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CROSSING, HERMITIAN ANALYTICITY AND THE CONNECTION BETWEEN SPIN AND STATISTICS

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ABSTRACT

The crossing and Hermitian analyticity properties of multiparticle scattering functions are derived within an S-matrix framework. The normal connection between spin and statistics is then shown to follow from a certain property of the paths of continuation connecting crossed and Hermitian conjugate points. The analyticity properties assumed are that the pole singularity at a location corresponding to the exchange of a physical particle is associated with one-particle-exchange type Landau diagrams (i.e. there is no superimposed pole singularity not associated with a Landau diagram of this type) and that certain discontinuity functions have the pole-factorization property that they would have if they were given by a formula of the Cutkosky type. Also, confluences of infinite numbers of Landau singularity surfaces are assumed not to invalidate results established for the various Landau surfaces considered individually.

I. INTRODUCTION

An earlier S-matrix proof of the normal connection between spin and statistics given by this author[1] depended on an assumption that self-conjugate combinations of particle and antiparticle amplitudes were in principle observable. The assumption is objectionable because it has no experimental basis in the case of charged particles, and in fact conflicts with a conjectured super-selection rule.[2]

In that original paper the beginning of a second proof not depending on this special assumption was also given.[3] This alternative proof depended on an apparent conflict between abnormal statistics and the crossing and Hermitian analyticity properties of scattering functions. The crossing property of (multiparticle) scattering functions is the property whereby the scattering function describing one reaction is connected by analytic continuation to the scattering function describing certain other reactions, called crossed reactions. The Hermitian analyticity property is the property whereby the scattering function representing a given process is analytically connected to the complex conjugate of the scattering function for the transposed process, at certain real boundary points.

This second argument was not a full proof, because it was incomplete on two counts. In the first place the statistics involved was the sign change under interchange of variables describing relative antiparticles, whereas the spin-statistics connection involves the sign change under interchange of two variables describing particles of the same type (identical particles). And in the second place the required properties of crossing and Hermitian analyticity were not derived, but simply assumed.

The first defect was partially remedied in a later paper,[3] where it was shown that a certain assumption on the phase factors occurring in the cluster decomposition expansion implied that the sign change under interchange of identical-particle variables was the same as the sign change under the interchange of conjugate-particle variables. This assumption on the phases was that they be such that the disconnected contributions to a unitarity equation be equivalent to a product of the unitarity equations in the various disconnected sectors considered by themselves. This assumption, although reasonable, is replaced in the present work by direct physical requirements.

The main object, however, of the present paper is to give proofs of the crossing and Hermitian analyticity properties. The work is a development of a line of approach initiated by Gunson[5] and explored by Olive,[6,7] and is based on an exploitation of the pole-factorization property of scattering functions. This is the property whereby the residues of poles of scattering functions at certain points in the physical regions of multiparticle processes are given essentially as the products of scattering functions for certain other reactions involving fewer particles. More specifically, the work is an elaboration of an unpublished work of this author[8] in which was developed the procedure, subsequently adopted by others,[7,9] whereby the paths of continuation to crossed (or Hermitian conjugate) points are defined by distorting certain paths originally lying in the physical region of the larger process, and running between different parts of the pole manifold $S_\nu = \mu_p^2$, into paths lying completely within the pole manifold $S_\nu = \mu_p^2$. These latter paths specify the mass-shell continuations between the relevant

points. The present work goes beyond earlier attempts[7,9] to exploit this idea in that it covers all possible reactions and all Landau singularities. The work of Ref. 7 treated only the simplest reactions and ignored all but the but the trivial normal-threshold singularities, and the work of Ref. 9 referred to a simple reaction with special mass ratios.

Having established the required crossing and Hermitian analyticity properties we give a new version of the remainder of the proof of the normal connection between spin and statistics. This new version is more simple and direct than the one given earlier.[3,4] It avoids completely the introduction of the notion of a phase change induced by an interchange of conjugate variables (variables that refer to relative antiparticles). The need to introduce this notion was a disagreeable feature of the earlier proofs, for this phase change, unlike the sign change under the interchange of like variables (variables referring to identical particles), apparently has no direct physical significance. And in order to deal with the interchange of unlike variables, certain stipulations had to be introduced relating the phases in the cluster decomposition equation to special orders of variables. The present proof circumvents these difficulties.

This new proof of spin and statistics, which is given in Section VII, is largely independent of the details of the work preceding it. That section is therefore designed to be largely self-contained. It depends, in fact, only on a very gross feature of the proof of the pole-factorization theorem, and on a rather trivial-sounding property of the paths connecting crossed reactions and Hermitian conjugate points.

under the interchange of like variables (variables representing identical particles). Moreover the sign (change) under interchange of a particular type of variable is a universal quantity that is independent both of the particular scattering function in which the variables appear, and of the particular location of the variables among the arguments of these functions.

The fact that parastatistics is precluded here is a direct consequence of our basic assumption[1] that the observables of the theory are squares of amplitudes. This is not true in parastatistics models. Thus the work of these subsections is not to be construed as a general disproof of parastatistics but rather as a proof that, within the framework adopted, in which observables are squares of amplitudes, the continuation of the scattering function, through a region near physical points, from an original region of definition to some region where like variables are interchanged must give back the original function, apart from a sign that depends only on the type of particle involved.

Having established that interchange of like variables leads to a sign (change) σ_p that depends only on the type of particle p, one may then ask what the value of this sign is. The normal connection between spin and statistics is the relationship

$$\sigma_p = (-1)^{2j_p} ,$$

where j_p is the spin of particle p.

Section III is devoted to the proof of some physical region analyticity properties of functions represented by bubble diagrams. These functions are functions of the kind occurring in unitarity equations, and are formed by integrating products of scattering functions and their conjugates over the physical phase space associated with certain internal particles. The singularities of any such function are shown to be confined to a certain corresponding subclass of Landau surfaces. The singular parts of the surfaces are not always the positive - α parts, however. The rules that determine which parts of the Landau surfaces are singular, and the $i\epsilon$ rules for continuing around these surfaces are derived.

The fact that the singularities of these functions are confined to Landau surfaces is a result somewhat similar to one obtained by Polkinghorne.[14] The result of Polkinghorne does not refer specifically to the physical region, however, and the possibility of non-Landau (i.e. second type) singularities arises. The $i\epsilon$ rules derived here for detouring around physical region singularities generalize results about physical region singularities recently obtained by Landshoff and Olive.[15]

Section IV is devoted to a general proof of the pole-factorization theorem. The original S-matrix proof by Olive[7] was for a simple case and was based on an assumption (called a theorem) that has recently been shown by Branson[16] to be not valid in general. Branson's counterexample also contradicts an assumption made in an earlier proof by this author.[17] That assumption was that almost all singularities lying on the "pole manifolds" $S_\nu = \mu_p^2$ are associated with "pole-type" (i.e., one-particle-exchange type) Landau diagrams, these being Landau diagrams that can be

reduced by contraction to connected Landau diagrams having just two vertices and just a single internal line connecting these two vertices. Branson has shown that other types of singular Landau surfaces can lie at $S_\nu = \mu_p^2$. Our earlier assumption is therefore weakened to the assertion that almost all pole [or worse] singularities lying on the manifold $S_\nu = \mu_p^2$ are associated with pole-type Landau diagrams. (That the assumption in Ref. 17 should be weakened in this way was already suggested there.)

This "pole assumption" is a fundamental assumption in the present work. It is believed that it can be verified by an examination of the nature of the possible Landau singularities lying at $S_\nu = \mu_p^2$, but this verification is not attempted here.

The derivation of the pole-factorization theorem given in Section IV is different from the one given in Ref. 17. The form given here is useful because essentially the same technique can be used to derive the general normal-threshold discontinuity equation, as will be discussed in a later paper. Also, the present derivation is given in greater detail than the earlier one and covers particles with spin (a trivial extension in the M-function formalism). More important, the phase factors occurring in the cluster decomposition are taken into account. These will play an important role in the discussion of spin and statistics.

Section V contains a proof of the Hermitian analyticity property. The "pole assumption" is again fundamental, and now it is extended to points lying outside the physical region. The essential idea of the proof is to consider a larger process from which the scattering function of interest can be extracted as a factor in the residue of a product of poles.[6,7,8] The unitarity equation for the larger process at a point

corresponding to null energy-momentum vectors for the reaction of interest is effectively continued to the pole position by exploiting the fact that most contributions to the larger process do not contribute to the residue.

The arguments in Section V deal individually with individual Landau surfaces. There is a tacit assumption that results that hold for the Landau surfaces individually will hold for them collectively. In particular it is assumed that no natural boundaries formed from confluences of infinite numbers of Landau surfaces invalidate results that are valid when the surfaces are considered individually.

In Section VI the reader is first referred to the proof of crossing given in Ref. 17. That proof is then extended in such a way as to obtain a compatibility condition on the paths of continuation connecting crossed and Hermitian conjugate points. This compatibility condition, which says essentially that the Hermitian conjugate points for crossed reactions are connected by the complex conjugate of the crossing path, plays a key role in the proof of spin and statistics. It is also shown that this compatibility requirement carries over to the case in which the paths of continuation jump across various cuts, rather than detouring around them, provided the pole-factorization property carries over to the relevant discontinuity functions, as it would do if these functions were given by a Cutkosky rule.

The proof of the normal connection between spin and statistics is given in Section VII. It is quite simple. It is noted that the residue function in the pole-factorization property was obtained from a corresponding pole term in the unitarity equation for the process in which

the pole appears, and that the phase factor in the residue formula is consequently precisely the phase factor of this contribution to unitarity. In special cases this contribution to unitarity is just one of the absolute-value-squared contributions to a "forward scattering" process, apart from some signs coming from interchanges of certain identical particles. Thus the phase factors in these pole contributions are determined by the statistics of certain particles. The residue functions associated with crossed reactions, which are connected by analytic continuation, are compared and shown to have a sign incompatibility in case any scattered particle has abnormal statistics.

In Section VIII the phase factor in the crossing relation is shown to be unity for the functions $M_c(K)$: these particular functions, without any added phase factors, give, when continued along the paths of continuation connecting crossed regions, the scattering functions for the various crossed process.

To obtain this result a special stipulation relating phases in the decomposition equation to order of variables is invoked. This stipulation is, in effect, used in the proof of spin and statistics given by Lu and Olive. Since this stipulation is of a formal rather than physical nature it renders that proof, like the proof of Ref. 4, not completely satisfactory from the pure S-matrix view point.

This stipulation, though objectionable as an element of a proof of spin and statistics, is quite natural and is adopted in the final specification of the formalism. It eliminates an indeterminant factor

in the crossing relations and also places conditions on the phases induced by interchange of variables associated with different particles. In particular it implies, as is shown in Section VIII, that the interchange of adjacent conjugate variables induces the same sign change as the interchange of the corresponding like variables.

$$L(V) = \prod_i L_i(v_i) \quad . \tag{2.7}$$

The $L_i(v_i)$ is a matrix in the spin space of particle i that transforms spin functions from values coordinated to a rest frame Σ_i of particle i to values coordinated to the general coordinate frame Σ. For a particle of spin $\frac{1}{2}$,

$$\begin{aligned} L_i^{\,j}(v_i) &\equiv L_i^{\frac{1}{2}}(v_i) \\ &\equiv (\sigma_i \cdot v_i)^{\frac{1}{2}} \\ &\equiv (\sigma_{i_\mu} v_i^{\,\mu})^{\frac{1}{2}} \\ &\equiv (v_i^{\,0} + \underset{\sim}{\sigma} \cdot \underset{\sim}{v})^{\frac{1}{2}} \\ &\equiv \cosh \frac{\alpha_i}{2} + (\underset{\sim}{\sigma} \cdot \underset{\sim}{v} \sinh \frac{\alpha_i}{2})/|v_i| \\ &\equiv (v_i^{\,0} + 1 + \underset{\sim}{\sigma} \cdot \underset{\sim}{v})(2\, v_i^{\,0} + 2)^{-\frac{1}{2}} \quad , \end{aligned} \tag{2.8}$$

where v_i is the covariant velocity vector

$$v_i = p_i / |(p_i \cdot p_i)^{\frac{1}{2}}| \tag{2.9}$$

and

$$\sinh \alpha_i \equiv |\underset{\sim}{v}_i| \quad . \tag{2.10}$$

The matrix $L_i^{\,j}(v_i)$ for particles of spin $j > \frac{1}{2}$ is obtained by extracting by means of Clebsch-Gordan coefficients the spin- j part of a tensor product of $2j$ factors $L^{\frac{1}{2}}(v_i)$. (See Eq. C14 of Ref. 12).

The rule for contraction of the spin indices in (2.6) is not always the matrix rule of contraction of adjacent indices. The exact rule is given in Subsection 8 below.

4. <u>Covariance Property.</u> From the assumed relativistic invariance of probability correlations one derives the covariance property

$$M(K'; K'') = \Lambda_s^{-1} M(\Lambda K'; \Lambda K'') , \tag{2.11}$$

where

$$\Lambda K = \{\Lambda p_i, m_i, t_i\} \quad . \tag{2.12}$$

Here Λ is any element of the real, orthochronous, proper, homogeneous Lorentz group $L_+^{\uparrow}$, and Λ_s is a corresponding p_i-independent spin-space transformation (see Ref. 12). To obtain (2.11) the weight factor $\rho(k)$ has been taken to have covariant form. In particular we take

$$\sum_K = \Sigma' \int dK = \Sigma' \prod_i \int \frac{d^4 p_i}{(2\pi)^4} \, \theta(p_i^{\,o}) \, 2\pi \, \delta(p_i^{\,2} - \mu_i^{\,2}) , \tag{2.13}$$

where Σ' is the sum over discrete indices.

5. Physical Irrelevancy of Order of Variables. The experimental result labeled by the set $K = \{p_i, m_i, t_i\}$ is assumed to be completely specified by the values of the arguments p_i, m_i, and t_i. In particular, no additional information having to do with the ordering of the variables is needed to determine the experimental result. This assumption, in conjunction with our quantum postulate of Ref. 17, means that in the integration (2.13) one should include only once the contribution from each value of K, considered as an unordered set of variables.

6. Fundamental Analyticity Property of M Functions. We introduce the following definitions.

Definition 2.1. A function $F(p_i)$ defined only over a subset W of the space of complex numbers p_i will be said to be an analytic function of the p_i at point P of W if and only if for every mapping $p_i(z_j)$ from an open set in a space of complex numbers z_i into W the functions $F'(z_j) \equiv F(p_i(z_j))$ are analytic functions of the z_j in the usual sense at all points $\{\bar{z}_j\}$ satisfying $\{p_i(\bar{z}_j)\} = P$ for which the functions $p_i(z_i)$ are analytic at $\{\bar{z}_j\}$.

Definition 2.2. A point K is a set $\{p_i, t_i\}$. It is distinguished from an argument $K = \{p_i, m_i, t_i\}$.

Definition 2.3. $F(K)$ is analytic at a point K will mean that the functions $F(K)$ corresponding to the various values of the spin indices m_i are all analytic functions of the momentum-energy vectors p_i at point K of W, where W is the set of points K over which $F(K)$ is defined.

A consequence[18] of the covariance property (2.11) is that if $M(K)$ is analytic at a point $\bar{K}$ then $M(K)$ can be extended to a function that is analytic at all points K of the set generated from $\bar{K}$ by application of any element of the proper homogeneous complex Lorentz group $L_+^{\uparrow}(C)$, which is the subgroup of $L(C)$ continuously connected to the identity. The property of $M(K)$ of being analytic at a point K is therefore Lorentz invariant. This is not the case for the S functions, since the functions $L(V)$ have singularities whose positions depend on the coordinate frame, as is seen from (2.8). The M functions have, in this sense, simpler analyticity properties and are the more convenient functions to use in a relativistic theory based on analyticity.

7. Expressions for Observables. It is advantageous to express observables directly in terms of M functions, rather than passing to S functions, for in this way manifest covariance is maintained. Let s_i be the spacelike four-vector satisfying

$$s_i \cdot s_i = -1 \tag{2.14a}$$

and

$$s_i \cdot p_i = 0 \tag{2.14b}$$

that specifies the axis relative to which the spin quantum number m_i is measured. Equation (2.14b) says that s_i is purely spacelike in any rest frame of particle i. With M functions one uses in place of the usual spin projection operators $P_i(m_i, s_i)$ rather the covariant spin operators $\tilde{P}_i(m_i, - s_i, v_i)$. For spin $- \frac{1}{2}$ particles

$$\begin{aligned} \tilde{P}_i^{\dot{\alpha}\beta}(m_i, - s_i, v_i) &= (\tfrac{1}{2} v_i - m_i s_i) \cdot \tilde{\sigma}_i^{\dot{\alpha}\beta} \\ &= (\tfrac{1}{2} v_i^{\mu} - m_i s_i^{\mu}) \tilde{\sigma}_{i\mu}^{\dot{\alpha}\beta} , \end{aligned} \tag{2.15}$$

where

$$\tilde{\sigma}_{\mu} = (1, - \underset{\sim}{\sigma}) , \tag{2.16}$$

and $\underset{\sim}{\sigma}$ is the usual Pauli spin-matrix vector. Notice that the $\tilde{P}_i(m_i, - s_i, v_i)$ in (2.15) reduces the usual $j = \frac{1}{2}$ projection operator in a rest frame of particle i. For $j > \frac{1}{2}$ the $\tilde{P}_i(m_i, - s_i, v_i)$ is obtained by extracting by means of Clebsch-Gordan coefficients the spin - j part of the symmetrized tensor product of $2j$ spin-$\frac{1}{2}$ spin operators $\tilde{P}_k(m_k, - s_i, v_i)$ subject to

$$\Sigma m_k = m_i .$$

The $2j$ upper dotted (undotted)indices of the $\tilde{P}_k^{\dot{\alpha}\beta}$ are combined to give the $(2j + 1)$ - valued upper dotted (undotted) index of $\tilde{P}_i^{\dot{\alpha}\beta}$ (more details are given in Ref. 12).

Each spin index m_i of $M(K'; K'')$ and $M^*(K'; K'')$ is defined to be lower dotted or lower undotted according to whether it is contracted in the calculation of observables with an upper dotted or upper undotted index of $\tilde{P}_i^{\dot{\alpha}\beta}$. Define, accordingly, a quantity λ_i :

$\lambda_i = +1$ if the spin index m_i of $M(K'; K'')$ is undotted, (2.17a)

$\lambda_i = -1$ if the spin index m_i of $M(K'; K'')$ is dotted . (2.17b)

Define ϵ_i by

$\epsilon_i = +1$ if particle i is final, (2.18a)

$\epsilon_i = -1$ if particle i is initial . (2.18b)

Then m_i is the projection of physical spin angular momentum on the "physical" spin direction

$$s_i^{phys} = \epsilon_i \lambda_i s_i , \tag{2.19}$$

where s_i is the "mathematical" spin vector s_i appearing in $\tilde{P}_i(m_i, -s_i, v_i)$. The result (2.19) follows from the covariance property (2.11) and the requirement that spin angular momentum plus orbital angular momentum be a conserved quantity (see Ref. 12).

In certain other formalisms the initial and final particles are associated with kets and bras respectively and one always gets $\epsilon_i \lambda_i = +1$. This special condition does not naturally occur in the development of the M function formalism from basic physical postulates, and it is advantageous not to introduce it. For in the development of the theory we shall be led to analytically continue our functions to

regions where ϵ_i is reversed. Under analytic continuation the transformation property (2.11), hence the index type, and hence λ_i, necessarily remains unaltered. To resolve this conflict with a condition $\epsilon_i \lambda_i = +1$ new functions would have to be introduced into the theory. This unneccessarily complicates the formalism and leads to possible phase ambiguities. It therefore is better never to introduce the artificial condition $\epsilon_i \lambda_i = +1$. Then a single function will describe both the direct and crossed reactions. However, the connection between the physical spin vector s_i^{phys} and the mathematical spin vector s_i will be reversed under continuation to crossed channels. This relationship between the physical and mathematical spin vectors is completely analogous to the one that will be obtained for the momentum-energy vectors.

8. <u>Contraction Rule in the Definition of $M(K'; K'')$.</u> In the general development of the theory the M Functions are originally defined by their connection to observables through contractions with the covariant spin operators, and Eq. (2.6) emergies as a consequence. The index m_1 of $S(K'; K'')$ turns out to be contracted with the adjacent index of $L(V')$ or $L(V'')$ if $\lambda_i \epsilon_i = +1$, and the nonadjacent index otherwise. The λ_i can be specified at will be specifying the index of $\tilde{P}^{\dot{\alpha}\beta}$ with which the index m_i of $M(K'; K'')$ is contracted in the calculation of observables.

9. <u>Unitarity for M Functions.</u> One can specify the conventions for λ_i so that the $\epsilon_i \lambda_i$ for each individual particle, whether occurring initially or finally, is a fixed sign depending only on the particle type. (This specification relates initial particle to final particle--not to final antiparticle; the crossing concept is not involved.) If the λ_i are specified in this way then unitarity takes the form

$$\sum_K M(K';\, K)\, \tilde{G}(V)\, M^*(K'';\, K) - G(V')\, \delta(K';\, K'') = 0 \;, \tag{2.20}$$

where

$$\tilde{G}(V) = \prod_i \tilde{G}_i(v_i) = [G(V)]^{-1} \tag{2.21}$$

and

$$\tilde{G}_1(v_i) = [L_i(v_i)]^{-2} \;. \tag{2.22}$$

For spin-$\frac{1}{2}$ particles,

$$\tilde{G}_i(v_i) = v_i \cdot \tilde{\sigma} = v_i^{\,o} - \underset{\sim}{v} \cdot \underset{\sim}{\sigma} \;, \tag{2.23}$$

while for spin $j > \frac{1}{2}$ the $\tilde{G}_1(v_i)$ is obtained by extracting by means of Glebsch-Gordan coefficients the spin-j part from a tensor product of $2j$ spin $\frac{1}{2}$ matrices (2.23). Thus $\tilde{G}_i(v_i)$ is of degree $2j_i$ in the vectors v_i and

$$\tilde{G}_i(-\, v_i) = (-\,1)^{2j_i}\, \tilde{G}_i(v_i) \;. \tag{2.24}$$

10. <u>Momentum-Energy Conservation.</u> The M functions are nonzero only at points satisfying

$$\Sigma\; p'_i = \Sigma\, p''_i \;. \tag{2.25}$$

This conservation-law constraint is equivalent to the statement of translational invariance if space and time are introduced by Fourier

transformation. In order to give finite effects, an M function must have a conservation-law δ function $(2\pi)^4\,\delta(\Sigma\, p'_i - \Sigma\, p''_i)$ as a factor.

