πN partial wave dispersion relations relative to past treatments, it will clearly show the origin of the symmetry and allow an extension to the arbitrary spin case with no extra effort. We consider only physical J in this section. Let us take first the πN case. The covariant helicity amplitude can be written:

$$T_{\lambda\mu} = \bar{u}_\lambda(p')\left[A(s,t) + \frac{\gamma}{2}\cdot(k_1 + k_2)\, B(s,t)\right]u_\mu(p) \qquad \text{II.1}$$

This form is a consequence of parity conservation and Lorentz invariance. From this the partial wave amplitude is easily derived:

$$T^J_{J\mp\frac{1}{2}}(W) \equiv T^J_{++} \pm T^J_{+-} = \frac{W}{q}\, e^{i\delta} \sin\delta \quad (\ell = J \mp \tfrac{1}{2})$$

$$= \frac{(W+M)^2 - \mu^2}{32\pi W}\left[A_{J\mp\frac{1}{2}}(s) + (W-M)\, B_{J\mp\frac{1}{2}}(s)\right] + \qquad \text{II.2}$$

$$+ \frac{(W-M)^2 - \mu^2}{32\pi W}\left[-A_{J\pm\frac{1}{2}}(s) + (W+M)\, B_{J\pm\frac{1}{2}}(s)\right]$$

where $A_\ell(s) = \int_{-1}^{+1} dz\, A\left(s, -2q^2(1-z)\right) P_\ell(z)$; etc. for $B_\ell(s)$.

From this we easily see that

$$T^J_{J\mp\frac{1}{2}}(W) = T^J_{J\pm\frac{1}{2}}(-W) \qquad \text{II.3}$$

which is the Mac Dowell symmetry. The fact that amplitudes of opposite parity are coupled to each other in this way forces one to treat them together and to work in the W plane. The original paper of Mac Dowell stated that the symmetry was a consequence of PT invariance. We shall see that it depends only on Lorentz invariance and a simple property of rotations through angle π in cases where J is a half integer versus cases where J is an integer. Let us generalize our discussion to any two body reaction:

$$1 + 2 \rightarrow 3 + 4$$

where particle i has spin S_i . We consider the covariant helicity

amplitude $T_{\lambda_3\lambda_4\,;\,\lambda_1\lambda_2}(p_3p_4\,;\,p_1p_2)$. Lorentz invariance requires:

$$T_{\lambda_3\lambda_4\,;\,\lambda_1\lambda_2}(p_3p_4\,;\,p_1p_2) = \qquad \text{II.4}$$

$$= \sum_{\lambda_1'} D^{S_3\,*}_{\lambda_3'\lambda_3}(R_3)\, D^{S_4*}_{\lambda_4'\lambda_4}(R_4)\, T_{\lambda_3'\lambda_4';\lambda_1'\lambda_2'}(\Lambda p_3\Lambda p_4;\,\Lambda p_1\Lambda p_2)\, D^{S_1}_{\lambda_1'\lambda_1}(R_1) D^{S_2}_{\lambda_2'\lambda_2}(R_2)$$

where Λ is some real proper Lorentz transformation. R_i is the Wigner rotation given by:

$$R_i = B^{-1}_{\Lambda\bar{p}_i}\,\Lambda\, B_{\bar{p}_i}$$

where

$$B_{\bar{p}_i} = e^{-i\phi_i J_3}\, e^{-i\theta_i J_2}\, e^{i\phi_i J_3}\, e^{-i\lambda_i K_3}$$

and

$$\cosh\,\lambda_i = \frac{E_i}{m_i}$$

$B_{\bar{p}_i}$ takes particle i from rest to momentum $\bar{p}_i$. Given $T_{\lambda_3\lambda_4\,;\,\lambda_1\lambda_2}(p_3p_4\,;\,p_1p_2)$, we can construct a simpler quantity. We introduce the spinor functions $\Psi^S_\lambda(p)$ which are 2S+1 dimensional column vectors which transform according to the (OS) representation of the homogeneous Lorentz group.

$$\psi_\lambda^S(p) = D^{OS}(B\bar{p}_i)\, \psi_\lambda^S(m)$$

$$D^{OS}(\Lambda)\, \psi_\lambda^S(p) = \sum_{\lambda'} D^S_{\lambda'\lambda}(R)\, \psi_{\lambda'}^S(\Lambda p) \qquad \text{II.5}$$

We define $T(K)$ as follows:

$$T_{\lambda_3\lambda_4\,;\,\lambda_1\lambda_2}(p_3p_4\,;\,p_1p_2) = \psi_{\lambda_3}^{+S_3}(p_3)\, \psi_{\lambda_4}^{+S_4}(p_4)\, T(K)\, \psi_{\lambda_1}^{S_1}(p_1)\, \psi_{\lambda_2}^{S_2}(p_2) \qquad \text{II.6}$$

where K denotes the set of momenta $\{p_i\}$. $T(K)$ is a generalized matrix with appropriate row and column indices to match onto the ψ's. Relativistic invariance of $T_{\lambda_3\lambda_4\,;\,\lambda_1\lambda_2}(p_3p_4\,;\,p_1p_2)$ implies the simpler transformation for $T(K)$:

$$T(K) = D^{OS_3}(\Lambda)^+\, D^{OS_4}(\Lambda)^+\, T(\Lambda K)\, D^{OS_1}(\Lambda)\, D^{OS_2}(\Lambda) \qquad \text{II.7}$$

The function $T(K)$ is essentially the same as the M function introduced by Stapp.[14] The unitarity condition for $T(K)$ is free of extraneous non-analytic factors and thus it is this function which has only those singularities determined by unitarity itself in any theory which uses unitarity to determine singularity structure.

Now let us extend (II.7) to the complex Lorentz group. The transformation Λ in equation (II.7) can be parameterized as follows:

$$\Lambda = e^{-i\bar{\theta}\cdot\bar{J}}\, e^{-i\bar{\lambda}\cdot\bar{K}} \qquad \text{II.8}$$

The J_i and K_i are the usual 4×4 matrices and therefore Λ is a holomorphic function of θ_i and λ_i. Similarly the $D^{OS}(\Lambda)$ are holomorphic in θ_i and λ_i. The right hand side of the equation is independent of the θ_i and λ_i and is therefore trivially holomorphic in θ_i and λ_i. Therefore this equation provides an extension of $T(K)$ as a function of θ_i and λ_i over the full complex planes of these variables. Thus the validity of (II.7) for real Lorentz transformations implies its validity for complex Lorentz transformations. This extended Lorentz invariance allows the calculation of $T(K')$ at any point K' related to K by a real or complex Lorentz transformation, if only $T(K)$ is known. Furthermore it is easy to show that if $T(K)$ is holomorphic in the momenta in some original domain, it is also holomorphic in the extended domain generated by the action of the complex Lorentz group.[15] Our approach to the Mac Dowell symmetry will be to use this extended Lorentz invariance to relate the partial wave helicity amplitude at values of W which are negatives of each other. For this consider the set of complex Lorentz transformations $\Lambda(\psi)$:

$$\Lambda(\psi) = e^{-i\psi J_3} e^{-i(i\psi)K_3} \qquad \text{II.9}$$

where ψ is real. If $\psi = \pi$, $\Lambda(\pi)$ maps any four vector into its negative, i.e.

$$\Lambda(\pi)\, k = -k$$

Furthermore,

$$D^{OS}\Big(\Lambda(\pi)\Big)\psi^S_\lambda(p) = \psi^S_\lambda(-p)\, e^{-i\pi\lambda} \qquad \text{II.10}$$

where $\psi^S_\lambda(-p) = e^{-i\phi J_3}\, e^{-i\theta J_2}\, e^{i\phi J_3}\, e^{-i(\lambda+i\pi)K_3}\, \psi^S_\lambda(m)$

This equation follows because K_3 is the same matrix as $i\,J_3$ in the OS representation. Now let us introduce partial wave helicity amplitudes:

II.11

$$T^J_{\lambda_3\lambda_4\,;\,\lambda_1\lambda_2}(W) = \frac{4\pi}{16\pi^2}\int d(R)\, D^J_{\lambda_1-\lambda_2\,;\,\lambda_3-\lambda_4}(R)\, T_{\lambda_3\lambda_4\,;\,\lambda_1\lambda_2}(p_3p_4;p_1p_2)$$

where $T_{\lambda_3\lambda_4\,;\,\lambda_1\lambda_2}(p_3p_4\,;\,p_1p_2)$ is taken in the center of mass frame, $\bar{p}_3 = R\bar{p}_1$, and $W = E_1 + E_2 = E_3 + E_4$. Similarly,

II.12

$$T^J_{\lambda_3\lambda_4\,;\,\lambda_1\lambda_2}(-W) = \frac{4\pi}{16\pi^2}\int d(R)\, D^J_{\lambda_1-\lambda_2;\lambda_3\lambda_4}(R)\, T_{\lambda_3\lambda_4;\,\lambda_1\lambda_2}(-p_3-p_4;\,-p_1-p_2)$$

where

II.13

$$T_{\lambda_3\lambda_4;\,\lambda_1\lambda_2}(-p_3-p_4;\,-p_1-p_2) = \psi^{+S_3}_{\lambda_3}(-p_3)\,\psi^{+S_4}_{\lambda_4}(-p_4)\,T(-K)\,\psi^{S_1}_{\lambda_1}(-p_1)\,\psi^{S_2}_{\lambda_2}(-p_2)$$

To get a relation between these two quantities, we use equation II.7 for $\Lambda = \Lambda(\pi)$ and II.10 . From these it is easy to see that

$$T_{\lambda_3\lambda_4;\,\lambda_1\lambda_2}(-p_3-p_4;\,-p_1-p_2)\,(-1)^{\lambda_3+\lambda_4-(\lambda_1+\lambda_2)}$$

II.14

$$= T_{\lambda_3\lambda_4;\,\lambda_1\lambda_2}(p_3p_4;\,p_1p_2)$$

therefore

$$T^J_{\lambda_3\lambda_4;\,\lambda_1\lambda_2}(-W)\,(-1)^{\lambda_3+\lambda_4-(\lambda_1+\lambda_2)} = T^J_{\lambda_3\lambda_4;\,\lambda_1\lambda_2}(W) \qquad \text{II.15}$$

Now the combinations

$$T^J_{\lambda_3\lambda_4;\,\lambda_1\lambda_2}(W)_{\pm} \equiv \tfrac{1}{2}\left\{\left[T^J_{\lambda_3\lambda_4;\,\lambda_1\lambda_2}(W) + T^J_{-\lambda_3-\lambda_4;\,-\lambda_1-\lambda_2}(W)\right] \pm \left[T^J_{\lambda_3\lambda_4;\,-\lambda_1-\lambda_2}(W) + T^J_{-\lambda_3-\lambda_4;\,\lambda_1\lambda_2}(W)\right]\right\},$$

and

$$T'^J_{\lambda_3\lambda_4;\,\lambda_1\lambda_2}(W)_{\pm} = \tfrac{1}{2}\left\{\left[T^J_{\lambda_3\lambda_4;\,\lambda_1\lambda_2}(W) - T^J_{-\lambda_3-\lambda_4;\,-\lambda_1-\lambda_2}(W)\right] \pm \left[T^J_{\lambda_3\lambda_4;\,-\lambda_1-\lambda_2}(W) - T^J_{-\lambda_3-\lambda_4;\,\lambda_1\lambda_2}(W)\right]\right\}$$

represent transitions between states of definite parity. Using relation II.15 above, we see that

II.16

$$T^J_{\lambda_4\lambda_4;\,\lambda_1\lambda_2}(-W)_{\pm} = \tfrac{1}{2}\left\{(-1)^{\lambda_3+\lambda_4-(\lambda_1+\lambda_2)}\left[T^J_{\lambda_3\lambda_4;\,\lambda_1\lambda_2}(W) + T^J_{-\lambda_3-\lambda_4;\,-\lambda_1-\lambda_2}(W)\right] \pm (-1)^{\lambda_1+\lambda_2+\lambda_3+\lambda_4}\left[T^J_{\lambda_3\lambda_4;\,-\lambda_1-\lambda_2}(W) + T^J_{-\lambda_3-\lambda_4;\,\lambda_1\lambda_2}(W)\right]\right\}$$

and therefore

$$T^J_{\lambda_3\lambda_4;\,\lambda_1\lambda_2}(-W)_{\pm} = (-1)^{\lambda_1+\lambda_2-(\lambda_3+\lambda_4)}\,T^J_{\lambda_3\lambda_4;\,\lambda_1\lambda_2}(W)_{\mp} \qquad \text{II.17}$$

for boson fermion scattering and

$$T^{J}_{\lambda_3\lambda_4;\ \lambda_1\lambda_2}(-W)_{\pm} = (-1)^{\lambda_1+\lambda_2-(\lambda_3+\lambda_4)}\, T^{J}_{\lambda_3\lambda_4;\ \lambda_1\lambda_2}(W)_{\pm} \qquad \text{II.18}$$

for boson boson or fermion fermion scattering. The same relations hold for the amplitudes $T'^{J}_{\lambda_3\lambda_4;\ \lambda_1\lambda_2}(W)_{\pm}$. The first case above is the Mac Dowell symmetry generalized to arbitrary spins and parity non-conserving transitions, if they occur. The difference in the two cases follows from the fact that

$$(-1)^{2(\lambda_3+\lambda_4)} = \begin{cases} -1 & \text{for the boson-fermion case} \\ +1 & \text{for the boson-boson and fermion - fermion case.} \end{cases}$$

This factor arose of course from the rotation $e^{i\pi J_3}$ in $\Lambda(\pi)$. So we have seen that the Mac Dowell symmetry is quite fundamental and follows from extended Lorentz invariance which is itself a consequence of ordinary Lorentz invariance. No use need be made of P or T invariance.

Let us consider the meaning of our definition of $T^{J}_{\lambda_3\lambda_4;\ \lambda_1\lambda_2}(-W)$ in more detail. In the usual treatment $T(K)$ is expanded as

$$T(K) = \sum_i A_i(s,t)\, Y_i(K) \qquad \text{II.19}$$

The A_i are scalar amplitudes analogous to the A and B in the πN case. The $Y_i(K)$ are spinor basis functions which are polynomials in momentum components. Upon making the projection II.11, one has an expression which is a sum of terms of the form:

$$g(W)\, A_\ell(s)\,, \text{ where } A_\ell(s) = \int_{-1}^{+1} dz\, A\big(s, t(z)\big)\, P_\ell(z)$$

$g(W)$ may in general have kinematic singularities at the points $W = \pm (M_1 \pm M_2)$, $\pm (M_3 + M_4)$, 0 .
One can always choose the cuts caused by these singularities to lie on straight lines connecting these points. Now considering the set of transformations $\Lambda(\psi)$ acting on the set of momenta K and defining $W = E_1 + E_2 = E_3 + E_4$, $T^J_{\lambda_3\lambda_4;\,\lambda_1\lambda_2}(-W)$ is gotten by allowing ψ to reach π starting from 0 . In a term of the form above, this means of course s is constant all along this path while W moves from a point to the right of the kinematic singularities to a point to the left of them on a path which does not cross any of the kinematic cuts as defined above. At the beginning and end of the path $W^2 = s$ is satisfied. This specifies the sheet in W on which the Mac Dowell symmetry holds. Note that one cannot set $s = W^2$ at the outset and continue directly to the Mac Dowell symmetric point $-W$, since one in general always meets a cut of $A_\ell(s)$ on the way. Any two points related by the Mac Dowell symmetry are always on the physical sheet of the scalar amplitudes $A_\ell(s)$.

III. CONTINUATION IN TOTAL J AND FERMION POLES

In this section we continue the πN amplitude in total angular momentum and establish some simple properties of the Fermion Regge poles, assuming the amplitude is meromorphic in the region considered. Continuation in total angular momentum for πN scattering has previously been considered by Singh.[16] Our discussion of the properties of the Fermion poles is based on the method of Barut and Zwanziger[17] who considered spinless particles.

