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GEAR RATIOS

Berkeley, California

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For Machine Design

UCRL-16226 Rev.

UNIVERSITY OF CALIFORNIA

Lawrence Radiation Laboratory
Berkeley, California

AEC Contract No. W-7405-eng-48

GEAR RATIOS

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August 1966

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Berkeley, California

October 20, 1965

ABSTRACT

A systematic procedure is described for finding suitable gear combinations to generate a given ratio to greater precision than permitted by gear ratio tables. Useful methods of extending factor tables for gear ratio calculations are also presented.

The calculation of gear ratios for obtaining a desired change in shaft rotation rates is generally a straightforward procedure. The simplest ratios can be obtained by a cut-and-try approach. Extensive tables have been prepared for obtaining many ratios to within a few parts per million.[1,2,3,4,5,6] Occasionally it is desired to obtain a given ratio within much smaller limits, and if the desired ratio is not simple, a great deal of labor may be required in finding suitable gear combinations. The following summarizes some techniques which have been found useful in systematizing the search for gear ratios.

In this type of work an extensive factor table is highly desirable. Mention may be made of AMS 55,[7] British Math Tables, Vol. 5,[8] Kavan,[9] and Lehmer.[10] The first three give all the prime factors of numbers up to 10,000, 100,000 and 256, 000 respectively. The fourth is less convenient to use but lists the smallest prime factor for all the numbers up to 10,017,000, excluding those divisible by 2, 5 or 7. Merritt[1] gives a table of composite numbers up to 200,000 which consist entirely of prime factors not greater than 127. Numbers divisible by 2, 3 or 5 are omitted.

The general gear ratio problem is to determine the values of a, b, c, d, ..., such that $R = (a/b)(c/d) \ldots$, is equal to a desired ratio within prescribed limits. An additional restriction is that a, b, ..., are to be less than some number, generally one or two hundred, because gears with more teeth are not usually convenient to use, especially in complicated trains. No restriction will be placed on the number of gear pairs to be used.

The method to be used in finding ratios is a modification of one described by Flanigan[11] and others using continued fractions. Two elementary texts on continued fractions of the type used here are given in Refs. 12 and 13. The present method is one which is systematic and restricts the search to ratios which are within the prescribed limits.

Let it be required to find a gear train that has a ratio R equal to a specified value R_o within certain limits, say $\pm\epsilon$. There are two requirements that must be met: 1) a rational fraction must be produced which represents the value R_o within the specified limits, and 2) the numerator and denominator of this fraction must be factorable into numbers sufficiently small to be useful in gear trains. The first requirement is met immediately by the procedure of this paper. The second depends on the particular values of numerator and denominator. Factor tables are consulted to determine the factors, if any.

As shown in texts on continued fractions,[12,13] any number expressed as an integer and decimal may be converted to a continued fraction of the form

$$a_o + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

which is usually written

$$a_o + \frac{1}{a_1+} \; \frac{1}{a_2+} \; \frac{1}{a_3+} \; \cdots$$

The a_o term corresponds to the integer and the remainder to the decimal part. If the decimal terminates or is a repeating decimal, the continued fraction will terminate, otherwise not.

The continued fraction is formed by dividing the decimal into 1, giving an integer a_1 (always 1 or larger), plus a second decimal. The latter is divided into 1 to give a_2 plus a third decimal. This procedure is continued as long as necessary. If two such fractions are generated, corresponding to the two allowed limits of the desired ratio R, it will be found that the first few values of a_i are identical, after which they diverge. Suppose divergence occurs after a_k. The desired continued fraction is then

$$a_o + \frac{1}{a_1+} \; \frac{1}{a_2+} \; \cdots \; \frac{1}{a_k+A}$$

where A can have any value between the two decimal remainders. Sometimes it is convenient to let A take negative values, as shown in examples 1 and 3. The continued fraction is now converted to an ordinary fraction by means of successive convergents. The first convergent is $\frac{b_o}{c_o} = \frac{a_o}{1}$ (a_o is 0 if R<1), the second is $\frac{b_1}{c_1} = \frac{a_o}{1} + \frac{1}{a_1} = (a_o a_1 + 1)/a_1$. The remaining convergents are obtained from the recurrence formula

