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NONFORWARD DISPERSION RELATIONS
AND THE ALGEBRA OF CURRENT COMMUTATORS

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March 8, 1966

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ABSTRACT

A general relation between the integral over energy of the discontinuity of a scattering amplitude and a form factor, obtained from the algebra of current commutators, is discussed. The relation is analyzed as a function of (momentum transfer)^2 t and of the external masses. It is shown how the high energy part of the discontinuity generates the singularities in t of the form factor and also how, when the external masses are those of strongly interacting particles, the relation reduces to a purely strong-interaction condition. The case of isotopic spin currents is discussed in detail and illustrations involving spin zero and spin one-half particles are given.

I. INTRODUCTION

The suggestion by Gell-Mann[1] that broken symmetry problems may be studied by examining the equal time commutators of the currents whose algebra defines the underlying symmetry has recently proven very successful. If we assume there are no additional terms, such as gradients of delta functions,[2] which contribute to relations between physical matrix elements of current commutators, these commutators are of the form

$$\langle p_1 \,|\, [J_0^{\,i}(\vec{x},0),\, J_\nu^{\,j}(0)] \,|\, p_2 \rangle = i\, c^{ijk}\, \delta(\vec{x}) \langle p_1 \,|\, J_\nu^{\,k}(0) \,|\, p_2 \rangle, \tag{1}$$

where i, j, and k are indices labeling the internal symmetry transformation properties of the currents and c^{ijk} is a structure constant of the underlying group. By the use of dispersive techniques, one of us has recently shown[3] that Eq. (1) yields a relation of the form*

$$\int_0^\infty a^{ij}(\nu',\, q_1^{\,2},\, q_2^{\,2},\, t)\, d\nu' = c^{ijk}\, G^k(t), \tag{2}$$

where $G^k(t)$ is the form factor (assuming for the moment that there is only one) of the current $J_\nu^{\,k}$ between states of momentum p_1 and p_2, $(p_1 - p_2)^2 = t$; a^{ij} is a scalar amplitude obtained from the expansion into invariants of $t_{\mu\nu}^{\,ij}$,

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* This relation has been derived independently by R. Dashen and M. Gell-Mann[4] by considering matrix elements of Eq. (1) in the $|\vec{p}_1| \to \infty$ frame.

$$\int \langle p_1 | \left[J_\mu{}^i(x),\ J_\nu{}^j(0) \right] | p_2 \rangle\, e^{iq_1 \cdot x}\, d^4x = s_{\mu\nu}^{ij} + t_{\mu\nu}^{ij} \tag{3}$$

$$t_{\mu\nu}^{ij} = a^{ij} P_\mu P_\nu + d^{ij} P_\mu Q_\nu + e^{ij} Q_\mu Q_\nu + \cdots \tag{4a}$$

$$s_{\mu\nu}^{ij} = h^{ij} P_\mu P_\nu + \cdots \tag{4b}$$

where $P = \dfrac{p_1 + p_2}{2}$, $Q = \dfrac{q_1 + q_2}{2}$, and $\Delta = \dfrac{p_1 - p_2}{2}$. The quantities $t_{\mu\nu}^{ij}$ are odd under the interchange of i and j while $s_{\mu\nu}^{ij}$'s are even. Our sum rules will always be for the $t_{\mu\nu}^{ij}$'s ; the $s_{\mu\nu}^{ij}$'s correspond in practice to the presence of additional terms like those of Ref. 2 being present in the commutation relations.

The a^{ij} in Eq. (2) is just the coefficient of $P_\mu P_\nu$ upon expanding $t_{\mu\nu}^{ij}$. In the general case there are several such relations that may be obtained from any given set of commutation relation matrix elements.

Now $a^{ij}(\nu, q_1{}^2, q_2{}^2, t)$ can be related to the imaginary part of a transition amplitude for a particle with quantum numbers i and momentum q_1, coupled to the current $J_\mu{}^i$, colliding with a particle p_1 (we have suppressed for convenience the internal quantum numbers of this particle) to give a two-particle state, one with momentum $q_2(j)$ and the other with momentum p_2. Schematically one can view this as in Fig. 1.

We wish to emphasize that the figure is merely illustrative and not to be interpreted in terms of Feynman diagrams.

Several questions immediately arise upon analyzing (2). Is it valid for all values of q_1^2, q_2^2, and t, or are there restrictions on the range of these variables? This leads to the related problems of what are the values of these three variables for which the integral converges most rapidly as a function of ν? This is crucial if one wishes to approximate the contribution of intermediate states by a few low-lying resonances, as is usually done in practice. In fact the integral presumably does not converge at all in certain cases. This of course implies that the validity of the current commutation relations is intimately connected with the asymptotic behavior of scattering amplitudes, and conversely that the latter may be able to restrict the class of commutation relations with physical meaning.

Yet another question is how do the known singularities of the form factor $G(t)$ develop out of the integral of a^{ij}?

We are not able to answer any of these questions completely, but will try to give indications of lines of approach for studying them and possible partial resolutions in some cases.

The simplest kinematical configuration is that of $q_1^2 = q_2^2 = t = 0$, corresponding to the commutator of two charges. In the case of axial vector charges, this leads, by use of PCAC,[5] to the well-known Adler-Weisberger[6] relation between weak axial vector current renormalization and pion-nucleon scattering. The general case of charge commutators has been studied extensively by the present authors in collaboration with J. D. Walecka,[7] so we shall say nothing about it and turn our attention immediately to more general kinematical configurations.