From equation II.2 , the partial wave amplitude can be written:

$$T^{J}_{J\mp\frac{1}{2}}(W) = \frac{(W+M)^2-\mu^2}{32\pi W}\left[A_{J\mp\frac{1}{2}}(s) + (W-M)\, B_{J\mp\frac{1}{2}}(s)\right]$$

$$+ \frac{(W-M)^2-\mu^2}{32\pi W}\left[-A_{J\pm\frac{1}{2}}(s) + (W+M)\, B_{J\pm\frac{1}{2}}(s)\right]$$

We assume that as $t(u) \to \infty$ at fixed s that A and B are bounded by some finite power of $t(u)$. Therefore for $\mathrm{Re}\,\ell$ large enough:

$$A_\ell(s) = \frac{1}{\pi q^2}\left\{\int_{4\mu^2}^{\infty} dt'\, A_t(s,t')\, Q_\ell\left(1+\frac{t'}{2q^2}\right) - \int_{(M+\mu)^2}^{\infty} du'\, A_u(s,u')\, Q_\ell\left(1+\frac{2(M^2+\mu^2)-s-u'}{2q^2}\right)\right\} \qquad \text{III.1}$$

and likewise for $B_\ell(s)$. We seek a continuation of III.1 away from integer values of ℓ that allows the Sommerfeld-Watson transformation to be made. Such a continuation has the desired property that its singularities in ℓ are directly related to the asymptotic behavior of the amplitude in t and u. For $q^2 > 0$, the first term of III.1 is holomorphic in ℓ for $\mathrm{Re}\,\ell > N$ and if $\ell = |\ell|\, e^{i\phi_\ell}$, it is nonincreasing as $|\ell| \to \infty$ for $-\pi/2 \leqslant \phi_\ell \leqslant \frac{\pi}{2}$. The second term is holomorphic for $\mathrm{Re}\,\ell > N$, but is badly behaved as $\ell_i \to \infty$ and will not allow the Sommerfeld-Watson transformation to be made. The standard cure for this difficulty is to define continuations from even integer ℓ and odd integer ℓ separately. So we define:

$$A_\ell^{e\,o}(s) = \frac{1}{\pi q^2}\left\{\int_{4\mu^2}^{\infty} dt'\, A_t(s,t')\, Q_\ell\left(1+\frac{t'}{2q^2}\right) \pm \int_{(M+\mu)^2}^{\infty} du'\, A_\mu(s,u')\, Q_\ell\left(-1-\frac{2(M^2+\mu^2)-s-u'}{2q^2}\right)\right. \qquad \text{III.2}$$

and similarly for $B_\ell^{e\,o}(s)$. For $q^2 > 0$ both $A_\ell^{e\,o}(s)$ and $B_\ell^{e\,o}(s)$ are well behaved and allow the Sommerfeld-Watson transformation to be made. These quantities specify the continuation of $T^J_{J+\frac{1}{2}}(W)$ away from physical values of J.

$$T^{Je_o}_{J-\frac{1}{2}}(W) = \frac{(W+M)^2 - \mu^2}{32\pi W}\left[A^{e_o}_{J-\frac{1}{2}}(s) + (W-M)\, B^{e_o}_{J-\frac{1}{2}}(s)\right] \qquad \text{III.3}$$

$$+ \frac{(W-M)^2 - \mu^2}{32\pi W}\left[-A^{o_e}_{J+\frac{1}{2}}(s) + (W+M)\, B^{o_e}_{J+\frac{1}{2}}(s)\right] , \qquad \text{III.4}$$

and $T^{J e_o}_{J+\frac{1}{2}}(W)$ is given by : $T^{J e_o}_{J+\frac{1}{2}}(W) = T^{J e_o}_{J-\frac{1}{2}}(-W)$

The e amplitude agrees with the physical amplitude at values of J given by $J = \frac{4n+1}{2}$, n an integer, the o amplitude at $J = \frac{4n-1}{2}$.

So far our discussion has been restricted to high values of J . The only assumption made so far is the Mandelstam representation with at most a finite number of subtractions necessary. At this point we make some further hypotheses. First, we assume that the amplitude can be analytically continued in J to the left to some point below all physical values of J and that the continued amplitude agrees with the physical amplitude at all physical values of J . Second, we assume that the only singularities in J to the right of this point are simple poles. These assumptions require some comment.

Even in theories which allow a continuation to a point below all physical values of J , the assumption that only poles appear will not hold in general. For example, Mandelstam[18] has considered a case in which if poles occur, so do cuts if account is

taken of certain complicated processes involving three and four particle intermediate states. More complicated singularities may also occur in higher order processes. The motivation for attempting to ignore these singularities is based mainly on a desire to see if a sensible approximation scheme can be built in the order of processes. That is, one attempts to handle explicitly reactions involving say at most N particles and to ignore the complications of $N + 1$ and higher particle systems. The strip approximation, with which we are dealing, represents the lowest order approximation in such a scheme in that only two particle systems are involved. If the scheme makes sense, then meaningful results can be obtained ignoring such intrinsically three particle effects as cuts. Two particle systems with reasonable forces generally give rise to amplitudes meromorphic in J in the right half plane. Since there are no nonsense states in elastic πN scattering and we do not introduce CDD poles, the continued amplitude will be the same as the physical amplitude in this theory.

Now under these assumptions, let us establish some simple properties of the Fermion poles. For definiteness, let us consider the amplitude $T^{Je}_{J-\frac{1}{2}}(W)$. We can break the quantities $A^{e\,o}_{J\mp}(s)$, $B^{e\,o}_{J\mp\frac{1}{2}}(s)$ into two parts. For example:

$$A^{e}_{J-\frac{1}{2}}(s) = \frac{1}{\pi q^2}\left\{\int_{4\mu^2}^{t_1} dt'\, A_t(s,t')\, Q_{J-\frac{1}{2}}\left(1+\frac{t'}{2q^2}\right)\right.$$

$$+\int_{t_1}^{\infty} dt'\, A_t(s,t')\, Q_{J-\frac{1}{2}}\left(1+\frac{t'}{2q^2}\right) + \int_{(M+\mu)^2}^{u_1} du'\, Au(s,u')\, Q_{J-\frac{1}{2}}\left(-1-\frac{2(M^2+\mu^2)-s-u'}{2q^2}\right)$$

$$\left. + \int_{u_1}^{\infty} du'\, A_u(s,u')\, Q_{J-\frac{1}{2}}\left(-1-\frac{2(M^2+\mu^2)-s-u')}{2q^2}\right)\right\}$$

The finite integrals are holomorphic in the entire J plane except for fixed poles at $J = -1/2, -3/2$, etc. This allows us to break $T^{Je}_{J-\frac{1}{2}}(W)$ into two terms, one coming from the finite integrals and the other from the infinite integrals.

$$T^{Je}_{J-\frac{1}{2}}(W) = b_1(J,W) + b_2(J,W) \qquad \text{III.5}$$

The Regge poles that exist must all be contained in the second term, $b_2(J,W)$. Let us consider the analytic properties of $b_2(J,W)$. First of all there are winding point singularities at $W = \pm(M\pm\mu)$. Let us divide them out and consider the function $\tilde{b}_2(J,W)$:

$$\tilde{b}_2(J,W) = \frac{b_2(J,W)}{(q^2)^{J-\frac{1}{2}}} \cdot \frac{1}{(E+M)} \qquad \text{III.6}$$

This function is real analytic and has various cuts from the A and B amplitudes and the Q functions on the $\pm$ real and $\pm$ imaginary axes. (We choose $t_1 > 4M^2$ so the usual circular cut does not appear in $b_2(J,W)$.) The function $\tilde{b}_2(J,W)$ is not the only real analytic function obtainable from $b_2(J,W)$. One could imagine replacing

$(q^2)^{J-\frac{1}{2}}$ by

$$\left\{\left[s - (M+\mu)^2\right]\left[s - (M-\mu)^2\right]\right\}^{J-\frac{1}{2}}$$

or $\left[s - (M+\mu)^2\right]^{J-\frac{1}{2}}$ in the above definition. However for the success of the argument which follows it is quite important that the function $\tilde{b}_2(J,W)$ as we have defined it be used. The cuts of $\tilde{b}_2(J,W)$ are as follows:

(a) $W = \pm\left[M+\mu\,,\ \infty\right]$

(b) $W = \pm\left[i\,\infty,\ i\left[t_1 + u(t_1) - 2(M^2+\mu^2)\right]^{\frac{1}{2}}\right]$

where $u(t_1) \to (M+\mu)^2$ as $t_1 \to \infty$

(c) $W = \pm\left[i\,\infty,\ i\left[u_1 + t(u_1) - 2(M^2+\mu^2)\right]^{\frac{1}{2}}\right]$

where $t(u_1) \to 4\mu^2$ as $u_1 \to \infty$

(d) $W = \pm\left[0, \frac{i}{\sqrt{2}}\left[t_1 - 2(M^2+\mu^2) - \left[(t_1 - 2M^2 - 2\mu^2)^2 - 4(M^2-\mu^2)^2\right]^{\frac{1}{2}}\right]^{\frac{1}{2}}\right]$

(e) $W = \pm\left[i\,\infty, \frac{i}{\sqrt{2}}\left[t_1 - 2(M^2+\mu^2) + \left[(t_1 - 2M^2 - 2\mu^2)^2 - 4(M^2-\mu^2)^2\right]^{\frac{1}{2}}\right]^{\frac{1}{2}}\right]$

(f) $W = \pm\left[0\,,\ \frac{M^2-\mu^2}{\sqrt{u_1}}\right]$

(g) $W = \pm\left[i\,\infty,\ i\left[u_1 - 2(M^2+\mu^2)\right]^{\frac{1}{2}}\right]$

Cuts (a), (b) and (c) arise from the A and B amplitudes. The rest are from the Q functions. Now the Regge poles are contained in $\tilde{b}_2(J,W)$ regardless of the values of u_1, t_1. These can be chosen to be arbitrarily large. If this is done, then cuts (d) and (f) shrink to zero, while cuts (b), (c), (e) and (g) move infinitely far away. Therefore the position and reduced residue of any pole which remains in the domain of meromorphy connected with high J for all energies are both analytic functions of W with only cut (a), as long as the pole does not intersect other poles. We expect that these conditions will be met for the leading trajectory. Of course the same results carry over for the poles and residues of the o amplitude.

If a factor having a J dependence other than $(q^2)^{J-\frac{1}{2}}$ is used in defining $\tilde{b}_2(J,W)$, a function having cuts in addition to (a) - (g) is obtained. These cuts do not shrink to zero or recede as u_1, $t_1 \to \infty$ and thus the reduced residue of the function so defined has cuts in addition to cut (a). This result is quite important for choosing the kinematic factor in the N/D equation which will be done in Section IV.

Let us now investigate the properties of the poles and residues near $W = \pm (M + \mu)$. We first consider the point $W = (M + \mu)$. We define

$$\tilde{b}(J,W) = \frac{T^J_{J-\frac{1}{2}}(W)}{(q^2)^{J-\frac{1}{2}}(E + M)} \qquad \text{III.7}$$

This function is real analytic near $W = (M+\mu)$ and has no zero at $W = (M+\mu)$. The generalized unitarity relation is :

$$\frac{\tilde{b}(J,W+i\epsilon) - \tilde{b}(J,W-i\epsilon)}{2i} = \frac{q(q^2)^{J-\frac{1}{2}}(E+M)}{W}\,\tilde{b}\,(J,W+i\epsilon)\,\tilde{b}(J,W-i\epsilon) \qquad \text{III.8}$$

Therefore we can write $\tilde{b}(J,W)$ as follows:

$$\tilde{b}(J,W) = \frac{\sin \pi J}{Y(J,W) - i\,q(q^2)^{J-\frac{1}{2}}\,\frac{E+M}{W}\,e^{-i\pi(J-\frac{1}{2})}} \qquad \text{III.9}$$

where $Y(J,W)$ is real analytic and has no branchpoint at $W = (M+\mu)$. The Regge poles are solutions of

$$Y(J,W) - i\,q(q^2)^{J-\frac{1}{2}}\,\frac{E+M}{W}\,e^{-i\pi(J-\frac{1}{2})} = 0 \qquad \text{III.10}$$

From this it is easily seen that if $\mathrm{Re}\,\alpha(M+\mu) > 0$ then $\mathrm{Im}\,\alpha(M+\mu) = 0$. Consequently,

$$Y_{\alpha(M+\mu)}\Big(\alpha(M+\mu),\ M+\mu\Big)\ \mathrm{Im}\,\alpha(W+i\epsilon) \xrightarrow[W \to (M+\mu)^+]{} q(q^2)^{\alpha(M+\mu)-\frac{1}{2}}\,\frac{E+M}{W}\,\sin\pi\,\alpha(M+\mu) \qquad \text{III.11}$$

where

$$Y_{\alpha(M+\mu)}\Big(\alpha(M+\mu),\ M+\mu\Big) = \frac{\partial}{\partial J}\,Y(J,M+\mu)\,/_{J\,=\,\alpha(M+\mu)}$$

This shows the nature of the branchpoint in $\alpha(W)$ near this point. The residue of the pole in $\tilde{b}(J,W)$ is

$$\frac{\sin \pi\, \alpha(M+\mu)}{Y_{\alpha(M+\mu)}\left(\alpha(M+\mu), M+\mu\right)}$$

and a glance at III.11 shows that this has the same sign as $\mathrm{Im}\, \alpha(W+i\epsilon)$ near $W = M+\mu$. Quite analogously the point $W = -(M+\mu)$ can be treated. This point refers to the $(J,J+\frac{1}{2})$ amplitude. We will establish the convention of associating the right hand physical cut with the $(J,J-\frac{1}{2})$ amplitude and the left hand physical cut with the $(J,J+\frac{1}{2})$ amplitude. The physical amplitude is reached by approaching the cut from positive imaginary values on the right and from negative imaginary values on the left. To summarize, the pole position $\alpha(W)$ is real analytic in the W plane with only cut (a). If it passes through or near a physical value of J for $W > 0$, it corresponds to a bound state or resonance in the $(J,J-\frac{1}{2})$ state. If it does so for $W < 0$, it corresponds to a bound state or resonance in the $(J,J+\frac{1}{2})$ state. If $\mathrm{Re}\, \alpha(M+\mu) > 0$, then

$$\frac{\alpha(W+i\epsilon) - \alpha(W-i\epsilon)}{2i} \xrightarrow[W \to (M+\mu)^+]{} (q^2)^{\alpha(M+\mu)} \frac{E+M}{W}$$

and if $\mathrm{Re}\,\alpha(-M-\mu) > -1$,

$$\frac{\alpha(W-i\epsilon) - \alpha(W+i\epsilon)}{2i} \xrightarrow[W \to -(M+\mu)^-]{} (q^2)^{\alpha(-M-\mu)} \frac{E+M}{W}$$

The residue $\beta(W) = \left[J-\alpha(W)\right] \tilde{b}(J,W) \big/_{J = \alpha(W)}$ is also real analytic in the cut W plane with only the cut (a) and has the same sign as $\mathrm{Im}\,\alpha(W+i\epsilon)$ near $W = M+\mu$ and $\mathrm{Im}\,\alpha(+W-i\epsilon)$ near $W = -(M+\mu)$.