$$\frac{b_{i+1}}{c_{i+1}} = \frac{b_i a_{i+1} + b_{i-1}}{c_i a_{i+1} + c_{i-1}}$$

where b_i and c_i are the numerator and denominator of the ith convergent. The final convergent is obtained by taking $a_{i+1} = a_k + A$. This produces a fraction of the form

$$\frac{g_1+g_2A}{h_1+h_2A}$$

in which g_1, g_2, h_1, h_2 are integers, and will provide a rational value of R within the limits allowed providing A is rational and limited to values previously indicated. It is usually convenient to let $A = \frac{n}{m}$, so that

$$R = \frac{mg_1+ng_2}{mh_1+nh_2} .$$

By assigning suitable values to m and n, a rational fraction is obtained whose numerator and denominator can be examined for factors useful for gear trains.

In general, when the limits of error are tightened the numerator and denominator in the generated fraction become larger. They also become larger as m and n (A=n/m) are made larger.

Eventually a point may be reached where the numerator or denominator exceeds the capacity of the available factor tables. In this case the numbers may be examined laboriously for factors by trial division of the prime numbers up to, say, 199. It is usually better to avoid this tedious procedure by the

simple expedient of generating a new ratio R_1 by multiplying R by an arbitrary fraction a/b. The previous method is now applied to obtain a new fraction for R_1. If this new value can be factored satisfactorily, then a valid value of R is obtained from R_1 by multiplying R_1 by b/a. Example 3 uses this method.

Factoring

When it is desired to factor numbers which exceed the capacity of available tables the range may be extended using the following method, provided the number is not greater than twice the range of the tables:

Let the number for which the factors are desired be N. It is assumed that N is odd. Then, if P is a factor of N, P is also a factor of N-P = N-1-(P-1). Both N-1 and P-1 are even and may be divided by 2. Then (N-1)/2-(P-1)/2 still contains the factor P if N contains P. Start with P=3 and look for the factor 3 in (N-1)/2-(3-1)/2 = (N-1)/2-1, then look for the factor 5 in (N-1)/2-2, 7 in (N-1)/2-3, etc., until a factor is found or the possible factors exhausted. This method is rapid and need not be continued beyond $P = \sqrt{N}$. For gear ratios it can be terminated when P exceeds the maximum allowable number of teeth. In normal practice, N will be examined directly for the factors 3, 5, 9, 11. Thus, if 5 is a factor, N (assumed odd) will end in 5. If 3 is a factor, the digits of N will add up to a multiple of 3. If 9 is a factor the digits will add up to a multiple of 9. The number can be checked quickly for the factor 11 by adding

alternate digits and taking the difference. If the difference is zero or a multiple of 11, then N is a multiple of 11. See example 4.

Even if the factors 3, 5, 9, 11 are absent from N, these should be looked for merely to keep the search orderly and to avoid "losing one's place" in the table.

The principle can be extended to larger multiples of the factor table, but it becomes more cumbersome to use. A method to increase the range of tables by a multiple M=30 is described next.

It is assumed that the factors 2, 3, 5 (these are all the prime factors of 30) are absent from N. If N is divided by 30, the possible remainders are R = 1, 7, 11, 13, 17, 19, 23, 29. If N contains P, then N-kP = N-R-(kP-R) also contains P, as does (N-R)/30-(kP-R)/30, where k is an integer chosen to make (kP-R)/30 = n, another integer. Only values of k which are less than 30 and relatively prime to 30 need be considered. Thus use k = 1, 7, 11, 13, 17, 19, 23, 29. Table I has been prepared listing the prime factors less than 200 which correspond to various values of n, given the remainder R obtained by dividing the original number N by 30 to get a new number N' = (N-R)/30, plus a remainder R. The parameter k does not appear in the table explicitly. For example, let R=7 . Under the heading R=7 in Table I are listed all the primes P less than 200, excluding 2, 3, 5, and the corresponding values of n to subtract from N'. The values of n are subtracted from N' and the difference examined for the factors P. If a P is found as a factor, then the original number N contains the factor P. See Example 5.