II. SUM RULES FOR SPINLESS TARGETS

(a) We shall illustrate our arguments by treating the algebra of isotopic spin currents and considering the matrix elements of the commutators defining the algebra between particle states of spin zero. This will greatly simplify the discussion though none of the general features will be lost; we shall in addition restrict ourselves to the case of $q_1^2 = q_2^2 = q^2$. Consider therefore the amplitude $t_{\mu\nu}^{ij}$ as defined in Eq. (3), where the initial and final states may be taken to be pions of momentum p_1 and p_2 respectively. Neglecting electromagnetic corrections, the isotopic spin current is conserved, so $t_{\mu\nu}^{ij}$ obeys the subsidiary conditions

$$q_{1\mu}\, t_{\mu\nu}^{ij} = q_{2\nu}\, t_{\mu\nu}^{ij} = 0 . \tag{5}$$

Before discussing the physical consequences of the current commutation relations, we shall find it helpful to give a short discussion on invariants. To begin with, let us introduce polarization vectors $\epsilon_{1\mu}$, $\epsilon_{2\nu}$ and use them to construct the scalar amplitude

$$A^{ij} = \epsilon_{1\mu}\, \epsilon_{2\nu}\, t_{\mu\nu}^{ij}\, (4E_1E_2)^{1/2} . \tag{6}$$

A^{ij} can now be decomposed into invariant amplitudes

$$A^{ij} = I_1 A_1^{ij} + I_2 A_2^{ij} + I_3 A_3^{ij} + I_4 A_4^{ij}, \tag{7}$$

the I's being chosen as follows:

$$I_1 = (q_1\cdot q_2)(\epsilon_1\cdot P)(\epsilon_2\cdot P) - P\cdot Q\left[(\epsilon_1\cdot q_2)(\epsilon_2\cdot P) + (\epsilon_2\cdot q_1)(\epsilon_1\cdot P)\right] + (\epsilon_1\cdot\epsilon_2)(Q\cdot P)^2, \tag{8a}$$

$$I_2 = (\epsilon_1\cdot\epsilon_2)(q_1\cdot q_2) - (\epsilon_1\cdot q_2)(\epsilon_2\cdot q_1), \tag{8b}$$

$$I_3 = (\epsilon_1\cdot\epsilon_2)q^4 - q^2\left[(\epsilon_1\cdot q_2)(\epsilon_2\cdot q_1) + (\epsilon_1\cdot q_2)(\epsilon_2\cdot q_2)\right] + (q_1\cdot q_2)(\epsilon_1\cdot q_1)(\epsilon_2\cdot q_2), \tag{8c}$$

$$I_4 = q^4(\epsilon_1\cdot P)(\epsilon_2\cdot P) - (P\cdot Q)q^2\left[(\epsilon_1\cdot q_1)(\epsilon_2\cdot P) + (\epsilon_2\cdot q_2)(\epsilon_1\cdot P)\right] + (P\cdot Q)^2(\epsilon_1\cdot q_1)(\epsilon_2\cdot q_2). \tag{8d}$$

It is particularly convenient, however, to consider the "transverse system" in which $\epsilon_1\cdot P = \epsilon_2\cdot P = 0$. In it the amplitude A^{ij} is expressed by:

$$\begin{aligned} A^{ij} &= \epsilon_{1\mu}\,\epsilon_{2\nu}\,t_{\mu\nu}^{ij}\,(4E_1E_2)^{1/2} \\ &= f_1^{ij}\,\epsilon_1\cdot\epsilon_2 + f_2^{ij}(\epsilon_1\cdot q_2)(\epsilon_2\cdot q_1) + f_3^{ij}(\epsilon_1\cdot q_1)(\epsilon_2\cdot q_2) \\ &\quad + f_4^{ij}\left[(\epsilon_1\cdot q_1)(\epsilon_2\cdot q_1) + (\epsilon_1\cdot q_2)(\epsilon_2\cdot q_2)\right]. \end{aligned} \tag{9}$$

The relation between the A's and the f's is

$$A_1^{ij}\nu^2 + A_2^{ij}(q_1 \cdot q_2) + A_3^{ij}\, q^4 = f_1^{ij}, \tag{10a}$$

$$- A_2^{ij} = f_2^{ij}, \tag{10b}$$

$$(q_1 \cdot q_2)A_3^{ij} + \nu^2 A_4^{ij} = f_3^{ij}, \tag{10c}$$

$$- q^2 A_3^{ij} = f_4^{ij}, \tag{10d}$$

and inverting, to express f's as functions of the A's ,

$$A_1^{ij} = \frac{1}{\nu^2}\left[f_1^{ij} + q_1 \cdot q_2 f_2^{ij} + q^2 f_4^{ij}\right], \tag{11a}$$

$$A_2^{ij} = - f_2^{ij}, \tag{11b}$$

$$A_3^{ij} = - f_4^{ij}/q^2, \tag{11c}$$

$$A_4^{ij} = \frac{1}{\nu^2}\left[f_3^{ij} + \frac{q_1 \cdot q_2}{q^2} f_4^{ij}\right]. \tag{11d}$$