SECTION IV. THE N/D EQUATION

In this section we consider the basic N/D equation. The amplitude which we will eventually use for this is $\tilde{b}\,(J,W)$, defined in section III. However, before we proceed to the treatment of the N/D equation, the behavior of the amplitude at $W = \pm\,(M-\mu)$ needs some attention. Let us rewrite the amplitude $T^{J\,e\,o}_{J-\frac{1}{2}}(W)$:

$$T^{J\,e\,o}_{J-\frac{1}{2}}(W) = \frac{(W+M)^2-\mu^2}{32\pi\,W}\left[A^{e\,o}_{J-\frac{1}{2}} + (W-M)\,B^{e\,o}_{J-\frac{1}{2}}\right]$$

$$+\ \frac{(W-M)^2-\mu^2}{32\pi\,W}\left[-A^{o\,e}_{J+\frac{1}{2}} + (W+M)\,B^{o\,e}_{J+\frac{1}{2}}\right]$$

The source of the complications at $W = \pm\,(M-\mu)$ is the integral over the u discontinuity in the amplitudes $A^{o\,e}_{\ell}$, $B^{o\,e}_{\ell}$. For example, the contribution of this integral to $A^{o\,e}_{\ell}$ is :

$$A^{e\,o}_{\ell,u}(s) = \pm\,\frac{1}{\pi q^2}\int_{(M+\mu)^2}^{\infty} du'\,A_u(s,u')\,Q_\ell\left(-1-\frac{2(M^2+\mu^2)-s-u'}{2q^2}\right)$$

where

$$q^2 = \frac{\left[s-(M+\mu)^2\right]\left[s-(M-\mu)^2\right]}{4s}$$

If the integral started at a lower limit $u_o > (M+\mu)^2$, the above amplitude would $\propto (q^2)^\ell$ near $s = (M-\mu)^2$. This follows from the $1/z^{\ell+1}$ behavior of $Q_\ell(z)$ at large z . However in our integral the region near $u' = (M+\mu)^2$ can cause the quantity $2(M^2+\mu^2)-s-u'$ to vanish near $s = (M-\mu)^2$ and this fact causes the behavior near $s = (M-\mu)^2$ to be somewhat more complicated. To investigate this in more detail let us split the integral into two parts, one from $(M+\mu)^2$ to u_2 and the other from u_2 to ∞ , where u_2 is slightly above $(M+\mu)^2$. Now the function $A_u(s,u')$ is analytic in the cut s plane with cuts $\left[(M+\mu)^2, \infty\right]$ and $\left[-\infty, 2(M^2+\mu^2) - u - t_o(u)\right]$ where $t_o(u) > 4\mu^2$. Consequently for u near $(M+\mu)^2$, $A_u(s,u)$ is analytic in s near $(M-u)^2$ and can be expanded about this point. This amounts to a rearrangement of the Legendre expansion of $A_u(s,u')$ in the πN u channel. This expansion for $A_u(s,u')$ can be expressed directly in terms of the πN partial wave amplitudes in the u channel. The appropriate formula for this is given in section V . Near $u = (M+\mu)^2$, the imaginary parts of the partial wave amplitudes are analytic in $\left[u-(M+\mu)^2\right]^{\frac{1}{2}}$ and consequently we can expand $A_u(s,u')$ in a double power series:

$$A_u(s,u') = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{A_u^{n,m}}{n!\ m!} \left[u - (M+\mu)^2\right]^{\frac{n}{2}} \left[s - (M-\mu)^2\right]^m \qquad \text{IV.2}$$

where for example $A_u^{10} = 2\pi \left(\frac{2M+\mu}{M+\mu}\right)\sqrt{\frac{\mu}{M}}\ a^2$

where a is the πN S wave scattering length. Let us introduce $y = s - (M-\mu)^2$, and denote the argument of the Q function by z. Then a term from the expansion of $A_u(s,u')$ contributes the following term to the integral from $(M+\mu)^2$ to u_2 :

$$\pm \frac{2}{\pi} \frac{A_u^{n,m}}{n!\ m!} \int_{-1+\frac{y}{2q^2}}^{-1+\frac{y}{2q^2}+\frac{u_2-(M+\mu)^2}{2q^2}} dz\ y^m \left[2q^2(z+1) - y\right]^{\frac{n}{2}} Q_\ell(z) \qquad \text{IV.3}$$

where

$$q^2 = -\frac{y}{4}\left[\frac{4M\mu - y}{(M-\mu)^2 + y}\right]$$

For small y the lower limit is always near the point $-1-\frac{(M-\mu)^2}{2M\mu}$, and the upper limit is near the point

$$\frac{\left[u_2 - (M+\mu)^2\right]\left[(M-\mu)^2\right]}{-2y\ M\mu}.$$

Now let us divide by $(q^2)^\ell$, keeping $y > 0$ for the moment. Then we have :

$$\mp \frac{2}{\pi} \frac{A_u^{nm}}{n!m!} \int_{-1+\frac{y}{2q^2}}^{-1+\frac{y}{2q^2}+\frac{u_2-(M+\mu)^2}{2q^2}} dz\, y^m \left[2q^2(z+1) - y\right]^{\frac{n}{2}} \frac{Q_\ell(-z)}{\left[\frac{y(4M\mu-y)}{4(M-\mu)^2+y}\right]^\ell} \qquad \text{IV.4}$$

Now we can see the nature of the branchpoint at $y = 0$ by taking the discontinuity of this expression as the negative y axis is approached from above and below. This gives:

$$\mp \frac{2}{\pi} \frac{A_u^{n,m}}{n!m!} y^m (-y)^{\frac{n}{2}-\ell} \left[\frac{(M-\mu)^2}{M\mu}\right]^\ell \left\{ -\sin\pi\ell \int_{-1-\frac{(M-\mu)^2}{2M\mu}}^{-1} dz \left[1 + \frac{2M\mu}{(M-\mu)^2}(1+z)\right] Q_\ell(-z) \right.$$

$$\left. + \frac{\pi}{2} \int_{-1}^{+1} dz \left[1 + \frac{2M\mu(1+z)}{(M-\mu)^2}\right]^{\frac{n}{2}} P_\ell(z) \right\} \qquad \text{IV.5}$$

where we have kept the lowest power of y. The dominant behavior near $y = 0$ will come from the term A_u^{10}. Consequently, to lowest order in y, the discontinuity is controlled by the S wave scattering length and it varies as $(-y)^{\frac{1}{2}-\ell}$. The terms which vanish more rapidly near $y = 0$ will involve the scattering lengths of higher waves as well and can be calculated in a straightforward way. Similarly for the $B_\ell^{e\,o}(s)$ amplitude, the discontinuity varies as

$(-y)^{\frac{1}{2}-\ell}$ and is proportional to a^2. Now for $\ell > 3/2$ or $J \geqslant 1$ the behavior $(-y)^{\frac{1}{2}-\ell}$ is not integrable. What this means is that if dispersion relations dealing directly with the discontinuity near $y = 0$ (or $W = \pm (M-\mu)$ are being used, the contributions from integrals on small circular contours around the points $W = \pm (M-\mu)$ must be retained. It is easily established by continuation from the region $\ell < 3/2$ that this does not introduce any arbitrary parameters into the problem. The same information that determined the discontinuity near $y = 0$ for $\ell < 3/2$, i.e. a^2 to lowest order in y, also is all that is needed to evaluate the small circular integrals for $\ell \geqslant 3/2$. Therefore, although the amplitude does not vary as $(q^2)^{\ell}$ near $y = 0$ as has often been claimed in the literature,[4,5] the operation of dividing by this factor causes no difficulty beyond the computational one of evaluating the small circular integrals. In our subsequent discussion we will avoid this computational difficulty by only dealing directly with the physical cuts. These points established, let us turn directly to the consideration of the appropriate amplitude for use in setting up an N/D equation. Let us start from the amplitude $T^{J\ e_o}_{J-\frac{1}{2}}(W)$. In order for the residues of the Regge poles to have the desired analyticity properties guaranteed, the discussion of section III requires that the kinematic factor which we use must contain $(q^2)^{J-\frac{1}{2}}$. Then the requirement that correct threshold behavior at $W = \pm (M+\mu)$ be guaranteed and that the possible kinematic pole at $W = 0$ be removed specifies the remaining factor to be $(E + M)$. The amplitude we propose to deal

with then is

$$\tilde{b}^{e_o}(J,W) = \frac{T^{J\,e_o}_{J-\frac{1}{2}}(W)}{(q^2)^{J-\frac{1}{2}}(E+M)} \qquad \text{IV.6}$$

It should be emphasized at this point that $(q^2)^{J-\frac{1}{2}}$ has the same value at the MacDowell symmetric points $W, -W$ or in other words is to be regarded as a function of s. This follows from the discussion at the end of section II and the fact that the behavior we are dividing out originates in the amplitudes

$$A_\ell^{e_o}(s),\ B_\ell^{e_o}(s) \quad .$$

To establish the N/D equation, we will start from high real values of J. The resulting integral equation will then be continued to the left in the J plane. We assume the function

$$F^{e_o}(J,W) = \tilde{b}^{e_o}(J,W) - \frac{1}{\pi}\int_{-W_1}^{-(M+\mu)} dW' \frac{\operatorname{Im}\tilde{b}^{e_o}(J,W')}{W'-W}$$

$$- \frac{1}{\pi}\int_{M+\mu}^{W_1} dW' \frac{\operatorname{Im}\tilde{b}^{e_o}(J,W')}{W'-W} \qquad \text{IV.7}$$

is known and we seek to generate the amplitude using two body unitarity in the strip regions $W \in \pm\left[W_1, M+\mu\right]$ where $W_1 = s_1^{\frac{1}{2}}$. For J large enough the amplitude has no poles in the physical sheet. In that case the existence of an N/D decomposition is guaranteed by the Omnes representation for $D(J,W)$:

$$D^{e_o}(J,W) = e^{-\frac{1}{\pi}\int_{M+\mu}^{W_1} \frac{\delta^{e_o}_{J-\frac{1}{2}}(W')dW'}{W'-W} + \frac{1}{\pi}\int_{-W_1}^{-(M+\mu)} \frac{\delta^{e_o}_{J+\frac{1}{2}}(-W')\,dW'}{(W'-W)}} \qquad \text{IV.8}$$

The requirement that $D^{e_o}(J,W)$ have no poles or zeroes at threshold implies the choice $\delta^{e_o}_{J-\frac{1}{2}}(M+\mu) = \delta^{e_o}_{J+\frac{1}{2}}(M+\mu) = 0$. $D^{e_o}(J,W)$ carries the phase of the amplitude in the two strip regions, or in other words, $\mathrm{Im}\left[\tilde{b}^{e_o}(J,W)\ D^{e_o}(J,W)\right] = 0$, $W \in \pm\left[M+\mu,\ W_1\right]$. Therefore in

$$N^{e_o}(J,W) = D^{e_o}(J,W)\,F^{e_o}(J,W) + \left[D^{e_o}(J,W)\left(\tilde{b}^{e_o} - F^{e_o}(J,W)\right)\right] \quad , \qquad \text{IV.9}$$

the term in brackets is

$$\begin{aligned} &-\frac{1}{\pi}\int_{-W_1}^{-(M+\mu)} \frac{dW'}{W'-W}\left[\mathrm{Im}\,D^{e_o}(J,W')\right] F^{e_o}(J,W') \\ &-\frac{1}{\pi}\int_{M+\mu}^{W_1} \frac{dW'}{W'-W}\left[\mathrm{Im}\,D^{e_o}(J,W')\right] F^{e_o}(j,W') \end{aligned} \qquad \text{IV.10}$$

where the definition of $F^{e_o}(J,W')$ and the fact that $D^{e_o}(J,W) \xrightarrow[W\to\infty]{} 1$ have been used. Unitarity for $\tilde{b}^{e_o}(J,W)$ gives

IV.11

$$\frac{D^{e}{}^{o}(J,W+i\epsilon) - D^{e}{}^{o}(J,W-i\epsilon)}{2i} = -(q^2)^J \frac{E+M}{W} N^{e}{}^{o}(J,W)$$

$$\text{for } W \in \left[M+\mu,\ W_1\right]$$

$$\frac{D^{e}{}^{o}(J,W+i\epsilon) - D^{e}{}^{o}(J,W-i\epsilon)}{2i} = -(q^2)^J \frac{E+M}{W} N^{e}{}^{o}(J,W) \qquad \text{IV.12}$$

$$\text{for } W \in \left[-W_1,\ -(M+\mu)\right]$$

where as mentioned previously, q^2 is a function of s. Therefore

$$D^{e}{}^{o}(J,W) = 1 - \frac{1}{\pi}\left[\int_{M+\mu}^{W_1} + \int_{-W_1}^{-(M+\mu)} \frac{dW'}{W'-W}\ \rho(J,W')\ N^{e}{}^{o}(J,W')\right] \qquad \text{IV.13}$$

where the combination $(q^2)^J \frac{E+M}{W}$ has been denoted by $\rho(J,W)$ and the fact that $D^{e}{}^{o}(J,W)$ is normalized to 1 at infinite W, has been used. Substituting the above expression for $D^{e}{}^{o}(J,W)$ into IV.9 for N gives:

$$N^{e}{}^{o}(J,W) = F^{e}{}^{o}(J,W)$$

$$+ \frac{1}{\pi}\int_{M+\mu}^{W_1} dW' \frac{F^{e}{}^{o}(J,W') - F^{e}{}^{o}(J,W)}{W'-W}\ \rho(J,W')\ N^{e}{}^{o}(J,W') \qquad \text{IV.14}$$

$$+ \frac{1}{\pi}\int_{-W_1}^{-(M+\mu)} dW' \frac{F^{e}{}^{o}(J,W') - F^{e}{}^{o}(J,W)}{W'-W}\ \rho(J,W')\ N^{e}{}^{o}(J,W')$$

which is the basic integral equation. The construction of $F^{e_o}(J,W)$ will be considered in section V and the difficulties associated with the logarithmic behavior near $W = \pm W_1$ will be considered in section VI. The analysis in section VI shows that the equation is one of essentially Fredholm character. That is, either the equation as written has a unique solution or the homogeneous equation has a solution. The result of section V will be to provide a function $F^{e_o}(J,W)$ holomorphic in J to the right of some point below all physical values of J. Thus except for certain isolated values of J where Fredholm poles exist, the function $N^{e_o}(J,W)$ is holomorphic in J to the right of this point. We assume there are no Fredholm poles to the right of some $J_o < \frac{1}{2}$. If this is so then the amplitude is meromorphic in J. The Regge poles that exist are solutions of $D^{e_o}(J,W) = 0$. This will mean in practice that the pole position $\alpha^{e_o}(W)$ has only the cuts $\pm\left[M+\mu,\ W_1\right]$ instead of the full physical cut. The same applies for the reduced residue given by:

$$\beta^{e_o}(W) = \frac{-\left\{\frac{1}{\pi}\int_{-W_1}^{-(M+\mu)} + \frac{1}{\pi}\int_{M+\mu}^{W_1} dW' \left[W'-W\right]^{-1}\left[\mathrm{Im}\, D^{e_o}(\alpha^{e_o}, W')\, F^{e_o}(\alpha^{e_o}, W')\right]\right\}}{D_\alpha^{e_o}(\alpha,W)} \qquad \text{IV.15}$$

Since the strip width is supposed to be chosen at a value of W_1 sufficiently large that the poles have turned back to the left half plane, the absence of the high energy cuts $\pm\left[W_1,\ \infty\right]$ may not seriously affect the values of the positon and residue in the low energy region.