This method does not eliminate the need to test each prime up to, say 200, one by one, but replaces trial divisions by shorter subtractions. If a factor is found it is divided into N and the remaining factors found directly in the factor table, or if necessary, the same procedure used on the reduced number. There is no obvious simple way to extend Lehmer's factor table by the above methods.

NUMERICAL EXAMPLES

Example 1

Let it be required to find a gear train that has a ratio R equal to π within one part per million. This means that the ratio must be at least 3.1415895 and not more than 3.1415958. These two numbers are converted to continued fractions until they "diverge." The terms are found by dividing successive decimal parts into 1:

3.1415895	3.1415958
7.0626706	7.0623564
15.95644 = 16 - .044	16.03685 = 16 + .037

Care should be taken to see that rounding off the decimals does not change the limits of allowed error. The continued fraction is

$$R = 3 + \cfrac{1}{7 + \cfrac{1}{16 + A}}$$

where $-.044 < A < +.037$, which may also be written $m/n < -23$ or $m/n > +27$, where m and n are integers and $A = n/m$. The above value for R is now converted to the rational fraction

$$R = \frac{355 + 22A}{113 + 7A} = \frac{355m + 22n}{113m + 7n} ,$$

by means of the successive convergents

$$\frac{3}{1}, \frac{22}{7}, \frac{355 + 22A}{113 + 7A} ,$$

or the continued fraction may be reduced by conventional methods. A may be given any rational value between $-.044$ and $+.037$ and a rational fraction will be obtained differing from π by not more than one part in one million. The simplest ratio is obviously the well-known one $355/113 = 5 \times 71/113$, obtained by putting $A = 0$, and is within one part in eleven million of π.

Example 8

Let R be required to be within one part in one hundred million of π:

3.1415926222	3.1415926850
7.0625148716	7.0625117392
15.99619378	15.99699532
1.00382077	1.00301373
261.727	331.815

Then $R = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{261+A}}}} = \dfrac{92988 + 355A}{29599 + 113A}$

where $0.7 < A < 70.8$. By trying successively $A = 1, 2, 3, \ldots,$ and examining the resulting numerators and denominators for suitably small factors, we find for $A = 57$,

$$R = \frac{113223}{36040} = \frac{3 \times 11 \times 47 \times 73}{2^3 \times 5 \times 17 \times 53} = 3.1415926748 \quad .$$

This is larger than π by 6.7 parts in one billion. If integer values of A up to 70 had not produced a suitable ratio, i.e., one consisting of sufficiently small factors in numerator and denominator, the next values of A to try would be half-integral, 1.5, 2.5, 3.5, ..., then perhaps 1-1/3, 1-2/3, 2-1/3, etc.

Example 3

Let it be required to find a gear train to generate the sideral ratio 1.0027379093[14] within one part in 10^8 (equivalent to an error of 1 second in 3 years). This ratio has been examined by many persons, and some very good approximations obtained. The best is probably that found by L. J. Comrie and mentioned by F. Hope-Jones,[15] and is

$$\frac{45 \times 71 \times 257}{29 \times 151 \times 187} = 1.002737909297$$

which is virtually exact. Olsen[16] used the ratio

$$\frac{80 \times 143 \times 173 \times 125}{118 \times 127 \times 163 \times 101} = 1.002737909408$$

and incorporated it into an elaborate clock. Several simple ratios were derived by Flanigan.[11]

