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ANALYSIS OF A PROPOSED ANALYTICITY TEST

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Graham H. Campbell
(Ph. D. Thesis)

August 10, 1965

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ABSTRACT

An analyticity test proposed by Regge and Viano* has been analyzed under conditions similar to possible applications of the test to experimental data. It has been shown that any results of such an application must be inconclusive. The class of functions to which this procedure may be applied has been determined. This class may be broadened if partial knowledge of the asymptotic behavior of the functions is available.

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* T. Regge, G. A. Viano; Nuovo Cimento 25, 709 (1962).
E. Albino, M. Bertero, G. A. Viano; Nuovo Cimento 32, 1269 (1964).

I. INTRODUCTION

In a series of two papers,[1] the following problem has been solved and the solution used as an analyticity test for proton-proton scattering amplitudes in the complex angular momentum plane. Given the knowledge of the values of a function $f(z)$ on the set of points $z = N + \lambda$ $(N = 0, 1, 2, \cdots; \lambda > 0)$ reconstruct the function in the right half plane if $f(z)$ is subject to the conditions:

1) $f(z)$ is analytic for $\mathrm{Re}\, z \geqslant 0$
2) $|f(z)|$ has certain bounds as $|z| \to \infty$ for $\mathrm{Re}\, z \geqslant 0$.

We shall comment on some aspects of this analysis and investigate the possibility of obtaining definite conclusions about experimental amplitudes by this reconstruction.

II. THE RECONSTRUCTION PROBLEM

The problem is solved in I by use of the Pollaczek polynomials, $P_n^{\lambda}(z)$, which are orthogonal under the weight function[2]

$$w_\lambda(z) = \frac{2^{2\lambda-1}}{\pi\Gamma(2\lambda)} \Gamma(\lambda + i z)\Gamma(\lambda - i z) . \tag{1}$$

We note that for $\lambda = \frac{1}{2}$, $w_\lambda(z)$ takes the convenient form

$$w_{\frac{1}{2}}(z) = \operatorname{sech} \pi z \tag{2}$$

Cauchy's theorem is used to write

$$f(z)\, w_\lambda(-i z) = \frac{1}{2\pi i} \int_C \frac{f(z')w_\lambda(-i z')}{z' - z} dz' \tag{3}$$

where C is shown in Fig. 1 and the point z is interior to C . Next we are instructed to evaluate this integral by summing over the residues at the poles of $w_\lambda(-i z')$ (which are on the real axis) and discarding the circular arcs at infinity. The first comment that we wish to make upon this analysis is that this step is valid only if we place stronger bounds on the asymptotic behavior than were explicitly stated in I . In particular we must at least require