Before turning to the construction of $F^{e}{}_{o}(J,W)$, let us make some remarks about the qualitative shape expected for trajectories in the πN system. We shall see in section V that the forces due to exchanges of Regge poles will bear a qualitative similarity to the forces treated in the usual un-Reggeized way. If so then in cases where the force is strongly attractive in one of the states $(J, J \mp \frac{1}{2})$, then it is either weakly attractive or repulsive in the MacDowell symmetric partner of this state. This behavior arises mainly because the dominant forces in the problem arise from the exchange of the particles in the u channel. The terms in such exchanges change sign in going from ℓ to $\ell \pm 1$ and when they add to give attraction in one case, will add up to a much weaker attraction or a repulsion in the opposite parity case. It is easy to see from the definition of the continued amplitude, that this behavior holds for unphysical as well as physical values of J. Consequently if conditions are favorable to a trajectory which passes near physical values of J in the $(J, J - \frac{1}{2})$ state for example, then we don't expect to see this trajectory near any physical values of J in the $(J, J + \frac{1}{2})$ state. In this example we would have a trajectory function $\alpha(W)$ with the following behavior: For $W > 0$ and increasing, the pole in the J plane will move to the right staying in the real axis until $W = (M+\mu)$, when it moves into the upper half plane continuing to move to the right for a time and then curving back to the left half plane. For $W < 0$ and decreasing, the pole would in the simplest case move to the left until $W = -(M+\mu)$ at which point it could move either up

or down from the real J axis and continue its motion until the endpoint of the trajectory. When the pole is in the right half plane, the sign of $\mathrm{Im}\,\alpha(W+i\epsilon)$ is determined by the requirement that resonances correspond to poles on unphysical sheets and the centrifugal barrier argument that puts a resonance with higher mass at a higher value of J. Neither of these requirements is operative in the case discussed above at negative W and therefore $\mathrm{Im}\,\alpha(W+i\epsilon)$ can have either sign. Similarly if a physically interesting trajectory occurs in the $(J,J+\frac{1}{2})$ state, it will be in the right half plane for $W < 0$ and will move to the right as W decreases. In this case $\mathrm{Im}\,\alpha(W+i\epsilon)$ is negative for $W < -(M+\mu)$. For $W > 0$ and increasing the pole would move to the left. We will see in section V that if this qualitative behavior holds true the construction of force terms is simplified considerably.

V. CONSTRUCTION OF $F^{e}{}_{o}(J,W)$

In this section we consider the term $F^{e}{}_{o}(J,W)$ which must be supplied as an input to the integral equation for $N^{e}{}_{o}(J,W)$. The contributions to $F^{e}{}_{o}(J,W)$ are of two types. The first is from all double spectral functions which are non-zero in regions outside the interval $s \in \left[(M+\mu)^2, s_1\right]$. These of course are never known exactly. As stated in section I the basic approximation scheme of this work is to parameterize these contributions in terms of Regge poles in the cross channels. The second contributions to $F^{e}{}_{o}(J,W)$ are from the regions $s \in \left[(M+\mu)^2, s_1\right]$ themselves. The double spectral functions for $s \in \left[(M+\mu)^2, s_1\right]$ contribute to the t and u discontinuities and thus to the force cuts of $F^{e}{}_{o}(J,W)$. These contributions are of course also unknown and there is no method available at present for reliably estimating them. They are ignored here. Their influence is predominately on the far away parts of the force cuts and thus may not have an appreciable effect on the low energy scattering. This difficulty is of course one which is always present when the N/D method is used. Let us turn then to the construction of $F^{e}{}_{o}(J,W)$ in terms of cross channel Regge poles, taking the crossed πN channel first. For this we need some of the standard formulas for partial wave projections in πN scattering:[16]

$$f_1 = \frac{E+M}{8\pi W}\left[A(u,s) + (W-M)\,B(u,s)\right]$$

$$f_2 = \frac{E-M}{8\pi W}\left[-A(u,s) + (W+M)\,B(u,s)\right] \qquad \text{V.1}$$

or

$$B(u,s) = 4\pi\left\{\frac{f_1}{E+M} + \frac{f_2}{E-M}\right\}$$

$$A(u,s) = 4\pi\left\{\frac{f_1(W+M)}{E+M} - \frac{f_2(W-M)}{E-M}\right\} \qquad \text{V.2}$$

where

$$f_1 = \frac{1}{W}\left\{\sum_J T^J_{J-\frac{1}{2}}(W)\,P'_{J+\frac{1}{2}}(z_u) - T^J_{J+\frac{1}{2}}(W)\,P'_{J-\frac{1}{2}}(z_u)\right\}$$

$$f_2 = \frac{1}{W}\left\{\sum_J T^J_{J+\frac{1}{2}}(W)\,P'_{J+\frac{1}{2}}(z_u) - T^J_{J-\frac{1}{2}}(W)\,P'_{J-\frac{1}{2}}(z_u)\right\} \qquad \text{V.3}$$

All energies in these formulas are understood to apply to the u channel. Our procedure is as follows. We first perform a Somerfeld-Watson transformation on V.3 , and obtain formulas for f_1 and f_2 in terms of u channel Regge poles. From these A and B are gotten and the term $F^{e\,o}(J,W_s)$ is calculated using III.3 . Now let us write f_1 and f_2 as integrals in the J plane:

$$f_1 = \frac{-i}{4W} \int_C \frac{dJ}{\cos \pi J} \left\{ T^{J\,e}_{J-\frac{1}{2}}(W) \left[P'_{J+\frac{1}{2}}(-z_u) + P'_{J+\frac{1}{2}}(z_u) \right] \right.$$

$$+ T^{J\,o}_{J-\frac{1}{2}}(W) \left[P'_{J+\frac{1}{2}}(-z_u) - P'_{J+\frac{1}{2}}(z_u) \right] + T^{J\,e}_{J+\frac{1}{2}}(W) \left[P'_{J-\frac{1}{2}}(-z_u) - P'_{J-\frac{1}{2}}(z_u) \right]$$

$$\left. + T^{J\,o}_{J+\frac{1}{2}}(W) \left[P'_{J-\frac{1}{2}}(-z_u) + P'_{J-\frac{1}{2}}(z_u) \right] \right\}$$

$$f_2 = -\frac{i}{4W} \int_C \frac{dJ}{\cos \pi J} \left\{ T^{J\,e}_{J+\frac{1}{2}}(W) \left[P'_{J+\frac{1}{2}}(-z_u) + P'_{J+\frac{1}{2}}(z_u) \right] \right. \qquad \text{V.4}$$

$$+ T^{J\,o}_{J+\frac{1}{2}}(W) \left[P'_{J+\frac{1}{2}}(-z_u) - P'_{J+\frac{1}{2}}(z_u) \right] + T^{J\,e}_{J-\frac{1}{2}}(W) \left[P'_{J-\frac{1}{2}}(-z_u) - P'_{J-\frac{1}{2}}(z_u) \right]$$

$$\left. + T^{J\,o}_{J-\frac{1}{2}}(W) \left[P'_{J-\frac{1}{2}}(-z_u) + P'_{J-\frac{1}{2}}(z_u) \right] \right\}$$

Here C denotes the usual contour encircling all physical values of J. The contour may now be opened up to give the usual integral over a vertical line in the J plane plus contributions from Regge poles. The validity of this representation depends on the vanishing of the integrand at infinite values of J. This is insured by the factor $1/\cos \pi J$ and the exponential decrease of the $Q_{J\mp\frac{1}{2}}(z)$ function in the J plane for z real and positive. Thus the first term in f_1 say, will be represented by the following expression:

$$- \frac{i}{4W_u} \int_{-i\infty+J_r}^{+i\infty+J_r} \frac{dJ}{\cos \pi J} \left\{ T^{J\,e}_{J-\frac{1}{2}}(W_u) \left[P'_{J+\frac{1}{2}}(-z_u) + P'_{J+\frac{1}{2}}(z_u) \right] \right\}$$

$$+ \frac{\pi}{2q_u} \frac{\rho(\alpha^e, W_u)\beta^e(W_u)}{\cos \pi\alpha^e} \left[P'_{\alpha^e+\frac{1}{2}}(-z_u) + P'_{\alpha^e+\frac{1}{2}}(z_u) \right]$$

where we have assumed only one Regge pole for simplicity. We assume that J_r lies to the left of any trajectory which emergies into the right half plane to make a bound state or a resonance. The Regge pole will dominate the expression for values of W_u such that $\alpha^e(W_u)$ is near $J \doteq 1/2$, $5/2$, etc. or for large values of z_u . So far we have tacitly assumed that W_u is real and above threshold. If this is so, then the integral term above converges for all z_u . However if one attempts to continue the expression to values of W_u below threshold where q_u^2 is negative, the domain in z_u for which the integral converges shrinks to zero. This follows from the fact that

$$Q_\ell(z) \xrightarrow[\ell \to \infty]{} O\left(\frac{1}{\sqrt{\ell}} \; e^{-\xi(\ell+\frac{1}{2})} \right)$$

where $z = \cosh \xi$,and the fact that $P'_\ell(z)$ increases exponentially as $\ell \to i\infty + \ell_r$. In an integral such as

$$\frac{1}{\pi q_u^2} \int_{4\mu^2}^{\infty} dt' \; Q_{J\mp\frac{1}{2}}\left(1 + \frac{t}{2q_u^2}\right) A_t(u,t')$$

if $q^2 < 0$, $\xi = \xi_r \pm i\pi$. This will cause $T^{J\,e}_{J-\frac{1}{2}}(W_u)$ to be $O\left(e^{|\pi J|}\right)$ along the imaginary J axis and thus the integral will diverge. This might lead one to suspect that the Regge pole term no longer gives the dominant behavior at large z_u or near $\alpha^e = 1/2$, $5/2$, etc. However one can easily establish that these results continue to hold by using a slightly modified continuation from physical values of J , which essentially amounts to replacing

$$Q_{J\mp\frac{1}{2}}\left(1 + \frac{t'}{2q_u^2}\right) \text{ by } \pm\, Q_{J\mp\frac{1}{2}}\left(-1 - \frac{t'}{2q_u^2}\right)$$

which is a true equation at physical values of J . This gives a convergent background integral and a slightly modified Regge pole term which agrees with the pole term above near the places where the pole dominates. Thus the failure of the background integral to converge for $q_u^2 < 0$ does not cause the usual features ascribed to the pole terms to lose their validity. Now our approximation will of course involve dropping the integral term entirely and keeping only the pole term. We also modify the pole term somewhat to remove some unwanted cuts. For this we need the following formula:

$$P'_{\alpha+\frac{1}{2}}(z) = \frac{\cos \pi\alpha}{\pi}\left\{\int_1^\infty dz' \frac{P'_{\alpha+\frac{1}{2}}(z')}{z'+z} + \frac{1}{z+1}\right\} \qquad \text{V.6}$$

where the right hand side is defined by continuation when the integral fails to converge. In the Regge pole term in V.5 , we make the following replacements:

$$P'_{\alpha^e+\frac{1}{2}}(-z_u) \to \frac{\cos \pi\alpha^e}{\pi}\left\{\int_{t_1}^{\infty} \frac{P'_{\alpha^e+\frac{1}{2}}\left(1+\frac{t'}{2q_u^2}\right)dt'}{t'-t}\right\}$$

$$= P'_{\alpha^e+\frac{1}{2}}\left(-1-\frac{t}{2q_u^2}\right) - \frac{\cos \pi\alpha^e}{\pi}\left\{\int_{0}^{t_1} \frac{P'_{\alpha^e+\frac{1}{2}}\left(1+\frac{t'}{2q_u^2}\right)dt'}{t'-t} - \frac{2q_u^2}{t}\right\} \qquad \text{V.7}$$

where the equality holds for q_u^2 real and positive. Similarly

$$P'_{\alpha^e+\frac{1}{2}}(z_u) \to \frac{\cos \pi\alpha^e}{\pi}\int_{s_1}^{\infty} P'_{\alpha^e+\frac{1}{2}}\left(-1-\frac{2(M^2+\mu^2)-s'-u}{2q_u^2}\right)\frac{ds'}{s'-s}$$

$$= P'_{\alpha^e+\frac{1}{2}}\left(1+\frac{2(M^2+\mu^2)-s-u}{2q_u^2}\right) - \frac{\cos \pi\alpha^e}{\pi}\left\{\int_{\frac{(M^2-\mu^2)^2}{u}}^{s_1} P'_{\alpha^e+\frac{1}{2}}\left(-1-\frac{2(M^2+\mu^2)-s'-u}{2q_u^2}\right)\frac{ds'}{s'-s} + \frac{1}{2+\frac{2(M^2+\mu^2)-s-u}{2q_u^2}}\right\} \qquad \text{V.8}$$

where again the equality holds for q_u^2 real and positive. What we have done here is remove the cuts of the $P'_{\alpha+\frac{1}{2}}$ which intrude

on the s and t channel strip regions and defined the remaining cuts to be along the positive s and t axes as required by the Mandelstam representation. Note that the modified expressions reduce to the original ones near $\alpha^e = 1/2, 5/2,$ etc. or at large z_u, so that the modified term is accurate in any region where Regge poles dominate. Our reason for removing these cuts is that in our dynamical scheme the s and t discontinuities in their respective strip regions are generated by two body unitarity in these channels. Removal of the cuts from the u channel pole terms requires that the u channel scattering provides a real potential in these two strip regions. Let us return to our expression for the u channel pole term. It now reads:

$$\frac{\rho(\alpha^e, W_u)}{2q_u} \beta^e(W_u) \left[\int_{t_1}^{\infty} dt' \frac{P'_{\alpha^e+\frac{1}{2}}\left(1+\frac{t'}{2q_u^2}\right)}{t'-t} + \int_{s_1}^{\infty} ds' \frac{P'_{\alpha^e+\frac{1}{2}}\left(-1-\frac{2(M^2+\mu^2)-s-u'}{2q_u^2}\right)}{s'-s} \right]$$

$$\equiv R_+^{t_1,e}(W_u, t) + R_+^{s_1,e}(W_u, s)$$

where the subscript $+$ denotes that $P'_{\alpha+\frac{1}{2}}$ is involved. If we carry out a similar analysis on the remaining pole terms in f_1 and f_2, the following expressions are obtained:

$$f_1 = \sum_{\text{all poles}} \Bigg\{ \left[R_+^{t_1,\,e}(W_u,\,t) + R_+^{s_1,\,e}(W_u,\,s) \right] \qquad \text{V.10}$$

$$+ \left[R_+^{t_1,\,0}(W_u,\,t) - R_+^{s_1,\,0}(W_u,\,s) \right] + \left[R_-^{t_1,\,e}(-W_u,t) - R_-^{s_1,\,e}(-W_u,\,s) \right]$$

$$+ \left[R_-^{t_1,0}(-W_u,t) + R_-^{s_1,0}(-W_u,\,s) \right] \Bigg\}$$

$$f_2 = \sum_{\text{all poles}} \Bigg\{ - \left[R_+^{t_1,e}(-W_u,t) + R_+^{s_1,e}(-W_u,s) \right] \qquad \text{V.11}$$

$$- \left[R_+^{t_1,0}(-W_u,t) - R_+^{s_1,0}(-W_u,s) \right] - \left[R_-^{t_1,e}(W_u,t) - R_-^{s_1,e}(W_u,s) \right]$$

$$- \left[R_-^{t_1,0}(W_u,t) + R_-^{s_1,0}(W_u,s) \right] \Bigg\}$$

where we have made use of the Mac Dowell symmetry in writing the terms evaluated at $-W_u$. We keep here the convention established in section III of associating the physical cut in the right half W_u plane with the $(J,\ J-\frac{1}{2})$ amplitude and the physical cut in the left half W_u plane with the $(J,\ J+\frac{1}{2})$ amplitude. The sense of approach to the physical cuts in the W_u plane is as before from positive imaginary values in the right half plane and from negative imaginary values in the left half plane. Using V.10 and V.11 in V.2 gives the following formulas for A and B :

$$B^P(u,s) = 4\pi \sum_{\text{all poles}} \left\{ \frac{1}{E_u+M}\left[R_+^{t_1,e}(W_u,t) + R_+^{s_1,e}(W_u,s)\right] \right.$$

$$- \frac{1}{E_u-M}\left[R_+^{t_1,e}(-W_u,t) + R_+^{s_1,e}(-W_u,s)\right] + \frac{1}{E_u+M}\left[R_+^{t_1,o}(W_u,t) - R_+^{s_1,o}(W_u,s)\right]$$

$$- \frac{1}{E_u-M}\left[R_+^{t_1,o}(-W_u,t) - R_+^{s_1,o}(-W_u,s)\right] + \frac{1}{E_u+M}\left[R_-^{t_1,e}(-W_u,t) - R_-^{s_1,e}(-W_u,s)\right]$$

$$- \frac{1}{E_u-M}\left[R_-^{t_1,e}(W_u,t) - R_-^{s_1,e}(W_u,s)\right] + \frac{1}{E_u+M}\left[R_-^{t_1,o}(-W_u,t) + R_-^{s_1,o}(-W_u,s)\right]$$

$$\left. - \frac{1}{E_u-M}\left[R_-^{t_1,o}(W_u,t) + R_-^{s_1,o}(W_u,s)\right] \right\} \qquad \text{V.12}$$

$$A^P(u,s) = 4\pi \sum_{\text{all poles}} \left\{ \frac{W_u+M}{E_u+M}\left[R_+^{t_1,e}(W_u,t) + R_+^{s_1,e}(W_u,s)\right] \right.$$

$$+ \frac{W_u-M}{E_u-M}\left[R_+^{t_1,e}(-W_u,t) + R_+^{s_1,e}(-W_u,s)\right] + \frac{W_u+M}{E_u+M}\left[R_+^{t_1,o}(W_u,t) - R_+^{s_1,o}(W_u,s)\right]$$

$$+ \frac{W_u-M}{E_u-M}\left[R_+^{t_1,o}(-W_u,t) - R_+^{s_1,o}(-W_u,s)\right] + \frac{W_u+M}{E_u+M}\left[R_-^{t_1,e}(-W_u,t) - R_-^{s_1,e}(-W_u,s)\right]$$

$$+ \frac{W_u-M}{E_u-M}\left[R_-^{t_1,e}(W_u,t) - R_-^{s_1,e}(W_u,s)\right] + \frac{W_u+M}{E_u+M}\left[R_-^{t_1,o}(-W_u,t) + R_-^{s_1,o}(-W_u,s)\right]$$

$$\left. + \frac{W_u-M}{E_u-M}\left[R_-^{t_1,o}(W_u,t) + R_-^{t_1,o}(W_u,s)\right] \right\} \qquad \text{V.13}$$