It will be assumed that all the simpler ratios have been examined with available tables. The method of attack is then to obtain a new ratio by multiplying the desired value by some arbitrary ratio and proceed as above with the new value. In this case we shall multiply 1.0027379093 by 73/72 (numbers "picked out of the air"). The new ratio is 1.0166648247 with limits of 1.0166648145 and 1.0166648349. Then we have

1.0166648145	1.0166648349
60.00666854	60.00659508
149.958 = 150 - .042	151.628 = 150 + 1.628

Therefore $R_1 = 1 + \cfrac{1}{60 + \cfrac{1}{150 + A}} = \frac{9151 + 61A}{9001 + 60A} = \frac{9151m + 61n}{9001m + 60n}$

where $-.042 < A < 1.628$. If we try $A = 0$ we find $R_1 = 9151/9001$, and both 9151 and 9001 are prime. If $A = 1$,

$$R_1 = \frac{9212}{9061} = \frac{2^2 \times 7^2 \times 47}{13 \times 17 \times 41} = 1.0166648273$$

If $A = 1$ had not produced a suitable solution, then other values of A would be tried, using $A = n/m$, n and m integers to restrict A to the required limits of $-.042$ to $+1.628$.

The desired sidereal ratio is obtained by multiplying R_1 by 72/73:

$$R = \frac{72}{73} \times \frac{2^2}{13} \times \frac{7^2}{17} \times \frac{47}{41} = \frac{72}{73} \times \frac{56}{52} \times \frac{56}{68} \times \frac{47}{41} = 1.00273791184 \quad .$$

The ratio is too large by 2.5×10^{-9}, only one-fourth of the allowed limit, and corresponds to a clock error of 1 second in 12 years. The ratio derived is so compact that it can be combined with the 3600:1 reduction in some Telechron clock motors, for example, to give one revolution per sidereal minute when the clock is supplied with sixty-cycle mean solar frequency. Such a train could be

$$\frac{12}{65} \times \frac{47}{41} \times \frac{14}{51} \times \frac{14}{50} \times \frac{12}{48} \times \frac{12}{48} \times \frac{20}{73} \quad ,$$

in which the first 12-tooth pinion turns at 3600 revolutions per mean solar minute and the 73-tooth gear turns at one revolution per sidereal minute.

Example 4

Let the number N be 18187 and the available factor table go to 10,000. If 18187 contains the factor P, then

$$\frac{18187 - 1}{2} - \frac{P-1}{2} = 9093 - \frac{P-1}{2}$$

also contains P. Try P=3. 9093-(3-1)/2 = 9092. Look up 9092 and check for the factor 3. Since the factor 3 is not contained in 9092, try P=5, and look for the factor 5 in 9093-(5-1)/2 = 9091. Continue with P=7 for 9090, P=9 for 9089, P=11 for 9088, P=13 for 9087. The last number is found to contain the factor 13, therefore N contains this factor. 18187 ÷ 13 = 1399. The last number is prime, so the two factors are 13 and 1399.

Example 5

Assume Kavan's table is available, and the number N to be searched for factors is 2,222,221. Dividing by 30 gives N' = 74074 with a remainder R = 1. Under the heading R = 1 in Table I the first entry is n = 1, P = 31. N' - n = 74073. Look up this number in Kavan and inspect for the factor 31. Under 74074-2 = 74072, look for 61; under 74071 look for 7 or 13, and continue in this manner until a factor is found or the list under R = 1 is exhausted. In this example a match is obtained at n = 12, N' - 12 = 74062, P = 19. Dividing N by 19 gives 116959, which Kavan lists as prime. Hence, the two factors of 2,222,221 are 19 and 116,959. This method is faster than trial division by the primes, and can be carried out without the aid of a desk calculator.

If Lehmer's table is available, then of course, the factor 19 for the number 2,222,221 can be found directly.

Table I. Test factors for increasing a factor table by M = 30.