This completes our discussion of invariants. Let us now see how the assumed current commutation relations lead to a set of relations connecting form factors to integrals over imaginary parts of scattering amplitudes. To begin with let us use translational invariance to rewrite Eq. (3) as

$$t_{\mu\nu}^{ij}(\nu, q_1^2, q_2^2, t) = \int \langle p_1 \left| \left[J_\mu^i(x/2), J_\nu^i(-x/2)\right] \right| p_2\rangle e^{iQ\cdot x}\, d^4x, \tag{12}$$

and then observe that we can define

$$T_{\mu\nu}^{ij}(\nu, p_1^2, p_2^2, t) = i\int \langle p_1| \left[J_\mu^i(x/2), J_\nu^i(-x/2)\right] |p_2\rangle e^{iQ\cdot x}\, \theta(x_0) d^4x, \tag{13}$$

related to $t_{\mu\nu}^{ij}$ by

$$T_{\mu\nu}^{ij} = \mathcal{H}\, t_{\mu\nu}^{ij}, \tag{14}$$

where $\mathcal{H}$ symbolizes the process of taking the Hilbert transform with respect to the variable ν. Equation (14) means, as stressed in Ref. 3, that, if we develop $T_{\mu\nu}^{ij}$ and $t_{\mu\nu}^{ij}$ in the same set of invariants, the components of $T_{\mu\nu}^{ij}$ are Hilbert transforms of the components of $t_{\mu\nu}^{ij}$. Because of the $\theta(x_0)$ in Eq. (13), operating on $t_{\mu\nu}$ with Q_μ and $\mathcal{H}$ are noncommutative processes,

$$(Q_\mu\mathcal{H} - \mathcal{H}Q_\mu)t_{\mu\nu}^{ij} = \int \langle p_1| \left[J_\mu^i(x/2), J_\nu^j(-x/2)\right] |p_2\rangle \delta(x_0) e^{iQ\cdot x}\, d^4x$$

$$= i\, c^{ijk} \langle p_1|\ J_\nu^k(0)\ |\ p_2\rangle, \tag{15}$$

having used the current commutation relations, in the form of Eq. (1), for the last step. Equation (15), the fundamental identity we shall use, is true even if the currents are not conserved.

When p_1 and p_2 are pion states,

$$\langle p_1\ |\ J_\nu^k(0)\ |\ p_2\rangle = (4E_1E_2)^{-1/2}\ F_\pi^k(t)\ P_\nu\ , \tag{16}$$

there being only one form factor in the limit of conserved current. We

have not written explicitly the isotopic spin labels of the initial and final states; we are taking it as implicit in the subscript of the form factor.* It is now easy to see that only A_1 and A_4 will contribute to the sum rule arising from Eq. (15), as only I_1 and I_4 contain terms proportional to $\epsilon_1 \cdot P \; \epsilon_2 \cdot P$, or, dropping the polarization vectors, to $P_\mu P_\nu$. The necessity of the P_ν is clear, as the right-hand side of (15) is proportional to P_ν; the P_μ term needs to be present to give a nonvanishing result when applying the operator $R_\mu = Q_\mu \mathcal{H} - \mathcal{H} Q_\mu$ to the amplitude $t_{\mu\nu}^{ij}$. To illustrate this, consider applying R_μ to an arbitrary function $a_\mu(\nu) = P_\mu \, a_1(\nu) + Q_\mu \, a_2(\nu)$,

$$\mathcal{H} Q_\mu \, a_\mu(\nu) = \mathcal{H}\left(\nu \, a_1(\nu) + Q^2 \, a_2(\nu)\right) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\nu' a_1(\nu') + Q^2 a_2(\nu')}{\nu' - \nu} , \tag{17a}$$

$$Q_\mu \mathcal{H} \, a_\mu(\nu) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\nu a_1(\nu') + Q^2 \, a^2(\nu')}{\nu' - \nu} , \tag{17b}$$

$$(\mathcal{H} Q_\mu - Q_\mu \mathcal{H}) a_\mu(\nu) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(\nu' - \nu) a_1(\nu')}{\nu' - \nu} = \frac{1}{\pi} \int_{-\infty}^{\infty} a_1(\nu') , \tag{17c}$$

and we see that $R_\mu a_\mu = R \cdot P a_1$.

The commutator of two isospin currents taken between pion states, using (6), (7), (8), (15), and (16) and the fact that $A_1(-\nu) = A_1(\nu)$ and $A_4(-\nu) = A_4(\nu)$, leads to the sum rule

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* Note that $p_1^2 = p_2^2 = \mu^2$, $\Delta \cdot P = \Delta \cdot Q = 0$.

$$\frac{2}{\pi}\int_0^\infty \left[(q_1\cdot q_2)\, A_1^{ij}(\nu',\, t,\, q^2,\, q^2) + q^4\, A_4^{ij}(\nu',\, t,\, q^2,\, q^2)\right] d\nu'$$
$$= i\, c^{ijk}\, F_\pi^{\,k}(t)\ . \tag{18}$$

Let us calculate explicitly the contribution of the one-pion intermediate state. This is easily done, as the integrand of Eq. (18) is the coefficient of $\epsilon_1\cdot P\ \epsilon_2\cdot P$ in A^{ij}, as one immediately sees by referring back to Eq. (7) and (8). That this must be so can be seen, as we stated earlier, by the fact that only a term in $P_\mu P_\nu$ gives something proportional to P_ν after being operated on by $Q_\mu \mathcal{H} - \mathcal{H} Q_\mu$. The one-pion contribution is

$$F_\pi^{\,i}(q^2)\, F_\pi^{\,j}(q^2) \;-\; F_\pi^{\,j}(q^2)\, F_\pi^{\,i}(q^2)\ .$$