$$\left| \frac{f(z)}{z} \right| \to 0$$

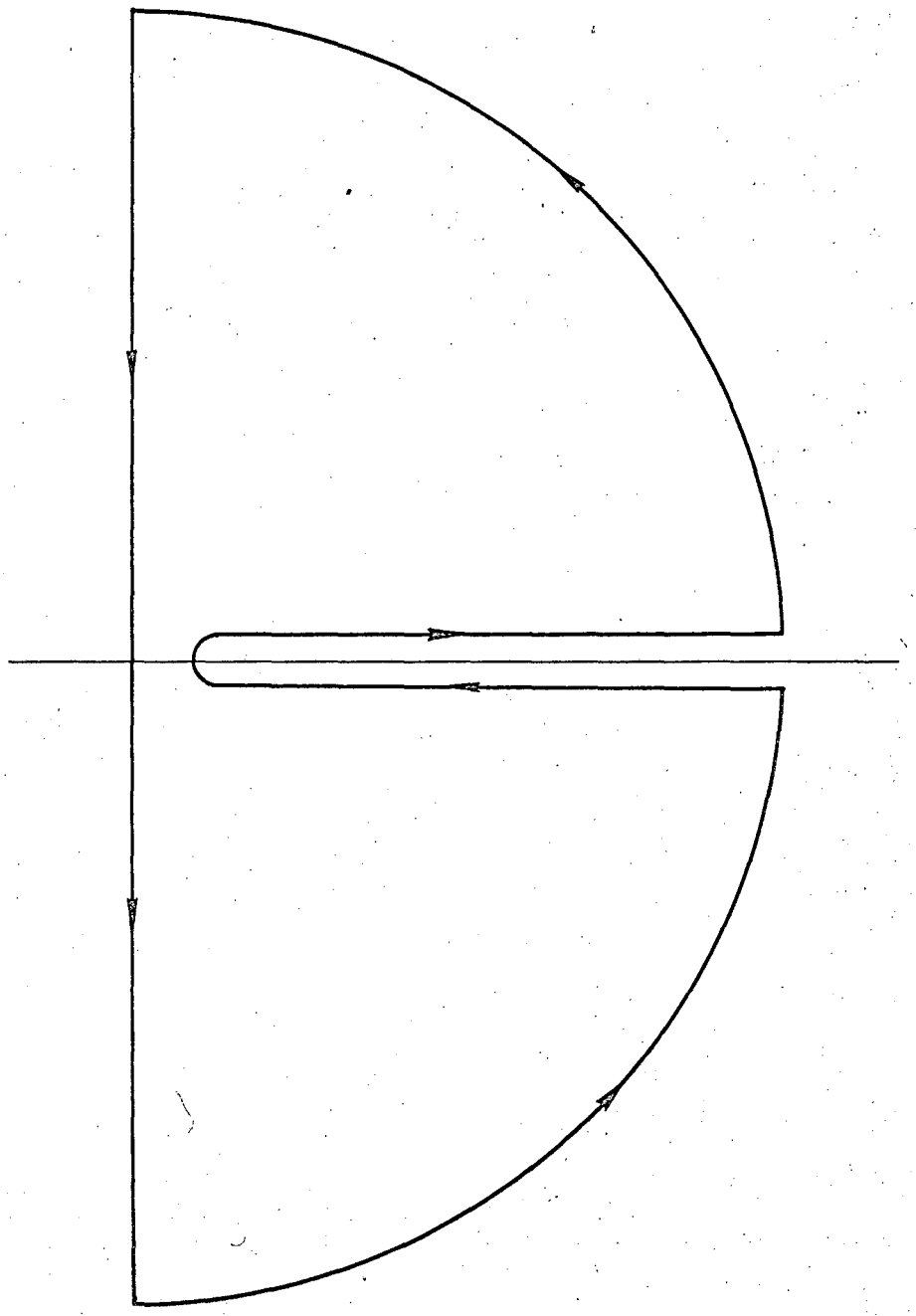

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Fig. 1. Contour of integration for equation 3.

as $|z| \to \infty$ in an angular region containing the real axis. If we now assume an asymptotic behavior that validates this step and write $F(y) = f(iy)$ we have

$$f(z)\, w_\lambda(-i\, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dy \; \frac{F(y)\, w(y)}{z - i\, y} - \sum_{n=0}^{\infty} (-1)^n \frac{2^{2\lambda-1}}{\pi\Gamma(2\lambda)} \frac{\Gamma(n+2\lambda)}{\Gamma(n+1)} \frac{f(n+\lambda)}{z - (n+\lambda)} \; . \tag{4}$$

To evaluate the remaining integral we express both $F(y)$ and $(z - i\, y)^{-1}$ as series of Pollaczek polynomials. It was pointed out in I that the asymptotic conditions insure the convergence of the above integral and also insure the existence of the expansion of $F(y)$. The expansion of $(z - i\, y)^{-1}$ was carefully considered in II . Thus if we write

$$F(y) = \sum_n a_n\, P_n^{\lambda}(y) \tag{5}$$

we have

$$f(z)\, w_\lambda(-i\, z) = \frac{1}{\pi} \sum_{n=0}^{\infty} a_n\, Q_n^{\lambda}(-i\, z) - \sum_{n=0}^{\infty} (-1)^n \frac{2^{2\lambda-1}}{\pi\,\Gamma(2\lambda)} \frac{\Gamma(n+2\lambda)}{\Gamma(n+1)} \frac{f(n+\lambda)}{z-(n+\lambda)} \tag{6}$$

where $Q_n^{\lambda}(z)$ is the associated Pollaczek function introduced in I . The a_n are given by

$$a_n = \frac{\Gamma(n+1)\,\Gamma(2\lambda)}{\Gamma(n+2\lambda)} \int_{-\infty}^{\infty} dy\; w(y)\, F(y)\, P_n^{\lambda}(y) \quad . \tag{7}$$