These formulas represent the contribution of the u channel poles to the s channel potential. The way in which we propose to use them is as follows:

$$F_u^{e_o}(J,W) = (\beta_{su}) \frac{1}{(q^2)^{J-\frac{1}{2}}(E+M)} \left\{ \frac{(W+M)^2-\mu^2}{32\pi\, W} \left[A^{P\,e_o}_{J-\frac{1}{2}}(s) - (W-M) B^{P\,e_o}_{J-\frac{1}{2}}(s) \right] \right.$$
$$\left. + \frac{(W-M^2-\mu^2}{32\pi\, W} \left[- A^{P\,e_o}_{J+\frac{1}{2}}(s) - (W+M)\, B^{P\,e_o}_{J+\frac{1}{2}}(s) \right] \right\} \qquad \text{V.14}$$

Here (β_{su}) is a symbolic notation for multiplying by the I spin crossing matrix

$$(\beta_{su})\, I'I = \begin{pmatrix} -1/3 & 4/3 \\ 2/3 & 1/3 \end{pmatrix}$$

the A^P and B^P being understood to refer to a definite I spin in the u channel. $F_u^{e_o}(J,W)$ represents the part of the potential coming from the u channel Regge poles. The quantities $A^{P\,e_o}_{J\mp\frac{1}{2}}$, $B^{P\,e_o}_{J\mp\frac{1}{2}}$ are given by :

$$A_\ell^{P\,e_o}(s) = \pm \frac{1}{\pi q^2} \int_{(M+\mu)^2}^{\infty} du'\, \mathrm{Im}(A^P(s,u')\, Q_\ell \left[-1 - \frac{2(M^2+\mu^2) - s - u'}{2q^2} \right] \qquad \text{V.15}$$

and similarly for $B_\ell^{P^{e_o}}(s)$. In addition to the integral from $(M+\mu)^2$ to ∞ , $B_\ell^{P^{e_o}}(s)$ contains a term

$$\pm \frac{(2\pi)^2 \beta_N(-M)}{\alpha_N'(-M)} \frac{1}{\pi q^2} Q_\ell \left(-1 - \frac{2(M^2+\mu^2)-s-M^2}{2q^2}\right) \qquad \text{V.16}$$

for $(I_u = \frac{1}{2})$, which comes from the nucleon trajectory which contributes the usual pole term at $u = M^2$. It will be noted that we have not included any contribution to the t discontinuity of the u channel Regge poles. There are two reasons for this. First, the maximum value of u which may appear in such contributions is $(M-\mu)^2 - t_1$, and since t_1 will normally be several times $(M-\mu)^2$, this will be a fairly large negative value of u . Thus any Regge pole in the u channel will be well to the left of $J = \frac{1}{2}$. Second, for $s \in \left[(M+\mu)^2, s_1\right]$, z_u is $\in \left[-1,1\right]$ for such contributions. Consequently, none of the criteria for dominance of a Regge pole are satisfied. Keeping terms which are of the same order as terms already ignored seems a dubious procedure, so we take the simpler course of dropping these as well. These terms influence predominantly the far away part of the force cut of $F^{e_o}(J,W)$ and just as the previously ignored s channel strip terms, should not appreciably affect low energy s channel scattering.

Let us examine some qualitative features of the force due to the u channel poles. First of all the expressions V.12 and V.13 will simplify considerably in practice. Let us take $W_u = +\sqrt{u}$.

Then suppose we are talking about the exchange of the N^*_{33} trajectory. If the fermion poles behave as expected on the basis of the discussion of section IV the terms evaluated at $-W_u$ will correspond to a pole to the left of any physical value of J, and will be small compared to the $+W_u$ terms and can be ignored. Thus A^P and B^P will contain only half as many terms as explicitly given in V.12 and V.13. A similiar result applies for exchange of the nucleon trajectory. We only keep the nucleon and N^*_{33} trajectories in the u channel since these are the only ones we can expect to generate in a one channel approximation. The other resonances such as the 600 MeV resonance probably require a multi channel treatment for even a qualitative understanding and therefore are not included in the one channel case we are considering here.

The qualitative features of the force can be understood if we notice that the Legendre expansions of A^P and B^P in the u channel converge if $s < s_1$. At low values of s, these expansions will converge rapidly and will be dominated by the u channel bound state and resonance terms. Thus at low values of s the exchange of a Regge pole will look like the exchange of the particles and resonances that lie on the trajectory, treated as fixed poles. At high values of s, the usual Regge behavior will take over. The high and low s behaviors are not sensitive to the value of s_1 as was noted in the discussion following V.7 and V.8. Except for the weak logarithmic singularity near $s = s_1$, the transition between these two regimes is smooth and therefore

we expect a force which is relatively insensitive to s_1. The fact that a reasonable high energy behavior of the force is assured no matter what s_1 is, is one of the major differences between the forcess as treated here and in previous un-Reggeized treatments. This will allow the theory to depend on s_1 in a much less sensitive way than the cutoffs of these previous un Reggeized treatments.

Before turning to the discussion of the force due to the t channel poles, let us comment on an alternate procedure which is useful in the $\pi\pi$ case. That is to make use of the dispersion relation satisfied by the $Q_\ell(z)$ function.

$$Q_\ell(z) = \frac{\sin \pi\ell}{\pi} \int_{-\infty}^{-1} \frac{Q_\ell(-z')\, dz'}{z' - z} + \frac{1}{2} \int_{-1}^{+1} \frac{P_\ell(z')dz'}{z - z'} \qquad \text{V.17}$$

Using this in an expression like

$$\frac{1}{\pi q^2} \int_{(M+\mu)^2}^{\infty} du'\, A_u(s,u')\, Q_\ell\left(-1 - \frac{2(M^2+\mu^2)-s-u'}{2q^2}\right)$$

leads to

$$\left\{ \int_{-1}^{+1} dz\, P_\ell(z)\, A'\left(s,\ 2(M^2+\mu^2) - s + 2q^2(z+1)\right) - \frac{2}{\pi} \sin \pi \int_{-\infty}^{-1} dz\, Q_\ell(-z)\, A'\left(s,\ 2(M^2+\mu^2) - s + 2q^2(z+1)\right) \right\} \qquad \text{V.18}$$

V.18 is usually known as the Wong formula. Here $A'(s,u)$ is given by:

$$A'(s,u) = \frac{1}{\pi} \int_{(M+\mu)^2}^{\infty} du' \frac{A_u(s,u')}{(u'-u)}$$

If the u channel were a Boson channel, the residues and positions of the Regge poles in this channel would be real for $u < (M+\mu)^2$ and V.18 is quite useful in this case since the integrals only involve $u < (M^2-\mu^2)^2/s$ and one can deal with real quantities in evaluating the potential. In our case the u channel is a Fermion channel and for $u < 0$, $W_u = \pm i \left|\sqrt{u}\right|$ and $\alpha(W_u)$, $\alpha(-W_u)$, $\beta(W_u)$, and $\beta(-W_u)$ are all complex. So from this point of view the Wong form is no easier. Furthermore, in practice this would mean carrying out the bootstrap on the part of the trajectories for which $W^2 = u < (M^2-\mu^2)^2/s$. Nothing is known experimentally about the trajectories in πN scattering for this range of energies. They have so far not manifested themselves in high energy backward scattering which is the only means of observing this range of energies. Finally, the presence of unequal masses makes it difficult to construct forms for u channel Regge pole terms which have correct analyticity properties for $u < \frac{(M^2-\mu^2)^2}{s}$. Thus in the πN case using the Froissort-Gribov form as we have done seems to be the best procedure.

Now let us turn to the construction of the part of the potential that comes from the Regge poles in the t channel, $F_t^{e_o}(J,W)$. The relevant poles in this case are the ρ and Pomeranchuk. The positions and residues of these poles are of course not determined by the solution to the πN problem and thus there is no bootstrap for these parameters within the πN problem alone. The positions of these poles are determined by the solution of the $\pi\pi$ bootstrap equations and we will imagine that we have such a solution at hand. Given the solution to the $\pi\pi$ problem, the residues of the poles as they couple to the πN system can be calculated. We shall show below how to do this. It might also be noted that a good deal is known about these parameters experimentally and in a less ambitious program, this information could be employed. To proceed with the construction of $F_t^{e_o}(J,W)$ we have to make use of some of the standard formulas for partial wave projections and I spin analysis in the $\pi\pi \to N\bar{N}$ system.[19] First we note that the I spin decomposition of A and B can be written as follows:

$$A = A^+ \delta_{\beta,\alpha} + A^- \tfrac{1}{2} \left[\tau_\beta, \tau_\alpha\right] \qquad \text{V.19}$$

and similarly for B , where β refers to the final π in the s channel and α to the initial. The relations of the $A^\pm$ to definite I spin amplitude in the s and u channels are given by:

$$
\begin{aligned}
s: \quad A^{1/2} &= A^{+} + 2A^{-} \\
A^{3/2} &= A^{+} - A^{-} \\
u: \quad A^{1/2} &= A^{+} - 2A^{-} \\
A^{3/2} &= A^{+} + A^{-}
\end{aligned}
\qquad \text{V.20}
$$

and similarly for $B^{\pm}$. In the t channel, A^{+}, B^{+} couple only to $I = 0$ systems while B^{-}, A^{-} couple only to $I = 1$ systems. Bose statistics requires that

$$
\begin{aligned}
A^{\pm}(t, -\cos\theta) &= \pm A^{\pm}(t, \cos\theta) \\
B^{\pm}(t, -\cos\theta) &= \mp B^{\pm}(t, \cos\theta)
\end{aligned}
\qquad \text{V.21}
$$

where $\cos\theta$ is defined as the angle between the final $\bar{N}$ and π_{α}. If we define the total helicity with respect to the momentum of the $\bar{N}$, then the helicity amplitudes are given by :

$$
\begin{aligned}
f_{o}^{\pm} &= \frac{1}{4\pi W}\left[\frac{p}{q}\right]^{\frac{1}{2}}\left[-A^{\pm}p + mq\,B^{\pm}\cos\theta\right] \\
f_{-1}^{\pm} &= \frac{1}{4\pi W}\left[\frac{p}{q}\right]^{\frac{1}{2}} B^{\pm} qE \sin\theta
\end{aligned}
\qquad \text{V.22}
$$

where $\frac{d\sigma}{d\Omega} = |f|^2$, p is the three-momentum magnitude in the $N\bar{N}$ system and q is the three-momentum magnitude in the $\pi\pi$ system. Partial wave decompositions are given by :

$$f_o^{\pm}(t) = \frac{1}{q} \sum_J (2J+1)\, P_J(\cos\theta)\, T_o^{\pm J}(t) \qquad \text{V.23}$$

where

$$T_o^{\pm J}(t) = \frac{\left[pq\right]^{\frac{1}{2}}}{8\pi W} \left[-p\, A_J^{\pm}(t) + \frac{mq}{2J+1} \left[(J+1)\, B_{J+1}^{\pm}(t) + JB_{J-1}^{\pm}(t)\right]\right] \qquad \text{V.24}$$

and $$f_{-1}^{\pm}(t) = \frac{1}{q} \sum_J (2J+1) \frac{\sin\theta\, P_J'(\cos\theta)}{\left[J(J+1)\right]^{\frac{1}{2}}} T_{-1}^{\pm J}(t) \qquad \text{V.25}$$

where

$$T_{-1}^{\pm J}(t) = \frac{\left[pq\right]^{\frac{1}{2}}}{8\pi W} \frac{qE}{(2J+1)} \left[J(J+1)\right]^{\frac{1}{2}} \left(B_{J-1}^{\pm}(t) - B_{J+1}^{\pm}(t)\right) \qquad \text{V.26}$$

Now

$$A^{\pm}(t,s) = \frac{1}{\pi} \int_{(M+\mu)^2}^{\infty} \frac{ds'\, A_s^{\pm}(t,s')}{s'+p^2+q^2-2pq\cos\theta} + \frac{1}{\pi} \int_{(M+\mu)^2}^{\infty} \frac{du'\, A_u^{\pm}(t,u')}{u'+p^2+q^2+2pq\cos\theta}$$

V.27

where from V.21, $A_s^{\pm}(t,s') = \pm A_u^{\pm}(t,s')$. A similar formula holds for $B^{\pm}$, where $B_s^{\pm}(t,s') = \mp B_u^{\pm}(t,s')$. Therefore

V.28

$$A_J^{\pm}(t) = \frac{2}{\pi pq}\int_{(M+\mu)^2}^{\infty} ds'\, A_s^{\pm}(t,s')Q_J\left(\frac{p^2+q^2+s'}{2pq}\right) = \pm\frac{2}{\pi pq}\int_{(M+\mu)^2}^{\infty} du'\, A_u^{\pm}(t,u')Q_J\left(\frac{p^2+q^2+u'}{2pq}\right)$$

This defines the continuation of $A_J^{\pm}(t)$ away from the $\begin{pmatrix}\text{even}\\ \text{odd}\end{pmatrix}$ integers, which allows the Sommerfeld-Watson transformation to be made for $t > 4\mu^2$. Similarly,

$$B_J^{\pm}(t) = \frac{2}{\pi pq}\int_{(M+\mu)^2}^{\infty} ds'\, B_s^{\pm}(t,s')\, Q_J\left(\frac{p^2+q^2+s'}{2pq}\right) \qquad \text{V.29}$$

$$= \mp\frac{2}{\pi pq}\int_{(M+\mu)^2}^{\infty} du'\, B_u^{\pm}(t,u')\, Q_J\left(\frac{p^2+q^2+u'}{2pq}\right)$$

defines the correct continuation of $B_J^{\pm}(t)$ away from $\begin{pmatrix}\text{odd}\\ \text{even}\end{pmatrix}$ integers. It should be emphasized at this point that the continued amplitude may not equal the physical amplitude at $J = 0$. This is due to the presence of the nonsense state $J = 0$, total helicity equals 1 . A glance at V.26 for $T_{-1}^{\pm J}(t)$ shows that in the continued amplitude this unphysical state is coupled in through the pole of Q_{J-1} at $J = 0$ which more than cancels the $[J]^{\frac{1}{2}}$ factor, whereas in the physical partial wave coupling to this unphysical state should not be included. The general problem of coupling to nonsense states

has been considered by Mandelstam[20] who showed that the continued amplitude and the physical amplitude differ in general by being different CDD solutions to the same dispersion relations. So unless the coupling to the nonsense state happens to vanish in the continued amplitude, one may expect in general that it will not be equal to the physical amplitude at $J = 0$. While this difficulty must be dealt with in any attempt to treat full $\pi\pi - N\bar{N}$ problem, it may not cause trouble in constructing the force for πN scattering. The reason for this is that recent calculations[21] indicate that the endpoints of both the ρ and Pomeranchuk trajectories lie to the right of $J = 0$. If so then the above difficulty will cause no trouble in constructing $F_t^{e_o}(J,W)$. We shall assume that this is the case and not consider the $J = 0$ behavior further.