If we now let i and j be the indices corresponding respectively to isotopic spin raising and lowering, and let the pions in the initial and final states both be π^+'s, only $F_\pi^{\,i}\, F_\pi^{\,j}$ is different from zero, and our sum rule (18) becomes

$$F_\pi^{\,+}(q^2)\, F_\pi^{\,-}(q^2) + \frac{2}{\pi}\int_{\nu_0}^\infty d\nu' \Big[(q_1\cdot q_2)\, A_1^{+-}(\nu',t,q^2,q^2)$$
$$+ q^4\, A_4^{+-}(\nu',t,q^2,q^2)\Big] \;=\; F_\pi^{\,3}(t) \tag{19}$$

or equivalently, by (11)

$$F_\pi^{\,+}(q^2)\, F_\pi^{\,-}(q^2) + \frac{2}{\pi}\int_{\nu_0}^\infty \frac{d\nu'}{\nu'^2}\Big[(q_1\cdot q_2)\, f_1^{+-}(\nu',t,q^2,q^2)$$
$$+ (q_1\cdot q_2)^2\, f_2^{+-}(\nu'\cdots) + q^4\, f_3^{+-}(\nu'\cdots) + 2q^2(q_1\cdot q_2) f_4^{+-}(\nu'\cdots)\Big]$$
$$= F_\pi^{\,3}(t)\ , \tag{20}$$

where $F_\pi^3(q^2)$ is the pion form factor normalized to unity at $q^2 = 0$ and ν_0 is the threshold for A_1 and A_4. Remembering that $q_1 \cdot q_2 = -\frac{t}{2} + q^2$, we see that for $t = q^2 = 0$ we recover the statement of current conservation, namely, that $F^+(0) = F^3(0)$. By taking $t = 0$ and then the derivative of (20) with respect to q^2 at $q^2 = 0$, we obtain a sum rule, first derived by Cabibbo and Radicati,[8] which relates the isovector charge radius of the pion to the integral over energy of the isovector photon-pion total cross section, the latter being obtained from the integrand in (20) by use of the optical theorem. The same sum rule follows by setting $q^2 = 0$ and then taking the derivative with respect to t at $t = 0$, remembering the definition of $q_1 \cdot q_2$. It is hard to estimate the validity of this sum rule, however, because of lack of data on the form factors and cross sections. The more readily verifiable case, in which the current commutator is evaluated between nucleon states, is discussed in Section III, so we shall defer the discussion of sum rules of the above-mentioned type until Section III.

(b) Until now we have discussed only the kinematical region of the infinitesimal neighborhood of q^2 and t equal to zero. What do the commutation relations lead to for finite q^2 and t? Let us begin by analyzing the t dependence; the first and most striking fact that meets the eye is that the right-hand side of Eq. (20),

proportional to $F(t)$, has well-defined singularities in t, namely a pole at the ρ mass, a cut, etc., while the one-pion contribution on the left-hand side of Eq. (20) is independent of t. It is in fact clear that, in the neighborhood of $t = m_\rho^2$, we cannot approximately saturate the sum rule by a few low-lying resonances, as these will not generate a pole. We are then faced by two possibilities to obtain consistency: either say that the large ν contributions are negligible and that essentially we have a subtraction constant of the form $t/(t-m_\rho^2)$, or else really analyze the large ν contributions. Clearly the second of the two is more attractive; it is also reasonable to suppose that we have to consider large ν, corresponding to taking into account many partial waves in the scattering amplitude, in order to obtain a pole in t. To see how this pole could indeed be generated by the large ν behavior of the scattering amplitude, let us use a Regge pole representation;[9] the poles presumably dominate the possible Regge cuts for time-like t, so this is a reasonable approximation. Remembering that $\nu \approx s/2$ for large ν, s being the (center-of-mass energy)^2 variable, a Regge representation would say that in Eq. (19), for large ν,

$$f^{ij} \sim \nu^{\alpha(t)} b^{ij}(t,q^2), \tag{21}$$

the f's being the discontinuity in ν of scattering amplitudes.*

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* It is easy to see that the f's have asymptotic behavior $\nu^{\alpha(t)}$ so $A \sim \nu^{\alpha(t)-2}$.

If we assume that Regge behavior sets in around some value of ν, which we shall call ν_R, the contribution of the "asymptotic tail" of (19) has the form, keeping, e.g., only the first term in the integral,

$$q_1 \cdot q_2 \int_{\nu_R}^{\infty} (\nu')^{\alpha(t)-2} \; b^{ij}(t,q^2) d\nu' = \frac{q_1 \cdot q_2 \; b^{ij}(t,q^2)}{\alpha(t) - 1} (\nu_R)^{\alpha(t)-1} , \tag{22}$$

where $\alpha(t)$ is the leading Regge trajectory contributing to f_1^{+-}. In the case under consideration in Eq. (20), namely that of isotopic spin, where f^{+-} is actually a difference of two terms corresponding to the two terms in the commutator, the leading trajectory is presumably that of the ρ meson. As $\alpha_\rho(t)$ has a real part which equals unity at $t = m_\rho^2$ and an imaginary part proportional to the width of the ρ meson, we see from (22) that the "asymptotic tail" is in fact generating the desired pole in t.[10 *]

We can then draw two conclusions: the first is that assuming the commutation relations to be saturated by a few intermediate states may be a very bad approximation in certain kinematical configurations,

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* The scattering amplitude is proportional to $[\sin \pi \, \alpha_\rho(t)]^{-1} \exp[\alpha_\rho(t) \log \nu]$. The discontinuity of the logarithm introduces a factor which cancels the denominator, leading to the form (21).

the second that the high-energy behavior of the scattering amplitude provides an "asymptotic tail" in the dispersion integral which restores consistency.