11. M(K) Functions. We define M(K) [without the semicolon] by

$$M(K) = M(K', -\tilde{K}'') = M(K'\,;\,K''), \tag{2.26}$$

where

$$(-\tilde{K}'') \equiv \{-p_i, m_i, -t_i\}. \tag{2.27}$$

For later convenience the order of writing the variables of $(-\tilde{K}'')$ is reversed relative to K''. (See Subsection 18 below and Section VIII.) The momentum-energy arguments of M(K) will be called the "mathematical" momentum-energy vectors k_i, where

$$k_i = \epsilon_i\, p_i \tag{2.28}$$

and ϵ_i is $+1$ or -1 according to whether particle i is final or initial. In terms of the k_i the momentum-energy conservation law δ function becomes

$$(2\pi)^4\,\delta^4(\Sigma\, k_i) \equiv (2\pi)^4\,\delta^4(\Sigma\, p'_i - \Sigma\, p''_i). \tag{2.29}$$

12. Cluster Decomposition. M(K) is assumed to satisfy the cluster property[4]

$$M(K) = \sum_p M_p(K), \tag{2.30a}$$

where

$$M_p(K) = \alpha_p \prod_s M_1(K_{ps}) \quad . \tag{2.30b}$$

Here K_{ps} is the sth subset of the pth partition of K. The first (and only) subset of the first partition of K is K itself,

$$K_{11} \equiv K \quad , \tag{2.30c}$$

and the function $M_1(K)$ is asserted to have no conservation-law δ function aside from the overall one given by (2.29). The α_p are phase factors depending on the ordering of variables of K and of the K_{ps} but not on the values of the momentum-energy arguments k_i. The phase of $M_1(K)$ is defined by

$$\alpha_1 = 1 \quad . \tag{2.30d}$$

The other phase factors α_p must evidently depend on the orders of the variables. They are asserted to be restricted by the following two conditions: E2. Let K_{ps} be an ordered set of variables consisting of the variables of the sth subset of the pth partition of the ordered set K. Let $\tilde{K}$ be some ordered subset of the variables of K. Suppose there are two partitions $p = a$ and $p = b$ of $\tilde{K}$, and also two partitions $p = a$ and $p = b$ of the set K, such that these two partitions of K coincide with the corresponding partitions of $\tilde{K}$ over the set $\tilde{K}$, and coincide with each other over the remaining variables. That is, for some arrangement of the indices s,

$$\tilde{K}_{as} = K_{as} \quad \text{for} \quad s \leqslant \tilde{s}_a , \tag{2.30e}$$

$$\tilde{K}_{bs} = K_{bs} \quad \text{for} \quad s \leqslant \tilde{s}_b \tag{2.30f}$$

$$K_{as} = K_{bs'} \quad \text{for} \quad s - \tilde{s}_a = s' - \tilde{s}_b > 0 \tag{2.30g}$$

where $\tilde{s}_p$ is the number of terms of partition p of $\tilde{K}$. Then the α_p and $\tilde{\alpha}_p$ in the cluster decompositions of K and $\tilde{K}$ satisfy

$$\frac{\tilde{\alpha}_a}{\tilde{\alpha}_b} = \frac{\alpha_a}{\alpha_b} . \tag{2.30h}$$

<u>E3</u>. Let a and b denote two initial sets of particles and let c and d denote two final sets. Suppose each of these four sets is divided into n subsets. Suppose the first $n-1$ subsets of sets a are identical to the first $n-1$ subsets of set b and the first $n-1$ subsets of set c are identical to the first $n-1$ subsets of set d. Let ac, ad, bc, and bd be values of p that denote the partitions of the four sets of variables $a + c$, $a + d$, $b + c$, and $b + d$, respectively, into n subsets, with the first initial subset of a or b grouped with the first final subset of c or d, etc. Then the four α_p satisfy

$$\frac{\alpha_{ac}}{\alpha_{ad}} = \frac{\alpha_{bc}}{\alpha_{bd}} . \tag{2.30i}$$

Postulate E2 asserts that the phase difference between two different cluster terms of a given scattering amplitude that differ only over a certain subset of variables is independent of the remaining set of variables R over which they are identical. These phase differences are observable quantities, according to B2 of Ref. 12. If they were not independent of R then observable phenomena would depend on effects associated with disconnected bubbles in a manner contrary to the physical decomposition principle; phenomena would depend on "unconnected" phenomena, where "unconnected" means unconnected by energy-momentum transfer.

Postulate E3 asserts that the ratio $\alpha_{ac}/\alpha_{ad}:\alpha_{bc}/\alpha_{bd}$ takes the same value (unity) that it would take if just the nth subsets alone were present. This ratio is an observable quantity (provided the various M functions are all non-zero--otherwise one of the phases can be defined at will). This observable corresponds to an interference effect in a transition from a combination of a and b to a combination of c and d. The postulate asserts that this observable quantity is independent of "unconnected" phenomena, as required by the physical decomposition principle.

It is easy to verify that postulates E2 and E3 imply that, in a unitarity equation, the sum of contributions having a given connectedness structure (i.e. having a given set of unintegrated conservation-law δ functions) combine to give a product of the connected parts of the unitarity equations for the appropriate subsectors. (The connected part of the unitarity equation is the part having only one unintegrated conservation-law δ function. The terminology comes from the diagrammatic representation discussed in Section III.)

13. Scattering Function $M_c(K)$. The function

$$M_c(K) \equiv M_1(K)/(2\pi)^4 \delta^4(\Sigma k_i) \tag{2.31}$$

is called a scattering function.

14. Landau Diagrams. A Landau diagram is a set

$$D = \{L_j, V_n, \epsilon_{jn}\}$$

consisting of several directed line segments L_j and two or more vertices V_n. Each V_n contains end points of three or more of the L_j, but only one end point of any single L_j. The structure of D is defined by the set of numbers ϵ_{jn} defined by

$$\begin{aligned} \epsilon_{jn} &= +1 \quad \text{if} \quad L_j^+ \subset V_n , \\ \epsilon_{jn} &= -1 \quad \text{if} \quad L_j^- \subset V_n , \\ \epsilon_{jn} &= 0 \quad \text{otherwise} , \end{aligned} \tag{2.32}$$

where L_j^+ and L_j^- are the leading and trailing end points of L_j respectively. With each L_j is associated a type of particle t_j whose mass is μ_j. If particles of type t_j carry a_j units of an additively conserved quantum number "a" then the conditions

$$\sum_j a_j \epsilon_{jn} = 0 \quad (\text{all } n) \tag{2.33}$$

are required of D.

The lines L_j are characterized as being initial, final, or internal according to the following rules:

$$L_j \text{ is final if } \epsilon_{jn} \leqslant 0 \text{ for all } n , \tag{2.34a}$$

$$L_j \text{ is initial if } \epsilon_{jn} \geqslant 0 \text{ for all } n , \tag{2.34b}$$

$$L_j \text{ is internal if neither of the above holds.} \tag{2.34c}$$

The initial and final lines are called external lines.

A Landau diagram $D(K)$ is a Landau diagram whose initial and final lines can be placed in one-to-one correspondences with the initial and final particles, respectively, associated with the set $K = (K', -\tilde{K}'')$. A $D_c(K)$ is a connected Landau diagram $D(K)$.

15. <u>Landau Surfaces $\mathfrak{M}[D]$.</u> Consider an association

$$L_j \leftrightarrow (\alpha_j, p_j) \tag{2.35}$$

between lines of a Landau diagram D and pairs consisting of a nonzero number α_j and a (positive energy) energy-momentum vector p_j. The Landau surface $\mathfrak{M}[D]$ is the set consisting of the p_j associated with the external lines of all associations (2.35) satisfying the conditions

$$p_j^2 = \mu_j^2 \quad (\text{all } j) , \tag{2.36a}$$

$$\sum_j p_j \epsilon_{jn} = 0 \quad (\text{all } n) , \tag{2.36b}$$

$$|\omega_n - \omega_m| \equiv \left((\omega_n - \omega_m)\,(\omega_n - \omega_m)\right)^{\frac{1}{2}} , \qquad (2.40)$$

which is necessarily positive, since $\omega_n - \omega_m$ is timelike. The parameters ω_n can be interpreted as vectors from an arbitrary origin to the vertices V_n of the energy-momentum diagram $\bar{D}$ associated with $\mathfrak{M}^+[D]$.[21]

Since (2.39) is invariant under translations and dilations, every point of $\mathfrak{M}^+[D]$ is achieved by a fivefold continuum of sets $\{\omega_n\}$. Sets $\{\omega'_n\}$ not exhibiting these degeneracies are therefore introduced:

<u>Definition 2.8 .</u> A set $\{\omega'_n\}$ is a set $\{\omega_n\}$ satisfying $\Sigma\, \omega'_n = 0$ and $\sum_{n > m} \sum_i |\omega'_n - \omega'_m|\, |\epsilon_{in}|\, |\epsilon_{im}| = 1$.

<u>Definition 2.9 .</u> A <u>simple point</u> of $\bar{\mathfrak{M}}_c{}^+(K)$ is a point $\bar{K}$ of $\bar{\mathfrak{M}}_c{}^+(K)$ such that all points of $\bar{\mathfrak{M}}_c{}^+(K)$ in some neighborhood of $\bar{K}$ are points of just a single surface $\bar{\mathfrak{M}}^+[D_c(K)]$, and such that the inverse functions $\omega'_n(K)$ are single-valued, continuous functions of $K \in \bar{\mathfrak{M}}_c{}^+(K)$ in some neighborhood of $\bar{K}$.

16. <u>Landau Condition for Physical Region Singularities.</u>

<u>Definition 2.10 .</u> The <u>Landau condition for physical region singularities</u> is the condition that $M_c(K)$ be analytic at points K of $\mathcal{P}(K) - \bar{\mathfrak{M}}_c{}^+(K)$.

This condition was derived in Ref. 13 from an asymptotic causality condition formulated within the mass-shell S-matrix framework.

17. <u>The $i\epsilon$ Rules.</u> The second chief result of Ref. 13 is the "basic $i\epsilon$ rule" defined as follows.

<u>Definition 2.11 .</u> The <u>basic $i\epsilon$ rule</u> is the assertion that for any simple point $\bar{K}$ of $\bar{\mathfrak{M}}^+_c(K) \cap \mathcal{P}(K)$ there is a neighborhood $N(\bar{K})$ of $\bar{K}$

and a function $M_c^N(K)$ defined and analytic at points of $[N(\bar{K}) \cap \operatorname{Im} \sigma(K;\bar{K}) \geqslant 0] - \bar{\mathfrak{M}}^+_c(K)$, and coinciding with $M_c(K)$ in $[\mathcal{P}(K) \cap N(\bar{K})] - \bar{\mathfrak{M}}^+_c(K)$. The function $\sigma(K;\bar{K})$ is

$$\sigma(K;\bar{K}) \equiv \Sigma\, q_n(K)\, \omega'_n(\bar{K}), \tag{2.41}$$

where $q_n(K)$ and $\omega'_n(\bar{K})$ are the quantities defined in Subsection 15. Furthermore, the contributions from small neighborhoods of points of $N(\bar{K}) \cap \mathcal{P}(K) \cap \bar{\mathfrak{M}}^+_c(K)$ to a summation over physical points can be represented by an integration of $M_c^N(K)$ over a contour that passes around these points by detours into the domain of definition of $M_c^N(K)$. The $M_c^N(K)$ is an analytic extension of $M_c(K)$, and the superscript N is usually omitted. [Actually, $\operatorname{Im} \sigma(K;\bar{K})$ should, according to the result obtained in Ref. 13, be replaced by its minimum as the ω'_n in (2.41) range over arbitrarily small neighborhoods of the points $\omega'_n(\bar{K})$. This slight complication does not materially affect our arguments, and it will be ignored.]

The points of $\bar{\mathfrak{M}}^+_c(K) \cap \mathcal{P}(K)$ that are not simple points fall into various classes:

<u>Definition 2.12</u>. An <u>almost-simple point</u> of $\bar{\mathfrak{M}}^+_c(K)$ is a point $\bar{K}$ of $\bar{\mathfrak{M}}^+_c(K)$ such that in some neighborhood $N(\bar{K})$ of $\bar{K}$ there is a function $\sigma'(K;K')$, defined and continuous in both K and K', when both K and K' are in $N(\bar{K})$, such that for every K in $N(\bar{K})$ and K' in $N(\bar{K}) \cap \bar{\mathfrak{M}}^+_c(K) \cap \mathcal{P}(K)$

$$\sigma'(K;K') = \sigma(K;K'),$$

where $\sigma(K;K')$ is as defined in (2.41).

At an almost-simple point $\bar{K}$ of $\bar{\mathfrak{M}}^+_c(K) \cap \mathcal{P}(K)$ there is evidently no conflict between the $i\epsilon$ rules associated with different surfaces $\mathfrak{M}^+[D_c(K)]$; the distortions around the various singularities are mutually compatible.

<u>Definition 2.13 .</u> A <u>simply multiplicative point of $\bar{\mathfrak{M}}^+_c(K)$</u> is a physical point lying on several $\bar{\mathfrak{M}}^+[D_c(K)]$ the $D_c(K)$ of each of which is obtained from one single larger $D_c(K)$ by contracting to points all but one of various "independent parts" contained in it. An independent part of a $D_c(K)$ is a part having an independent dilation parameter, in the energy-momentum diagram $\bar{D}$ associated with $\mathfrak{M}^+[D_c]$. The various independent parts of a $D_c(K)$ touch each other only at single points, and the Feynman loops can all be confined to individual independent parts; i.e., no loop need pass through several independent parts.

Because the dilation parameters of independent parts are independent it follows from some algebra that the corresponding distortions can be made in independent combinations of the δp_i . The $i\epsilon$ rules for all of the surfaces $\bar{\mathfrak{M}}^+[D_c(K)]$ passing through a simply multiplicative point can therefore be simultaneously satisfied. Thus there is no difficulty extending the basic $i\epsilon$ rule to simply multiplicative points of $\bar{\mathfrak{M}}^+_c(K) \cap \mathcal{P}(K)$.

<u>Definition 2.14 .</u> The <u>extended $i\epsilon$ rule</u> is the extension of the basic $i\epsilon$ rule to cover combinations of surfaces that are related in the manner of surfaces at almost-simple or simply multiplicative points.

It is argued in Ref. 13, on the basis of physical considerations, that surfaces $\mathfrak{M}^+[D_c(K)]$ in the neighborhood of a point $\bar{\mathfrak{M}}^+_c(K) \cap \mathcal{P}(K)$ that are not related in the manner of singularity surfaces at an almost-

simple or simply multiplicative point are in fact completely independent, in the sense that they contribute additively to $M_c(K)$. That is, $M_c(K)$ in the neighborhood of any point of $\overline{\mathfrak{M}}^+_c(K) \cap \mathcal{P}(K)$ can be expressed as a sum of terms each containing sets of singularity surfaces related in the manner of those at almost-simple or simply multiplicative points. Continuation is made by continuing independently in each term. This provides for the "physical continuation" past any point of $\overline{\mathfrak{M}}^+_c(K) \cap \mathcal{P}(K)$.

Definition 2.15 . The general $i\epsilon$ rule is the assertion that in some neighborhood $N(\overline{K})$ of any point $\overline{K}$ of $\mathcal{P}(K) \cap \overline{\mathfrak{M}}^+_c(K)$ the function $M_c(K)$ breaks up into a finite number of terms to each of which the extended $i\epsilon$ rule applies. In particular, for each term the extended $i\epsilon$ rules specifies a region of continuation connecting points of $[\mathcal{P}(K) \cap N(\overline{K})] - \overline{\mathfrak{M}}^+_c(K)$, and summations over physical points of $N(\overline{K})$ are represented by integrals along contours distorted slightly into this region. Moreover, the decompositions in nearby neighborhoods $N(\overline{K})$ are "compatible in the sense that the distortions of contours can be extended globally by patching together distortions allowed in nearby neighborhoods. A detailed discussion is given in Ref. 13.

Remark 2.1 . It has not as yet been demonstrated that the number of terms in the above decomposition is indeed finite. This stipulation constitutes a special assumption of the present work.

Definition 2.16 . An essentially real path is a path that remains at real points except for arbitrarily small distortions around points of $\overline{\mathfrak{M}}^+_c(K)$ made in accordance with the general $i\epsilon$ rules. The physical functions $M_c(K)$ at points of $\mathcal{P}(K) - \overline{\mathfrak{M}}^+_c(K)$ are analytically connected by essentially real paths, according to the general $i\epsilon$ rules, and summations over physical points are represented by contour integrals over essentially real paths.

18. Persistence and the Interchange of Like Variables. A variable $\overline{V}_i$ is the triplet

$$\overline{V}_i = (k_i, m_i, t_i) .$$

Thus

$$K = \{\overline{V}_i\} .$$

Two variables $\overline{V}_i$ and $\overline{V}_j$ are called like variables if and only if

$$t_i = t_j , \tag{2.42a}$$

$$m_i = m_j , \tag{2.42b}$$

and

$$k_i^0 k_j^0 > 0 . \tag{2.42c}$$

[Equation (2.42c) is in fact implied by (2.42a), since according to (2.27) the sign of t_i is the same as the sign of k_i^0. However, (2.42c) is included for emphasis]. Like variables refer to particles differing only in their energy-momentum vectors k_i.

The assumption was made (Subsection 5) that a complete set of experimental results is labeled by the various possible sets K considered as unordered sets of variables. However, the variables of the analytic function $M_c(K)$ must originally be placed in some specific order. Let the set of points K for which $M_c(K)$ originally represents the physical function be called $\mathcal{P}$.

Suppose $K_1 \epsilon \mathcal{P}$ has two like variables and suppose these occupy positions i and j. Let $\mathcal{E}_{ij}$ be the operator that exchanges the variables that occupy positions i and j. The point K_2 is defined by

$$K_2 = \mathcal{E}_{ij} K_1 . \tag{2.42}$$

One may now inquire whether analytic continuation of $M_c(K)$ along essentially real paths from $K_1 \epsilon \mathcal{P}$ to K_2 is possible, and if so what significance the so-defined function $M_c(K_2)$ has, if any. The object of the remainder of this section is to show that the physical significance of the function $M_c(K)$ must persist when continued along essentially real paths to outside the original region of definition $\mathcal{P}$, and that $M_c(K_2)$ has, consequently, the same physical significance as $M_c(K_1)$. It will further be shown that $M_c(K_2)$ must be equal to $M_c(K_1)$ up to a possible sign, and that this sign must be the same for all M functions in which these two like variables appear, and must moreover be independent of the positions of these like variables within the sets K. The sign is therefore a universal quantity depending only on the type of variables interchanged. Once this is proved the remainder of the spin-statistics problem is to establish the connection between this universal sign and the spin of the particles corresponding to the interchanged like variables.

The problem of proving the universality of this sign under interchange does not generally arise in field theory, because there one generally assumes that the interchange of like operators gives at most a change of sign, and that this sign under interchange is independent of

states upon which the operators are acting. The reader willing to accept the corresponding proposition that the sign under interchange of like variables is a universal quantity, depending only on the particle type, may proceed to Section III.

Our natural idea of the connection between physical functions and analytic functions is that if a certain physical function is represented by a function analytic in some region, then this correspondence should "persist" as the variables move through a region where the physical function is defined and the mathematical function remains analytic; there should be no break in the correspondence so long as the mathematical function remains analytic at real points.

This persistence property follows, in fact, from the considerations of Ref. 13. There the M functions were considered initially to be distributions defined over test functions of compact support in momentum space. In the case of identical particles these test functions can be initially restricted to those having supports containing no pairs of distinct points related by an interchange of like variables. This restriction is imposed to avoid possible ambiguities associated with indistinguishability.

For a given process (specified by $\{m_i\}$ and $\{t_i\}$) a distribution is defined over this restricted space of test functions. This distribution is defined by the set of physical transition amplitudes between initial and final systems represented by the allowed set of test functions. According to B2 of Ref. 12 these physical transition amplitudes are well defined up to a possible overall phase.

It follows from the work of Ref. 13 that over the support of any allowed test function the distribution can be represented by a function analytic except at points of $\overline{\mathfrak{M}}^{+}_{c}(K)$. Furthermore, the functions defined in the intersection of two support regions must agree up to an overall phase, since they both represent the same physical process and hence must give the same relative for various test functions defined over the overlap regions. By patching these functions together one obtains a single function defined over the union of the allowed support regions. This function is single valued, since with all but two momenta fixed the regularity region is simply connected. It will continue to represent the single specified physical process even when continued outside of some original region $\mathcal{P}$. That is, the physical significance <u>persists</u> under analytic continuation, so long as the real path of continuation reaches no point of $\overline{\mathfrak{M}}^{+}_{c}$, where the analyticity property fails.

By the very same argument the result extends past the points of $\overline{\mathfrak{M}}^{+}_{c}(K)$, provided continuation is made along the essentially real paths; one simply patches together the functions over the various support regions, in each of which the result follows from the work of Ref. 13. Thus the physical significance of $M_c(K)$ cannot suddenly change; when continued along essentially real paths $M_c(K)$ continues to represent the correspondingly continued physical function.

The physical continuation from K_1 to K_2 has the effect of exchanging the detectors of the two like particles. That this has no effect on the experimental observables is just the content of the assumption of Subsection 5: the experimental results were there assumed to be specified by the sets $\{k_i, m_i, t_i\}$ considered as unordered sets of

variables, and, in particular, no additional information having to do with order of variables is supposed to be needed to identify the experimental result. Such information would be required if the experimental results depended on which piece of apparatus detected which particle.

One concludes from the above arguments that if K_1 is such that the overall conservation law is the only one satisfied at K_1, so that $M(K_1)$ is proportional to $M_c(K_1)$, then the experimental correlations are unaltered by the replacement of $M_c(K_1)$ by the $M_c(K_2)$ obtained by continuation along any essentially real path from K_1. In particular, at such a point K, we have

$$M_c(K_1) = \alpha M_c(\mathcal{E}_{ij} K_1) , \tag{2.43}$$

where $M_c(\mathcal{E}_{ij} K_1)$ is defined by analytic connection from $K_1 \in \mathcal{P}$ along any essentially real path, and α is a phase factor depending on the arguments other than spin indices displayed in

$$\alpha = \alpha_{ij}(K) . \tag{2.44}$$

The fact that the α are independent of the spin indices follows from the completeness of the set of spin matrices $\tilde{P}^{\dot{\alpha}\beta}$ in spin space: interference effects between amplitudes labeled by different spin quantum numbers are observable (see Ref. 12).