The next step is to evaluate this integral in terms of values of $f(n+\lambda)$ by reintroducing the variable $z = i\,y$ and closing the contour in the right half plane. To insure that there is no contribution from the circular arcs at infinity we must again impose more stringent asymptotic bounds than are mentioned in I and II . We must now require that

$$|f(z)\, P_n^{\lambda}(-i\,z)| \to 0$$

as $|z| \to \infty$ in an angular region containing the real axis. In order for this to hold for all n we must have $f(z)$ decreasing exponentially in this region. With appropriate assumptions we now have

$$a_n = 2^{2\lambda} \frac{\Gamma(n+1)}{\Gamma(n+2\lambda)} \sum_{m=0}^{\infty} (-1)^m \frac{\Gamma(m+2\lambda)}{\Gamma(m+1)} P_n^{\lambda}\Big(-i(m+\lambda)\Big) f(m+\lambda) \tag{8}$$

and the problem is solved. The following assumptions are sufficient[3] to insure the validity of the above procedure

1) $|f(z)|$ decreases exponentially for $z \to \infty$ on the positive real axis.
2) $|f(z)|$ increases less rapidly than $\exp\Big(\pi|y|\Big)$ as $y \to \pm\infty$ where $z = i\,y$.

The uniqueness of any reconstruction procedure such as the above is insured for a very large class of functions by Carlson's theorem which is given by Boas[4] as:

If $f(z)$ is regular and of exponential type in the half plane $\mathrm{Re}\, z \geqslant 0$ and $h(\frac{\pi}{2}) + h(-\frac{\pi}{2}) < 2\pi$ then $f(z) = 0$ if $f(n) = 0$ $n = 0, 1, 2, \cdots$ where

$$h(\theta) = \limsup_{r \to \infty} r^{-\rho} \log |f(r\, e^{i\theta})| \tag{9}$$

and $r = |z|$, ρ is the order of the function,[5] and the limit is taken for $\mathrm{Re}\, z \geqslant 0$.

Roughly speaking, this asymptotic condition on $f(z)$ excludes only those functions which contain asymptotically a <u>mixture</u> of $\exp(\pm i a z)$ that can be combined to form $\sin Bz$ with $B \geqslant \pi$. This allows a far wider class of functions than the conditions assumed in I allow. Indeed the conditions assumed in I and II do not allow functions of the form $\exp(-a z)$ which are used there to test the reconstruction procedure. We note that assumptions 1) and 2) above imply that the functions for which the reconstruction procedure is valid are insured of a unique reconstruction by Carlson's theorem. If information is available on the asymptotic behavior of $f(z)$ then assumption 1) may be relaxed. We simply apply the reconstruction procedure to the function $g(z) = \exp(-a z) f(z)$ with <u>a</u> chosen sufficiently large to insure the validity of 1) for $g(z)$. This will not affect the uniqueness

of the reconstruction since (even with a complex) it will add nothing to the sum of $h(\frac{\pi}{2})$ and $h(-\frac{\pi}{2})$. Hence this procedure can be extended to apply to any function of exponential type which is subject to assumption 2) (which may be stated as $h(\pm\frac{\pi}{2}) < \pi$). This procedure breaks down for the remainder of the functions which have unique reconstructions, since these functions cannot be expanded as a series of Pollaczek polynomials.

III. THE COMPUTATIONAL PROCEDURE

The previous applications of this technique have centered about the following procedure: If we use as input the function $[z - (k+\lambda)]\, f(z) = h_k(z)$ (which vanishes for $z = k+\lambda$) and evaluate the derivative of the reconstructed function for $z = k+\lambda$ we obtain $f(k+\lambda)$ without including it in the input. If this process is repeated for all k we obtain an infinite set of simultaneous linear equations which the values $f(n+\lambda)$ must satisfy. For numerical work we must obviously truncate the infinite sums appearing here. Let us denote by M the index of the last coefficient a_m which we calculate and let us denote by N the index of the last input value of $f(n+\lambda)$. It was shown in II that the value of λ is irrelevant for the analysis in the complex angular momentum plane, and that if we fix λ to be $\frac{1}{2}$ we obtain