The situation is much more drastic in the case of the commutator of a charge and a current,[11] e.g.,

$$\langle p_1 \mid \left[Q_5^{\,i}, J_\mu^{\,j}(0) \right] \mid p_2 \rangle = i\, c^5_{ijk} \langle p_1 \mid J_{5\mu}^{\,k} \mid p_2 \rangle , \qquad (24)$$

where Q_5 and $J_{5\mu}$ are axial vector charges and currents, because the relation corresponding to (19) and (20) (see Reference 3) then become of the form, with $q \to 0$,

$$\lambda^i F^j(t) + \frac{2}{\pi} \int_{\nu_0}^{\infty} \left[\frac{w_5^{\,ij}(\nu',t)}{\nu'} \right] d\nu' = i\, c_5^{\,ijk} F_5^{\,k}(t) , \qquad (25)$$

where λ is a constant and $w_5^{\,ij}$ can be related to the discontinuity in ν' of a scattering amplitude. When one does the integration over ν' the "asymptotic tail" contributes a term of the form $\frac{1}{\alpha(t)}\, g(t,q^2)$, so if the pion trajectory contributes, we have a singularity in t very close to the physical region.*

(c) Let us conclude this section with a discussion of the behavior of the sum rules on the variable q^2 . The most striking feature is the independence of the right-hand side of Eq. (20) on q^2 . This implies that there must necessarily be some strong cancellations

* We hope to give soon a more thorough analysis of the t dependence of current commutator matrix elements using Regge pole notions.

occurring in the scattering amplitudes. To illustrate this point, let us, assuming $F_\pi(q^2)$ to have a simple pole at $q^2 = m_\rho^2$, with residue R_π, multiply both sides of Eq. (20) by $(q^2 - m_\rho^2)^2$ and then let $q^2 \to m_\rho^2$. A_1 and A_4 presumably have a double pole at $q^2 = m_\rho^2$ whose residue, $\mathcal{A}_{1,4}(\nu,t,m_\rho^2, m_\rho^2)$, can be related to the absorptive part of an amplitude for $\pi - \rho$ scattering.* R_π is in turn proportional to the $\rho\pi\pi$ coupling constant, and in this limit we obtain a condition on $\pi - \rho$ scattering,

$$R_\pi^+ R_\pi^- + \frac{2}{\pi} \int_{\nu_0}^{\infty} d\nu' \left[q_1 \cdot q_2 \, \mathcal{A}_1^{+-}(\nu',t) + q^4 \mathcal{A}_4^{+-}(\nu',t) \right] = 0 . \tag{26}$$

Denoting by $\mathcal{F}^{ij}$ the residue of the f functions of Eq. (19) at the double pole in q^2, (26) can also be written as,[12] with

$$\bar{F}^{ij} = \mathcal{H} \mathcal{F}^{ij} \quad \text{and} \quad \bar{A}^{ij} = \mathcal{H} \mathcal{A}^{ij}$$

$$\lim_{\nu \to \infty} \nu \left[(q_1 \cdot q_2) \bar{A}_1^{ij}(\nu,t) + m_\rho^4 \bar{A}_4^{ij}(\nu,t) \right]$$

$$= \lim_{\nu \to 0} \frac{1}{\nu} \left[(q_1 \cdot q_2) \left(\bar{F}_1^{ij}(\nu,t) + (q_1 \cdot q_2) \bar{F}_2^{ij} + 2m_\rho^2 \bar{F}_4^{ij} \right) + m_\rho^4 \bar{F}_3^{ij} \right]$$

$$= 0 \tag{27}$$

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* See appendices of Ref. 7.

An elegant way of stating that the current commutation relations do not place a restriction on the strong interactions is to write the analogue of Eq. (15) for $q^2 = m_\rho^2$, with $t_{\mu\nu}^{ij}\Big|_{STRONG}$ being the residue of $t_{\mu\nu}^{ij}$ at the double pole in q^2. It reads

$$(Q_\mu \mathcal{H} - \mathcal{H} Q_\mu)\, t_{\mu\nu}^{ij}\Big|_{STRONG} = 0 \quad . \tag{28}$$

In this strong interaction configuration, it may well be that the "asymptotic tail" is not as important as we saw it to be previously. The reason for supposing this is that the whole integral no longer has t channel singularities, as the right-hand side of (28) is zero, so the "asymptotic tail" is not required by consistency for $t \to m_\rho^2$. Of course the question of how one should simultaneously let $t \to m_\rho^2$ and $q^2 \to m_\rho^2$ is still open.

Adopting now as a working hypothesis the neglect of all but low-lying resonances, we see that an approximate relation can immediately be derived from (20). Of possible intermediate states in the commutator the ρ is forbidden by G parity, the A_1 is highly questionable, and the $\phi \to \rho\pi$ seems to be very small experimentally and forbidden theoretically in various models. The only remaining low-lying state with the correct quantum numbers is the ω; keeping only it, Cabibbo and Radicati[8] found a relation between the pion charge radius and the rate for $\omega_0 \to \pi_0 + \gamma$ from a sum rule like (20) with q^2 infinitesimally close to zero. Now, however, we are free to extrapolate in q^2; letting in fact $q^2 \to m_\rho^2$ and separating out explicitly the ω pole from Eq. (20), as is easily done once again by just keeping the coefficient

of $\epsilon_1 \cdot P \; \epsilon_2 \cdot P$, we find, in this approximation, a relation between $g_{\rho\pi\pi}$ and $g_{\omega\rho\pi}$, the ω-ρ-π coupling constant. If we define these constants by the following effective Lagrangians

$$\mathcal{L}_{\rho\pi\pi} = g_{\rho\pi\pi} \, \vec{\rho} \cdot (\vec{\pi} \times \partial_\mu \vec{\pi}) \; , \tag{29a}$$

$$\mathcal{L}_{\omega\rho\pi} = \frac{i g_{\omega\rho\pi}}{m_\rho} \, \epsilon_{\mu\nu\lambda\sigma} \, \partial_\mu \omega_\nu \, \partial_\lambda \vec{\rho}_\sigma \cdot \vec{\pi} \; , \tag{29b}$$

the condition is quite simply

$$4 g^2_{\rho\pi\pi} - g^2_{\omega\rho\pi} = 0 \; . \tag{30}$$

Taking $g^2_{\rho\pi\pi}/4\pi = 2.5$, corresponding to a width[13] of 124 MeV, we find, using the Gell-Mann - Sharp-Wagner[14] model, that the partial width $\Gamma(\omega \to 3\pi)$ equals 6.8 MeV, as compared with the experimental value[13] of $(12 \pm 1.7) \times 0.88 = 10.6 \pm 1.7$ MeV. Considering the fact that several approximations have been made to obtain this comparison, agreement is not too bad.