19. <u>The Sign Change Under Interchange of Like Variables.</u> By virtue of postulate <u>B2</u> of Ref. 12, linear combinations of amplitudes labeled by

K differing only in the values of k_i are observable; these are just the usual interference phenomena. This implies, as a generalization of (2.43), which we now rewrite as

$$M_c(K';K'') = \alpha_{ij}(K';K'')\, M_c(\mathcal{E}_{ij}K';K'') , \tag{2.45}$$

that

$$\begin{aligned} &M_c(K';\, K'') + M_c(K''';\, K'') \\ &\qquad = \alpha_{ij}(K',K''';K'')[M_c(\mathcal{E}_{ij}K';K'') + M_c(\mathcal{E}_{ij}K''';K'')] , \end{aligned} \tag{2.46}$$

where K' and K''' are sets differing only by values of the k_i. Substitution of (2.45) into (2.46) gives

$$\begin{aligned} &\alpha_{ij}^{-1}(K',K''';\, K'')\, [M_c(K';\, K'') + M_c(K''';\, K'')] \\ &= \alpha_{ij}^{-1}(K';\, K'')\, M_c(K';\, K'') \\ &+ \alpha_{ij}^{-1}(K''';\, K'')\, M_c(K''';\, K'') . \end{aligned} \tag{2.47}$$

If $M_c(K''';\, K'')$ is zero but $M_c(K';\, K'')$ is not, then $\alpha^{-1}(K',\, K''';\, K'') = \alpha^{-1}(K';\, K'')$. If both functions are nonzero, then there are two possible solutions of (2.47). The first is

$$\alpha_{ij}(K',K'''; K'') = \alpha_{ij}(K'; K'') = \alpha_{ij}(K'''; K'') . \tag{2.48}$$

This implies that $\alpha_{ij}(K'; K'')$ is independent of K'. On the other hand, we know that $\alpha_{ij}(K'; K'')\,\alpha_{ij}(\mathcal{E}_{ij}K'; K'') = 1$, since a double interchange is the identity. The nondependence on K' then implies

$$\alpha_{ij}(K'; K'') = \alpha_{ij}(K'') = \pm 1 . \tag{2.50}$$

The alternative solution to (2.47) gives

$$\frac{M_c(\mathcal{E}_{ij}K'; K'')}{M^*_c(K'; K'')} = \frac{M_c(\mathcal{E}_{ij}K'''; K'')}{M^*_c(K'''; K'')} . \tag{2.51}$$

This says that $M_c(\mathcal{E}_{ij}K)$ equals $M^*_c(K)$ up to a phase factor $\alpha_{ij}(K'; K'')$ that is independent of K'. This K'-independent phase factor must again be $+1$ or -1, as before.

If $M_c(K)$ had singularities at real points, then (2.51) would contradict the $i\epsilon$ rules. Thus solution (2.48) must hold for M functions having singularities at real points. But the unitarity equations demand there be singularities at least at normal thresholds. Thus only the case (2.48) is possible, and we have

$$M_c(K'; K'') = \alpha_{ij}(K'')\, M_c(\mathcal{E}_{ij}K'; K'') , \tag{2.52}$$

where $\alpha_{ij}(K'')$ is either $+1$ or -1.

20. Equality of Sign Changes for Interchange of Like Variables. If the set K' of $M(K'; K'') = M(K)$ contains several like variables located at positions $i, j, k, \cdots$, then there will be corresponding signs $\alpha_{ij}(K'')$, $\alpha_{ik}(K'')$, $\alpha_{jk}(K'')$, etc. These signs must all be equal. To see this let the exchange $\mathcal{E}_{jk}$ be applied to both sides of (2.52). This gives, suppressing the K'' dependence of the α's,

$$\alpha_{jk}\, M(\mathcal{E}_{jk} K) = \alpha_{jk}\, \alpha_{ij}\, M(\mathcal{E}_{jk}\, \mathcal{E}_{ij} K) , \tag{2.53}$$

which with the replacement of K by $\mathcal{E}_{jk} K$ and cancellation of α_{jk} becomes

$$\begin{aligned} M(\mathcal{E}_{jk}\, \mathcal{E}_{jk}\, K) &\equiv M(K) \\ &= \alpha_{ij}\, M(\mathcal{E}_{jk}\, \mathcal{E}_{ij}\, \mathcal{E}_{jk}\, K) \qquad (2.54) \\ &= \alpha_{ij}\, M(\mathcal{E}_{ik}\, K) , \end{aligned}$$

since, as may be readily confirmed,

$$\mathcal{E}_{jk}\, \mathcal{E}_{ij}\, \mathcal{E}_{jk} = \mathcal{E}_{ik} . \tag{2.55}$$

But (2.52), with j replaced by k, together with (2.54) gives

$$\alpha_{ik} = \alpha_{ij} . \qquad (2.56$$

This implies equations like

$$\alpha_{ik} = \alpha_{ij} = \alpha_{gj} \quad . \tag{2.57}$$

Hence all the $\alpha(K'')$ referring to exchanges of this particular kind of like variables are equal.

21. Order-Independence of Sign Changes Under Interchange of Like Variables. The sign $\alpha_{ij}(K'')$ is independent of the order of any like variables occurring in K'', for the relation (2.52) can be continued along essentially real paths to the point where the like variables of K'' are exchanged. That is, $\alpha_{ij}(K'') = \alpha_{ij}(\mathcal{E}_{kg}K'')$.

22. Persistence of Unitarity Equation. Initially arbitrary phases can be specified so that the no-scattering part is unity.

$$S_0(K'; K'') = \delta(K'; K'') \quad . \tag{2.58a}$$

This convention is uniformly adopted in each of the original regions arising in the proof of the persistence property. Thus the unitarity equation takes the form

$$\begin{aligned} & M(K'; K'') + M^*(K''; K') \\ & = - \sum_k M(K'; K)\, \tilde{G}(V)\, M^*(K''; K) \qquad (2.58b) \\ & = - \sum_k M(K\ ; K'')\, \tilde{G}(V)\, M^*(K\ ; K') \end{aligned}$$

at all real points connected to a physical point by a real path. That is, the form (2.58) of unitarity "persists." Of course individual terms continue in different ways around various singularities, but the entire equation nonetheless remains true.

23. Sign Change for Interchange of Related Like Variables. The sign change $\alpha_{ij}(K'')$ for the interchange of two like variables of K' in $M_c(K'; K'')$ is the same as the sign change $\bar{\alpha}_{ij}(K')$ for the interchange of the two corresponding like variables of K'', in the special case where K' and K'' are originally equal. One sees this applying both interchanges to the unitarity equations (2.58). The right-hand sides become the right-hand sides of unitarity at the new point. The two terms on the left, which are complex conjugates in the case $K' = K''$, become the terms on the left of this equation only if $\alpha_{ij}(K') = \bar{\alpha}_{ij}(K')$.

24. Universality of Sign Change Under Interchange of Like Variables. By virtue of the result of the above section the sign change $\alpha_{ij}(K'')$ is in fact independent of K''. If one interchanges like variables of K', but not K'', for the case in which K' originally equals K'', then the left-hand side of (2.58) is multiplied by $\alpha_{ij}(K'') = \bar{\alpha}_{ij}(K')$. Since the right-hand side is a sum of positive numbers each of these must undergo this same sign change in order that the equation remain valid. That is, $\alpha_{ij}(K'') = \bar{\alpha}_{ij}(K') = \alpha_{ij}(K) = \bar{\alpha}_{ij}(K)$ for all K such that $M(K'; K)$ is physical. The sign change is therefore a universal number depending on the type of variables interchanged but not on the position that these variables occupy in $M_c(K)$ or on the particular $M_c(K)$ in which the variables occur.

Remark 2.2 . No interchange of variables between initial and final sets has been discussed. In terms of the variables $K \equiv \{K', -\tilde{K}''\}$ we consider only interchanges of variables having the same type of variables, including sign. However, the interchange of two like variables of type t_i induces the same sign change as the exchange of two like variables of type $-t_i$. This is a rephrasing of the result of Subsection 23.

Remark 2.3 . The sign change holding for the $M_c(K)$ must evidently hold for the $M(K)$ as well: since the phase factors α_p in the cluster expansion equation are independent of the values of the arguments k_i , a continuation that interchanges the k_i of two like variables can give no change in α_p , and hence the sign changes in $M_c(K)$ must carry over to $M(K)$.

III. STRUCTURE THEOREMS

Some properties of the functions $B(K)$ represented by bubble diagrams are derived in this section.

A <u>bubble diagram</u> B is a collection of directed line segments L_i and signed circles called bubbles. The L_i are directed leftward and each one either issues from the left side of some bubble or terminates on the right side of some bubble, or does both. In this last case the line L_i is called an internal line of B. In the other two cases the line is called a final or an initial line of B, respectively. The bubbles of B are partially ordered by the requirement that each internal line terminate on a bubble standing left of the bubble when it issues.

Each line L_i of a bubble diagram B represents a variable (p_i, m_i, t_i) and each bubble b represents a function $F_b(K'_b ; K''_b)$, where K'_b is the set of variables represented by the lines issuing from the left of b and K''_b is the set of variables represented by the lines terminating on the right of b. The function $M^B(K)$ represented by B is a function of the variables represented by the external (noninternal) lines of B and is defined by

$$M^B(K) = \sum_{\text{int}} \prod_{b \in B} F_b(K'_b ; K''_b) \prod_i \tilde{G}_i(v_i) , \qquad (3.1)$$

where the summation is over all physical values of the variables represented by the internal lines of B, and the product over i runs over the indices i of all the internal lines L_i of B. The $\tilde{G}_i(v_i)$ are the spin-space factors (2.22) associated with the internal lines L_i,

and their spin indices are covariantly contracted on corresponding indices of the F_b. The function $F_b(K'_b ; K''_b)$ is either $M_1(K'_b ; K''_b)$ or $M^*_1(K''^*_b ; K'^*_b)$ according to whether the sign of b is plus or minus.

Remark 3.1. The summation over physical points is represented by integrations over contours that are distorted about singularities of the F_b in accordance with the $i\epsilon$ rules described in the earlier sections. Our first task will be to determine when the distortions prescribed by the various relevant $i\epsilon$ rules are mutually compatible.

Remark 3.2. The decomposition principle is, apart from the phase factors α_p, graphically exhibited by representing $M(K'; K'')$ as a sum of bubble diagrams. Each term in the sum consists of a column of plus bubbles such that every line represented by K' issues from the left of some bubble and every line represented by K'' terminates on the right of some bubble. The summation is over all different ways that the external lines can be connected to a column of bubbles. The contributions from certain of these terms will vanish due to the conservation law and mass constraints. Unitarity in the one-particle system requires that the "trivial" two-line bubble associated with an unscattered line be the "unit" operator $G_i(v_i)\,\delta(k'_i ; k''_i)$, aside from a phase factor that can be defined to be unity. (This definition fixes relative initial and final phases.)

Definition 3.1. With respect to $M^B(K)$ the physical points $\mathcal{P}(K)$ will mean the original points of definition of $M^B(K)$. At these points all the occurring $M_1(K'_b ; K''_b)$ are evaluated at physical points, or at points infinitesimally removed from them in the manner prescribed by the $i\epsilon$ rules. Analytic continuations from these original (physical) points will be discussed later.

Definition 3.2 . A $D' \subset B$ is a Landau diagram D' that can be constructed by replacing each bubble b of the bubble diagram B by either a connected Landau diagram D_c^b or by a point vertex V^b . The D_c^b is required to be a $D_c^b(K)$ such that $\mathcal{M}^+[D_c^b(K)]$ is a Landau surface corresponding to b .

Definition 3.3 . A contraction $D \supset D'$ of a Landau diagram D' is a Landau diagram D that can be obtained by shrinking to points certain internal line segments L_i of D' , and then removing all the line segments that terminate at their own origin points. D' is considered a trivial contraction of itself.

Definition 3.4 . A $D \supset\subset B$ is a Landau diagram D that is a contraction of some Landau diagram $D' \subset B$. If $D \supset\subset B$, then B is said to support D , and conversely.

Definition 3.5 .

$$\mathcal{M}^B(K) \equiv \bigcup_{D(K)\supset\subset B} \mathcal{M}[D(K)] \quad , \tag{3.2a}$$

$$\overline{\mathcal{M}}^B(K) \equiv \text{closure of } \mathcal{M}^B(K) \quad , \tag{3.2b}$$

$$\mathcal{M}_c^B(K) \equiv \bigcup_{D_c(K)\supset\subset B} \mathcal{M}[D_c(K)] \quad , \tag{3.2c}$$

$$\overline{\mathcal{M}}_c^B(K) \equiv \text{closure of } \mathcal{M}_c^B(K) \quad , \tag{3.2d}$$

$$\mathcal{P}(K) = \{K : K \text{ is a physical point}\} \ . \tag{3.2e}$$

Remark 3.2 . Every $M^B(K)$ contains a factor $(2\pi)^4\,\delta^4(\Sigma\, k_i)$.

Definition 3.6 .

$$M_c{}^B(K) \equiv M^B(K) \,/\, (2\pi)^4\,\delta^4(\Sigma\, k_i) \,. \tag{3.3}$$

Theorem 1. (First Structure Property)

If the functions $M_c(K)$ are analytic at points K of $\mathcal{P}(K) - \overline{\mathcal{M}}^+{}_c(K)$, and if the general $i\epsilon$ rules are valid, then the function $M_c{}^B(K)$ represented by a connected diagram B is analytic at points K of $\mathcal{P}(K) - \overline{\mathcal{M}}_c{}^B(K)$.

Proof. Define $\mathcal{R}(B,K)$ to be the set of physical points represented by the internal lines of B when the variables represented by the external lines of B are fixed at the physical point K . The general $i\epsilon$ rules then imply that the summation over physical points occurring in the definition of $M_c{}^B(K)$ can be represented as an integral over a contour that coincides with $\mathcal{R}(B,K)$ except for infinitesimal distortions into the appropriate upper or lower half σ_b planes near the points of the sets $\overline{\mathcal{M}}^+{}_c(K_b)$ corresponding to the functions $M_c(K'_b\,;\,K''_b)$ or $M^*{}_c(K''^*{}_b\,;\,K'^*{}_b)$, respectively. The general $i\epsilon$ rules also assure that the various distortions associated with various surfaces $\mathcal{M}^+[D_c{}^b]$ corresponding to a single bubble b do not conflict with one another. However, the distortions associated with surfaces $\mathcal{M}^+[D_c{}^b]$ corresponding to different bubbles b must also be compatible if the $i\epsilon$ rules are to assure a representation of $M_c{}^B(K)$ in which the contours can be made to avoid all singularities. We therefore examine the compatibility requirements on the distortions associated with a set of surfaces $\mathcal{M}^+[D_c{}^b]$, one for each bubble b of B .

The integration region is constrained by conservation-law and mass constraints. The conservation-law requirements are automatically satisfied if the integration variables are taken to be Feynman loop momenta h_f. Variations of these parameters h_f are subject, however, to the various mass constraints $\delta p_i^2 = 0$, where p_i is the momentum-energy vector associated with the internal line L_i of B. The variations δp_i^2 are given by

$$\delta p_i^2 = 2 \sum_f p_i n_{if} \delta h_f , \tag{3.4}$$

where n_{if} is the number of times loop f passes along line L_i in the positive sense minus the number of times f passes along L_i in the negative sense.

To calculate the variations $\delta \sigma_b$ one may write, using Eqs. (2.41) and (2.38), and suppressing the prime on ω'_n,

$$\begin{aligned}
\sigma_b &= \sum_n q_n^b \omega_n^b \\
&= - \sum_n \sum_{ex\ i} p_i^b \epsilon_{in} \omega_n^b \\
&= \sum_n \sum_{int\ j} p_j^b \epsilon_{jn} \omega_n^b \\
&= \sum_{int\ j} p_j^b \sum_n \epsilon_{jn} \omega_n^b \\
&= \sum_{int\ j} p_j^b \Delta_j^b ,
\end{aligned} \tag{3.5}$$

where p_j^b is the energy-momentum vector of the jth internal line L_j^b of D_c^b, and Δ_j^b is the difference of the two end points of the line L_j^b of the energy-momentum diagram $\bar{D}_c^b$ associated with $\mathcal{M}^+[D_c^b]$. The variation $\delta\,\sigma_b(K\,;\,\bar{K})$ for K in the neighborhood of a fixed point $\bar{K}$ of $\mathcal{M}^+[D_c^b]$ is then

$$\delta\,\sigma_b = \sum_{\text{int } j} \delta\, p_j^b\ \Delta_j^b(\bar{K})$$

$$= \sum_{\text{int } j} n_{jf}^b\ \delta\, h_f\, \Delta_j^b\ , \tag{3.6}$$

where n_{jf}^b is the algebraic number of times loop f passes along line L_j^b of the diagram D_c obtained by replacing the bubbles b of B by the connected Landau diagrams D_c^b. The particular path within D_c^b taken by the loop f is irrelevant to $\delta\,\sigma_b$, since the sum of $n_{jf}^b\,\Delta_j^b$ around any closed loop of D_c^b is zero: this sum is just the sum of vectors around a closed loop of the energy-momentum diagram $\bar{D}_c^b$.

By the theory of linear equations the variations $\delta\,\sigma_b$ and $\delta\,p_i^2$ can be specified in any desired manner by an appropriate choice of the $\delta\,h_f$ unless there is a set of α's, not all zero, such that

$$\sum_i \alpha_i\, p_i\, n_{if} + \sum_{b,j} \alpha_b\, \Delta_j^b\ n_{jf}^b = 0 \tag{3.7}$$

for all f. Now the vectors Δ_j^b can be expressed as

$$\Delta_j^b = \alpha_j^b\ p_j^b\ , \tag{3.8}$$

since both sides represent the vector of $\bar{D}_c^{\,b}$ corresponding to the line $L_j^{\,b}$ of $D_c^{\,b}$ [see Def. 2.4]. Thus (3.7) can be written

$$\sum_j \alpha'_j \; p_j \; n_{jf} = 0 \; , \tag{3.7'}$$

where the sum now runs over all internal lines L_j of the $D_c \subset B$. The α'_i is α_i for internal lines L_i of B and α'_j is $\alpha_b \, \alpha_j^{\,b}$ for internal lines $L_j^{\,b}$ of $D_c^{\,b}$.

The Eqs. (3.7') are just the Landau loop equations for D_c. Since the conservation-law and mass conditions are satisfied by the construction, the various $i\epsilon$ distortions associated with the various $\mathcal{M}^+[D_c^{\,b}]$ are mutually compatible at a point $\bar{K}$ of $\mathcal{R}(B,K)$ lying on the intersection of these surfaces $\mathcal{M}^+[D_c^{\,b}]$ unless the Landau equations are satisfied at $\bar{K}$. That is, the required distortions are mutually compatible at every point $\bar{K}$ of $\mathcal{R}(B,K)$ for every combination of surfaces $\mathcal{M}^+[D_c^{\,b}]$, one for each b of B, for all points K of $\mathcal{P}(K) - \bar{\mathcal{M}}_c^{\,B}(K)$.

It is also required, for analyticity, that the $i\epsilon$ distortion to imaginary values be a continuous function of the real point $\bar{K}$ of $\mathcal{R}(B,K)$. If $\bar{K}$ is a simple point, or almost-simple point, of all the $\bar{\mathcal{M}}^+[K_b]$, then each $\sigma_b(K;\bar{K})$ can be extended to a function $\sigma'_b(K;K')$ continuous in both variables in a neighborhood of $\bar{K}$. The space of the allowed $\delta \, h_f$ can be solved for in terms of the $\delta \, \sigma_b$. Since the equations are nonsingular, this space of the allowed $\delta \, h_f$ will be a continuous function of the $\delta \, \sigma_b$, and hence also of the point $\bar{K}$ in $\mathcal{R}(B,K)$, for simple and almost-simple points.

For a simply multiplicative point one can carry out an analysis similar to the one above by using a modified bubble diagram: a bubble b of B in which a simply multiplicative point occurs can be regarded as a cluster of bubbles connected to each other at single points. With each bubble of the cluster there will be a separate $\delta\,\sigma_b$. The analysis is then just the same as before and the results are the same.

The considerations for general singular points can be reduced to those for simple, almost-simple, or simply multiplicative points by means of the additivity property asserted by the general $i\epsilon$ rules. (The smooth fitting together of the decompositions defined in neighboring regions $N(\overline{K})$ is assured by the construction given in Ref. 13, as is discussed there).

It follows from the above argument that the contours can be distorted so as to remain in regions of analyticity for all K in $\mathcal{P}(K) - \overline{\mathfrak{m}}_c{}^B(K)$. To prove the theorem one needs, however, also to establish the analytic character of the surface $\mathcal{R}(B,K)$. To examine this question consider the transformation from the N_f variables h_f to the set of N_m variables p_i^2 and $N_f - N_m$ other variables x_j, where the N_m variables p_i^2 are the squares of the momentum energy carried by the internal lines of B. At points K of $\mathcal{P}(K) - \overline{\mathfrak{m}}_c{}^B(K)$ the variations $\delta\,p_i^2$ and $\delta\,\sigma_b$, considered as functions of the $\delta\,h_f$ are linearly independent. Thus the $\delta\,p_i^2$ are themselves linearly independent. Hence it is possible to choose, for any K in $\mathcal{P}(K) - \overline{\mathfrak{m}}_c{}^B(K)$ and any $\overline{K}$ in $\mathcal{R}(B,K)$, a set of $N_f - N_m$ variables x_j, linear in the h_f, so that $\partial(p_i^2, x_j)/\partial h_f$ is nonzero in a neighborhood of $\overline{K}$. This implies,[22] for K in $\mathcal{P}(K) - \overline{\mathfrak{m}}_c{}^B(K)$, that every point of $\mathcal{R}(B,K)$ is

an "interior point" of $\mathcal{R}(B,K)$. In fact, $\mathcal{R}(B,K)$ is an analytic manifold, which means that each point of $\mathcal{R}(B,K)$ is contained in an open set of points of $\mathcal{R}(B,K)$ that is the image of an open set in the space of the $N_f - N_m$ real variables x_j, under an analytic mapping $k_i = k_i(x_j)$.

Since $\mathcal{R}(B,K)$ is defined as the common zeros of a finite set of analytic functions it is necessarily a closed set. But a closed set consisting of interior points can have no boundary points. Thus, for K in $\mathcal{P}(K) - \overline{\mathcal{M}}_c{}^B(K)$, the set $\mathcal{R}(B,K)$ is a closed $N_f - N_m$ dimensional surface without edges (i.e. a cycle). This surface is confined to a bounded region in k_i space, and is easily shown to be of finite measure. Moreover the functions $k_i(x_j)$ are analytic (in fact linear) in K. Thus for K in $\mathcal{P}(K) - \overline{\mathcal{M}}_c{}^B(K)$, $\mathcal{R}(B,K)$ is a real analytic manifold of finite measure depending analytically on the variables of K. Moreover, as shown earlier, the contour can be distorted so that the integrand is analytic at all points K' on the contour. It therefore follows from Theorem A of Appendix A that the integral $M_c{}^B(K)$ is analytic at points K of $\mathcal{P}(K) - \overline{\mathcal{M}}_c{}^B(K)$. Since all the relevant quantities are well defined and depend analytically on the relevant variables it is, of course, highly plausible that the integral $M_c{}^B(K)$ should be analytic, though the proof is not completely trivial.

<u>Definition 3.7.</u> A <u>simple point of $\overline{\mathcal{M}}_c{}^B(K)$</u> is a point $\overline{K}$ of $\overline{\mathcal{M}}_c{}^B(K)$ such that in some neighborhood $N(\overline{K})$ of $\overline{K}$ all points of $\overline{\mathcal{M}}_c{}^B(K)$ belong to the $\mathcal{M}[D_c]$ of only one $D_c \supset\subset B$, and such that the α'_j in (3.7'), when subjected to the constraint $\sum_j |\alpha'_j\, p_j| = 1$, are uniquely defined continuous functions of the K in $N(\overline{K}) \cap \overline{\mathcal{M}}_c{}^B(K)$.