$$f'_k = \sum_{n=0}^{n} \alpha_{kn}^{M} f_n \tag{10}$$

where $f_n = f(n + \frac{1}{2})$, and

$$\alpha_{kn}^{M} = (M+1)(-1)^{n+k+1}(1-\delta_{nk}) \left\{ (M,n)\ \langle M+1,k\rangle + \langle M,k\rangle\ (M+1,n) \right\}$$

$$(M,n) = i^{M} P_{M}^{\frac{1}{2}} \left[-1\ (n+\tfrac{1}{2}) \right] \tag{11}$$

$$\langle M,n\rangle = (-1)^{M}\ Q_{M}^{\frac{1}{2}} \left[-1\ (n+\tfrac{1}{2}) \right] \quad .$$

These equations are used by computing f_k' from equation (10) and then adjusting M to yield the best agreement with f_k. To this end an error function, $\delta\ f_n$, is defined by

$$\delta\ f_n = \frac{f_n' - f_n}{f_n} \quad . \tag{12}$$

This error is then minimized in some sense (e.g. the root mean square of the $\delta\ f_n$ is minimized). Although the analytic procedure is linear in $f(z)$, the computational procedure is not. That is, if $f_n = f_{1n} + f_{2n}$ then the computational procedure would not yield $f(z) = f_1(z) + f_2(z)$ since the optimal value of M may be different for $f(z)$, $f_1(z)$, and $f_2(z)$ individually.

IV. APPLICATIONS

Any application of this computational procedure must be based on the assumption that the restrictions 1) and 2) on $f(z)$ are necessary and sufficient conditions for the values f_n to satisfy Eq. (10). We will show by counter-example that this is not true.

Since the computational procedure is non-linear in $f(z)$, one has the option, in analyzing experimental data, of augmenting the input by a theoretical asymptotic behavior (e.g. use OPEC for higher partial waves). If this is not done then one must use a severely restricted set of input points since J - parity[6] implies that only even or odd partial waves belong to a given function. The computational procedure is very unreliable for small numbers of input points (e.g. if $f_n = 2^{-n}$ and we use 3 input points, the RMS error is 67.5%) hence, no information can be extracted in this case. If we do include a theoretical asymptotic behavior in the input (which in general will satisfy 1) and 2)) we are then faced with detecting a breakdown of assumptious 1) and 2) in a function which is small compared to the input (and unknown) over most of the range of the input.

In order to have an example for comparison, we first use the input function $f_n = 2^{-n}$. The RMS error as a function of the number of input points is displayed in Table I. Examples of the individual errors involved are in II where we note that most of the RMS error comes from the first two and last several points.

TABLE I

Errors in the application of Eq. (10) to the function $f_n = 2^{-n}$ as a function of the number of input points.

N	M	RMS error (%)
1	0	21.2
2	0	67.5
3	0	5.9
4	1	54.9
5	0	27.0
6	1	33.3
7	1	14.0
8	1	27.1
9	1	4.5
10	2	22.6
11	1	13.4
12	2	17.2
13	2	10.9
14	2	17.2
15	2	5.4
16	3	13.8
17	2	5.8
18	3	12.5
19	3	9.2

Let us now consider the possibility of detecting[7] a breakdown of assumption 2). It was noted in II that this method breaks down completely when applied to a Gaussian type function (for $f_n = 3^{-n^2}$, using 8 input points gives an RMS error > 10^{19}%). If we now "hide" such a function under a normal asymptotic form by using

$$f_n = 2^{-n} + 3^{-n^2} \tag{13}$$

we obtain the results in Table II. These errors are comparable to those in Table I and hence we are unable to detect such a function. If we concede the impossibility of checking asymptotic behavior in this manner, we are free to modify the input points, as was recommended in I and done in II, by multiplication by C^{-n}. This generally is expected to allow higher values of M and hence better convergence.

The effect of a breakdown of assumption 1) was investigated in II both analytically and numerically. For a pole in the right half plane it was demonstrated there that δf_n for n corresponding to the points to the left of the pole was excessive (analytically the series diverges for these points). We wish to examine functions in which the pole is masked by an additive term and also to allow for the modification procedure mentioned above, thus we consider the two parameter family of functions

$$f_n = e^{-an} + \frac{e^{-bn}}{1 + (n-2)^2} \tag{14}$$

TABLE II

Errors in the application of Eq. (10) to the function of Eq. (13).