III. GENERALIZATION TO SPIN $\frac{1}{2}$ TARGETS

Let us now turn our attention to the case in which the initial and final states of Eq. (1) are spin-$\frac{1}{2}$ baryons. As is well known, the presence of spin is a nontrivial complication in the decomposition into invariants; we hope to treat in the future the general case, but shall limit ourselves here to the particularly simple kinematical configuration of $q_1^2 = q_2^2 = q^2$ and $t = 0$, averaging over the initial and final spins.

There are then only two independent invariant amplitudes,

$$B^{ij} = \epsilon_{1\mu}\,\epsilon_{2\nu}\,t^{ij}_{\mu\nu}\,(E_1E_2/M_1M_2)^{1/2} = I_{n1}\,B_1^{ij} + I_{n2}\,B_2^{ij}\,. \tag{31}$$

The invariants are

$$I_{n1} = q^2(\epsilon_1\cdot P)(\epsilon_2\cdot P) - \nu\left[(\epsilon_1\cdot q)(\epsilon_2\cdot P) + (\epsilon_1\cdot P)(\epsilon_2\cdot q)\right] + (\epsilon_1\cdot\epsilon_2)\nu^2 \tag{32a}$$

$$I_{n2} = (\epsilon_1\cdot\epsilon_2)q^2 - (\epsilon_1\cdot q)(\epsilon_2\cdot q)\,, \tag{32b}$$

where, since $t = 0$, $q_1 = q_2$. Once again we can go to the "transverse system" and there decompose B^{ij} as follows:

$$B^{ij} = b^{ij}\,\epsilon_1\cdot\epsilon_2 + b^{ij}(\epsilon_1\cdot q)(\epsilon_2\cdot q)\,, \tag{33}$$

the relation between the b^{ij}'s and the B^{ij}'s being

$$B_1^{ij}\,\nu^2 + q^2\,B_2^{ij} = b_1^{ij}\,, \tag{34a}$$

$$B_2^{ij} = -b_2^{ij}\,, \tag{34b}$$

and conversely,

$$B_1^{ij} = \frac{1}{\nu^2}\left[b_1^{ij} + q^2 b_2^{ij}\right] , \tag{35}$$

where of course the functions are all evaluated at $q_1^2 = q_2^2 = q^2$ and $t = 0$. By then applying the operator $Q_\mu \not{N} - \not{N} Q_\mu$ to $t_{\mu\nu}^{ij}$ we obtain, analougously to (19) and (20), the sum rule

$$\frac{2}{\pi}\int_0^\infty q^2 B_1^{ij}(\nu',q^2,q^2,0)d\nu' = \frac{2}{\pi}\int_0^\infty \frac{(q^2 b_1^{ij} + q^4 b_2^{ij})}{\nu'^2} d\nu'$$

$$= i\, c_{ijk} F^k(0) . \tag{36}$$

Once again we separate out the one-nucleon state contribution to the sum rule. This is easily done, as it is the coefficient of $\epsilon_1 \cdot P\ \epsilon_2 \cdot P$ in the evaluation of B^{ij} at the nucleon pole. When this contribution is made explicit, the sum rule becomes

$$\left[F_1^i(q^2)F_1^j(q^2) - \frac{q^2}{4M^2} F_2^i(q^2)F_2^j(q^2)\right] - [i \leftrightarrow j]$$

$$+ \frac{2}{\pi}\int_{\nu_0}^\infty q^2 B_1^{ij}(\nu',q^2,q^2,0)d\nu' = i\, c^{ijk} F_1^k(0) , \tag{37}$$

where F_1^i and F_2^i are the Dirac and Pauli isovector form factors and ν_0 is the inelastic threshold corresponding to the one-pion and one-nucleon state. Let us now take the particles of momentum p_1 and p_2 contained in respectively the initial and final states to both be

protons, and let i and j denote isotopic spin raising and lowering, as in Section II. Our sum rule then takes on the form, with F^v being the isovector form factor

$$F_1^{v2}(q^2) - \frac{q^2}{4M^2} F_2^{v2}(q^2) + \frac{2q^2}{\pi} \int_{\nu_0}^{\infty} B_1^{+-}(\nu',q^2,q^2,0)d\nu' = 1 , \tag{38}$$

or alternatively, using (35),

$$1 - F_1^{v2}(q^2) = -\frac{q^2}{4M^2} F_2^{v2}(q^2) + \frac{2q^2}{\pi} \int_{\nu_0}^{\infty} \frac{b_1^{+-} + q^2 b_2^{+-}}{\nu'^2} , \tag{39}$$

where B_1^{+-} and $b_{1,2}^{+-}$ have two terms corresponding to the two terms in the commutator of currents. This sum rule has already been derived in one form or another by several authors,[15] but we would like to discuss some of its features, in particular its dependence on q^2. At $q^2 = 0$, we have the identity $F_1^2(0) = 1$; taking the derivative with respect to q^2 at $q^2 = 0$, we obtain

$$\frac{\langle r_v^2 \rangle}{3} = \frac{F_2^{v2}(0)}{4M^2} + \frac{1}{2\pi^2\alpha} \int_{\nu_0}^{\infty} \frac{d\nu'}{\nu'} (2\sigma_{1/2}^v - \sigma_{3/2}^v) , \tag{40}$$