Definition 3.8 . $\mathfrak{M}_c^{B+}(K)$ is the subset of $\mathfrak{M}_c^{B}(K)$ that can be achieved by restricting the α'_j in (3.7') to be positive or negative for lines L_j of D_c contained in D_c^b's corresponding to plus or minus bubbles b of B , respectively. The remaining lines of D_c , which are just the lines occurring in B itself, can be either positive or negative.

Definition 3.5'. $\overline{\mathfrak{M}}_c^{B+}(K)$ = closure of $\mathfrak{M}_c^{B+}(K)$.

Definition 3.9 . $\tilde{\mathfrak{M}}_c^{B}(K)$ = {K:K is a point of $\overline{\mathfrak{M}}_c^{B}(K)$ that is not a simple point of $\overline{\mathfrak{M}}_c^{B}(K)$} .

Theorem 2. (Second Structure Property)

In Theorem 1 the set $\mathcal{P}(K) - \overline{\mathfrak{M}}_c^{B}(K)$ can be replaced by the set $\mathcal{P}(K) - \overline{\mathfrak{M}}_c^{B+}(K) - \tilde{\mathfrak{M}}_c^{B}(K)$.

Proof. For points on $\mathfrak{M}_c^{B}(K)$ it is not possible to arbitrarily specify all the variations $\delta\,\sigma_b$ and $\delta\,p_i^2$. But it may nonetheless be possible to find variations that keep $\delta\,p_i^2 = 0$ and $\eta_b \operatorname{Im} \delta\,\sigma_b > 0$, where η_b is the sign of bubble b of B . This is a sufficient condition for regularity, since it allows one to keep the contour in the region of analyticity.

The variations are subject to the condition (3.7). If $\overline{K}$ is a simple point of $\overline{\mathfrak{M}}_c^{B}(K)$ then there is only one such condition (3.7), since each such condition gives either another $\mathfrak{M}[D]$ or another set of α's. When there is only one condition (3.7), all but one of the variations $\{\delta\,p_i^2, \delta\,\sigma_b\}$ can be specified, and this remaining one depends continuously on the specified ones. Suppose for some pair of b the sign of the ratio of the α_b of the unique (3.7) differs from the ratio of the corresponding η_b . Consider a variation in which the

$\delta\,\sigma_b$ for one of these b's is the dependent variation and the $\delta\,\sigma_b$ of the other one of these b's is large compared to the remaining independent ones, which can be considered relatively infinitesimal. If the $\delta\,p_i^{\,2}$ are taken zero, then Eq. (3.7), together with continuity, assures that $\eta_b\,\mathrm{Im}\,\delta\,\sigma_b > 0$ is satisfied for the one dependent variation $\delta\,\sigma_b$ if it is satisfied for all the independent ones. Thus the function $M_c^{\,B}(K)$ is analytic at simple points of $\mathcal{P}(K)\cap\overline{\mathfrak{M}}_c^{\,B}(K)$ that are not on $\overline{\mathfrak{M}}_c^{\,B+}(K)$, which is what the theorem says. The signs of the α_b in (3.7) carry directly over to the corresponding signs in (3.7') because the $\alpha_j^{\,b}$ in (3.8) are all positive. Theorem 2 goes beyond Theorem 1 only if the single linear dependence relation (3.7) involves at least one α_b contribution. Thus the $\delta\,p_i^{\,2}$ contributions will still be linearly independent and $\mathcal{R}(B,K)$ will be an analytic manifold, just as in Theorem 1.

<u>Definition 3.10 .</u> Let D_c be the Landau diagram corresponding to a simple point $\overline{K}$ of $\overline{\mathfrak{M}}_c^{\,B+}(K)$. The corresponding energy-momentum diagram $\overline{D}_c$ is the diagram obtained by replacing each L_j of D_c by the energy-momentum vectors $\Delta_j = \alpha'_j\,p_j$ of Eq. (3.7'), with $\eta_b\,\alpha_b \geqslant 0$. (η_b is still the sign of bubble b .) The $\sigma(K;\overline{K})$ for $\overline{D}_c$ is now defined exactly as in (2.41).

<u>Definition 3.11 .</u> The <u>basic $i\epsilon$ rule for the functions $M_c^{\,B}(K)$</u> is the same as the "basic $i\epsilon$ rule" defined in Def. 2.11 except that $M_c^{\,B}(K)$ replaces $M_c(K)$ and $\overline{\mathfrak{M}}_c^{\,B+}(K)$ replaces $\overline{\mathfrak{M}}^{+}{}_c(K)$. That is, this rule asserts that $M_c^{\,B}(K)$ is analytic at points of the upper half $\sigma(K;\overline{K})$ plane near a simple point $\overline{K}$ of $\overline{\mathfrak{M}}_c^{\,B+}(K)$. However, there is one proviso: at least one line of the diagram D_c must correspond to an internal line of some bubble of B ; the basic $i\epsilon$ rule for the function $M_c^{\,B}(K)$ asserts (by definition) nothing about the case in which every line of D_c is a line of B .

Theorem 3. (Third Structure Theorem)

If the assumptions of Theorem 1 are satisfied then the basic $i\epsilon$ rule for the function $M_c^B(K)$ is valid.

Proof. The arguments leading to (3.6) and (3.7) give, similarly,

$$\begin{aligned}
\delta\,\sigma &= \sum_{\text{int } j} n_{jf}\,\delta\,h_f\,\Delta_j \\
&= \sum_{\text{int } j} n_{jf}\,\delta\,h_f\,\alpha'_j\,p_j \\
&= \sum_{\text{int } i} \alpha_i\,p_i\,n_{if}\,\delta\,h_f + \sum_{b,j} \alpha_b\,\Delta_j^{\,b}\,n_{jf}^{\,b}\,\delta\,h_f \\
&= \sum_{\text{int } i} \alpha_i\,p_i\,n_{if}\,\delta\,h_f + \sum_b \alpha_b\,\frac{\partial\,\sigma_b}{\partial h_f}\,\delta\,h_f \\
&= \sum_{\text{int } i} \alpha_i\,\delta\,p_i^2 + \sum_b \alpha_b\,\delta\,\sigma_b\,, \qquad (3.9)
\end{aligned}$$

where now the h_f include also the momentum-energy vectors carried along some paths similar to Feynman loops, but unclosed, that enter D_c at certain external vertices and leave at others. These vectors provide for the variations of the external variables, and will be called the external parameters h_f. The actual paths they take along the lines of D_c are not relevant to our argument.

The basic $i\epsilon$ rule to be proved refers only to simple points $\overline{K}$ of $\mathcal{P}(K) \cap \overline{\mathfrak{M}}_c^B(K)$. At these points there is only one equation (3.7') and consequently all the $\delta\,p_i^2$ and $\delta\,\sigma_b$ except one can be fixed arbitrarily by appropriate choice of the internal $\delta\,h_f$, as mentioned in Theorem 2. Therefore if we shift $\sigma(K;\overline{K})$ into the upper half plane by variations of the external $\delta\,h_f$ then the internal $\delta\,h_f$ can be adjusted

so that all δp_i^2 and all but one of the $\delta \sigma_b$ vanish. This last $\delta \sigma_b$, when multiplied by α_b, must therefore be shifted into the upper half plane, since σ is. But then by a slight adjustment of the internal δh_f the remaining $\alpha_b \sigma_b$ can also be shifted into their upper half planes, keeping the $\delta p_i^2 = 0$. This achieves the required result of moving all $\eta_b \sigma_b$ into their upper half planes, which are the regularity regions, while keeping all the $\delta p_i^2 = 0$.

This argument depends on the assumption that D_c contains some line that is an internal line of the Landau diagram D_c^b corresponding to some bubble b of B, since otherwise the contributions $\delta \sigma_b$ in (3.9) all vanish. If D_c has only the lines L_i that are the lines occurring already in B itself then continuation past the singularity is not possible in general. In fact, the phase-space factor in $M_c^B(K)$ vanishes at such a point. Thus the physical function $M_c^B(K)$ is zero on one side of such a singularity-manifold. On the other hand, if D_c contains any line that comes from the interior of any bubble b of B, then theorem 3 gives the rule for continuation past this singularity surface unless $\delta \sigma$ is necessarily zero. This circumstance can occur only if the external vertices of $\bar{D}_c$ all coincide, or all lie on a single line that is parallel to every external line incident upon all but one of the external vertices. (The external vertices are vertices upon which external lines end.) In this situation the mass constraints on the external lines force all variations $\delta \sigma$ to vanish, and hence no rule for continuation past the singularity is provided by Theorem 3.

IV. THE POLE-FACTORIZATION THEOREM

This section is devoted to a proof of the pole-factorization theorem. Some definitions are first introduced.

Definition 4.1 . A pole diagram D_p is a connected Landau diagram having precisely two vertices V_ν and $V_{\nu'}$, and precisely one internal line L_p . Each vertex therefore contains exactly one of the two endpoints of L_p :

$$\epsilon_{p\nu} = +1 \ , \tag{4.1a}$$

$$\epsilon_{p\nu'} = -1 \ . \tag{4.1b}$$

Definition 4.2 . The sets ν and ν' defined by a pole diagram D_p are the sets of external lines connected to the vertices V_ν and $V_{\nu'}$, respectively. The $\overline{\nu}$ and $\overline{\nu'}$ represent the corresponding index sets

$$\overline{\nu} = \{i : \epsilon_{i\nu} \neq 0 \ , \ i \neq p\} \ , \tag{4.2a}$$

$$\nu' = \{i : \epsilon_{i\nu'} \neq 0 \ , \ i \neq p\} \ . \tag{4.2b}$$

Definition 4.3 .

$$q_\nu = - \sum_{ex} p_i \, \epsilon_{i\nu} = \sum_{i\epsilon\overline{\nu}} k_i \ , \tag{4.3a}$$

$$q_{\nu'} = - \sum_{ex} p_i \, \epsilon_{i\nu'} = \sum_{i\epsilon\overline{\nu'}} k_i = - q_\nu \tag{4.3b}$$

$$S_\nu = q_\nu^2 = S_{\nu'} \ . \tag{4.3c}$$

Remark 4.1 . For a pole diagram D_p one has $\mathcal{M}^+[D_p] = \mathcal{M}[D_p] = \{K: S_\nu(K) = \mu_p^2\}$, where μ_p is the mass of the particle associated with line L_p of D_p . For $M_c(K)$, the condition of analyticity near $\bar{K}$ of $\mathcal{M}[D_p] \cap \mathcal{P}(K)$ in $\mathrm{Im}\ \sigma(K;\bar{K}) > 0$ implies analyticity near $\bar{K}$ in $\mathrm{Im}\ S_\nu(K) > 0$, as is shown by some simple algebra. For the function M_c^* represented by a minus bubble this region of analyticity is switched to $-\ \mathrm{Im}\ S_\nu(K) > 0$.

Definition 4.4 . A quasi-simple point $\bar{K} \in \mathcal{M}[D_p] \cap \mathcal{P}(K)$ of a function of the form

$$F(K) = \sum_{B \in \mathcal{B}} M_c^B(K) \tag{4.4}$$

is a point $\bar{K} \in \mathcal{M}[D_p] \cap \mathcal{P}(K)$ such that $F(K)$, considered as a distribution over a real neighborhood of $\bar{K}$, admits a decomposition into analytic functions[24]

$$\begin{aligned} F(K) &= f\big(S_\nu(K),\ W(K)\big) \\ &= \lim_{\epsilon \to 0} \Big[f_+\big(S(K) + i\epsilon,\ W(K)\big) - f_-\big(S(K) - i\epsilon,\ W(K)\big)\Big] \end{aligned} \tag{4.5}$$

where f_+ and f_- are analytic functions of their arguments in the region corresponding real K in a neighborhood of $\bar{K}$ and $0 \leqslant \epsilon < \eta > 0$, except possibly at points where $S(K) = \mu_p^2$ and $\epsilon = 0$. The set $W(K)$ is some set of arguments such that $\big(S_\nu(K),\ W(K)\big)$ gives a one-to-one analytic mapping of a neighborhood of $\bar{K}$ (in the domain of definition of $F(K)$) into a bounded open set in (S_ν, W) .

Definition 4.5 . A function $F(K)$ of the form (4.4) is said to have no pole [or worse] singularity at a quasi-simple point $\bar{K} \in \mathcal{M}[D_p) \cap \mathcal{P}(K)$ only if

$$\lim_{\epsilon \to 0} \epsilon f_{\pm} \left(S_\nu(\bar{K}) \pm i\epsilon , W(\bar{K})\right) = 0 . \qquad (4.6)$$

Definition 4.6 . The pole assumption P_1 for a simple-point $\bar{K}$ of $\mathcal{M}^+(K)$ lying on $\mathcal{M}[D_p] \cap \mathcal{P}(K)$ is the assumption that all pole singularities of functions $F(K)$ of the form (4.4) are associated with pole diagrams in the limited sense that if $\bar{K}$ is a quasi-simple point $\bar{K} \in \mathcal{M}[D_p] \cap \mathcal{P}(K)$ of $F(K)$ then $F(K)$ has no pole [or worse] singularity at $\bar{K}$ unless some $B \in \mathcal{B}$ supports D_p . Furthermore, the $i\epsilon$ rule for the part of $F(K)$ contributing to the residue (4.6) at such a point $\bar{K}$ is the same as the $i\epsilon$ rule for the various $D_p \supset\subset B \in \mathcal{B}$, provided these $i\epsilon$ rules are all the same (i.e., all have the same sign in $\pm \operatorname{Im} S_\nu > 0$) .

Definition 4.7 . The stability condition on physical-particle masses is the condition that the mass of any (physical) particle is less than the sum of the masses of any set of particles into which is allowed by selection rules to decay. Thus any (nontrivial) bubble b that represents a nonvanishing $M_1(K_b)$ must have at least two initial lines and at least two final lines. And correspondingly, each vertex V_n of any Landau diagram must contain the leading end points of at least two lines and the trailing end points of at least two lines: formal Landau diagrams not satisfying this condition are spurious and can be ignored.

Theorem 4 (Pole-Factorization Theorem)

Assumptions:

(1) Unitarity [Eq. (2.20)]

(2) Cluster Decomposition [Eq. (2.30)]

(3) Landau Conditions for Physical Region Singularities [Def. 2.10]

(4) Basic $i\epsilon$ Rules [Def. 2.11]

(5) Stability Conditions for Physical-Particle Masses [Def. 4.7]

Consequences: Let $\bar{K}$ be a simple point of $\mathcal{M}^+(K)$ lying on $\mathcal{M}[D_p] \cap \mathcal{P}(K)$ such that the pole assumption P_1 is valid at $\bar{K}$. Then $M_c(K)$ has a pole singularity at $\bar{K}$ whose residue

$$r(\bar{K}) = \lim_{\substack{K \to \bar{K} \\ \mathrm{Re}\, S_\nu(K) = \mu_p^2 \\ \mathrm{Im}\, S_\nu(K) > 0}} \left(S_\nu(K) - \mu_p^2\right) M_c(K) \tag{4.7}$$

is

$$r(K) = i\,\alpha\, M_c(K_\nu)\, \tilde{G}_p(v_p)\, M_c(K_{\nu'}) \,. \tag{4.8}$$

The sets K_ν and $K_{\nu'}$ are the sets of variables associated with the lines, both internal and external, incident on vertices V_ν and $V_{\nu'}$ respectively, of $D_p(K)$. The indices associated with particle p are covariantly contracted with the corresponding indices of $\tilde{G}_p(v_p)$, which is the spin matrix (2.22). The factor α is

$$\alpha = \alpha_a / \alpha_b\, \sigma_p \,, \tag{4.9}$$

where σ_p is the sign induced by interchange of two like variables p,

and α_a and α_b are the phase factors occurring in the decomposition

$$M(K_{\nu\nu'}) = \alpha_a M_1(K_\nu) M_1(K_{\nu'})$$

$$+ \alpha_b M_1(K) G_p(v_p) \delta(k'_p ; k''_p) \qquad (4.10)$$

$$+ \sum_{p \neq a,b} \alpha_p \prod_s M_1(K_{ps}) .$$

Here $K_{\nu\nu'}$ is the set of variables consisting of all those in either K_ν or $K_{\nu'}$. This set is just the set K plus two variables, one for an initial particle p and one for a final particle p. That α is independent of the order of variables in $K_{\nu\nu'}$ is assured by E2.

Proof. The functions $M(K'; K'')$ and $M^*(K''^*; K'^*)$ will be represented by plus and minus boxes, respectively, with the sets of lines issuing from the left and terminating on the right of these boxes being the lines representing the sets K' and K'' respectively. Then the cluster property is the assertion that the plus (minus) box is equal, apart from the phases α_p, to a sum of bubble diagrams, each consisting of a column of plus (minus) bubbles, with the sum being over all ways of connecting the given external lines to a column of bubbles. In this notation unitarity takes the form shown in Fig. 1.

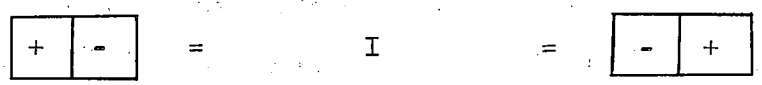

Fig. 1. Unitarity in box notation. The external lines are suppressed and a summation over all possible sets of internal lines is understood. The unit operator I is a product of factors $G(v_i)\,\delta(k'_i\,;\,k''_i)$.

Multiplication of $M(K)$ by unitarity gives the equation represented by Fig. 2.

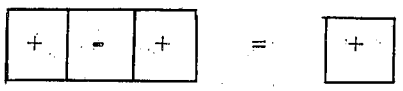

Fig. 2. Result of multiplying $M(K)$ by unitarity.

The terms on the left of Fig. 2 that support D_p will be classified with the aid of the following two lemmas, which depend upon the idea of "key bubble":

<u>Definition 4.8 .</u> A <u>key bubble</u> (relative to D_p) of a bubble diagram B is a bubble b of B such that every path in B from a line in the set ν defined by D_p to a line in the set ν' defined by D_p passes through b .

<u>Lemma 1.</u> If B supports D_p then B has a key bubble (relative to D_p) .

<u>Proof.</u> B supports D_p means there is a $D \subset B$ having an internal line $\bar{L}$ such that the contraction to points of all other internal lines of D gives D_p , with $\bar{L}$ of D becoming L_p of D_p . Every path in D from ν to ν' must pass along $\bar{L}$, for if there were one not passing along $\bar{L}$ then the contraction would give a path in D_p from ν to ν' not passing

along L_p, which is not possible. Since every path from ν to ν' passes along $\bar{L}$, each of these path passes also through any $\bar{b}$ such that $\bar{L}$ is an internal or external line of the D_c^b replacing $\bar{b}$ of B. At least one such $\bar{b}$ must exist, and any such $\bar{b}$ is a key bubble.

This argument proves, in addition to the lemma, the result asserted in the following corollary.

<u>Corollary.</u> Any $\bar{L}$ of $D \subset B$ that becomes the L_p of $D_p \supset D \subset B$ upon contraction of the other lines of D is an internal or external line of the D_c^b replacing some key bubble of B.

<u>Convention 4.1 .</u> In this section all trivial (two-line) bubbles will be considered absent: the unscattered particles of $M(K)$ will be represented by single lines containing no bubbles.

<u>Convention 4.2 .</u> Bubble diagrams that correspond to functions $B(K)$ that vanish because of combinations of mass constraints, conservation laws, and stability conditions will be considered not to exist.

<u>Definition 4.9 .</u> A <u>direct path</u> connecting two bubbles is a path that touches these two bubbles at, but only at, its two end points.

<u>Lemma 2.</u> A B that supports D_p has precisely one or two key bubbles (relative to D_p, always). In the first case no L of any $D \subset B$ can become the L_p of $D_p \supset D \subset B$ unless L is an internal line of the D_c^b that replaces the one key bubble. In the second case the two key bubbles are connected by a line $\bar{L}$ of B. Moreover, the removal of $\bar{L}$ from B disconnects the part of the diagram connected to ν from the part connected to ν'. This line $\bar{L}$ of B becomes L_p of $D_p \supset \subset B$ upon contraction of the rest of the diagram. No other L of any $D \subset B$ can become the L_p of a $D_p \supset D \subset B$, in this case of two key bubbles.

Proof. By lemma 1 there is at least one key bubble. If there is precisely one key bubble then any L of $D \subset B$ that becomes the L_p of $D_p \supset D \subset B$ upon contraction is an internal or external line of the D_c^b replacing this key bubble, by the corollary of lemma 1. If L were an external line of D_c^b then it would have to be an internal line of B itself. But then the bubble of B lying on the other end of L would also be a key bubble, contradicting the supposition that there is only one key bubble. Thus L must be an internal line of the D_c^b that replaces the one key bubble, in this case of just one key bubble.

If there is more than one key bubble, then pick two. These two are connected by some path in B, since B must be connected in order to support the connected D_p. This path can be taken to be a direct path, by removing closed loops. If this direct path touches some other bubble b', then any path from b' to any external line, L_e, of B must pass through one of the two key bubbles. Otherwise L_e could be connected to any specified external line of B by a path passing through at most one key bubble: one could pass via b' directly to the last of these two bubbles lying on some original path to that specified external line. But then all external lines of B would belong to the same set, ν or ν', to which that L_e belongs, since a path from ν to ν' must pass through all key bubbles, by definition. But, by virtue of our definitions, all external lines of B cannot belong to a single one of the two sets ν or ν', and hence any path from any b' to any L_e must pass through one of the two key bubbles.

This implies, in turn, that every b' lying on any direct path connecting the two key bubbles must stand to the right of one of these two key bubbles and must stand to the left of the other of these two key

bubbles; otherwise the rightmost of the b''s could have no initial lines or the leftmost of the b''s could have no final lines, which is not possible because of the conservation-law requirement. Thus the two key bubbles must be ordered, with one standing to the right of the other, and this is (trivially) true also if there is no b', since the connecting path is then simply a single line segment L_j, which is directed. If b_r is a key bubble that stands right of a key bubble b_ℓ, then all the lines of b_r lying on direct paths connecting b_r to b_ℓ must be final lines of b_r (which issue from the left of b_r), since otherwise either b_ℓ or some b' on some path from b_r to b_ℓ would have to stand right of b_r, contrary to assumption or to the above result.

If b is a key bubble of B its removal must give a diagram $B - b$ in which the parts connected to ν and ν', respectively, are relatively disjoint. The external lines of b belonging to these two parts will be called b^ν and $b^{\nu'}$ respectively. They are disjoint, and all external lines of b must belong to their union, since every one of these lines is connected in $B - b$ to some external line of B, by virtue of the fact that each bubble of B has both initial and final lines.