K	N=6 M=1 %	N=8 M=2 %	N=10 M=2 %	N=12 M=3 %	N=14 M=4 %	N=16 M=4
0	-10.0	-5.43	18.9	12.1	-52.2	23.2
1	4.89	2.79	-7.26	-2.81	14.1	- 4.75
2	-5.64	-3.14	6.82	1.60	-9.44	2.48
3	5.71	3.01	-5.78	- .828	5.86	-1.21
4	-6.52	-3.20	5.65	.493	-4.32	.723
5	8.30	3.78	-6.28	-.325	3.72	-.507
6	-11.4	-4.81	7.63	.219	-3.57	.402
7		6.50	-9.92	-.133	3.76	-.352
8		-9.17	13.6	.041	-4.24	.331
9			19.4	.080	5.06	-.331
10			28.6	-.261	-6.32	.347
11				.547	8.22	-.378
12				-1.01	-11.0	.423
13					15.3	-.485
14					-21.6	.563
15						-.659
16						.767
RMS	7.84	5.06	13.8	3.51	16.5	5.81

which have a pole at the same place as those functions considered in II. In order to have an appropriate comparison we have also considered the family of functions

$$f_n = \frac{6}{5}\left[e^{-an} + \frac{1}{6}\left(\frac{5}{2} - e^{(b-a)}\right) n e^{-bn}\right] \tag{15}$$

which are entire and agree with (14) for $n = 0, 1$. The RMS average of the δf_n using 8 input points for Eq. (14) is recorded in Table III and for Eq. (15) in Table IV. There is nothing in these two tables to distinguish these two cases. With prior knowledge of the existence of a pole, one can see its effects in the δf_n although there are very few cases in which the pole can be identified solely by comparison with the δf_n for Eq. (15). A typical example of this is shown in Table V. We also note that some of the cases in Table IV with large errors appear to have a pole. That this general behavior persists when more input points are used is demonstrated in Table VI which is similar to Table III except 18 points were used in the input.

Thus we can state that, under conditions similar to the analysis of experimental data, poles will generally not be detected and functions without poles can imitate the behavior of functions with poles.

TABLE III. RMS error in % for the functions of equation 14 with 8 input points.

b \ a	0.5000	0.6000	0.7000	0.8000	0.9000	1.0000	1.1000	1.2000	1.3000	1.4000	1.5000
2.5000	18.13	19.26	7.61	9.05	11.11	1.91	9.65	1.01	4.74	6.53	5.33
2.4000	17.43	19.48	7.91	8.49	11.09	1.86	9.13	2.26	3.33	4.61	4.02
2.3000	16.74	19.64	8.15	8.05	10.91	1.85	8.30	3.56	1.54	2.17	1.11
2.2000	16.07	19.74	8.28	7.79	10.52	2.05	7.10	5.58	1.76	1.59	3.41
2.1000	15.45	19.74	8.28	7.75	9.87	2.78	5.46	8.16	5.04	5.49	5.68
2.0000	14.89	19.63	8.11	8.01	8.89	4.24	3.33	5.41	5.73	4.20	1.09
1.9000	14.43	19.37	7.71	8.67	7.51	6.52	1.29	2.16	1.78	1.76	6.42
1.8000	14.11	18.92	7.03	9.84	5.68	9.71	3.74	2.88	4.16	7.71	9.64
1.7000	13.96	18.23	6.03	11.66	3.37	7.75	8.24	7.94	10.27	9.11	1.43
1.6000	14.05	17.25	4.66	14.28	1.62	4.08	6.36	7.36	5.80	1.46	9.03
1.5000	14.43	15.92	3.05	11.59	4.53	1.89	1.80	1.74	3.29	9.31	19.04
1.4000	15.20	14.18	2.50	8.27	9.34	6.71	6.67	8.28	12.26	19.20	20.84
1.3000	16.46	11.95	4.92	4.37	10.65	13.18	14.28	17.16	22.00	25.37	17.36
1.2000	18.34	9.17	9.04	2.16	4.68	10.93	17.59	22.05	23.07	22.16	24.10
1.1000	20.96	5.83	14.36	7.12	3.86	3.08	8.35	13.89	19.56	29.45	31.50
1.0000	22.78	2.46	12.08	13.76	11.41	7.59	2.38	9.53	26.04	32.75	2.09
0.9000	18.86	4.50	5.92	21.38	19.99	16.22	6.74	13.80	38.76	4.63	31.95
0.8000	14.25	10.11	3.20	13.29	28.72	23.22	5.73	36.88	15.00	39.68	41.77
0.7000	9.01	16.75	10.01	5.90	36.53	25.53	11.30	30.97	48.41	24.33	12.93
0.6000	3.60	24.07	17.67	3.22	41.47	17.43	47.96	17.70	3.66	11.99	16.65
0.5000	4.54	20.08	24.42	3.35	39.98	13.68	16.73	30.98	38.09	41.87	44.11