where $\langle r_v^2 \rangle$ is the isovector charge radius, $F_2^v(0) = \mu_P - \mu_N$, and $\sigma_{1/2}$, $\sigma_{3/2}$ are the total cross sections for isovector photon production on protons of $I = 1/2$ and $I = 3/2$ states, the cross sections having been obtained by the optical theorem applied to $b_1^{+-}(\nu,0,0,0)$. If we try to saturate the integral by a few low-lying

resonances, the agreement of (40) with experiment is not exceptionally good. Roughly speaking, $F_2^v(0)/4M^2$ is equal to $\langle r_v^2\rangle/3$, and the contribution of the N^* to the integral is negative and of the order of magnitude of one half the anomalous magnetic moment term. Low-energy photoproduction is dominated by the N^*; other resonant low-energy states, say up to energies of 2 BeV, contribute to the first term in the integrand of (40), but do not appear to be more than 20-30% of the N^* contribution, so we are still left with a discrepancy of the order of 1.5 to 2 between the two sides of (40). This does not mean that the relation is not valid and consequently that the equal time commutation relations of isotopic spin currents are suspect; it may just signify that the intermediate states with energies greater than $\approx$ 2 BeV make up the difference.

There is still an interesting question, however, and that is the dependence of Eq. (39) on q^2. We shall illustrate this point by assuming the integral to be approximately saturated by the N^*, whose contribution we calculate by use of an isobar model,[16] and comparing the two sides of (39) as functions of q^2. The matrix element of the current between proton and N^* is defined as

$$\langle \text{prot.}, p_1 \,|J_\mu^3|\, N^{*+}, p_1+q\rangle =$$

$$= \left(\frac{MM^*}{EE^*}\right)^{1/2} \frac{C_3(q^2)}{M}\, \bar{u}(p_1)(q_\lambda\gamma_\mu - \gamma_\lambda q_\mu)\gamma_5 w_\lambda(p_1 + q)\,, \tag{41}$$

where w_λ is a Rarita-Schwinger spin-3/2 wave function, $C_3(0) = 2.0$ by experiment,[17] and we assume for $C_3(q^2)$ the momentum transfer dependence[18]

$$C_3(q^2) = C_3(0) \frac{G_M^{\ v}(q^2)}{G_M^{\ v}(0)} , \tag{42}$$

where $G_M^{\ v}$ is the Sachs magnetic form factor. When we use the relation between Sachs and Dirac form factors,[19]

$$F_1^{v2}(q^2) - \frac{q^2}{4M^2} F_2^{v2}(q^2) = \frac{G_E^{v2}(q^2) - \frac{q^2}{4M^2} G_M^{v2}(q^2)}{1 - \frac{q^2}{4M^2}} , \tag{43}$$

Eq. (39), in the isobar approximation, becomes

$$1 - \frac{G_E^{\ v2}(q^2)}{1 - \frac{q^2}{4M^2}} = - \frac{G_M^{\ v2}(q^2)}{4M^2} q^2 \left[\frac{G_M^{\ v2}(q^2)}{1 - \frac{q^2}{4M^2}} - \frac{4C_3^{\ 2}(0)}{9\, G_M^{\ v2}(0)} \left\{ \frac{3M^{*2} + M^2 - q^2}{2M^{*2}} \right\} \right] . \tag{44}$$

We shall plot the ratio between the right- and left-hand sides of Eq. (44), which we call $\gamma(q^2)$, as a function of q^2 ; for the form factor we shall use a recent fit by means of two poles and a hard core,[20] one of the poles being at $q^2 = m_\rho^{\ 2}$. At $q^2 = m_\rho^{\ 2}$, as explained in Sec. II, we lose all trace of the algebra of current commutators in (39) and obtain a consistency condition on the residues of the form factors at the ρ pole which is presumably testing ρ-N

scattering. The plot of $\gamma(q^2)$ is given in Fig. 2; we believe the remarkable agreement at $q^2 = m_\rho^2$ to be largely accidental. However, the dependence on q^2 is perhaps significant; what we are probably testing is how rapidly the integral in Eq. (39) converges as a function of the mass (q^2) of the scattering particle. The eventual convergence is guaranteed by Pomeranchuk-theorem-like arguments, but it is not implausible that there are regions of q^2 for which the convergence is most rapid and consequently allows us to best approximate the integral by its low-energy part. This is very likely to be associated with some strong interaction mechanism and consequently is most marked at $q^2 = m_\rho^2$, which places us in a strong-interaction kinematical configuration.

In conclusion we hope to have at least shown what are a few of the problems that lie ahead on the road to a thorough understanding of current commutation relations. First of all there is the complete kinematic analysis for the case of initial and final particles with spin.* The independence of the form of the sum rules on the values of q_1^2 and q_2^2, on the other hand, is still a puzzle to a certain extent, as is the dynamical significance of the connection between form factors and integrals of scattering amplitude discontinuities. We have tried to indicate lines of approach for tackling these problems, showing how the high-energy behavior of the discontinuity of scattering amplitudes generates the cross-channel singularities, how the strong-interaction

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* One immediate result of such an analysis is a second sum rule like (39), which presumably will be for the isovector magnetic form-factor radius.

configuration differs from that of arbitrary q^2, and so on, but have probably raised more questions than we have answered. We hope this to be the consequence of a healthy, thriving theory rather than the authors' confusion.