All external lines of b_r lying on direct paths to some other (fixed) key bubble b_ℓ must belong to a single one of the two sets b_r^ν or $b_r^{\nu'}$; otherwise ν and ν' would be connected by a path that passes through b_ℓ but not through b_r, which is impossible, since b_r is a key bubble. Moreover, all the external lines of b_r not lying on any direct path to the (fixed) key bubble b_ℓ must belong to a single one of the two sets b_r^ν or $b_r^{\nu'}$; otherwise ν and ν' could be connected by a path passing through b_r but not through b_ℓ, which is impossible

since b_ℓ is a key bubble. Thus the set of lines of b_r lying on direct paths to b_ℓ constitute one of the two sets b_r^{ν} or $b_r^{\nu'}$. This set consists of only final lines of b_r if b_r stands right of b_ℓ. On the other hand, we have

<u>Proposition 1.</u> An internal line L of a diagram D_c^b replacing a key bubble b of B can become an L_p of $D_p \supset\subset B$ only if b^{ν} and $b^{\nu'}$ both contain both initial and final lines of B.

<u>Proof.</u> The removal of L must disconnect b^{ν} from $b^{\nu'}$ in D_c^b, since otherwise ν and ν' would not become disconnected by the removal of L, as is required if L is to become an L_p of $D_p \supset\subset B$. If b^{ν} or $b^{\nu'}$ consisted of only initial lines or only final lines then the energy-momentum carried by L would have to be the energy-momentum carried by this set of initial or final lines. This conflicts with the stability requirements unless b^{ν} consists of a single line. But this possibility is precluded by the requirement that vertices of Landau diagrams must contain end points of three or more lines, together with the mass, conservation, and stability conditions and the positive $-\alpha$ condition imposed on the D_c^b by definition (3.2).

Combining Proposition 1 with the result stated just before it, we conclude that an L of $D \subset B$ that becomes L_p of D_p cannot be an internal line of the D_c^b replacing the key bubble b_r or, by exactly similar arguments, b_ℓ. Thus any L of $D \subset B$ that becomes L_p of D_p must be an external line of some key bubble, hence an internal line of B, in this case in which there is more than one key bubble.

If there is more than one key bubble then the L of D that becomes L_p of D_p can only be an internal L of B, as just shown. The two bubbles on either end of this internal line of B are both key bubbles. Since the line L connecting them becomes L_p of D_p, its removal must leave the parts of the diagram conneced to ν and ν' relatively disjoint. This proves the lemma for the case of precisely two key bubbles, since the existence of another L that becomes and L_p of $D_p \subset B$ would imply the existence of other key bubbles. It remains to be shown that there can be no other key bubbles.

Let b_r and b_ℓ be the right-hand and left-hand key bubbles on the two ends of some L of B that becomes L_p of D_p, in the case of more than one key bubble. Suppose there is another key bubble. If this other key bubble stands right of b_r then the lines of b_r lying on the direct paths to this other key bubble all lie on the right side of b_r and constitute one of the two sets b_r^{ν} or $b_r^{\nu'}$. The single line L, which is the only line of b_r lying on the direct path to b_ℓ, also constitutes one of the two sets b_r^{ν} or $b_r^{\nu'}$, and in fact the other one of these two sets, since it lies on the left of b_r. Thus L is the only final line of b_r. This contradicts stability. Thus this other key bubble cannot stand right of b_r. Neither can it stand left of b_ℓ. Nor can it stand to the right of b_ℓ and to the left of b_r, since this would imply the existence of a direct path between b_r and b_ℓ that parallels L, and hence precludes the possibility that L becomes L_p of D_p as required. Thus there can be at most two key bubbles. This concludes the proof of lemma 2.

Lemma 2 allows a classification of the terms on the left of Fig. 2 that can support D_p. First there are terms having only one key bubble. This single bubble can belong to any one of the three columns. Then there are the various terms having two key bubbles connected by a line of B, the removal of which separates B into two disjoint parts, one connected to the set ν and the other connected to the set ν'. And this line must become the L_p of D_p upon contraction of the rest of the diagram. The various terms supporting D_p are indicated in Fig. 3.

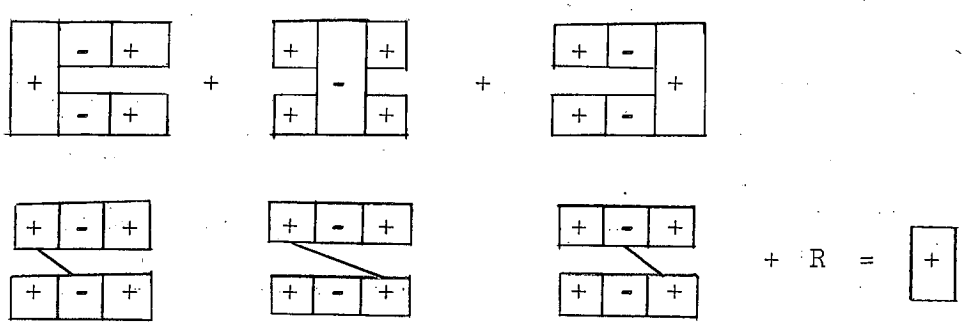

Fig. 3. Decomposition of left side of Fig. 2 into the six possible types of terms that support D_p plus a remainder term R that does not support D_p. The line terminating at the top or bottom of a box is suppose to end on some (nontrivial) bubble within that box.

The protruding products of little plus and minus boxes in the first and third terms are just the identity, by virtue of unitarity, and can be dropped.

For the next terms we make use of the identity shown in Fig. 4.

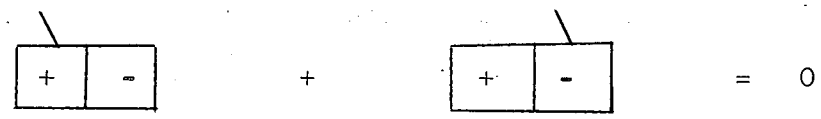

Fig. 4. Modified form of unitarity equation. The singled-out line emerging from the left side of the unitarity equation can come either from some nontrivial plus bubble, from some nontrivial minus bubble, or from the incident lines of the right. Terms of this third kind cancel the unity on the right of unitarity, leaving the equation represented in Fig. 4.

The equation represented by Fig. 4 allows the line leaving the lower minus box of term four of Fig. 3 to be shifted to the lower right plus box. Then unitarity can be used to cancel the protruding products of plus and minus boxes. An equation similar to that represented in Fig. 4 allows the fifth and sixth terms to be cancelled. Extraction of the connected part then gives Fig. 5.

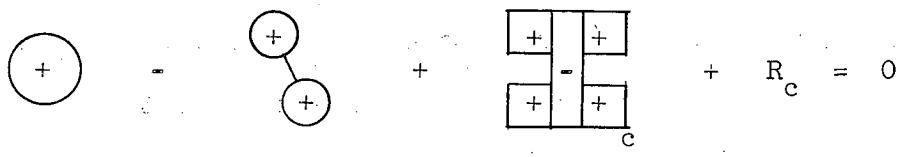

Fig. 5. Result of applying unitarity in lower-order sectors to Fig. 3. The subscript c denotes connected part. The phases α_p, here assumed to be unity, are discussed in the text.

The result claimed in the theorem now follows essentially from the fact that first and third terms in Fig. 5 are analytic in the upper and lower half S_ν planes respectively, as far as the pole contribution is concerned. The detailed argument is as follows: Near a simple point $\overline{K}$

of $\mathcal{M}[D_p] \cap \mathcal{P}(K)$ the first term of Fig. 5 is (after the conservation-law δ function is factored out) the limit of a function analytic at points K near $\bar{K}$ in the upper half S_ν plane, according to the basic $i\epsilon$ rule. The second term, which has a factor $2\pi\,\delta(S_\nu - \mu^2)$ coming from the phase-space factor (2.13), can be decomposed into a sum of two functions, one analytic at points K near $\bar{K}$ in the upper half S_ν plane and the other analytic at points K near $\bar{K}$ in the lower half S_ν plane. These two functions both have pole singularities at $S_\nu = \mu_p^2$, but are otherwise analytic at points K near $\bar{K}$, since the two M function factors can have no singularities at a simple point $\bar{K}$.

Since the sum of the first two terms of Fig. 5 can be decomposed into functions analytic in the upper and lower half S_ν planes the same must be true of the sum of the second two terms. These decompositions into upper and lower half parts are unique up to a function analytic at $\bar{K}$, by virtue of Theorem C of Appendix C. Thus the residue at $\bar{K}$ of the sum over all four terms in Fig. 5 of either the upper or the lower half-plane parts must separately be zero. However, the residue at $\bar{K}$ of the sum of the last two terms is zero for the upper-half-plane parts by virtue of the pole assumption P_1, which says that this sum is the limit of a function analytic in the lower half plane, so far as the pole contribution is concerned. Thus the residues of the upper-half-plane-parts of the first two terms of Fig. 5 must cancel. This gives just the desired result (4.8), apart from the effect of the phase factors α_p.

To complete the proof the case $\alpha_p \neq 1$ must be considered. Then one must be careful about the switching of lines on the boxes of Fig. 3 by means of Fig. 4. The equation to be used is shown in Fig. 6.

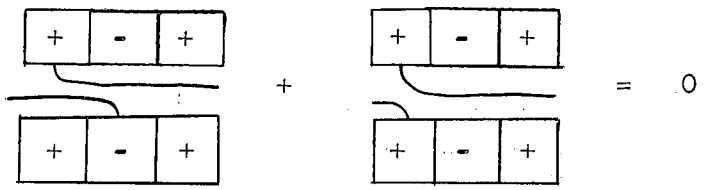

Fig. 6. Diagram indicating unitarity in the lower sector. Note that the left-hand column of the first term has an extra unscattered line that is not present in the fourth term in Fig. 3. The other two columns also contain extra unscattered lines. These lines induce phase changes.

In order to bring the fourth term in Fig. 3 into a form where Fig. 6 can be applied it must be multiplied by the phase factors α_p from the decomposition law that multiply a contribution to the left-hand term in Fig. 6 and divided by the phase factors that multiply this contribution in Fig. 3. By virtue of E2 and E3 this ratio of phases is a single phase that is independent of the particular contribution considered. It is in fact just σ_p times the α_b of (4.10), as is shown in Appendix B. After multiplication by this phase one can apply Fig. 6, which gives the product of the two plus bubbles appearing with the phase that they have in the second term of Fig. 6. This phase is just the α_a of (4.10). Dividing now by the unwanted phase $\alpha_b\ \sigma_p$ one obtains the required (4.9).

Concluding Remarks

(1) If the point $\bar{K}$ were a simple point of both $\overline{\mathfrak{M}}_c^{+}(K)$ and $\overline{\mathfrak{M}}_c^{B+}(K)$ for all B occurring in the third term of Fig. 5, then the second part of the pole assumption P_1 would not be necessary; one could use Theorem 3 instead.

(2) It will be assumed in what follows that $\mathcal{M}^+[D_p] \cap \mathcal{P}(K)$ has a dense set of simple points so that the pole-factorization property is valid for almost all points of $\mathcal{M}^+[D_p] \cap \mathcal{P}(K)$. This assumption should ultimately be confirmed by a study of the locations of the Landau surfaces $\mathcal{M}^+[D]$.

V. HERMITIAN ANALYTICITY

Hermitian analyticity is the property of scattering functions whereby $M_c(K'; K'')$ and $-M^*_c(K''^*; K'^*)$ are different boundary values of a single analytic function. The central idea of the present proof is to justify, within a strictly mass-shell framework, an effective continuation in external masses.

Instead of the original M_c one considers the M_c of a "larger" process, the external lines of which are those of a diagram constructed by connecting to each line L_i of the bubble representing the original M_c an "outer bubble" b_i, which is connected to other bubbles only along L_i. The M_c of the larger process will have poles at $S_i = \mu_i^2$, corresponding to these lines L_i, and will contain the original M_c as a factor of the residue of the product of these poles.

The unitarity equation for the larger process will be considered at a point P_0 where all $S_i = 0$. Certain continuations will then be made to points P where all $S_i = \mu_i^2$, and the residue of the product of the poles examined.

The unitarity equations at P_0 will consist of a sum of terms each represented by a bubble diagram. According to our basic pole assumption P_1 a function represented by a bubble diagram B can have a pole singularity at $S_i = \mu_i^2$ only if this diagram will support a corresponding pole diagram. In order to have a pole in each of the S_i the diagram must support each of the corresponding pole diagrams. Thus, according to lemma 1 of Theorem 4, the function represented by the connected bubble diagram B can have pole singularities in each of the set of channel energies S_i only if for each i, individually, the removal of some single

bubble b_i of B completely disconnects the external lines of one of the two complementary sets associated with S_i from those of the other of these two sets.

Definition 5.1 . A line of a bubble diagram is said to be directly connected to another line if and only if these two lines end on a common bubble. A set of lines, consisting of one or more lines, is said to be directly connected to another set of lines if and only if some line of one set is directly connected to some line of the other set. A bubble is said to be directly connected to a line that ends on it, or to a set containing a line that ends on it.

Definition 5.2 . The bubbles of a bubble diagram B that represents a contribution to unitarity can be classed as initial or final according to whether they contain end points of initial or final lines of B , respectively.

Convention (5.1). In this section trivial two-line bubbles will be considered inserted into the unscattered lines of each factor of a unitarity diagram, so that no bubble touches both initial and final lines of the diagram. Accordingly, each bubble is either an initial bubble or a final bubble, but not both. And each line is either initial, final, or internal.

Lemma 1. Let B be a connected bubble diagram representing a term in a unitarity equation. Suppose the initial lines $\mathcal{I}$ of B are divided into $n > 1$ disjoint sets $\mathcal{I}_i$, and the final lines $\mathcal{F}$ of B are divided into n disjoint sets $\mathcal{F}_i$. And suppose B is such that for each i , individually, there is a bubble b_i of B such that the removal of b_i from B completely disconnects the set $\mathcal{F}_i \cup \mathcal{I}_i$ from the complementary

set $\mathcal{F} \cup \mathcal{J} - \mathcal{F}_i \cup \mathcal{J}_i$. Then the internal lines of B can be divided into n disjoint sets I_i, plus a remainder set I^R, such that for each i either

(a) I_i is directly connected to every line of $\mathcal{F}_i$ but to no line of $\mathcal{F} - \mathcal{F}_i$, or

(b) I_i is directly connected to every line of $\mathcal{J}_i$ but to no line of $\mathcal{J} - \mathcal{J}_i$.

The set $\mathcal{E}_i$ is defined to be $\mathcal{F}_i$ or $\mathcal{J}_i$ in cases (a) and (b), respectively, and I_i contains every internal line directly connected to $\mathcal{E}_i$. (A - B is the set of elements belonging to A but not to B).

<u>Proof.</u> If for some i the set $\mathcal{J}_i$ is directly connected to the set $\mathcal{J} - \mathcal{J}_i$ (necessarily by an initial bubble) and also the set $\mathcal{F}_i$ is directly connected to the set $\mathcal{F} - \mathcal{F}_i$ (necessarily by a final bubble), then the removal of no single bubble can completely disconnect $\mathcal{F}_i \cup \mathcal{J}_i$ from $\mathcal{F} \cup \mathcal{J} - \mathcal{F}_i \cup \mathcal{J}_i$. Thus the stipulations of the lemma assure, for each i, either that $\mathcal{J}_i$ is not directly connected to $\mathcal{J} - \mathcal{J}_i$ or that $\mathcal{F}_i$ is not directly connected to $\mathcal{F} - \mathcal{F}_i$. If for any i only one of these two conditions is satisfied then we define $\mathcal{E}_i$ to be the set $\mathcal{J}_i$ or $\mathcal{F}_i$ that is not directly connected to $\mathcal{J} - \mathcal{J}_i$ or $\mathcal{F} - \mathcal{F}_i$, respectively, and define I_i to be the set of all internal lines of B directly connected to lines of $\mathcal{E}_i$. On the other hand, if both conditions are satisfied, for some i, then we have two sets $\mathcal{E}_i$ and $\mathcal{E}'_i$, one $\mathcal{J}_i$ and one $\mathcal{F}_i$, and two corresponding sets I_i and I'_i.

In this latter case it is impossible that $I'_i - I_i$ and $I_i - I'_i$ both be nonempty. If $I'_i - I_i$ is nonempty then b_i must lie at one end or the other of this set in order that the removal of b_i completely

disconnect $\mathcal{E}'_i$ from $\mathcal{F} \cup \mathcal{J} - \mathcal{E}'_i \cup \mathcal{E}_i$. And if $I_i - I'_i$ is nonempty then b_i must lie on one end or the other of $I_i - I'_i$. Both these sets' being nonempty would therefore require both that b_i either touch $\mathcal{F}_i$ or $\mathcal{J}$ but not $\mathcal{J}_i$ and that b_i either touch $\mathcal{J}_i$ or $\mathcal{F}$ but not $\mathcal{F}_i$. This is impossible, and hence one of the two sets $I'_i - I_i$ or $I_i - I'_i$ is empty. We adjust the definitions so that $I_i - I'_i$ is empty, which defines the I_i for this case. The definition is unique because $I_i - I'_i$ and $I'_i - I_i$ cannot both be empty, as this would make $I_i = I'_i$, and the diagram would not be connected.

Having defined the I_i we must now prove them disjoint. Two I_i corresponding to two initial sets $\mathcal{E}_i = \mathcal{J}_i$ must evidently be disjoint. For if they contained a common line then the initial bubble connected to this line would directly connect these two sets $\mathcal{E}_i = \mathcal{J}_i$. But the defining characteristic of these sets $\mathcal{E}_i = \mathcal{J}_i$ is that no initial bubble connect a line of $\mathcal{J}_i$ to a line of $\mathcal{J} - \mathcal{J}_i$. Similarly two I_i corresponding to two final sets $\mathcal{E}_i = \mathcal{F}_i$ are disjoint.

Finally the I_i corresponding to an initial $\mathcal{E}_i = \mathcal{J}_i$ must be disjoint from the I_j corresponding to a final $\mathcal{E}_j = \mathcal{F}_j$. For suppose $I_i \cap I_j$ were nonempty. The bubble b_i would then have to lie on one end or the other of the set $I_i \cap I_j$, in order that its removal disconnect $\mathcal{J}_i$ from $\mathcal{F}_j$. Suppose, first, that b_i were a final bubble directly connected to $\mathcal{F}_j = \mathcal{E}_j$, and hence not directly connected to $\mathcal{F}_i$. In order that its removal completely disconnect the

lines of $\mathcal{F}_i$ from those of $\mathcal{F} - \mathcal{F}_i$ it must be true that $\mathcal{F}_i$ is not directly connected within B to $\mathcal{F} - \mathcal{F}_i$. For any bubble directly connecting them could not be b_i, since b_i is not directly connected to $\mathcal{F}_i$. But if the lines of $\mathcal{F}_i$ are not originally directly connected to those of $\mathcal{F} - \mathcal{F}_i$ then we must have $\mathcal{F}_i = \mathcal{E}'_i$, since $\mathcal{E}_i = \mathcal{D}_i$. There is, then, a set I'_i, and the set $I'_i - I_i$ is nonempty, by construction. Thus b_i must lie one one end or the other of $I'_i - I_i$, as mentioned before. This means that b_i must either be a final bubble directly connected to $\mathcal{E}'_i = \mathcal{F}_i$ or it must be an initial bubble not directly connected to $\mathcal{E}_i = \mathcal{D}_i$. This contradicts the assumption that b_i was a final bubble directly connected to $\mathcal{F}_j = \mathcal{E}_j$, hence directly connected to no lines of $\mathcal{F} - \mathcal{F}_j$.

Suppose, alternatively, that b_i were an initial bubble directly connected to $\mathcal{E}_i = \mathcal{D}_i$. Again we must have $\mathcal{F}_i = \mathcal{E}'_i$ and $I'_i - I_i$ nonempty. Thus again b_i would have to be either a final bubble directly connected to $\mathcal{E}'_i = \mathcal{F}_i$ or an initial bubble not directly connected to $\mathcal{E}_i = \mathcal{D}_i$. This is again contradictory. Thus $I_i \cap I_j$ must be empty, which proves the lemma.

It follows from the definition of I_i that the energy carried by the internal set I_i is equal to the energy carried by the external set $\mathcal{E}_i$. Moreover, the energy carried by the set of all internal particles is the sum of the energies carried by all of the final sets $\mathcal{F}_i$ (or by all of the initial sets $\mathcal{D}_i$). Thus the energy carried by the set I^R of lemma 1 is $E^R = \Sigma\, q_i^{\,0}$, where $q_i^{\,0}$ is the energy part of the vector q_i that is the energy-momentum carried by $\mathcal{F}_i$ minus the energy-momentum carried by $\mathcal{D}_i$, and the sum is over those i for which $\mathcal{E}_i = \mathcal{D}_i$. The point P_0 is taken to be a point where $q_i = 0$ for all i. Thus E^R is zero at P_0. This means I^R is empty at P_0.

Lemma 2. A connected diagram B such that the I^R of lemma 1 is empty has either $\mathcal{E}_i = \mathcal{D}_i$ for all i or $\mathcal{E}_i = \mathcal{F}_i$ for all i.

Proof. No bubble can be directly connected to a line of an I_i corresponding to an $\mathcal{E}_i = \mathcal{D}_i$ and also to a line of an I_j corresponding to an $\mathcal{E}_j = \mathcal{F}_j$. For any initial bubble directly connected to a line of I_i cannot be directly connected to any internal line not in I_i, by virtue of the definition of I_i. Similarly any final bubble directly connected to a line of I_j cannot be directly connected to any internal line not in I_j. Thus, since $I_i \cap I_j$ is empty, neither an initial nor a final bubble can be directly connected both to a line of I_i and to a line of I_j.

Let C_i be the set of bubbles of B directly connected to any line of any I_i corresponding to an $\mathcal{E}_i = \mathcal{D}_i$. And let C_j be the set of bubbles of B directly connected to any line of any I_j corresponding to an $\mathcal{E}_j = \mathcal{F}_j$. According to the above result the sets C_i and C_j are disjoint. If I^R is empty then every bubble of B must be in either C_i or C_j. And moreover no line of B can connect a bubble of C_i to a bubble of C_j. Since B is connected it follows that either C_i or C_j must be empty.

Lemma 3. If the I^R of lemma 1 is empty, then all the b_i of B in lemma 1 must be one and the same bubble $b = b'$.

Proof. It was shown at the end of lemma 1 that b_i cannot be directly connected to $\mathcal{E}_i$. The same argument show that b_i must be directly connected to the other one of the two sets $\mathcal{F}_i$ or $\mathcal{D}_i$.

Consider, in view of lemma 2, the case in which all $\mathcal{E}_i = \mathcal{F}_i$. Then each b_i is an initial bubble directly connected to $\mathcal{J}_i$. If b_i is directly connected also to either $\mathcal{J}_j$ or I_j then $b_i = b_j$, for the removal of no other bubble could then disconnect either $\mathcal{J}_j$ or $\mathcal{F}_j$ from $\mathcal{J}_i$. But b_i must be directly connected to some $\mathcal{J}_j$ or I_j with $j \neq i$ for B to be connected: if b_i were directly connected to no $\mathcal{J}_j$ or I_j with $j \neq i$ then the part of B connected to $\mathcal{F}_i \cup \mathcal{J}_i$ could not be connected to the rest of B, since I^R is empty.