TABLE IV. RMS error in % for the functions of equation 15 with 8 input points.

b \ a	0.5000	0.6000	0.7000	0.8000	0.9000	1.0000	1.1000	1.2000	1.3000	1.4000	1.5000
2.5000	22.06	15.52	3.52	15.27	7.72	4.51	8.80	1.65	7.98	1.14	6.27
2.4000	21.27	16.41	4.22	13.27	8.92	2.92	10.07	1.98	9.35	2.04	7.80
2.3000	19.81	17.33	5.17	11.24	10.14	2.04	11.30	3.55	8.87	3.92	7.56
2.2000	18.29	18.27	6.27	9.21	11.34	2.66	12.48	5.24	5.86	5.69	4.50
2.1000	16.72	19.23	7.46	7.21	12.52	4.09	9.66	6.85	3.17	7.23	1.94
2.0000	15.12	20.18	8.69	5.32	13.65	5.66	6.80	8.32	1.22	8.50	0.87
1.9000	13.50	21.13	9.93	3.73	14.72	7.19	4.23	9.58	2.01	9.47	2.33
1.8000	11.89	22.05	11.16	2.89	15.69	8.61	2.18	10.62	3.62	10.16	3.77
1.7000	10.33	22.93	12.34	3.26	16.18	9.86	1.76	11.42	5.00	7.54	5.21
1.6000	8.86	23.76	13.45	4.42	13.11	10.91	3.01	12.00	6.20	4.05	7.03
1.5000	7.56	24.51	14.47	5.77	10.28	11.75	4.42	9.91	7.44	0.99	9.53
1.4000	6.52	25.17	15.36	7.08	7.69	12.41	5.79	5.82	9.12	4.58	1.39
1.3000	5.86	25.74	16.12	8.26	5.29	13.01	7.32	1.70	11.62	9.09	6.54
1.2000	5.66	23.08	16.74	9.33	3.16	13.87	9.37	4.52	2.08	7.06	12.45
1.1000	5.90	20.09	17.30	10.41	2.75	12.07	12.40	9.61	6.99	4.51	2.58
1.0000	6.51	16.82	18.00	11.82	5.59	3.71	11.20	15.90	14.53	13.16	11.72
0.9000	7.42	13.02	19.40	14.10	9.90	6.13	3.31	3.74	6.62	9.93	13.31
0.8000	8.76	8.60	22.74	18.35	15.66	13.62	11.80	10.04	8.29	6.62	5.14
0.7000	10.88	5.55	11.87	21.35	25.57	24.60	23.82	23.04	22.17	21.22	20.18
0.6000	14.43	9.16	5.98	6.21	8.74	11.83	15.00	18.14	21.22	24.23	27.16
0.5000	20.84	17.39	15.05	13.21	11.58	10.05	8.63	7.38	6.43	5.93	5.98

TABLE V

δf_n for $a = .7$, $b = 2.0$ for the functions of Eqs. (14) and (15).