R = 1		R = 7		R = 11		R = 13		R = 17		R = 19		R = 23		R = 29	
n	P	n	P	n	P	n	P	n	P	n	P	n	P	n	P
1	31	0	7	0	11	0	13	0	17	0	19	0	23	0	29
2	61	1	37	1	41	1	43	1	47	1	7	1	53	1	59
3	7,13	2	67	2	71	2	73	2	7,11	2	79	2	83	2	89
4	11	3	97	3	101	3	103	3	107	3	109	3	113	3	7,17
5	151	4	127	4	131	4	7,19	4	137	4	139	4	11,13	4	149
6	181	5	157	5	7,23	5	163	5	167	5	13	5	173	5	179
10	43	6	11,17	6	191	6	193	6	197	6	199	6	7,29	6	11,19
12	19	7	7,31	7	13,17	8	11,23	9	41	8	37	10	17,19	9	13,23
13	17,23	8	13,19	11	31	13	31	12	13,29	9	17	13	59	10	47
15	41	14	61	12	53	16	17,29	13	37	10	11,29	15	43	20	37
16	37	17	47	18	19,29	18	79	14	19,23	15	67	17	41	22	53
17	73	22	23,29	19	83	19	53	16	71	17	23	20	89	24	107
24	103	23	41	20	47	23	37	17	31	18	43	23	31	25	41
26	71	27	43	22	61	25	109	23	101	19	31	26	73	28	79
28	29	34	79	24	43	26	61	24	67	21	59	29	47	29	31
29	67	35	151	26	113	30	83	25	59	22	97	30	71	31	137
30	53	39	107	28	37	31	41	30	131	26	47	34	149	32	43
36	47	40	71	37	59	32	139	33	53	29	127	35	37	35	83
37	101	42	181	40	173	33	59	34	61	31	73	37	103	37	67
38	163	45	59	41	73	41	113	35	97	32	89	41	179	38	167

Table I. Continued

R = 1		R = 7		R = 11		R = 13		R = 17		R = 19		R = 23		R = 29	
n	P	n	P	n	P	n	P	n	P	n	P	n	P	n	P
42	97	46	73	46	107	42	67	38	89	36	157	43	101	39	109
45	193	47	109	51	67	45	47	41	43	38	61	44	79	44	71
47	83	50	137	55	151	46	199	44	191	39	41	46	61	45	197
48	131	51	53	56	89	50	89	46	127	40	53	56	131	48	113
50	79	57	101	58	103	54	71	52	83	44	103	59	163	50	139
55	127	60	139	59	137	61	97	57	157	54	149	61	109	54	97
57	59	61	167	66	181	63	173	60	79	60	107	64	67	55	73
64	113	65	103	72	167	65	151	64	149	63	83	67	107	58	61
68	157	68	89	74	97	77	101	70	73	65	179	70	193	63	101
69	109	72	197	76	79	78	181	71	113	68	71	78	139	71	127
70	191	74	131	85	197	80	127	77	179	70	163	82	191	72	199
82	107	80	83	92	163	84	149	83	109	77	137	86	137	74	173
86	89	86	199	94	149	99	157	85	151	83	193	93	97	78	103
88	139	103	163	97	127	100	131	99	103	86	113	105	167	82	131
98	173	108	191	105	109	101	179	102	181	94	167	112	199	88	157
105	137	109	113	109	193	103	107	106	139	95	151	115	151	120	191
126	199	114	149	113	179	132	137	109	173	97	101	122	127	124	163
128	167	122	193	120	157	146	191	152	199	111	197	124	197	145	151
144	149	137	179	134	139	161	167	157	163	114	181	138	181	147	193
151	197	167	173	192	199	190	197	186	193	126	131	151	157	174	181
173	179									132	173				
										184	191				

ACKNOWLEDGMENT

The author wishes to thank Mrs. Elinor Potter, Mrs. Martha Lawrence, and Mrs. R. Kingsbury for assistance with the literature search, and Mrs. Potter for help with Table I.

FOOTNOTES AND REFERENCES

*This work was done under the auspices of the U. S. Atomic Energy Commission.

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