By this argument b_i must in fact be directly connected to either I_j or $\mathcal{J}_j$ for every j. For if it were directly connected to one of these two sets only for a proper subset J of the j's, then the replacement of b_i to give back B could not reconnect the part of B connected to the $\mathcal{F}_j \cup \mathcal{J}_j$ for j in J to the rest of B. Thus either I_j or $\mathcal{J}_j$ is directly connected to b_i for every j and one has $b_i = b_j$ for every j. The case in which all $\mathcal{E}_i = \mathcal{J}_i$ is essentially the same. This completes the proof.

According to the above lemmas and discussion the unitarity equations (for the larger process) at P_0 has only two kinds of terms that will contribute, when continued to P, to the residue of the product of all the poles. The first is the kind in which all $\mathcal{E}_i = \mathcal{F}_i$ and the second is the kind in which all $\mathcal{E}_i = \mathcal{J}_i$. The sums of the terms of these two kinds are represented by the two diagrams of Fig. 7.

First Kind
A^+

Second Kind
A^-

Fig. 7. Diagrams representing the sums of terms of the first and second kind. In this figure n, the number of outer bubbles, is 4. The small boxes represent the sums of sets of bubbles of the indicated sign connected to, and only to, the indicated set $\mathcal{E}_i$. The large rectangle consists of the sum of all sets of bubbles of the indicated sign such that the overall diagram is connected, and such that the removal of some single bubble b' of this rectangle disconnects each set $\mathcal{F}_i \cup \mathcal{J}_i$ from every other one. Only connected diagrams are included because we consider here only the connected part of the unitarity equation, which is itself a valid equation, since the disconnected parts themselves give valid equations, by virtue of postulates E2 and E3.

The sums of terms of the first and second kinds will be denoted by A^+ and A^-, respectively. The functions A^+ and A^- will be continued from P_0 to points P^+ and P^-, respectively, by detouring around singularities of the terms of these functions in accordance with the basic $i\epsilon$ rules for the various functions $M_c^B(K)$ constituting these terms. Near the points $P^\pm$ the pole terms indicated in Fig. 8 become dominant.

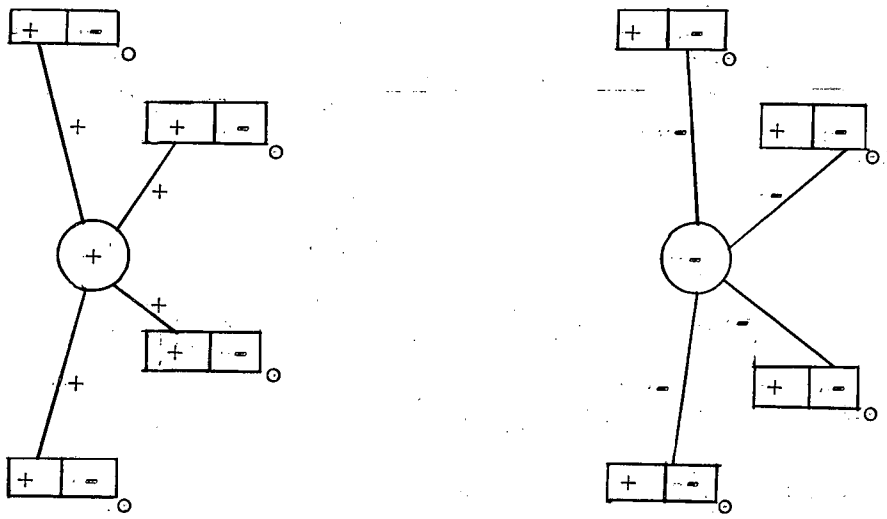

Fig. 8. Diagrams representing the dominant pole contributions near $P^{\pm}$. The multipole contributions come from insertion of the pole contributions into the key bubble b' of one of the large rectangle. The subscripts 0 denote connected parts, and also the possible limitations in the set of diagrams arising from the fact that the terms occurring in $A^{\pm}$ are those present at P_0.

Figure 8 can be considered to represent the residue of the product of the poles at $P^{\pm}$, then the plus and minus lines represent the factors $\pm i\,\tilde{G}_i(v_i)$. In particular the residues have the forms

$$r^{\pm} = M^{\pm}_c(K) \prod_i [\pm i\,\tilde{G}_i(v_i)\, F_i^{\pm}(K_i)], \tag{5.1}$$

where the $F_i^{\pm}(K_i)$ are the functions represented by the outer boxes of the right- and left-hand diagrams of Fig. 8 and $M^{\pm}_c(K)$ represent the original $M_c(K'; K'')$ and its conjugate $M^*_c(K''^*; K'^*)$.

The unitarity equation at P_0 is $A^+ + A^- + A^0 = 0$, where A^0 is the sum of the terms of the unitarity equation appearing in neither A^+ nor A^-. One can think that A^+ is continued from P^+ to P_0; at P_0 one adds the function A^0 and obtains, according to unitarity, the result $-A^-$; and function $-A^-$ is then continued to P^-. One has in this way a path of continuation leading from P^+ to P_0 to P^-, but there is a "jump" across a certain cut at P_0. The discontinuity across this cut at P_0 is A^0.

Following the method of Ref. 8, we now attempt to shift (distort) this path of continuation $P^+ P_0 P^-$ to a new path $P^+ P'_0 P^-$ that lies completely in the manifold $S_i = \mu_i^2$ for all i. This will be done by constructing a set of manifolds $S_i = a_i^2$ and gradually increasing the a_i^2 from zero to μ_i^2. The original path $P^+ P_0 P^-$ touches $S_i = 0$ (all i) at P_0. As one increases the a_i^2 the part of the path in $S_i \leqslant a_i^2$ is shifted into the manifold $S_i = a_i^2$ while the part of the original path $P^+ P_0 P^-$ in $S_i \geqslant a_i^2$ is left as originally. The original path $P^+ P_0 P^-$ then defines the end points of the part of the path lying at constant a_i.

In this distortion of the path there is the jump originally at P_0 to consider. If the singularities of the function A_0 are confined, locally, to a finite number of singularity manifolds, then A_0 can be continued to a (continually shifting) point on the shifting path. Then one simply adds at this point the jump defined by the continued function A_0. When one arrives finally at the situation where all $a_i^2 = \mu_i^2$ and considers the residue of the product of poles, this jump will not contribute, since A^0, by construction, is the set of terms not having all the requisite

poles. The pole assumption P_1 is now considered extended to nonphysical points, as will be discussed momentarily.[25]

In the process of distortion, various singularities may be met by the shifting path. One can perhaps distort the path away from them. Alternatively one can jump across the cut trailing such a singularity s by adding the discontinuity across the cut, just as one did at P_0. If the singularities of this discontinuity function are confined to a finite number singularity manifolds, then the function can be continued to a (continually shifting) point P_s on the shifting path. Then in the analytic continuation from P^+ to P^- along the shifting path one simply adds at the point P_s this discontinuity function. The contribution from such a discontinuity will not affect the residue at $a_i^2 = \mu_i^2$ unless the discontinuity has all the required poles.

To discuss which singularities have cuts having discontinuities having the requisite poles some definitions will be introduced.

<u>Definition 5.3 .</u> $D' \subset D$ is a Landau diagram D' that can be constructed by replacing some of the vertices V_n of D by connected Landau diagrams D_c^n.

<u>Remark 5.1 .</u> It is easily confirmed that if D' is a $D' \subset D$ then D is a contraction $D \supset D'$ of D' and conversely (see Def. 3.3).

<u>Definition 5.4 .</u> $D_1 \supset\subset D$ is a Landau diagram D_1 that is a contraction $D_1 \supset D'$ of some Landau diagram $D' \subset D$. D <u>supports</u> D_1 means $D_1 \supset\subset D$.

<u>Lemma 1D.</u> A Landau diagram D supports a pole diagram D_p only if D has a key vertex V_p such that every path in D connecting a line in the set of external lines ν defined by D_p to a line in the conjugate set ν' passes through V_p. Such a vertex V_p is called a key vertex of D (relative to D_p).

Proof. The proof is essentially identical to the proof of lemma 1 of Theorem 4.

Definition 5.5 . The pole assumption P_2 is the assumption that the discontinuity across the cut connected to a singularity surface $\mathfrak{M}$ of a function $M_c^B(K)$ represented by a connected bubble diagram B has no pole [or worse] singularity at $\mathfrak{M}[D_p]$ unless the surface $\mathfrak{M}$ is a surface $\mathfrak{M}[D]$ such that $D \subset B$ supports the pole diagram D_p [or possibly some other pole diagram D'_p with $\mathfrak{M}[D_p] = \mathfrak{M}[D'_p]$] . In this sense, all pole singularities at $\mathfrak{M}[D_p]$ are associated with the pole diagram D_p .

Remark 5.2 . Pole assumption P_2 is similar to pole assumption P_1 . It is more general in that it is not restricted to physical points $\mathcal{P}(K)$. [The functions $M_c^B(K)$ and the sets $\mathfrak{M}[D]$ are, of course, now analytically extended to include points not in $\mathcal{P}(K)$.] Aside from this difference the assumptions P_1 and P_2 would be very similar if the discontinuity across a surface $\mathfrak{M}[D]$ were given by a Cutkosky rule. This would make the discontinuity function essentially a bubble diagram function $M_c^{B'}(K)$ represented by a bubble diagram B' obtained by replacing each vertex of $D \subset B$ by an appropriate bubble. Then pole assumption P_1 , generalized to nonphysical points, would say that the discontinuity function has no pole [or worse] singularity unless $D \subset B$ supports D_p . By no pole [or worse] singularity we mean, here, no singularity that affects the residue of the pole.

The quantity of interest to us is the residue of the product over i of the poles at $S_i = \mu_i^2$. The only singularities having discontinuities contributing to this residue are, according to pole assumption P_2 , those associated with diagrams D supporting each of the corresponding pole

diagrams D_p^i. According to Lemma 1D a Landau diagram D can support a pole diagram D_p^i only if it contains a corresponding key vertex V_p^i. Thus the earlier arguments now show that A_0 will not contribute to the residue at either physical or nonphysical points.

The contributions to A^+ and A^- are represented by bubble diagrams B each having only one key bubble b'. If $D \subset B$ supports D_b^i then the key vertex V_p^i of D must be a vertex of the diagram $D_c^{b'}$ replacing b' in the construction of D from B; the removal of a vertex from the D_c^b of a nonkey bubble cannot effect the required separation, since the removal of the entire bubble does not. Thus, for a B representing a term in A^+ or A^-, which we write as $B \in A^{\pm}$, any key vertex of any $D \subset B \in A^{\pm}$ must be a vertex of the diagram $D_c^{b'}$ replacing b' in the construction of D from B. It then follows that any $D \subset B \in A^{\pm}$ containing all the required key vertices V_p^i can be constructed by replacing the bubbles in one of the terms indicated in Fig. 8 by Landau diagrams D_c^b and then contracting certain lines. At points $S_i < \mu_i^2$ (all i) all the lines L_i explicitly shown in Fig. 8 must be contracted to points, since the corresponding mass constraints are not satisfied at these points. The required key vertices V_p^i are then just these contracted lines L_i.

The Landau diagrams D associated with the singularities having all the required poles have, according to the above arguments, a very special structure. They consist of $n + 1$ "independent" parts connected only at key vertices. For each positive $i \leqslant n$ there is one "outer" independent part having the external lines $\mathcal{F}_i \cup \mathcal{J}_i$, and precisely one

of the n key vertices. For $i = 0$ there is the one inner part, which has no external lines but which has all n key vertices. These $n + 1$ parts are independent in the sense used in Section I : they have independent dilation parameters, and the Feynman loops can be confined to individual independent parts. Because of this, the Landau surfaces are just the Landau surfaces for these independent parts. That is, the singularity surface $\mathcal{M}[D]$ is just a sum of singularity surfaces $\mathcal{M}[D_i]$, where each surface $\mathcal{M}[D_i]$ is a surface in the variables associated with just one of the $n + 1$ independent parts. As a consequence, the path of continuation at fixed $S_i = a_i^2$ can be considered to be a product of paths $P_i(a_i)$, one in the variables associated with each of the $n + 1$ independent parts. For the outer parts there are mass constraints on each of the external lines, and there is one additional "mass" constraint $S_i = a_i^2$ associated with the vertex V_p^i . For the inner part there is a "mass" constraint $S_i = a_i^2$ associated with each vertex V_p^i . As the a_i increase, the motion of the singularity surfaces $\mathcal{M}[D_i]$ can be viewed as the motion of the Landau singularity surfaces for the individual parts under a continuation in the "masses" a_i . These "masses" a_i are, of course, not physical-particle masses, but rather variables of the larger process.

Consider now $P^+(a_i)\, P'_0(a_i)\, P^-(a_i)$, the part of the shifting path of continuation lying in the surfaces $S_i = a_i^2$. At $a_i = 0$ the two end points $P^+(a_i)$ and $P^-(a_i)$ of this part of the path coincide with the point P_0 . Then, as the a_i increase, the points $P^{\pm}(a_i)$ start moving along paths determined by the original paths $P_0\, P^+$ and $P_0\, P^-$. These

original paths lie in the physical region of the larger process and their detours around the physical region singularities are specified by the basic $i\epsilon$ rules for functions $M_c^B(K)$.

Apart from these $i\epsilon$ detours, the two paths $P_0\ P^+$ and $P_0\ P^-$ will be taken to be identical. Then $P^+(a_i)\ P'_0(a_i)\ P^-(a_i)$, becomes a closed loop, except for the small $i\epsilon$ gap between the two endpoints $P^+(a_i)$ and $P^-(a_i)$. Thus the only singularities that can get inside this loop are either physical region singularities that have, for some value of the a_i, entered through this gap, or singularities that have emerged from the cuts traiting physical region singularities that have entered through the gap.

One follows the motion of these singularities by a continuation in the "masses" a_i. Assuming, temporarily, that the paths can be kept away from the various singularities whose discontinuities contribute to the residue, one arrives finally at $a_i \simeq \mu_i$ (all i), and considers the residue of the product over all i of the poles at $S_i = \mu_i^2$. At the points P^+ and P^- one has the residues r^+ and r^- given in (5.1) and indicated in Fig. 8. By construction, continuation along the path $P^+\ P'_0\ P^-$ takes r^+ to $-r^-$.

As just discussed, $P^+\ P'\ P^-$ is effectively a product of paths P_i, with one path P_i in the variables associated with each factor in (5.1). Each factor in r^+ is, accordingly, continued along the corresponding path P_i. Under this continuation r^+ goes to r^h,

$$r^h = M_c{}^h(K) \prod_i i\,\tilde{G}(v_i)\, F_i{}^h(K_i) , \tag{5.2}$$

which we know to be $-r^-$.

Consider the paths P_i corresponding to the outer sets $\mathcal{F}_i \cup \mathcal{J}_i$. The path $P_0 P$ can, as we shall verify below, be chosen such that it crosses no singularity in these variables. In this case the outer paths P_i can be shrunk to points, which means that

$$F_i{}^h(K_i) = F_i{}^+(K_i) . \tag{5.3}$$

Moreover, in this case in which P_0P crosses no singularities in the outer variables, the terms contributing to the outer factors in Fig. 8 are the same at P_0 and P. Hence the subscript zero can be replaced by the subscript c denoting connected part. Then unitarity (Fig. 2) gives

$$F_i{}^+(K_i) + F_i{}^-(K_i) = 0 . \tag{5.4}$$

These equations convert the equation $r^h = -r^-$ into

$$M_c{}^h(K) = -M^-{}_c(K) , \tag{5.5}$$

which is just Hermitian analyticity. The path of continuation h, which is constructed by the procedure described above, is called the path of Hermitian analyticity.

The phases α of the pole-factorization theorem are incorporated into the functions $F^{\pm}(K)$. That the phases of the various contributions to (5.4) are then such as to ensure its validity, by virtue of unitarity, follows from the fact that the various contributions to unitarity associated with different connected structures, hence different conservation-law δ functions, must satisfy separately the unitarity equations, by virtue of postulates E2 and E3. The argument is similar to the one given in conjunction with these phases in the proof of the pole-factorization theorem, and need not be given again.

Equation (5.3) is valid provided $P_0 P$ is chosen so as to cross no singularities in variables associated with the outer processes. This can be achieved, for instance, by taking the outer processes to be simple two-particle scattering processes, and holding fixed, in the continuation from P_0 to P, the total energies E_i of these outer processes. The "masses" a_i are varied by varying the momenta of the particles of the outer processes. The only possible Landau singularities in the variables associated with these outer two-particle processes are normal-threshold singularities at constant E_i. This follows from a simple enumeration of possible physical region Landau diagrams for a two-particle process. These singularities at constant E_i will not be crossed because the E_i are held fixed (at values not at a normal threshold).

When the path $P^+ P_0 P^-$, which lies at physical points of the larger process, is shifted to $P^+ P'_0 P^-$, which lies at $S_i = \mu_i^2$, the values of (at least some of) the momentum vectors q_i associated with lines L_i must become complex (at some points on the new path). This is because the part of the path associated with the variables of the inner process is

forced to pass through the region below the physical threshold of the inner process, and such regions cannot be realized with real q_i. These complex values of the q_i can and will be obtained by performing appropriate complex Lorentz transformations on the corresponding outer parts; that is, the complexification of q_i will be obtained by a complex Lorentz transformation on all the vectors of the ith outer part. Since the singularity structure is not altered by a (real or complex) Lorentz transformation, the fact that q_i is complex will not reflect itself in the part of the path associated with the variables of the outer process; one can consider part of the path associated with the outer parts to lie at real values of the energy-momentum vectors, as far as the singularities in these parts themselves allow this.

In the above discussion it was assumed that the paths of continuation can be distorted so as to stay away from all singularity surfaces that contribute to the multipole residue. That this is possible follows in most cases from dimensional considerations: a one dimensional curve is, generally too "thin" to get trapped by a finite set of singularity surfaces. For instance, we know that contours of integration of real dimension n in a space of complex dimension n can get trapped at a point to give a singularity of an integral. But then for $n > 1$ a one dimensional curve will in general not be trapped: it can slide away from the usual pinch configuration in $n-1$ directions.

One can confirm this in a simple example: Consider the surfaces $z_1 = 0$, $z_1 + \alpha = z_2^2$ and the curve $x_1 = -\epsilon$, $x_2 = 0$, $y_1 = \tau$, $y_2 = \tau$, where $\alpha > \epsilon > 0$. The curve intersects the real plane at a point lying in the region $\mathcal{R}$ of the real plane bounded by the restriction to real points of the two surfaces. As α approaches zero this region $\mathcal{R}$ shrinks

to a point. But the curve can be moved away from the pinch, by shifting it, for example, to the curve $x_1 = -\epsilon + g(\tau-2)$, $x_2 = 0$, $y_1 = \tau$, $y_2 = \tau - 2$ where $g(\tau)$ is zero for $|\tau| \geqslant 1$ and greater than ϵ near $\tau = 0$.

This dimensional argument does not cover all cases, however. For example, a curve might get trapped between two surfaces that reduce to a single surface at a pinch configuration, for then the situation is essentially one dimensional. Though such possibilities can probably be ruled out, we do not pursue this tack, for in any case the curve might get pulled into an unphysical sheet by some singularity surface. In order to avoid this, the path of continuation will be taken to jump across the cuts trailing certain singularities, rather than detouring around them. In particular, if a singularity moves across the path of continuation then one can define the discontinuity function in the situation before the singularity leaves the physical sheet, or is pinched against another singularity. The path of continuation can then be taken to jump across the cut, by adding the discontinuity function at the cut. If these cuts are part of the boundary of the physical sheet then the path of continuation will remain always on the physical sheet. The definition of the physical sheet given in Refs. 1 and 12 was in terms of essentially this same procedure of continuation in external "masses", though there it was justified by comparison to perturbation theory, rather than by mass-shell considerations, as was done here. We shall return to a discussion of this matter after discussing the rules for constructing the paths connecting crossed reactions.

VI. A CONNECTION BETWEEN PATHS OF CROSSING AND HERMITIAN ANALYTICITY

A derivation of the crossing property of scattering functions is given in Ref. 17, and it will not be repeated here. This section gives an extension of that argument that leads to an important connection between the paths connecting crossed reaction and Hermitian conjugate points. This connection will play a key role in the proof of the normal connection between spin and statistics given in the next section.

The basic idea in the S-matrix derivation of crossing properties is similar to the one used in the above derivation of Hermitian analyticity: one considers a "larger process" having pole singularities with residues containing the scattering functions of interest as factors. In the study of crossing, the larger process is selected so that its physical region, which is a connected set, intersects the "pole manifold" $S_\nu = \mu_p^2$ in two different disjoint regions, with these two regions corresponding to the two different signs of the energy part of the vector k_p whose square is S_ν. Let K and $\bar{K}$ be points of $S_\nu = \mu_p^2$ lying in these two different regions. The corresponding residues are

$$r(K) = i\,\alpha\, M_c\,(K_\nu)\,\tilde{G}_p(v_p)\,M_c\,(K_{\nu'}) \tag{6.1}$$

and

$$r(\bar{K}) = i\,\bar{\alpha}\, M_c\,(\bar{K}_\nu)\,\tilde{G}_p(\bar{v}_p)\,M_c\,(\bar{K}_{\nu'}) \tag{6.2}$$

respectively, by virtue of Formula (4.8).

The sets K_ν and $\bar{K}_\nu$ both refer to the same subset ν of the external particles of the larger reaction, and the sets $K_{\nu'}$ and $\bar{K}_{\nu'}$ both refer to the same complementary subset ν' of the particles of the larger reaction. The additional particle referred to by both K_ν and $K_{\nu'}$, and associated with the pole at K, is denoted by p. The additional particle referred to by both $\bar{K}_\nu$ and $\bar{K}_{\nu'}$, and associated with the pole at $\bar{K}$, is denoted by $\bar{p}$. Since the poles at K and $\bar{K}$ lie on disjoint parts of the manifold $S_\nu = \mu_p^2$ the two particles p and $\bar{p}$ need not be identical, though their masses are equal. Indeed, the energy conservation-law requirement demands that the particles p and $\bar{p}$ have the opposite initial-final status and be therefore particles carrying opposite units of all additive quantum numbers. The particles p and $\bar{p}$ are called conjugate particles, or relative antiparticles. Use is made in this argument of the converse pole-factorization theorem, which asserts, under the same assumptions, that if there is a pole at $S_\nu = \mu_p^2$ in the physical region then there must be a corresponding physical particle contributing to unitarity summations; if there were no such particle then the δ - function contribution needed for the pole would be absent.