	Pole (M = 2)	No Pole (M = 2)
K	δf_n (%)	δf_n (%)
0	.83	2.39
1	-1.94	-2.89
2	2.55	3.21
3	-3.54	-3.98
4	4.96	5.38
5	-7.26	-7.77
6	11.06	11.73
7	-17.40	-18.36

TABLE VI. RMS error in % for the functions of equation 14 with 18 input points.

b \ a	0.5000	0.6000	0.7000	0.8000	0.9000	1.0000	1.1000	1.2000	1.3000	1.4000	1.5000
2.5000	13.94	1.75	5.18	4.85	2.65	1.88	4.55	2.99	1.31	2.15	1.43
2.4000	13.93	1.62	4.39	3.49	0.86	5.05	2.42	6.47	3.84	2.85	4.38
2.3000	13.82	1.21	3.17	1.69	2.46	6.33	1.72	6.00	8.03	7.63	10.10
2.2000	13.59	0.87	1.48	1.26	5.94	3.97	4.71	2.46	10.30	13.30	15.54
2.1000	13.21	2.01	1.34	4.34	7.65	1.84	8.82	3.28	4.87	10.10	8.35
2.0000	12.65	4.32	4.43	8.47	5.05	3.68	10.54	8.33	2.30	3.22	1.37
1.9000	11.86	7.66	9.60	7.87	2.44	7.80	5.56	14.34	8.74	4.93	7.41
1.8000	10.80	10.75	9.13	5.07	3.39	12.83	2.13	21.10	16.16	12.67	15.05
1.7000	9.41	8.86	6.57	2.47	7.61	16.73	7.52	15.00	24.00	20.43	21.97
1.6000	7.63	6.56	3.74	4.02	12.87	10.42	14.56	7.35	31.92	27.60	27.33
1.5000	5.43	3.92	2.54	8.56	18.97	3.91	22.18	2.05	27.17	33.43	29.91
1.4000	2.84	2.16	5.96	14.13	22.94	5.02	30.09	8.21	22.03	36.78	27.46
1.3000	1.92	4.91	10.98	20.51	15.62	12.44	37.41	14.24	19.87	35.58	13.47
1.2000	5.51	9.57	16.90	27.65	8.17	20.07	37.20	18.05	23.61	24.00	33.23
1.1000	10.38	15.16	23.62	25.99	2.81	27.13	33.93	17.43	43.60	23.79	18.82
1.0000	16.20	21.61	31.07	18.42	7.70	32.78	36.15	5.58	9.13	40.05	27.83
0.9000	23.00	28.88	29.62	11.33	13.26	35.31	49.24	45.41	5.36	14.66	16.37
0.8000	28.69	30.49	22.28	5.80	15.79	29.40	20.68	48.37	31.51	28.65	28.14
0.7000	21.99	23.14	15.88	3.98	8.79	10.31	32.34	39.86	41.29	41.65	41.86
0.6000	15.05	16.36	12.34	10.40	41.93	15.40	25.10	26.87	27.27	27.48	27.68
0.5000	8.51	11.87	19.36	58.72	4.61	13.97	16.01	16.46	16.67	16.88	17.14

V. CONCLUSIONS

We have demonstrated by working with known functions that the computational procedure outlined in I and II cannot be successfully applied to determine either the presence of poles or asymptotic bounds of an input function. In addition we have determined the class of functions to which this procedure can theoretically be applied and indicated how to broaden this class if information on asymptotic behavior is available.

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FOOTNOTES AND REFERENCES

1. T. Regge, G. A. Viano : Nuovo Cimento 25, 709 (1962), hereafter referred to as I .
2. Higher Transcendental Functions (Bateman Manuscript Project) II McGraw Hill, New York (1953) 218.
3. R. P. Boas: Entire Functions (New York, 1954) p. 66, Theorem 5.1.2.
4. Reference 3, p. 153.

5. The order ρ of a function regular in an angular sector is defined by

   1) Let $M(r)$ be the maximum modulus of $f(z)$ for $|z| \leq r$ and z in the sector.

   2) Then

      a) if $\limsup_{r \to \infty} M(r) < \infty$; $\rho = 0$

      b) otherwise

$$\rho = \limsup_{r \to \infty} \frac{\log \log M(r)}{\log r}$$

   A function is of exponential type if $\rho < 1$ or if $\rho = 1$ and

$$\limsup_{r \to \infty} \frac{\log M(r)}{r} < \infty$$

6. S. C. Frautschi, M. Gell-Mann, and F. Zachariasen, Phys. Rev. 126, 2204 (1962).

7. In all calculations we have considered the "small" function as being exactly known over the entire input range.

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