Let us introduce two modified partial wave amplitudes :

$$f_{-1}^{\pm J}(t) = \frac{4\pi W}{E}\left[\frac{q}{p}\right]^{\frac{1}{2}} \frac{1}{q^2} \frac{1}{(pq)^{J-1}} \frac{T_{-1}^{\pm J}(t)}{\left[J(J+1)\right]^{\frac{1}{2}}} \qquad \text{V.30}$$

$$f_{o}^{\pm J}(t) = 4\pi W\left[\frac{q}{p}\right]^{\frac{1}{2}} \frac{p^2}{(pq)^{J+1}} T_{o}^{\pm J}(t) \qquad \text{V.31}$$

These amplitudes have no zeroes at $t = 4M^2$ or $t = 4\mu^2$ and are real analytic functions of t whose cuts for $t > 4\mu^2$ originate directly in the t cuts of $A^{\pm}$ and $B^{\pm}$. Using formulas V.22, V.23 and V.25, we have

$$B^{\pm} = \sum_{J = \binom{\text{even}}{\text{odd}}} (2J+1) \; P'_J(\cos\theta) \; (pq)^{J-1} f_{-1}^{\pm J}(t) \qquad \text{V.32}$$

$$A^{\pm} - \frac{mq\cos\theta}{p} B^{\pm} = -\frac{1}{p^2} \sum_{J = \binom{\text{even}}{\text{odd}}} (2J+1) \; (pq)^J \; f_o^{\pm J}(t) \; P_J(\cos\theta) \qquad \text{V.33}$$

Let us write these quantities as integrals over the usual contour in the J plane.

$$B^{\pm}(s,t) = \frac{1}{4\pi i} \int_c \frac{\pi \, dJ}{\sin \pi J} (2J+1) \left[P'_J(\cos\theta) \mp P'_J(-\cos\theta) \right] (-pq)^{J-1} f_{-1}^{\pm J}(t)$$

V.34

$$A^{\pm}(s,t) - \frac{mq}{p} \cos\theta \, B^{\pm}(s,t) =$$

$$= \frac{1}{4\pi i} \int \frac{\pi \, dJ}{\sin \pi J} \; \frac{2J+1}{p^2} \left[P_J(\cos\theta) \pm P_J(-\cos\theta) \right] (-pq)^J f_o^{\pm J}(t)$$

V.35

Now in the case of the contributions of the t channel poles to the potential, it is much simpler to use the Wong formula rather than the Froissart-Gribov formula. The positions and reduced residues of the t channel poles are real throughout the negative t region and the presence of unequal masses causes no difficulty in this case. So we desire expressions for t channel Regge poles valid in the negative t region. This has been anticipated in writing V.34 and V.35 which are

appropriate for use in the negative t region where $pq < 0$. Opening up the contour and keeping only the Regge pole terms leads to the following results:

V.36

$$B^{\pm}(s,t) = \frac{\pi}{2} \frac{2\alpha^{\pm}+1}{\sin \pi\alpha^{\pm}} \left[P'_{\alpha^{\pm}}(\cos\theta) \mp P'_{\alpha^{\pm}}(-\cos\theta) \right] (-pq)^{\alpha^{\pm}-1} \beta^{\pm}_{-1}(t)$$

$$A^{\pm} - \frac{mq}{p} \cos\theta \, B^{\pm} = \frac{\pi}{2p^2} \frac{2\alpha^{\pm}+1}{\sin \pi\alpha^{\pm}} \left[P'_{\alpha^{\pm}}(\cos\theta) \mp P'_{\alpha^{\pm}}(-\cos\theta) \right] (-pq)^{\alpha^{\pm}-1} \beta^{\pm}_{-1}(t)$$

V.37

where

$$\cos\theta = \frac{p^2+q^2+s}{2pq} = -\frac{p^2+q^2+u}{2pq} .$$

From now on we understand the $+$ amplitude to represent the contribution of the Pomeranchuk, while the $-$ amplitude represents the contribution of the ρ . Since we desire a real potential from the t channel poles we modify the above terms in a way quite analogous to the already discussed u channel terms. We make the following replacements:

$$\left[P_{\alpha^{\pm}}(\cos\theta) \pm P_{\alpha^{\pm}}(-\cos\theta) \right] \frac{1}{\sin \pi\alpha^{\pm}} \qquad \text{V.38}$$

$$\longrightarrow \; - \frac{1}{\pi} \left[\int_{s_1}^{\infty} P_{\alpha^{\pm}}\left(-\frac{p^2+q^2+s'}{2pq}\right) \frac{ds'}{s'-s} \pm \int_{u_1}^{\infty} P_{\alpha^{\pm}}\left(-\frac{p^2+q^2+u'}{2pq}\right) \frac{du'}{u'-u} \right]$$

$$\frac{P'_{\alpha^\pm}(\cos\theta) \mp P'_{\alpha^\pm}(-\cos\theta)}{\sin \pi\alpha^\pm} \rightarrow \frac{1}{\pi}\left[\int_{s_1}^{\infty} P'_{\alpha^\pm}\left(-\frac{p^2+q^2+s'}{2pq}\right)\frac{ds'}{s'-s}\right.$$

$$\left. \mp \int_{u_1}^{\infty} P'_{\alpha^\pm}\left(-\frac{p^2+q^2+u'}{2pq}\right)\frac{du'}{u'-u}\right] \qquad \text{V.39}$$

where as usual the integrals are defined by continuation from regions where they converge. We denote the modified expressions obtained for $A^\pm$ and $B^\pm$ as follows:

$$B^\pm(s,t) = S^\pm_{-1}(s,t) \mp S^\pm_{-1}(u,t) \qquad \text{V.40}$$

$$A^\pm(s,t) - \frac{m(p^2+q^2+s)\,B^\pm(s,t)}{2p^2} = S^\pm_o(s,t) \pm S^\pm_o(u,t)$$

or

$$A^\pm(s,t) = S^\pm_o(s,t) + \frac{m(p^2+q^2+s)}{2p^2}\,S^\pm_{-1}(s,t) \qquad \text{V.41}$$

$$\pm\left[S^\pm_o(u,t) - \frac{m(p^2+q^2+s)}{2p^2}\,S^\pm_{-1}(u,t)\right]$$

Using V.20 to convert to amplitudes of definite I spin in the s channel, the contribution of the channel poles to the t channel potential is given by:

$$F_t^{e_o}(J,W) = \frac{1}{(q^2)^{J-\frac{1}{2}}(E+M)} \left\{ \frac{(W+M)^2-\mu^2}{32\ \pi W} \left[A_{J-\frac{1}{2}}^{P_t}(s) + (W-M)\ B_{J-\frac{1}{2}}^{P_t}(s) \right] \right.$$

$$\left. + \frac{(W-M)^2-\mu^2}{32\ \pi W} \left[- A_{J+\frac{1}{2}}^{P_t}(s) + (W+M)\ B_{J+\frac{1}{2}}^{P_t}(s) \right] \right\} \qquad \text{V.42}$$

where the I spin label is suppressed. $A_\ell^{P_t}(s)$ is given by

$$A_\ell^{P_t}(s) = \int_{-1}^{+1} dz'\ P_\ell(z')\ A^{P_t}\left(s, 2q^2(1-z')\right) - \frac{2\sin\pi\ell}{\pi} \int_{-\infty}^{-1} dz' Q_\ell(-z') A^{P_t}\left(s, 2q^2(1-z')\right)$$

$$\text{V.43}$$

and similarly for $B_{J+\frac{1}{2}}^{P_t}$. We have not attempted to include any contributions to the u discontinuity from the t channel Regge poles. The reasons for this are quite analogous to those given previously for dropping the contributions of the u channel poles to the t discontinuity. Before going on to calculate the residues $\beta_{-1}^{\pm}(t)$ and $\beta_o^{\pm}(t)$ we take note of a subtlety associated with the exchange of the Pomeranchuk.[22] In the case of the u channel contributions to the potential we argued that they could be well represented by the modified Regge pole terms. The reasoning was that these gave a good description of the u discontinuity in the low s region where the u channel bound states and resonances dominate and they match onto the correct Regge high energy behavior at large s. Although the same type of argument will apply for the case of the ρ, it does not for the Pomeranchuk. The Pomeranchuk

spends most of its time near $J = 1$ and consequently does not dominate the low t discontinuity in the $I = 0$ state in the low s region, until the region where the f_o occurs is reached. This could require a more elaborate treatment to be given to the low t region. We do not attempt to do so here, since as mentioned earlier we expect the cross channel Fermion poles to dominate the force and thus this inaccuracy may not be serious. Now let us consider the residue functions $\beta^{\pm}_{-1}$ and $\beta^{\pm}_{o}$. We shall calculate these using the strip approximation in the t channel. This consists of assuming that the amplitudes $f^{\pm J}_{-1}(t)$ and $f^{\pm J}_{o}(t)$ carry the phase of $\pi\pi$ scattering over the entire interval $\left[4\mu^2, t_1\right]$. This is exact of course for $t \in \left[4\mu^2, 16\mu^2\right]$ and probably represents a good approximation over the wider interval. As mentioned earlier we imagine that we have the solution of the $\pi\pi$ problem. In particular we need the $\pi\pi$ D function, which we denote by $D^{\pm}(J,t)$ for $I = \binom{0}{1}$. We assume that $D^{\pm}(J,t)$ has a cut from $4\mu^2$ to t_1 and is normalized to 1 at infinite t. The phase condition implies:

$$\operatorname{Im}\left\{ D^{\pm}(J,t)\, f^{\pm J}_{-1,0}(t) \right\}_{t \in \left[4\mu^2,\, t_1\right]} = 0 \qquad \text{V.44}$$

We assume that all the poles of $f^{\pm J}_{-1,0}(t)$ to the right of $J = 0$ are also zeroes of $D^{\pm}(J,t)$. Writing

$$f^{\pm J}_{-1,0}(t) = \frac{N^{\pm}_{-1,0}(t)}{D^{\pm}(J,t)} \qquad \text{V.45}$$

and introducing

$$C^{\pm J}_{-1,0}(t) = f^{\pm J}_{-1,0}(t) - \frac{1}{\pi}\int_{4\mu^2}^{t_1} \mathrm{Im}\, f^{\pm J}_{-1,0}(t')\,\frac{dt'}{t'-t} \qquad \text{V.46}$$

we have

$$N^{\pm}_{-1,0}(t) = D^{\pm}(J,t)\, C^{\pm J}_{-1,0}(t) \qquad \text{V.47}$$

$$+ D^{\pm}(J,t)\left[f^{\pm J}_{-1,0}(t) - C^{\pm J}_{-1,0}(t)\right]$$

The second quantity has no poles to the right of $J = 0$ by assumption. It has a cut from $4\mu^2$ to t_1 and goes to zero as $t \to \infty$. Therefore using the phase condition, we have:

$$N^{\pm}_{-1,0}(t) = D^{\pm}(J,t)\, C^{\pm}_{-1,0}(t) - \frac{1}{\pi}\int_{4\mu^2}^{t_1} \frac{\mathrm{Im}\, D^{\pm}(J,t)\, C^{\pm J}_{-1,0}(t)}{t'-t}\, dt' \qquad \text{V.48}$$

If we knew $C^{\pm J}_{-1,0}(t)$, we would have the answer for the residues $\beta^{\pm}_{-1,0}(t)$. For

$$\beta^{\pm}_{-1,0}(t) = \frac{-\frac{1}{\pi}\int_{4\mu^2}^{t_1} dt' \left[\mathrm{Im}\, D^{\pm}\left(\alpha^{\pm}(t),t'\right) C^{\pm\alpha^{\pm}(t)}_{-1,0}(t')\right]\left[t'-t\right]^{-1}}{D^{\pm}_{\alpha^{\pm}}\left(\alpha^{\pm}(t),\ t\right)} \qquad \text{V.49}$$

where

$$D_{\alpha}(\alpha,t) = \frac{\partial D^{\pm}}{\partial J}(J,t)/_{J=\alpha}$$

The problem is then, to calculate $C^{\pm J}_{-1,0}(t)$. If we ignore the contributions of the double spectral regions $t \in \left[4\mu^2,\ t_1\right]$ to the left cuts of $C^{\pm J}_{-1,0}(t)$, we can calculate $C^{\pm J}_{-1,0}(t)$ in terms of the Regge poles in the crossed πN channel. We have made this approximation in dealing with the πN system and we use it here as well. Using V.12 , V.13 and V.24, V.26, V.30 and V.31, this leads to the following results:

$$C^{\pm J}_{-1}(t) = \frac{1}{2}\frac{1}{(pq)^{J-1}}\ \frac{1}{2J+1}\left[B^{P^{\pm}}_{J-1}(t) - B^{P^{\pm}}_{J+1}(t)\right] \qquad \text{V.50}$$

and

$$C^{\pm J}_{o}(t) = \frac{1}{2}\frac{p}{(pq)^{J}}\left[-\,p\,A^{P^{\pm}}_{J} + \frac{mq}{2J+1}\left[(J+1)\,B^{P^{\pm}}_{J+1} + J\,B^{P^{\pm}}_{J-1}\right]\right] \qquad \text{V.51}$$

Where using V.20 and V.28, $A^{P^{\pm}}_{J}(t)$ is given by :

$$A_J^{P\pm}(t) = (\beta_{tu})\frac{2}{\pi pq}\int_{(M+\mu)^2}^{\infty} du' \; \mathrm{Im}\, A^P\left(u';2(M^2+\mu^2)-u'-t\right) Q_J\left(\frac{p^2+q^2+u'}{2pq}\right)$$

$$\beta_{tu} = \begin{array}{c} (+) \\ (-) \\ \\ \end{array}\begin{pmatrix} 1/3 & 2/3 \\ 1/3 & -1/3 \end{pmatrix} \qquad \text{V.52}$$

$$\begin{array}{cc} 1/2 & 3/2 \end{array}$$

the A^P taken as before to refer to definite I spin in the u channel. Similarly

$$B_J^{P\pm}(t) = (\beta'_{tu})\frac{2}{\pi pq}\int_{(M+\mu)^2}^{\infty} du' \; \mathrm{Im}\, B^P\left(u',2(M^2+\mu^2)-u'-t\right) Q_J\left(\frac{p^2+q^2+u'}{2pq}\right)$$

V.53

$$\beta'_{tu} = \begin{array}{c} (+) \\ (-) \\ \\ \end{array}\begin{pmatrix} -1/3 & -2/3 \\ -1/3 & 1/3 \end{pmatrix}$$

$$\begin{array}{cc} 1/2 & 3/2 \end{array}$$

This completes the discussion of the potential. $F^{e_o}(J,W)$ is given by :

$$F^{e_o}(J,W) = F_t^{e_o}(J,W) + F_u^{e_o}(J,W) \qquad \text{V.54}$$

Given the solution of the $\pi\pi$ problem, $F^{e_o}(J,W)$ is given entirely in terms of quantities which are outputs of the πN problem and thus a bootstrap situation exists.