Let C^o be a path from K to $\bar{K}$ that runs through the physical region of the larger process, passing around singularities in accordance with physical region $i\epsilon$ rules. Following the procedure of Ref. 8, we distort (if possible) this path C^o into a path C between K and $\bar{K}$ lying in the mass shell $S_\nu = \mu_p^2$. The continuation of $r(K)$ from K to $\bar{K}$ along path C is designated by $r(K^C)$. By virtue of the definition of C we have

$$r(K^c) = r(\overline{K}) \quad , \tag{6.3}$$

or equivalently,

$$i\,\alpha\, M_c(K_\nu{}^c)\, \tilde{G}_p(v_p{}^c)\, M_c(K_{\nu'}{}^c)$$

$$= i\,\overline{\alpha}\, M_c(\overline{K}_\nu)\, \tilde{G}_p(\overline{v}_p)\, M_c(\overline{K}_\nu) \quad . \tag{6.4}$$

In the distortion of the original path C^o into the path C one must, as in the case of Hermitian analyticity, avoid singularities having discontinuities with nonzero residue at $S_\nu = \mu_p{}^2$. The necessary distortions are examined by using an effective continuation in the mass of the pole particle. That is, the various paths of continuation intermediate between C^o and C are divided into three segments with the middle segment at constant $S_\nu = a_p{}^2$. The path C^o is carried to C by increasing a_p from zero to μ_p. The distortions required of the middle segment are those needed to avoid those Landau singularity surfaces of the larger process that become Landau singularity surfaces of one of the two subreactions associated with the residue when a_p reaches μ_p. The discontinuities across the cuts associated with the remaining singularities will not contribute to the residue, by virtue of pole assumption P_2.

The two end points of the middle segment lie on the portions of the original path C^o leading from the zero point to K and $\overline{K}$ respectively. As a_p increases these end points detour around any encountered singularities in the manner specified by the physical region $i\epsilon$ rules.

In the procedure just sketched the pole-factorization property was applied to just one particle of some original reaction of interest; the other particles were taken to be particles of the larger reaction. The crossing paths associated with these other particles can be constructed by applying this same procedure to each of these particles separately. However, to standardize the construction and obtain a connection to paths of Hermitian analyticity we shall apply the pole-factorization property simultaneously to all of the particles of the original reaction of interest. That is, the larger process will be chosen to be one having a pole singularity for each particle of this original reaction, so that the scattering function for this reaction occurs as a factor in the residue of the product of all these poles, much as in the case of the derivation of Hermitian analyticity.

Just as in the preceding case of Hermitian analyticity the continuation in the a_p now starts from a point where all the a_p, and also all their associated k_p, are zero. There are paths in the physical region of the larger process leading from this starting point to the regions associated with each of the n different crossed reactions associated with original reaction. Each of these n paths passes around any encountered singularities in accordance with the physical region $i\epsilon$ rules. If $C_i^{\,o}$ is a path in the physical region of the larger process leading from the starting point zero to the point K_i associated with the ith one of the crossed reaction, then $C_{ij}^{\,o} = C_i^{\,o} - C_j^{\,o}$ is a path in the physical region of the larger process leading from K_j to K_i. (Sums of paths are read from right to left.) The result of distorting $C_{ij}^{\,o}$ into the mass shell $a_p = \mu_p$ (all p) in such a way as to avoid singularities that contribute to the residue of the product of the poles is denoted by C_{ij}. Since

$$C_{ij}^{\,o} + C_{jk}^{\,o} = C_{ik}^{\,o} , \tag{6.5}$$

and since the distortions are such as to avoid the relevant singularity surfaces, we have also

$$C_{ij} + C_{jk} \simeq C_{ij} , \tag{6.6}$$

where the $\simeq$ sign in (6.6) means equivalence with respect to continuation of the residue $r(K)$ of the product of poles. Equation (6.6) expresses the compatibility of the various crossing paths C that connect the various crossed reaction regions.

As discussed in the preceding section, and also in Ref. 17, the pole assumption P_2 implies that the singularity surfaces that have cuts having discontinuities contributing to the residue are just the Landau surfaces corresponding to the individual scattering functions of the residue, but with the external mass μ_p shifted to a_p. (Only the Landau surfaces are extended off the mass shell, not the M functions.) Because of this special character of the relevant singularity surfaces the parts of the paths of continuation at constant a_p can be considered to be products of paths, with one factor for each process referred to by the residue. The distortion of each individual path is then followed by following the motion of the Landau surfaces corresponding to the appropriate process, as the a_p increase from zero to μ_p. Only those Landau surfaces need be avoided that are not forced to be nonsingular, by the positive-α requirement on the Landau singularities entering the physical region of the larger process.

The parts of the paths C_{ij} , C_{ij}^{o}, and C_i^{o} referring to the inner process will be represented by the corresponding lower case quantities. Then the part of (6.6) referring to the inner process reads

$$c_{ij} + c_{jk} \simeq c_{ik} , \qquad (6.6')$$

where $\simeq$ means equivalence with respect to analytic continuation of any function whose singularities are confined to those Landau surfaces of the inner process that are restrictions to $a_p = \mu_p$ of the surfaces avoided in the distortions of the c_{ij} into the mass shell. Equation (6.6') is certainly valid if in these distortions one retains the original structure of the paths wherein a single central point is connected to each of the various crossed reaction points. If it is possible to distort all the paths c_{ij}^{o} into the mass shell so that the relevant singularities are avoided, then this structure can certainly be retained. The case in which it is not possible to distort the paths so as to avoid all the relevant singularities will be discussed later.

For each path c_i^{o} there is a complex conjugate path $\bar{c}_i^{o}$ that coincides with c_i^{o} except that it detours around the physical region singularities in accordance with the $i\epsilon$ rules associated with the conjugate function. According to the previous section it is the path $\bar{c}_i^{o} - c_i^{o}$ that, distorted into the mass shell, gives the path of Hermitian analyticity h_i that takes the scattering function $M(K_i)$ into $- M^{+}(K_i)$. The rules for the distortion of the path $\bar{c}_i^{o} - c_i^{o}$ into the mass shell to give h_i are the same as the rules for the distortion of the path $c_j^{o} - c_i^{o}$ into the mass shell to give c_{ji} : one must avoid the points of a larger

process that lie on the formal extension off the mass shell of the Landau surfaces of the (inner) reaction of interest. However, one need not avoid those Landau surfaces that are required to be nonsingular by the positive-α requirement on singularities that enter the physical region of the larger process.

Since the paths $c_i^{\,o}$ and $\bar{c}_i^{\,o}$ leading to the n crossed reactions and their Hermitian conjugate points all start from a single point, and the rule for distortion of these paths into the mass shell is a uniform one, the same set of Landau surfaces being avoided in all cases (see below), the compatibility requirement (6.6') carries over also to paths connecting Hermitian conjugate points. In particular we obtain relations such as

$$h_i + c_{ij} \simeq \bar{c}_{ij} + h_j \,, \tag{6.7}$$

where $\bar{c}_{ij}$ is the result of distorting $\bar{c}_{ij}^{\,o} = \bar{c}_i^{\,o} - \bar{c}_j^{\,o}$ into the mass shell. Equation (6.7) says (reading from right to left) that the path from K_j to K_i to its conjugate point $\bar{K}_i$ is equivalent to the path from K_j to its conjugate point $\bar{K}_j$, to $\bar{K}_i$. Here equivalent means equivalent with respect to analytic continuation of a function having singularities only on Landau surfaces that are restrictions to mass shell $a_p = \mu_p$ of Landau surfaces associated with the inner process and that are not required to be nonsingular for the larger processes by the positive-α requirements. The scattering function of interest must have its singularities confined to these surfaces, since it is a factor of the residue of the larger process, and this factor contains all the dependence on the variables associated with these singularities.

Equation (6.7) is certainly valid if in the distortion into the mass shell one maintains the structure wherein all the n crossed reaction points

and their conjugate points are connected to a single central point. Alternatively, the n crossed reaction points can be connected to one central point, and the n conjugate points can be connected by conjugate paths to a conjugate central point that is connected by a single path to the unconjugate central point. More generally, Eq. (6.7) is certainly valid so long as no closed loops are introduced into the set of paths connecting the various points. Cases where closed loops are present require some additional discussion, which is given in Appendix D. However, there is no real need to introduce closed loops.

Because the Landau structure is invariant under Hermitian conjugation the paths $\bar{c}_{ij}$ can be taken to be the complex conjugates of the paths $\tilde{c}_{ij}$, where $\tilde{c}$ is the path of crossing for the transposed function. The relationship of Hermitian conjugateness is maintained if the two related functions are continued along conjugate paths. Thus from the Hermitian analyticity relationship

$$M_c(K^h) = - M_c^{\dagger}(K) = - M_c^{T}(K)^* , \tag{6.8}$$

where $M_c(K^h)$ is the result of continuing $M_c(K)$ along the path h to the conjugate point, and the superscript T represents transpose, one obtains

$$M_c(K^{h\bar{c}}) = - M_c^{T}(K^{\tilde{c}})^* , \tag{6.9}$$

where $M_c(K^{h\bar{c}})$ is the result of continuing $M_c(K)$ first along h and then along $\bar{c}$, and $M_c^{T}(K^{\tilde{c}})^*$ is the result of continuing $M_c^{T}(K)$ along

its path of crossing $\tilde{c}$, and then complex conjugating. The $\tilde{c}$ and $\bar{c}$ are conjugate paths. Applying (6.7) to the left-hand side of (6.9), we obtain our principal result,

$$M_c(K^{c\bar{h}}) = - M_c^{\,T}(K^{\tilde{c}})^* , \tag{6.10}$$

which says that the result of continuing $M_c(K)$ first along c and then along the path of Hermitian analyticity h associated with the crossed point $\bar{K}$ gives minus the complex conjugate of the function $M_c^{\,T}(K^{\tilde{c}})$.

Equation (6.10) would follow directly from the Hermitian analyticity relation at the crossed point,

$$M_c(\bar{K}^{\bar{h}}) = - M_c^{\,T}(\bar{K})^* , \tag{6.11}$$

if we were in possession of the crossing relationships $M_c(K^c) = M_c(\bar{K})$ and $M_c^{\,T}(K^{\tilde{c}}) = M_c^{\,T}(\bar{K})$. However, we have so far obtained only the weaker condition (6.4). Because (6.4) has a product of two M functions there is an ambiguity in the relative phase and normalization of $M_c(K^c)$ and $M_c(\bar{K})$, as was stressed by Olive.[7] And there are also the extra phase factors α and $\bar{\alpha}$ to be considered. These latter depend on the statistics of the particles and will be discussed in the next section.

In the discussion of crossing and Hermitian analyticity given so far it was assumed that the various physical region paths can actually be distorted into the mass shell without cutting across any of the singularity surfaces having cuts with discontinuities contributing to the residue. It

is conceivable, however, that these singularities might pinch together in such a way as to make impossible the distortion into the mass shell of some of these paths.

Rather than distorting the paths around the various singularities we can elect rather to jump across the associated cuts, by adding the corresponding discontinuity functions. This was in fact the procedure adopted for the various cuts whose discontinuities do not contribute to the residue. For cuts around which it is not always possible to detour, the discontinuity across the cut is defined for values of a_p for which the two sides are still connected, and this function is then continued to $a_p = \mu_p$.

The discontinuity functions associated with cuts around which it is possible to detour, within the mass shell, share with the original function the important pole-factorization property, since this property can be continued around these cuts. In particular if a singularity under consideration occurs in the variables associated with the inner reaction, then one can detour around this cut without changing the functions in the residue formula that are associated with the outer processes. One makes use here of the relativistic invariance property, which allows the momentum-energy transfered to the outer reactions to be altered without changing their invariants. Hence the functions associated with these outer reactions will remain unchanged under continuation in the inner variables, except for the alteration of certain polynomials associated with the expansion of spin states. These polynomials return to their original values when the continuation is brought back to the other side of the cut and hence the

outer factors return to their original values. The jump in the residue function across the cut is therefore represented by adding a certain discontinuity function to the factor associated with the inner process, upon whose variables the singularity was assumed to depend, the outer factors remaining unchanged. Moreover the discontinuity function for the factor associated with the inner reaction is independent of the particular larger process being considered. These properties of the discontinuity function will be called the pole-factorization property of discontinuity functions. It is the property whereby the discontinuity function of the residue across a cut in the variables associated with a given one of the functions occurring in the residue is obtained by adding a discontinuity to that particular one of these functions, this discontinuity being independent of the particular larger process under consideration.

The pole-factorization property of discontinuity functions is, as we have just said, automatically satisfied for cuts around which one can detour without leaving the mass shell. It is also satisfied for cuts with discontinuities given by a Cutkosky formula, for then the pole-factorization property of the discontinuity function is a consequence of the pole-factorization property of the individual functions occurring in the Cutkosky formula.

If the pole-factorization property of discontinuity functions is satisfied for all the cuts across which the paths of continuation jump, then the discussion of crossing and Hermitian analyticity given above is essentially unaltered. For then there are certain cuts across which the paths must jump, but the corresponding discontinuity functions are universal quantities that do not depend upon the particular larger process from which it is derived. Thus, the discontinuities that must be added as a path

jumps across the various cuts will be independent of the particular end points being connected by this path, and the compatibility conditions (6.6') and (6.7) still hold. Furthermore, the property whereby the Hermitian conjugateness relation is maintained when the related functions are continued along conjugate paths is also undisturbed by the cuts. For in the defining of the discontinuities on the two conjugate paths one can use for the larger processes two Hermitian conjugate reactions. Then the Hermitian conjugateness property will be valid for the discontinuity functions calculated at $a_p < \mu_p$, and will be carried into the mass shell by continuation in a_p to μ_p. Thus these discontinuities will not destroy the Hermitian conjugateness property and one still obtains (6.9) and hence (6.10).

VII. THE CONNECTION BETWEEN SPIN AND STATISTICS

The residue of the pole at a point K on the manifold $S_\nu = \mu_p^2$ in the physical region of the (larger) process described by the scattering function $M_c(K)$ is given according to (4.8) as

$$r(k) = i\,\alpha\, M_c(K_\nu)\, \tilde{G}_p(v_p)\, M_c(K_{\nu'}) , \tag{7.1}$$

where $\tilde{G}_p(v_p)$ is a metric tensor satisfying (2.24),

$$\tilde{G}(-v_p) = (-1)^{2j_p}\, \tilde{G}(v_p) , \tag{7.2}$$

and α is a phase factor given by (4.9). This phase factor is important to our considerations but the formula (4.9) will not be needed.

At a physical point $\bar{K}$ lying on the crossed-region part of the manifold $S_\nu = \mu_p^2$ the residue of the scattering function for this same larger process is

$$r(\bar{K}) = i\,\bar{a}\, M_c(\bar{K}_\nu)\, \tilde{G}_p(\bar{v}_p)\, M_c(\bar{K}_{\nu'}) . \tag{7.3}$$

The point $\bar{K}$ has the same set of variables as K, but the values of the momentum vector parts have been shifted. The sets of variables K_ν and $K_{\nu'}$ each contain, in addition to certain of the variables of K, a variable associated with the particle p associated with the pole at K. Similarly the sets of variables $\bar{K}_\nu$ and $\bar{K}_{\nu'}$ each contain, in addition to certain

of the variables of $\bar{K}$, a variable associated with the particle $\bar{p}$ associated with the pole at $\bar{K}$. Particle $\bar{p}$ is called the antiparticle of particle p.

The result of continuing from K along the mass-shell path of continuation c is represented by placing the superscript c on the set of arguments K. The path c is constructed so that $r(K^c) = r(\bar{K})$. This gives

$$\begin{aligned} & i\,\alpha\, M_c(K_\nu{}^c)\,\tilde{G}_p(v_p{}^c)\,M_c(K_{\nu'}{}^c) \\ & = i\,\bar{\alpha}\, M_c(\bar{K}_\nu)\,\tilde{G}_p(\bar{v}_p)\,M_c(\bar{K}_{\nu'}) \quad . \end{aligned} \tag{7.4}$$

Since the point $\bar{K}$ was assumed to lie on the crossed-region part of the manifold $S_\nu = \mu_p{}^2$ we have $v_p{}^c = -\bar{v}_p$, which gives with the help of (7.2), the result

$$\begin{aligned} & (-1)^{2j_p}\ \alpha\, M_c(K_\nu{}^c)\,\tilde{G}_p(\bar{v}_p)\,M_c(K_{\nu'}{}^c) \\ & = \bar{\alpha}\ M_c(\bar{K}_\nu)\,\tilde{G}_p(\bar{v}_p)\,M_c(\bar{K}_{\nu'}) \quad . \end{aligned} \tag{7.5}$$

This factor $(-1)^{2j_p}$ will be the origin of the normal connection between spin and statistics. It is also the origin of the connection between the intrinsic parities of particles and their conjugate antiparticles, as was shown in Ref. 26.

The relationship between α and $\bar{\alpha}$ is determined in certain cases by the statistics of the particles of the larger process.[27] Consider in particular the residue formulas in the special cases indicated in Fig. 9.

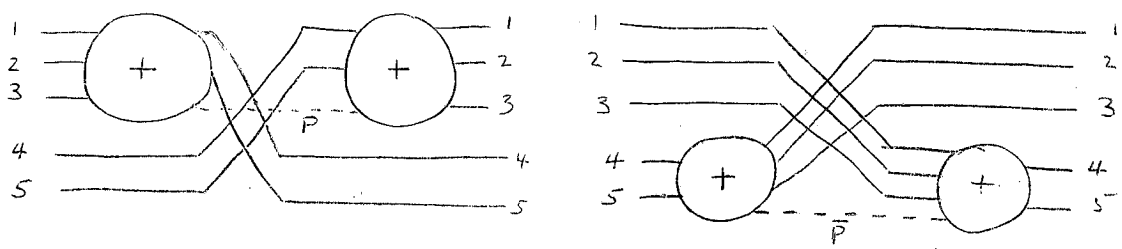

Fig. 9. Representation of direct and crossed residue formulas in a special case where the initial and final particles of the larger process are the same set of particles. The factors $i\alpha$ and $\bar{\alpha}$ must be added. The order of incident lines, reading from top to bottom, will indicate the order of variables of the functions $M_c(K'; K'')$ and $M(K'; K'')$.

The phase factors α and $\bar{\alpha}$ associated with these residues can be determined from statistics by a direct examination of the derivation of the residue formula. In the key step of the derivation the contributions to a unitarity equation represented in Fig. 10 were converted by means of the unitarity equations represented in Fig. 11 to the residues represented in Fig. 9.

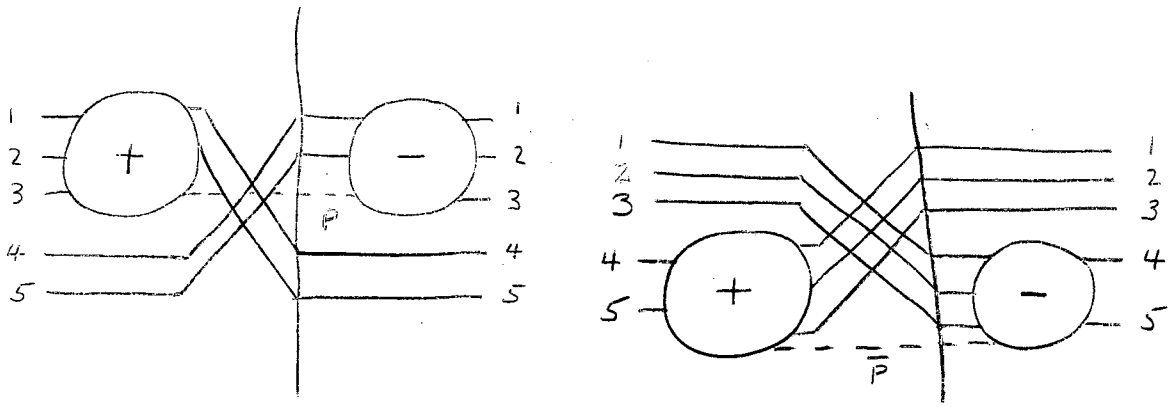

Fig. 10. The contributions to the unitarity equations of the larger process that are converted by means of the unitarity equations shown in Fig.11 into the residue formulas shown in Fig. 9.

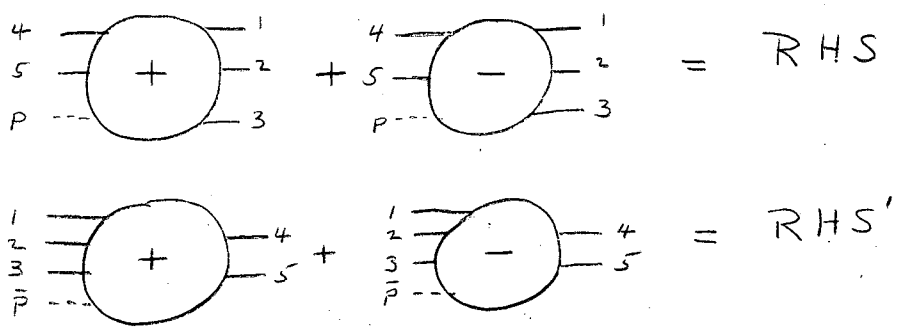

Fig. 11. Unitarity equations used to convert the pole contributions to the unitarity equations of the larger process shown in Fig. 10 into the residue formulas shown in Fig. 9. That the right-hand sides of these equations correctly cancel against terms of the unitarity equation not shown in Fig. 10 is shown in the proof of the pole-factorization theorem.

The two terms on the left of the equations in Fig. 11 are just the complex conjugate transposes (Hermitian conjugates) of each other, according to the conventions adopted in (2.30d) and (2.58). It is with these phase conventions that the Hermitian analyticity properties were proved.

Because there are no extra phases on the left side of the equations represented in Fig. 11, the phases α and $\bar{\alpha}$ of the residue functions shown in Fig. 9 must be precisely the phases of the corresponding contribution to unitarity shown in Fig. 10. This is a key point.

The phases of the contributions to unitarity shown in Fig. 10 are determined by the statistics of the external particles of these diagrams. In particular, interchange of the pairs of identical particles 4 and 5 in the first factor of the first diagram of Fig. 10, and the pairs of identical particles 1, 2, and 3 in the first factor of the second diagram of Fig. 10, leads to Fig. 12.

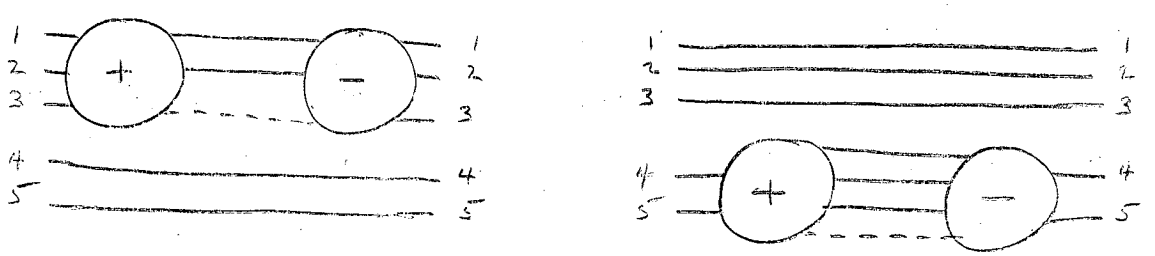

Fig. 12. Result of interchanging some pairs of identical particles in Fig. 10.