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FOOTNOTES AND REFERENCES

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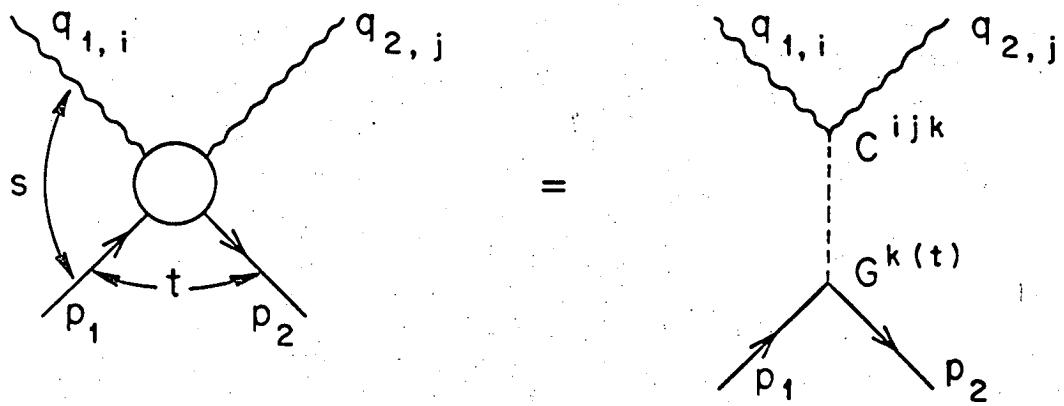
FIGURE CAPTIONS

Fig. 1. Scattering kinematics.

Fig. 2. $\gamma(q^2)$ as function of q^2 .



$$t = (p_1 - p_2)^2$$

$$s = (p_1 + q_1)^2$$

$$\nu = \frac{(p_1 + p_2)\cdot(q_1 + q_2)}{4} = P\cdot Q = \frac{1}{4}\left[2s + t - q_1^2 - q_2^2 - p_1^2 - p_2^2\right]$$

$$= \frac{1}{4}\left[2s + t - \sum_i m_i^2\right]$$

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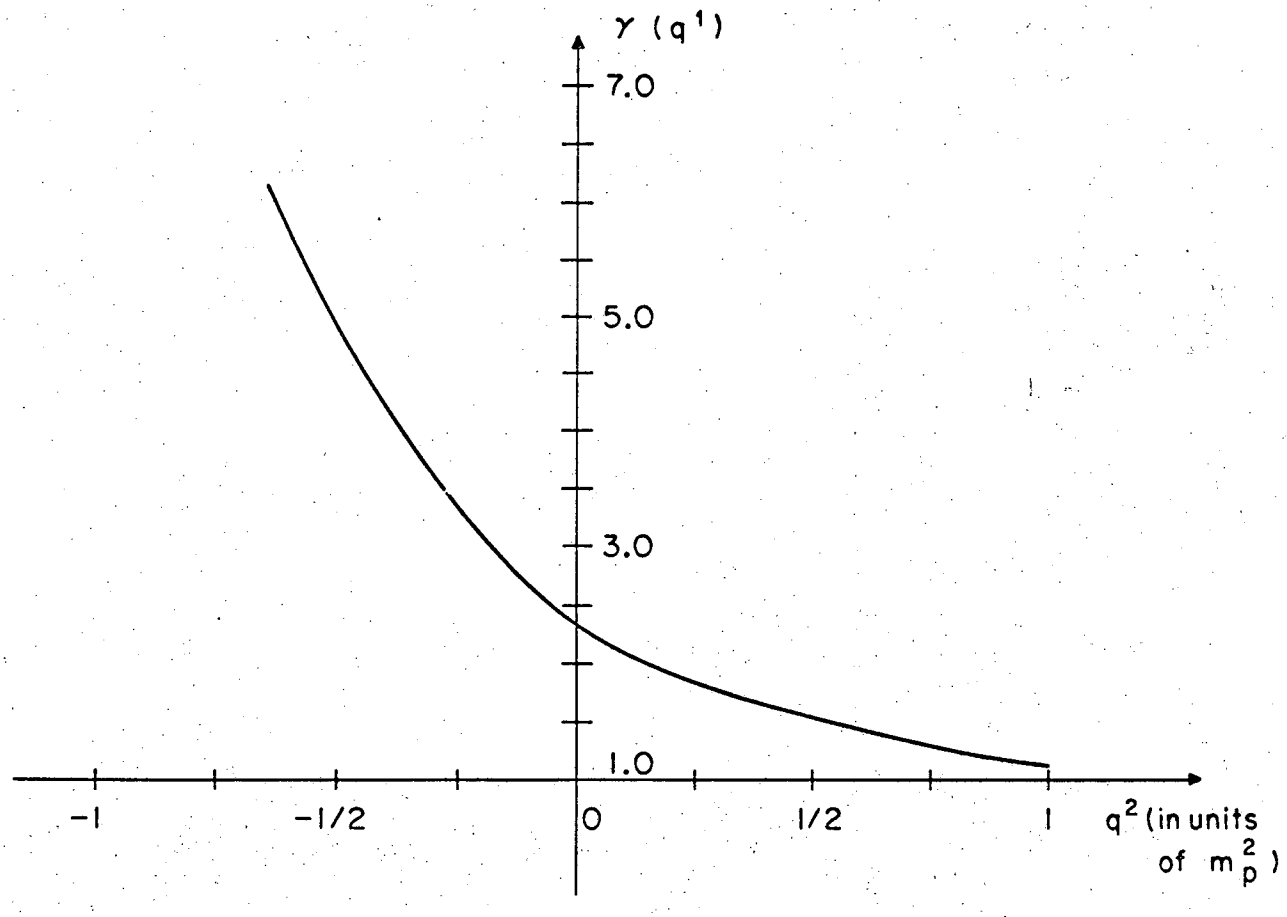

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