SECTION VI: SOLUTION OF THE INTEGRAL EQUATION

In this section we consider the solution of equation IV.14 . The integral equation as written is a singular equation. The kernel is not square summable and standard techniques such as matrix inversion cannot be directly applied. The purpose of this section is to split off the singular part of the kernel and treat it explicitly, leaving a non-singular Fredholm equation. The technique we use is a slight generalization of the Wiener-Hopf method as applied by Chew[23] in the $\pi\pi$ case. First, let us consider the singularity of the kernel of the equation for $N(J,W)$:

$$N(J,W) = F(J,W) + \frac{1}{\pi}\int_{M+\mu}^{W_1} dW' \, \frac{F(J,W') - F(J,W)}{W' - W} \, \rho(J,W') \, N(J,W')$$

$$+ \frac{1}{\pi}\int_{-W_1}^{-(M+\mu)} dW' \, \frac{F(J,W') - F(J,W)}{W' - W} \, \rho(J,W') \, N(J,W')$$

(We drop the signature superscript in this section. All equations are understood to be for a definite signature.) The singular behavior of this equation is caused by logarithmic branchpoints in $F(J,W)$ near $W = \pm W_1$.

$$F(J,W) \xrightarrow[W \to W_1]{} - \mathrm{Im} \, \frac{F_J(W_1 + i\epsilon)}{\pi} \, \ln \left[W_1 - W \right] \qquad \text{VI.1}$$

$$F(J,W) \xrightarrow[W \to -W_1]{} \mathrm{Im}\, \frac{F_J(-W_1+i\epsilon)}{\pi} \ln \left[W_1 + W\right] \qquad \text{VI.2}$$

Let us denote $\frac{1}{\pi} \frac{F(J,W')-F(J,W)}{W'-W} \rho(J,W')$ by $K_J(W,W')$.

If the generalized Fredholm theory were to apply, the following integrals would have to be finite:

(1) $\int_{M+\mu}^{W_1} dW' \int_{M+\mu}^{W_1} dW \; |K_J(W,W')|^2$

(2) $\int_{M+\mu}^{W_1} dW' \int_{-W_1}^{-(M+\mu)} dW \; |K_J(W,W')|^2$

(3) $\int_{-W_1}^{-(M+\mu)} dW' \int_{M+\mu}^{W_1} dW \; |K_J(W,W')|^2$

(4) $\int_{-W_1}^{-(M+\mu)} dW' \int_{-W_1}^{-(M+\mu)} dW \; |K_J(W,W')|^2$

The behavior of $F(J,W)$ near $W = \pm W_1$ causes integrals (1) and (4) to diverge, while (2) and (3) are finite. The fact that (2) and (3) remain finite is crucial to the success of the method we use here to treat the singular behavior. Before we attack the problem of treating

the singular behavior of IV.14 , we need to relate the behaviors of $N(J,W)$, $D(J,W)$ and $F(J,W)$ near $W = \pm W_1$ to physical requirements. The physical amplitude $\tilde{b}(J,W)$ can have no singular behavior near $W = \pm W_1$. This requires that $\mathrm{Im}\,\tilde{b}(J,W)$ be continuous near $W = \pm W_1$. Using elastic unitarity, this implies that

$$\frac{\sin^2 \delta_{J-\frac{1}{2}}(W_1)}{\rho(W_1)} = \mathrm{Im}\, F(J,W_1+i\epsilon) \qquad \text{VI.3}$$

and

$$\frac{\sin^2 \delta_{J+\frac{1}{2}}(W_1)}{\rho(-W_1)} = \mathrm{Im}\, F(J,-W_1+i\epsilon) \qquad \text{VI.4}$$

where

$$\delta_{J\mp\frac{1}{2}}(W_1) = \lim_{W\to W_1^-}\left[\delta_{J\mp\frac{1}{2}}(W)\right]$$

Using equation IV.8 for $D(J,W)$, we also have :

$$D(J,W+i\epsilon) \xrightarrow[W\to W_1^+]{} (W_1 - W)^{\delta_{J-\frac{1}{2}}(W_1)/\pi}\, e^{-i\delta_{J-\frac{1}{2}}(W_1)}$$

$$D(J,W+i\epsilon) \xrightarrow[W\to -W_1^+]{} (W_1 + W)^{-\delta_{J+\frac{1}{2}}(W_1)/\pi}\, e^{i\delta_{J+\frac{1}{2}}(W_1)} \qquad \text{VI.5}$$

Unitarity demands $N(J,W)/D(J,W)$ be bounded. Therefore

$$N(J,W) \xrightarrow[W\to W_1^-]{} (W_1 - W)^{-\delta_{J-\frac{1}{2}}(W_1)/\pi}$$

$$N(J,W) \xrightarrow[W\to -W_1^+]{} (W_1 + W)^{-\delta_{J+\frac{1}{2}}(W_1)/\pi} \qquad \text{VI.6}$$

Now, let us turn directly to the treatment of equation IV.14 . As it is written, it is an integral equation for $W\epsilon\left[M+\mu, W_1\right] \equiv I_1$ and $W\epsilon -\left[W_1, M+\mu\right] \equiv I_2$, and a definition otherwise. In the following we will be concerned only with $W\epsilon\, I_1$ or I_2 . We consider the following modified quantities:

$$N_1(W) = \begin{cases} N(J,W) & W \in I_1 \\ 0 & \text{otherwise} \end{cases}$$

$$N_2(W) = \begin{cases} N(J,W) & W \in I_2 \\ 0 & \text{otherwise} \end{cases}$$

$$K_1(W,W') = \begin{cases} K_J(W,W') & W'\epsilon\, I_1 \\ 0 & \text{otherwise} \end{cases} \qquad \text{VI.7}$$

$$K_2(W,W') = \begin{cases} K_J(W,W') & W'\epsilon\, I_2 \\ 0 & \text{otherwise} \end{cases}$$

The last two definitions are understood to apply for $W \epsilon I_1$ or I_2. With these definitions, the integral equation for $N(J,W)$ becomes equivalent to the following two equations:

$$N_1(W) = F(J,W) + \int_{M+\mu}^{W_1} dW' \, K_1(W,W') \, N_1(W') + \int_{-W_1}^{-(M+\mu)} dW' \, K_2(W,W') \, N_2(W') \qquad \text{VI.8}$$

$$N_2(W) = F(J,W) + \int_{M+\mu}^{W_1} dW' \, K_1(W,W') \, N_1(W') + \int_{-W_1}^{-(M+\mu)} dW' \, K_2(W,W') \, N_2(W') \qquad \text{VI.9}$$

Let us consider equation VI.8 first. The singular term is $\int_{M+\mu}^{W_1} dW' \, K_1(W,W') \, N_1(W')$. Let us split off the singular part of $K_1(W,W')$:

$$K_1(,W') = -\frac{\lambda_{J-\frac{1}{2}}}{\pi^2}\left[\frac{\ln\left[W_1 - W'\right] - \ln\left[W_1 - W\right]}{W' - W}\right] + \bar{K}_1(W,W') \qquad \text{VI.10}$$

where

$$\lambda_{J-\frac{1}{2}} = \rho(J,W_1)\ \mathrm{Im}\left[F(J,W_1)\right] = \sin^2\left[\delta_{J-\frac{1}{2}}(W_1)\right] \qquad \text{VI.11}$$

and we define

$$k_1(W,W') = \frac{\ln\left[W_1 - W'\right] - \ln\left[W_1 - W\right]}{W' - W} \qquad \text{VI.12}$$

Our equation for $N_1(W)$ now reads:

$$N_1(W) = F(J,W) + \int_{-W_1}^{-(M+\mu)} dW'\ K_2(W,W')\ N_2(W') + \int_{M+\mu}^{W_1} dW'\ \bar{K}_1(W,W')\ N_1(W') - \frac{\lambda_{J-\frac{1}{2}}}{\pi^2}\int_{M+\mu}^{W_1} dW'\ k_1(W,W')\ N_1(W') \qquad \text{VI.13}$$

Now let us write this as two coupled equations:

$$N_1^0(W) = F(J,W) + \int_{-W_1}^{-(M+\mu)} dW' \; K_2(W,W') \; N_2(W') + \int_{M+\mu}^{W_1} dW' \; \bar{K}_1(W,W') \; N_1(W') \qquad \text{VI.14}$$

$$N_1(W) = N_1^0(W) - \frac{\lambda_{J-\frac{1}{2}}}{\pi^2} \int_{M+\mu}^{W_1} dW' \; k_1(W,W') \; N_1(W') \qquad \text{VI.15}$$

Our first objective is to solve VI.15 for N_1 in terms of $N_1{}^0$. For this let

$$x = \ln\left[\frac{W_1 - M-\mu}{W_1 - W}\right]$$

$$n_1{}^+(x) = N_1\big(W(x)\big)$$

$$n_1{}^0(x) = N_1{}^0\big(W(x)\big) \qquad \text{VI.16}$$

then

$$n_1{}^+(x) = n_1{}^0(x) + \frac{\lambda_{J-\frac{1}{2}}}{\pi^2} \int_0^\infty \frac{dx'(x'-x)}{\left[e^{x'} - x_{-1}\right]} \; n_1(x') \qquad \text{VI.17}$$

As written this equation is just a transformation of equation VI.13 and therefore is restricted to $W \epsilon I_1$ or $x \epsilon [o, \infty]$. We can solve this equation by the Wiener-Hopf technique if we extend it to $x \epsilon [-\infty, o]$. We define $n_1^{\,0}(x) = n_1^{\,+}(x) = 0$ for $x \epsilon [-\infty, o]$. We introduce $n_1^{\,-}(x) = 0$ for $x \epsilon [o, \infty]$ and

$$n_1^{\,-}(x) = \frac{\lambda_{J-\frac{1}{2}}}{\pi^2} \int_o^\infty dx' \, \frac{x' - x}{\left[e^{x'-x} - 1\right]} \, n_1^{\,+}(x') \qquad \text{VI.18}$$

for $x \epsilon [-\infty, o]$

Therefore our equation now reads:

$$n_1^{\,+}(x) + n_1^{\,-}(x) = n_1^{\,0}(x) + \frac{\lambda_{J-\frac{1}{2}}}{\pi^2} \int_o^\infty dx' \, \frac{x' - x}{e^{x'-x} - 1} \, n_1^{\,+}(x') \qquad \text{VI.19}$$

Now referring back to equation VI.6, we see that

$$n_1^{\,+}(x) \xrightarrow[x \to \infty]{} e^{\delta_{J-\frac{1}{2}}(W_1)\, x/\pi}$$

In the case of physical interest, $0 < \delta_{J-\frac{1}{2}} < \frac{\pi}{2}$. Thus $n_1^{\,+}(x)$ can increase at ∞ no faster than $e^{x/2}$. Referring back to equation VI.14 and using $0 < \delta_{J-\frac{1}{2}} < \pi/2$, we see that $n_1^{\,0}(x)$ can grow at most linearly in x at ∞. Finally equation VI.18 above shows that $n_1^{\,-}(x) \xrightarrow[x \to -\infty]{} e^x$. Thus we have an equation of exactly the

same form as that considered by Chew and we can use his solution of it directly. Defining $\omega_{J-\frac{1}{2}} = \delta_{J-\frac{1}{2}}(W_1)/\pi$, we may take the Fourier transform of VI.19 anywhere in the strip $1 > \mathrm{Im}\, k > \omega_{J-\frac{1}{2}}$. Doing this, we get

$$g_1^{+}(k) + g_1^{-}(k) = g_1^{0}(k) + \frac{\lambda_{J-\frac{1}{2}}}{\sin^2(\pi i k)}\, g_1^{+}(k) \quad . \qquad \text{VI.20}$$

$g_1^{+}(k)$ is holomorphic in $\mathrm{Im}\, k > \omega_{J-\frac{1}{2}}$,

$g_1^{-}(k)$ is holomorphic in $\mathrm{Im}\, k < 1$,

$g_1^{0}(k)$ is holomorphic in $\mathrm{Im}\, k > 0$,

and $g_1^{+}(k)$, $g_1^{-}(k)$, and $g_1^{0}(k)$ all vanish as $k \to \infty$ along any ray in their respective domains of holomorphy. Now let us consider the term

$$1 - \frac{\lambda_{J-\frac{1}{2}}}{\sin^2(\pi i k)} = \frac{\sin^2(\pi i k) - \sin^2\left(\delta_{J-\frac{1}{2}}(w_1)\right)}{\sin^2(\pi i k)}$$

This has zeroes at

$$k = k_1 = i\, \omega_{J-\frac{1}{2}}$$

and $k = k_2 = i(1-\omega_{J-\frac{1}{2}})$.

The crucial step in the solution of VI.20 consists in writing

$$1 - \frac{\lambda_{J-\frac{1}{2}}}{\sin^2(\pi i k)} = \frac{\phi_2(k)}{\phi_1(k)} \qquad \text{VI.21}$$

where $\phi_2(k)$ is holomorphic and free of zeroes in $\operatorname{Im} k < 1 - \omega_{J-\frac{1}{2}}$, and $\phi_2(k)$ is holomorphic and free of zeroes in $\operatorname{Im} k > \omega_{J-\frac{1}{2}}$. As shown by Chew, this is accomplished by

$$\phi_2(k) = \frac{\Gamma^2(1+ik)}{\Gamma(1+ik-\omega_{J-\frac{1}{2}})\,\Gamma(1+ik+\omega_{J-\frac{1}{2}})} \qquad \text{VI.22}$$

and

$$\phi_1(k) = \frac{\Gamma(-ik+\omega_{J-\frac{1}{2}})\,\Gamma(-ik-\omega_{J-\frac{1}{2}})}{\Gamma^2(-ik)} \qquad \text{VI.23}$$

$\phi_1(k)$ and $\phi_2(k)$ approach constants at ∞ . Using this factorization and dividing by $\phi_2(k)$, we have :

$$\frac{g_1^{+}(k)}{\phi_1(k)} + \frac{g_1^{-}(k)}{\phi_2(k)} = \frac{g_1^{0}(k)}{\phi_2(k)}$$

$$= \frac{1}{2\pi i}\int_{-\infty+i\epsilon}^{+\infty+i\epsilon} \frac{dk'}{k'-k}\,\frac{g_1^{0}(k')}{\phi_2(k')} - \frac{1}{2\pi i}\int_{-\infty+i(1-\omega_{J-\frac{1}{2}}-\epsilon)}^{+\infty+i(1-\omega_{J-\frac{1}{2}}-\epsilon)} \frac{dk'}{k'-k}\,\frac{g_1^{0}(k')}{\phi_2(k')} \qquad \text{VI.24}$$

or

$$\frac{g_1^{+}(k)}{\phi_1(k)} - \frac{1}{2\pi i}\int_{-\infty+i\epsilon}^{+\infty+i\epsilon} \frac{dk'}{k'-k}\,\frac{g_1^{0}(k')}{\phi_2(k')}$$

$$= -\frac{g_1^{-}(k)}{\phi_2(k)} - \frac{1}{2\pi i}\int_{-\infty+i(1-\omega_{J-\frac{1}{2}}-\epsilon)}^{\infty+i(1-\omega_{J-\frac{1}{2}}-\epsilon)} \frac{dk'}{k'-k}\,\frac{g_1^{0}(k')}{\phi_2(k')} \qquad \text{VI.25}$$

The left hand side of the equation is holomorphic for $\operatorname{Im} k > \omega_{J-\frac{1}{2}}$, the right hand side is holomorphic for $\operatorname{Im} k < 1 - \omega_{J-\frac{1}{2}}$, they agree for $1 - \omega_{J-\frac{1}{2}} > \operatorname{Im} k > \omega_{J-\frac{1}{2}}$, and both sides approach zero at ∞ and their half planes of holomorphy. Thus we have an entire function which vanishes at ∞ and therefore it must be identically zero. This gives

$$g_1^{+}(k) = \frac{\phi_1(k)}{2\pi i} \int_{-\infty+i\epsilon}^{\infty+i\epsilon} \frac{dk'}{k'-k} \frac{g_1^{0}(k')}{\phi_2(k')} \qquad \text{VI.26}$$

Consequently,

$$n_1^{+}(x) = \frac{1}{(2\pi)^{3/2}} \frac{1}{i} \int_{-\infty+ik_i}^{+\infty+ik_i} dk\, e^{-ikx} \phi_1(k) \int_{-\infty+i\epsilon}^{\infty+i\epsilon} \frac{dk'}{k'-k} \frac{g_1^{0}(k')}{\phi_2(k')}$$

VI.27

$$= \frac{1}{(2\pi)^2} \frac{1}{i} \int_{-\infty+ik_i}^{+\infty+ik_i} dk\, e^{-ikx} \phi_1(k) \int_{-\infty+ik'_i}^{+\infty+ik'_i} \frac{dk'}{k'-k} \frac{1}{\phi_2(k')} \int_0^{\infty} dx'\, n_1^{0}(x') e^{ik'x'}$$

where $k_i > \omega_{J-\frac{1}{2}} > k'_i$.

We define

$$\theta_1(x,x') = \frac{1}{(2\pi)^2} \frac{1}{i} \int_{-\infty+ik_i}^{+\infty+ik_i} dk\, e^{-ikx} \phi_1(k) \int_{-\infty+ik'_i}^{\infty+ik'_i} \frac{dk'}{k'-k} \frac{e^{ik'\cdot x'}}{\phi_2(k')} \qquad \text{VI.28}$$

Then

$$n_1(x) = \int_0^{\infty} \theta_1(x,x')\, n_1^{0}(x')\, dx' \qquad \text{VI.29}$$

Using the analyticity properties of $\phi_1(k)$ and $\phi_2(k)$, in particular the fact that $\phi_1(k)$ has a simple pole at $k = i\omega_{J-\frac{1}{2}}$ and $1/\phi_2(k)$ has a simple pole at $k = i(1-\omega_{J-\frac{1}{2}})$, we have

$$\Theta_1(x,x') \underset{\substack{x\to\infty \\ x' \text{ fixed}}}{\longrightarrow} e^{(\omega_{J-\frac{1}{2}})x} \qquad \text{VI.30}$$

$$\Theta_1(x,x') \underset{\substack{x'\to\infty \\ x \text{ fixed}}}{\longrightarrow} e^{-(1-\omega_{J-\frac{1}{2}})x'} \qquad \text{VI.31}$$

Defining $$O_{J-\frac{1}{2}}(W,W') = \frac{\Theta_1\big(x(W),\ x'(W')\big)}{W_1 - W'}, \qquad \text{VI.32}$$

we have

$$N_1(W) = \int_{M+\mu}^{W_1} O_{J-\frac{1}{2}}(W,W')\, N_1^{\,0}(W')\, dW' \qquad \text{VI.33}$$

From equation VI.30 above $N_1(W)$ clearly has the correct behavior near $W = W_1$. Substituting our solution for $N_1(W)$ into equation VI.14, we have:

$$N_1^{\,0}(W) = F(J,W) + \int_{-W_1}^{-(M+\mu)} K_2(W,W')\, N_2(W')\, dW' + \int_{M+\mu}^{W_1} \bar{M}_1(W,W')\, N_1^{\,0}(W')\, dW' \qquad \text{VI.34}$$

where

$$\bar{M}_1(W,W') = \int_{M+\mu}^{W_1} \bar{K}_1(W,W'')\, O_{J-\frac{1}{2}}(W'',W')\, dW'' \qquad \text{VI.35}$$

Now we must treat equation VI.9 . The analysis of this equation is quite analogous to that just completed of VI.8 , and therefore we merely sketch the details. The first step is to split off the singular part of $K_2(W,W')$:

$$K_2(W,W') = \frac{\lambda_{J+\frac{1}{2}}}{\pi^2} \frac{\left[\ln\left[W_1 + W'\right] - \ln\left[W_1 + W\right]\right]}{W' - W} + \bar{K}_2(W,W') \qquad \text{VI.36}$$

where $\lambda_{J+\frac{1}{2}} = \sin^2\left[\delta_{J+\frac{1}{2}}(W_1)\right]$

Setting

$$k_2(W,W') = \frac{\ln\left[W_1 + W'\right] - \ln\left[W_1 + W\right]}{W' - W}$$

and

$$N_2^{\,0}(W) = F(J,W) + \int_{M+\mu}^{W_1} dW'\, K_1(W,W')\, N_1(W') + \int_{-W_1}^{-(M+\mu)} dW'\, \overline{K}_2(W,W')\, N_2(W') \qquad \text{VI.37}$$

we have

$$N_2(W) = N_2^{\,0}(W) + \frac{\lambda_{J+\frac{1}{2}}}{\pi^2} \int_{-W_1}^{-(M+\mu)} dW'\, k_2(W,W')\, N_2(W') \qquad \text{VI.38}$$

Letting

$$y = \ln\left[\frac{W_1 - M - \mu}{W_1 + W}\right],$$

$$n_2(y) = N_2\big(W(y)\big),$$

$$n_2^{\,0}(y) = N_2^{\,0}\big(W(y)\big),$$

and

$$\omega_{J+\frac{1}{2}} = \frac{\delta_{J+\frac{1}{2}}(W_1)}{\pi},$$

we have :

$$n_2(y) = n_2^{0}(y) + \frac{\lambda_{J+\frac{1}{2}}}{\pi^2}\int_0^\infty \frac{dy'(y'-y)\, n_2(y')}{\left[e^{y'-y}-1\right]} \qquad \text{VI.39}$$

which is of exactly the same form as II.17 . Thus

$$n_2(y) = \int_0^\infty \Theta_2(y,y')\, n_2^{0}(y')\, dy' \qquad \text{VI.40}$$

where $\Theta_2(y,y')$ is gotten from $\Theta_1(y,y')$ by replacing $\omega_{J-\frac{1}{2}}$ by $\omega_{J+\frac{1}{2}}$ in $\phi_1(k)$ and $\phi_2(k)$.

Defining

$$O_{J+\frac{1}{2}}(W,W') = \frac{\Theta_2\big(y(W),\ y(W')\big)}{(W_1+W')} \qquad \text{VI.41}$$

we have

$$N_2(W) = \int_{-W_1}^{-(M+\mu)} dW'\ \Theta_{J+\frac{1}{2}}(W,W')\, N_2^{0}(W')\, dW' \qquad \text{VI.42}$$

It is easily seen that $N_2(W)$ has the correct behavior near $W = -W_1$. Now let us summarize the results obtained so far.

$$N_1(W) = \int_{M+\mu}^{W_1} dW' \; O_{J-\frac{1}{2}}(W,W') \; N_1^{\,0}(W') \qquad \text{VI.43}$$

$$N_2(W) = \int_{-W_1}^{-(M+\mu)} dW' \; O_{J+\frac{1}{2}}(W,W') \; N_2^{\,0}(W') \; dW' \; , \qquad \text{VI.44}$$

$$N_1^{\,0}(W) = F(J,W) + \int_{-W_1}^{-(M+\mu)} dW' \; M_2(W,W') \; N_2^{\,0}(W') + \int_{M+\mu}^{W_1} dW' \; \overline{M}_1(W,W') \; N_1^{\,0}(W') \qquad \text{VI.45}$$

$$N_2^{\,0}(W) = F(J,W) + \int_{M+\mu}^{W_1} dW' \; M_1(W,W') \; N_1^{\,0}(W') + \int_{-W_1}^{-(M+\mu)} dW' \; \overline{M}_2(W,W') \; N_2^{\,0}(W') \qquad \text{VI.46}$$

where as before $N_1(W)$ and $N_1^{\,0}(W)$ vanish except when $W \epsilon I_1$, and $N_2(W)$ and $N_2^{\,0}(W)$ vanish except when $W \epsilon I_2$, and where

$$M_1(W,W') = \int_{M+\mu}^{W_1} dW''\, K_1(W,W'')\, O_{J-\frac{1}{2}}(W'',W')$$

$$\overline{M}_1(W,W') = \int_{M+\mu}^{W_1} dW''\, \overline{K}_1(W,W'')\, O_{J-\frac{1}{2}}(W'',W')$$

$$M_2(W,W') = \int_{-W_1}^{-(M+\mu)} dW''\, K_2(W,W'')\, O_{J+\frac{1}{2}}(W'',W') \quad ,$$

and $$\overline{M}_2(W,W') = \int_{-W_1}^{-(M+\mu)} dW''\, \overline{K}_2(W,W'')\, O_{J+\frac{1}{2}}(W'',W') \quad .$$

Now for the generalized Fredholm theory to be applicable to the coupled equations VI.45, VI.46 , we must have:

$$\int_{-W_1}^{-(M+\mu)} dW \int_{M+\mu}^{W_1} dW' \; | \overline{M}_1(W,W') |^2$$

$$\int_{M+\mu}^{W_1} dW \int_{M+\mu}^{W_1} dW' \; | \overline{M}_1(W,W') |^2$$

$$\int_{M+\mu}^{W_1} dW \int_{-W_1}^{-(M+\mu)} dW' \; | M_2(W,W') |^2$$

$$\int_{-W_1}^{-(M+\mu)} dW \int_{-W_1}^{-(M+\mu)} dW' \; | \overline{M}_2(W,W') |^2$$

all finite. Noting the behavior of $O_{J\mp\frac{1}{2}}(W,W')$ as given in equations VI.30, VI.31, and recalling the behaviors of $K_{1,2}(W,W')$, $\bar{K}_{1,2}(W,W')$ we see that the integrals are indeed all finite. Thus the equations VI.45, VI.46 can be solved by any of the usual techniques. The solution of the original equation IV.14 for $N(J,W)$ is accomplished as follows: Given $F(J,W)$, one constructs the kernels $K_{1,2}(W,W')$, $\bar{K}_{1,2}(W,W')$ and from them and the transformations $O_{J\mp\frac{1}{2}}(W,W')$, the kernels $M_{1,2}(W,W'), \bar{M}_{1,2}(W,W')$ are calculated. The coupled equations for $N_{1,2}^{0}(W)$ are solved and $N_{1,2}(W)$ are computed using VI.43 and VI.44. This gives $N(J,W)$ in I_1, I_2 and therefore the $D(J,W)$ function for all energies can be calculated. Finally to get $N(J,W)$ for values of W outside I_1, I_2, the original equation for $N(J,W)$ is used along with the known values of $N(J,W)$ in I_1, I_2. That such a series of steps is actually feasible numerically has been shown by Teplitz and Teplitz[11] for the $\pi\pi$ case. The labor here would be roughly quadrupled, but the procedure should still be feasible.

The kernels $M_{1,2}(W,W')$ and $\bar{M}_{1,2}(W,W')$ are holomorphic in J in the same domain as $F(J,W)$ as long as $0 < \lambda_{J\mp\frac{1}{2}} < 1$, and thus $N_{1,2}^{0}(W)$ and $N_{1,2}(W)$ are holomorphic in the same domain except for fixed (Fredholm) poles. These Fredholm poles will serve as the high energy limits of Regge poles which are the zeroes of $D(J,W)$.

VII. ASYMPTOTIC BEHAVIOR AND CONCLUSIONS

At the end of section VI it was stated that the fixed poles of $N(J,W)$ serve as the high energy limits of Regge poles. This is easily seen as follows: Suppose $N(J,W)$ has a simple pole at J_0. Consider a small circle around J_0, encircling no other singularities or zeroes of $N(J,W)$. For $|W|$ large enough, $|D(J,W)-1|$ can be made as small as desired on the circle. If we make the radius of the circle small enough, $|D(J,W)-1|$ can be made strictly increasing as one moves to the center of the circle and since the phase of $D(J,W)-1$ goes through 2π in encircling J_0, it follows that $D(J,W)$ has one zero for some J inside the circle at any fixed W for which $|W|$ is large enough. Thus the point J_0 will be the high energy limit of a Regge pole. The amplitude of course, has no pole at J_0. In the theory we are considering here, the poles of $N(J,W)$ are determined by the solution of the integral equation and thus their precise location cannot be determined a priori. However, it is quite likely that they will lie near the point $J = -\frac{1}{2}$. This is because the kernel of the integral equation has poles at $J = -\frac{1}{2}$ and is holomorphic to the right of $J = -\frac{1}{2}$. (We consider here the case where the residues and positions of the Regge poles in the u channel have finite cuts.) The residues of the poles near $J = -\frac{1}{2}$ are not of finite rank and thus we expect an infinite number of eigenvalues of the kernel near this point, or in other words $N(J,W)$ will have an infinite number of poles near this point.

The rightmost one of these will be the high energy limit of the leading trajectory. It is of course quite desirable that this limit lie as far to the left as possible, since this will tend to depress the interior regions of the double spectral functions, which have been ignored here. The limit must lie to the right of $J = -\frac{1}{2}$, but the theory should still allow all Regge poles to retreat to the left of $J = 0$ at high energies.

Now let us make some brief remarks about the practical implications of the theory presented here. Although the actual carrying out of the full solution of the set of bootstrap equations presented in sections IV, V, and VI is technically feasible, one could imagine attacking the equations in a somewhat less ambitious manner in a first attempt. Clearly the most essential new feature of the scheme presented here is the Reggeized treatment of the forces from the crossed πN channel. Keeping this feature, one could ignore completely the t channel poles and set $\delta_{J\mp\frac{1}{2}}(W_1) = 0$ in a first approximation. This would eliminate the need for the Wiener-Hopf transformation as well as simplify the calculation of $F(J,W)$. Only slightly more difficult would be retention of un-Reggeized ρ exchange, which has turned out to be a good approximation in the $\pi\pi$ case. The fully Reggeized theory with $\delta_{J\mp\frac{1}{2}}(W_1) = 0$ would represent the next level of approximation and finally the complete set of equations with $\delta_{J\mp\frac{1}{2}}(W_1) \neq 0$ could be attacked. The carrying out of any one of these approximations would represent a substantial improvement over the simple un-Reggeized

bootstrap calculations done so far. The masses of the recurrences of the nucleon and N^*_{33} might be given in a semiquantitative way in the present one channel approximation although their widths certainly would not be. Going beyond the one channel theory presented here is not possible even in principle at present until further insight is gained into the formulation of bootstrap equations for particles with high spin and the closely related question of the role of complex angular momentum in the three body problem.

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REFERENCES AND FOOTNOTES

1. G.F. Chew and F.E. Low, Phys. Rev. 101, 1570 (1956).
2. S. Mandelstam, Phys. Rev. 112, 1344 (1958).
3. S.W. Mac Dowell, Phys. Rev. 116, 774 (1959).
4. W.R. Frazer and J.R. Fulco, Phys. Rev. 119, 1420 (1960)
5. S.C. Frautschi and J.D. Walecka, Phys. Rev. 120, 1486 (1960).
6. G.F. Chew, Phys. Rev. Letters 9, 233 (1962).
7. E. Abers and C. Zemach, Phys. Rev. 131, 2305 (1963).
8. J.S. Ball and D. Y. Wong, Phys. Rev. 133, B179 (1964).
9. See A. Donnachie, J. Hamilton, and A.T. Lea, Phys. Rev. 135, B515 (1964), and previous work referred to in this paper.
10. G.F. Chew, Phys. Rev. 129, 2363 (1963).
11. See D.C. Teplitz and V.L. Teplitz, Phys. Rev. 137, B142 (1965) for a complete list of references.
12. G.F. Chew and S.C. Frautschi, Phys. Rev. 123, 1478 (1961).
13. T. Regge, Nuovo Cimento 14, 951 (1959).
14. H.P. Stapp, Phys. Rev. 125, 2139 (1962).
15. D. Hall and A.S. Wightman, Mat. Fys. Medd. Dan. Vid. Selsk. 31, no. 5 (1957).
16. V. Singh, Phys. Rev. 129, 1889 (1963).
17. A. O. Barut and D.E. Zwanziger, Phys. Rev. 127, 975 (1962).
18. S. Mandelstam, Nuovo Cimento 30, 1148 (1963).
19. W.R. Frazer and J.R. Fulco, Phys. Rev. 117, 1603 (1960).
20. S. Mandelstam, Phys. Rev. 137, B949 (1965).

21. Peter Collins - private communication.

22. I am indebted to Peter Collins for a discussion on this point.

23. G.F. Chew, Phys. Rev. 130, 1264 (1963) .

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