The contributions to unitarity shown in Fig. 12 are just absolute-value-squared contributions, and these must have no phase. More specifically, if the minus bubbles in Fig. 12 represent precisely the complex conjugates of the corresponding plus bubbles in this figure, then the phase factors of the terms in Fig. 12 are unity. This stipulation that the minus bubbles be the complex conjugates of the plus bubbles we record as

$$M_c^{\dagger}(K_{\nu'}) = M_c^{*}(K_{\nu}) \tag{7.6a}$$

and

$$M_c^{\dagger}(\bar{K}_{\nu'}) = M_c^{*}(\bar{K}_{\nu}) . \tag{7.6b}$$

In the passage from Fig. 10 to Fig. 11 no reordering of the variables of the connected parts is performed; these functions $M_c(K'; K'')$ are kept fixed. Only the orderings of the variables of the larger process on the left are altered. This interchange induces an overall sign in accordance with the statistics of the particles interchanged.

From the fact that the phase factors in Fig. 12 are unity, the phase factors in Fig. 11, hence Fig. 9, are immediately determined from the statistics of the five external particles. If the number of these particles obeying abnormal statistics $\left[\sigma_i = -(-1)^{2j_i}\right]$ is odd, then the quotient $a/\bar{a}$ is

$$-\prod_{i=1}^{5} (-1)^{2j_i} = -(-1)^{2j_p} , \tag{7.7}$$

where the equality (7.7) follows from the fact that the sum of the spins of the particles participating in a nonvanishing reaction is even, by virtue of the convariance condition (2.11). But if $a/\bar{a}$ is $-(-1)^{2j_p}$ then (7.5) becomes

$$\begin{aligned} &- M_c(K_\nu{}^c)\, \tilde{G}_p(\bar{v}_p)\, M_c(K_{\nu'}{}^c) \\ &= M_c(\bar{K}_\nu)\, \tilde{G}_p(\bar{v}_p)\, M_c(\bar{K}_{\nu'}) \ . \end{aligned} \tag{7.8}$$

Continuation of the right-hand factors in (7.8) along the path of Hermitian analyticity $\bar{h}$ associated with the crossed point $\bar{K}_{\nu'}$ converts (7.8) to

$$\begin{aligned} &M_c(K_\nu{}^c)\, \tilde{G}_p(\bar{v}_p)\, M_c{}^*(K_\nu{}^c) \\ &= - M_c(\bar{K}_\nu)\, \tilde{G}_p(\bar{v}_p)\, M^*(\bar{K}_\nu) \ , \end{aligned} \tag{7.9}$$

where use has been made of (6.10), (6.11), and (7.6).

The functions M and M^* are complex conjugate functions and the factor G is the square of a Hermitian matrix. Thus the right side of (7.9) is nonpositive and the left side is nonnegative. Therefore both sides must be zero, which implies immediately that

$$M_c(\bar{K}_\nu) = 0 \ , \tag{7.10}$$

for this case in which an odd number of the external particles of the set $\bar{K}_\nu$, exclusive of $\bar{p}$, are abnormal. This result was derived for the particular case of an M function with three initial particles and three final particles, but the argument holds also for the cases more than three. That is, any M_c function referring to a process with three or more initial particles and three or more final particles is zero if any subset of all but one of these particles has an odd number of particles with abnormal statistics. This immediately implies that all such M functions referring to a set of particles containing <u>any</u> abnormal particles must vanish except possibly for M functions referring to an odd number of particles all of which are abnormal. This last possibility is rule out by unitarity, since the nonvanishing process would contribute a term to the sum of positive terms giving the real part of the M function for a corresponding forward scattering process, which must, however, vanish because it involves an even number of abnormal particles.

The possibility that abnormal particles occur in a reaction involving only two initial or two final particles, but in no reactions involving three or more initial and final particles, conflicts with the pole-factorization property plus unitarity. (Unitarity guarantees that the transpose process is nonzero) Thus we conclude that the scattering function $M_c(K)$ vanishes if any of the particles referred to by K obey abnormal statistics: only particles obeying normal statistics can react.

VIII. THE PHASE FACTOR IN THE CROSSING RELATIONSHIP

Having established that all particles obey normal statistics we obtain instead of (7.8) and (7.9) the relationships

$$M_c(K_\nu{}^c)\,\tilde{G}_p(\bar{v}_p)\,M_c(K_{\nu'}{}^c) = M_c(\bar{K}_\nu)\,\tilde{G}_p(\bar{v}_p)\,M_c(\bar{K}_{\nu'}) \tag{8.1}$$

and

$$M_c(K_\nu{}^c)\,\tilde{G}_p(\bar{v}_p)\,M^*{}_c(K_\nu{}^c) = M_c(\bar{K}_\nu)\,\tilde{G}_p(\bar{v}_p)\,M^*{}_c(\bar{K}_\nu)\ . \tag{8.2}$$

From (8.2) it follows that

$$|M_c(K_\nu{}^c)| = |M_c(\bar{K}_\nu)|\ . \tag{8.3}$$

That is, the continuation of $M_c(K_\nu)$ is equal to $M_c(\bar{K}_\nu)$ up to a possible phase factor.

Equation (8.3) can be written as

$$M_c(K_\nu{}^c) = M_c(\bar{K}_\nu)\,\alpha(\bar{K}_\nu)\ , \tag{8.4}$$

where $\alpha(\bar{K}_\nu)$ is a spin-independent phase factor. [The spin independence of $\alpha(\bar{K}_\nu)$ follows immediately from the covariance conditions (2.11)[28]] . Insertion of (8.4) into (7.5) gives

$$\alpha(\overline{K}_\nu)\ \alpha(\overline{K}_{\nu'}) = Q(\overline{K}_\nu, \overline{K}_{\nu'}) \tag{8.5}$$

where $Q = (-1)^{2j_p}\ \overline{\alpha} / \alpha$.

All our equations are invariant under a transformation of the form

$$M_c(K) \longrightarrow \left[\exp(\Sigma \pm i\ \varphi_p)\right] M_c(K) , \tag{8.6}$$

where φ_p is a real number depending only on the particle type p . The sum in (8.6) is over the particles referred to by K , and the $\pm$ sign is plus for final particles and minus for initial. The numbers φ_p can be chosen so that for each particle p there is one particular M function that satisfies instead of (8.4) the more stringent condition

$$M_c(K_\nu{}^c) = M_c(\overline{K}_\nu) . \tag{8.7}$$

That is, the phases φ_p can be chosen so that $\alpha(\overline{K}_\nu) = 1$, which implies that for the particular $\overline{K}_{\nu'}$ associated with $\overline{K}_\nu$ in (8.1) one has also $\alpha(\overline{K}_{\nu'}) = 1$. If the phases α and $\overline{\alpha}$ in (7.5) were such that Q were always unity, then the above adjustment of phases to give $\alpha(\overline{K}_\nu) = 1$ would make $\alpha(\overline{K}_{\nu'}) = 1$ for all $\overline{K}_{\nu'}$, and the crossing relationship (8.4) would have no extra phase.

In order to discuss the value of Q , certain stipulations regarding the order of variables must apparently be made. In field theoretic models one has a cluster decomposition law that yields

$$M(K'_1, K'_2, \cdots K'_n; K''_1, K''_2, \cdots K''_n)$$

$$= M_c(K'_1 ; K''_1)\, M_c(K'_2 ; K''_2) \cdots M_c(K'_n ; K''_n) + \cdots \quad . \tag{8.8}$$

That is, if the variables in M are ordered according to a particular cluster term, in the manner shown, then the α_p for this particular cluster term is unity.

That this equation should continue to hold in a pure S-Matrix theory can be argued as follows.[29] Let all but one of the sets K'_i and K''_j be held fixed and let this one remaining set be denoted by A. Then (8.8) will be written in the abbreviated form

$$M(A) = M_c(A)\, \Pi + \cdots , \tag{8.9}$$

where Π stands for the product of the remaining factors on the right. [The $M(A)$ on the left is, of course, a quite different function from the $M_c(A)$ on the right.]

From our general cluster decomposition property we have instead of (8.9) the more general equation

$$M(A) = \alpha_A\, M_c(A)\, \Pi + \cdots , \tag{8.10}$$

where α_A is the phase factor α_p of (2.30). Let B be a set of variables labeling an amplitude in the same superselection class as the result labeled by A. We have then also

$$M(B) = \alpha_B M_c(B) \Pi + \cdots , \tag{8.11}$$

where all other variables are still fixed as before.

Consider now a superposition $C = aA + bB$. That is, C labels the amplitude such that

$$M_c(C) = a M_c(A) + b M_c(B) . \tag{8.12}$$

According to the general cluster property one should have, in analogy to (8.10) and (8.11), also

$$M(C) = \alpha_c M_c(C) \Pi + \cdots . \tag{8.13}$$

This is actually a slight extension of our postulate E1 of Ref. 12, which, as stated, referred only to amplitudes labeled by sets K, not to their superpositions. But exactly the same physical principle should apply to superpositions. This extension of the postulate E1, which we call E1', gives (8.13).

We need also a stipulation that

$$M(C) = a M(A) + b M(B) . \tag{8.14}$$

This requirement would be rather natural if we were dealing with a Hilbert space formalism in which the S matrix were regarded as a unitarity mapping of free-particle states onto free-particle states, with these

free-particle states regarded as tensor products of individual single-particle states, and in which the ordering of variables specifies the ordering of these states. Of course, since we know that the tensor products of these states in different orders are not all equal, it is not absolutely clear that (8.14) must be satisfied, since the addition of extra states might affect different states differently. Proofs of spin and statistics that depend on such extra stipulations are not completely satisfactory, since it is conceivable that a theory with abnormal statistics might be possible if one were to abandon the extra stipulations. This might be done in such a way as to leave the physical relationships of superposition, Lorentz invariance, etc., unaltered.

However, having proved the normal connection between spin and statistics without recourse to such stipulations, our objective now is to complete the specification of the basic formalism of a proposed S-Matrix theory. The stipulation (8.14) is therefore now adopted.

The stipulation (8.14) immediately gives the result

$$\alpha_A = \alpha_B \; ; \tag{8.15}$$

as one sees by taking special values a and b satisfying

$$a : b = - M_c(B) : M_c(A) \; . \tag{8.16}$$

Then the relevant term on the left side of (8.13) vanishes, which implies, by virtue of (8.10), (8.11), and (8.14), the result (8.15). The phase factor α_A therefore depends only on the superselection class of A, or

more generally on the superselection classes of the various sets K''_i. In that case, however, one can take the sets K'_i to be equal to the sets K''_i, without altering α_A. Since the relative phases of the connected parts $M_c(K''_i ; K''_i)$ and their corresponding no-scattering parts are fixed to be unity, by virtue of E2, and since the phase of the no-scattering contribution to $M(K''_1, K''_2, \cdots K''_\mu ; K''_1, K''_2, \cdots K''_\mu)$ is unity by virtue of our original conventions on the no-scattering parts (which we were free to choose), we find that the factor $\alpha_A = \alpha_B$ is also unity, and thus obtain (8.8).

From the fact that the phase factor α_p is unity for the decomposition of the type shown in (8.8) one can conclude that the factors α and $\bar{\alpha}$ in Q are independent of the external variables of the process containing the pole. In particular, if one writes the $M(K_{\nu\nu'})$ on the left of (4.10) as

$$M(K_{\nu\nu'}) = M(K'_{\nu'}, K'_\nu, K'_p ; K''_{\nu'}, K''_\nu, K''_p), \qquad (8.17)$$

where K''_ν and K'_ν contain the initial and final variables of K_ν, and similarly for $K_{\nu'}$, and K'_p and K''_p are the variables associated with the exchanged particle, then according to (8.8) one can say that α_b in (4.10) is unity.

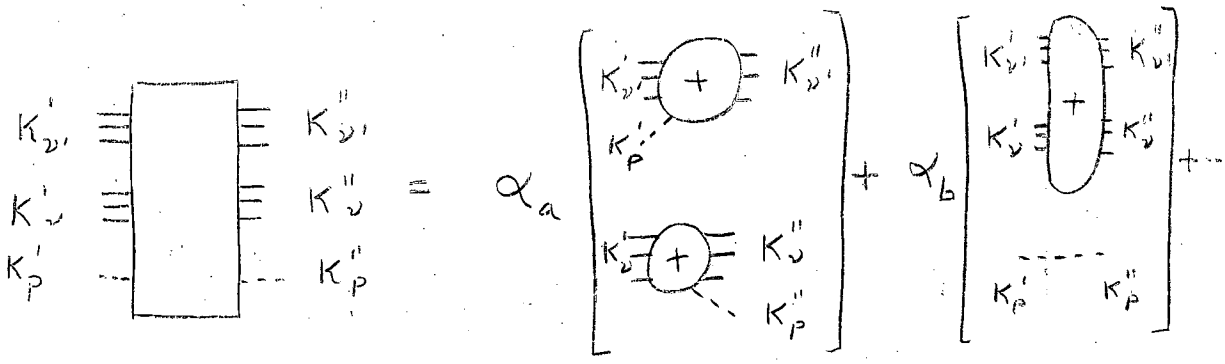

Fig. 13. Decomposition of (8.17) according to (4.10).

The phase α_a in (4.10) is the inverse of the phase change induced by moving K'_p through K'_ν, into the position where (8.8) is again applicable. But this phase change is independent of the variables associated with the external particles associated with $K_{\nu'}$. Here we are using the fact that the phase change induced by the interchange of any two adjacent variables is independent of the remaining variables of the M function. This is a consequence of (8.8) and unitarity. For unitarity ensures that the phase change induced by a reordering of the various final variables is independent of the particular initial variables (To see this consider the contributions to forward scattering, which is a sum of absolute values squared. Thus all contributions must suffer the same phase change under a reordering of the final external variables). But then (8.8) ensures that the interchange of two adjacent final variables must induce a phase change that is independent of all of the other variables, since one can consider a decomposition in which these two final variables are the only two final variables of one of the individual factors on the right of (8.8). This factor can appear as a contribution to various reactions.

For the analogous calculation of $\bar{\alpha}$ one uses (8.17) with $K_{\bar{p}}$ in place of K_p.

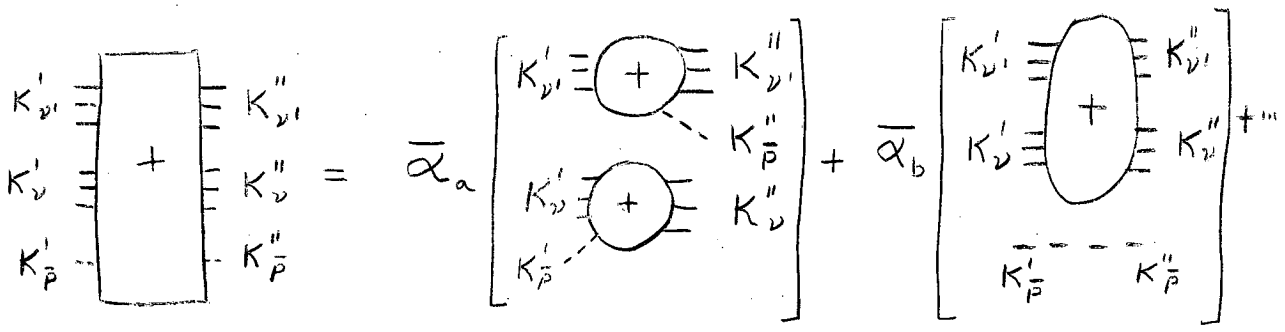

Fig. $\overline{13}$. Decomposition of analogous to the one in Fig. 13, but with $\bar{p}$ in place of p.

Again $\bar{\alpha}_a = 1$, and $\bar{\alpha}_b$ is independent of the external variables of $K_{\nu'}$: now one must commute $K''_{\bar{p}}$ through K''_{ν} to obtain the form where (8.8) is applicable. But then Q is independent of the external lines of $K_{\nu'}$, and one obtains from the special crossing relation (8.7) (with the variables ordered as in Figs. 13 and $\overline{13}$) the general crossing relation

$$M_c(K'^{\,c}_{\nu'}, K_p^{\,c}; K''^{\,c}_{\nu'}) = M_c(\bar{K}_{\nu'}; \bar{K}''_{\nu'}, \bar{K}_p). \tag{8.18}$$

That is, if the phases are adjusted so that (8.18) is valid for one particular ν', then it will be valid for every ν'.

In terms of the functions $M_c(K)$ defined in (2.26), the crossing relation (8.18) becomes

$$M_c(K_{\nu'}{}^c) = M_c(\overline{K}_{\nu'}) , \tag{8.19}$$

which is just the statement that $(-p_p , m_p , -t_p)$ is equivalent to $(p_p , m_p , t_{\overline{p}})$, with the understanding that the continuation from the original region of positive $(-p_p)$ to the new region of positive p_p has been made along the path of continuation c. That is, for the $M_c(K)$ functions the division between initial and final can be drawn arbitrarily: one has a single universal M function for all the crossed reactions.

The important fact that the sign change under interchange of adjacent conjugate variables is the same as the sign change under the interchange of the corresponding like variables follows immediately from (8.8) and the fact that the phase change under interchange of adjacent variables is independent of the other variables of M. One can consider (8.8) for two different orderings of the sets of variables of the M function on the left. Then to one of these sets of variables one adds a conjugate pair and finds that this pair must commute with all variables. In fact any set of variables of zero quantum numbers must commute with any other variable, if (8.8) is to be consistent with unitarity.

IX. CONCLUDING REMARKS - THE STATUS OF ANALYTIC S-MATRIX THEORY

The present work is basically a contribution to the development of the S-matrix framework begun in Ref. 1. The assumptions in that work were stronger than one would like in several respects. In the first place the possibility of superselection rules was not encompassed, and there was a special assumption, B', referring to phase factors. In the second place the crossing property was derived from a postulate of physical connection, which is an assumption of the general crossing principle without its specific detailed form. In the third place the analyticity postulate, though expressed entirely in terms of the unitarity equation, was guided in the definition of the physical sheet by previous experience with potential and perturbation-theory models.

The removal of the superselection rule restriction and of the phase assumption B' was the task accomplished in Ref. 12. The present work provides a derivation of crossing from weak analyticity assumptions of the general type postulated in Ref. 1, but without use of the specific assumptions regarding the form of physical sheet postulated there. The nature of the construction given here of paths of continuation connecting crossed and Hermitian analytic points makes it very likely that these points will be on the physical sheet as defined there; the constructions in both cases are in terms of formal extensions of the Landau surfaces off the mass shell and the requirement that the various region all are linked together when the external masses are zero. In Ref. 1 the Landau surfaces allowed to be singular in the region relevant to the continuation from the zero mass region were constructed so as to be just those Landau

surfaces that are singular in the corresponding region (defined as the physical sheet) in perturbation theory. In the present construction we have the condition that those singularities entering the physical region must be just those of perturbation theory. It is very likely that this correspondence with perturbation theory at physical points will ensure the correspondence with perturbation theory throughout the physical sheet, to the extent that only Landau singularities are allowed in this sheet. A proof of this would provide a justification of the particular construction for the physical sheet defined in Ref. 1. This problem is left for a later work.

Once the definition of the physical sheet given in Refs. 1 and 12 is confirmed by mass-shell considerations the central problem becomes to derive a general integral representation for multiparticle scattering functions analogous to the Mandelstam representation for the four-particle functions. The first part of this problem is to obtain the general discontinuity formulas for cuts entering the physical sheet. A paper on this subject is in preparation. That work should provide a basis for a proof of the pole assumptions P_1 and P_2 that were the basis of the present work.

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APPENDIX A. A Fundamental Theorem on Analyticity of Integrals

Theorem A'. Let $F(K)$ be defined by

$$F(K) = \int f(\bar{K};K) \prod_{i=1}^{m} \delta\big(g_i(\bar{K};K)\big)\, d\bar{K} , \tag{A.1}$$

where the g_i are single-valued real analytic functions of the sets of real variables $K = (k_1,\cdots,k_N)$ and $\bar{K} = (\bar{k}_1,\cdots,\bar{k}_n)$. Suppose for K in a set $\mathcal{S}$ that $\mathcal{R}(K) = \{\bar{K}:g_j(\bar{K};K) = 0 \text{ all } j\}$ is a bounded set over which $\partial g_j/\partial\bar{k}_i$ is of maximal rank $m \leqslant n$, and that $f(\bar{K};K)$ is analytic at points $(\bar{K};K)$ of $\{(\bar{K};K):K\epsilon\, \mathcal{S},\ \bar{K}\epsilon\, \mathcal{R}(K)\}$. Then $F(K)$ is analytic at points K of $\mathcal{S}$. The analyticity of $f(\bar{K};K)$ and $F(K)$ is in the sense of Def. 2.1 of Section II.

Proof. Let K be a fixed point in $\mathcal{S}$. Because the rank of $\partial g_j/\partial\bar{k}_i$ is maximal the set $\mathcal{R}(K)$ is a real analytic submanifold.[30,31] That is, for any $\bar{K}$ in $\mathcal{R}(K)$ there is a real function $\bar{K}'_{\bar{K}}(X)$, defined and analytic on $\bar{U}_{\bar{K}}(K)$, the closure of a bounded open set $U_{\bar{K}}(K)$, in the space of points labeled by the set of local coordinates $X = \{x_1,\cdots,x_{n-m}\}$, such that $\bar{K}'_{\bar{K}}(X)$ maps $U_{\bar{K}}(K)$ onto an open neighborhood $\mathcal{R}_{\bar{K}}(K)$ of $\bar{K}$ in the space $\mathcal{R}(K)$. One can, in fact, evidently take the x_j to be linear functions of the $\bar{k}_i$ in such a way that $\partial(x_j,g_j)/\partial\bar{k}_i \equiv \partial(X,G)/\partial\bar{K}$ is nonzero at $G = 0$ (all $g_j = 0$) for X in a sufficiently small $\bar{U}_{\bar{K}}(K)$. This ensures[30] that the inverse function $\bar{K}'_{\bar{K}}(X,G)$ will be unique and analytic in both arguments at $G = 0$ for X in $\bar{U}_{\bar{K}}(K)$. The function $\bar{K}'_{\bar{K}}(X)$ is then $\bar{K}'_{\bar{K}}(X,0)$.

The function $\overline{K}'_{\overline{K}}(X,G)$ depends also on K and will sometimes be written as $\overline{K}'_{\overline{K}}(X,G;K)$. It is, in fact, analytic in K, by virtue of the facts that it is analytic in G and that G is analytic in K. For one can write

$$d\overline{K}' = \frac{\partial \overline{K}'}{\partial X} \frac{\partial X}{\partial K} d\overline{K} + \frac{\partial \overline{K}'}{\partial G} \frac{\partial G}{\partial K} d\overline{K} + \frac{\partial \overline{K}'}{\partial G} \frac{\partial G}{\partial K} dK + \frac{\partial \overline{K}'}{\partial K} dK = d\overline{K} , \qquad (A.2)$$

which gives

$$\frac{\partial \overline{K}'}{\partial K} = - \frac{\partial \overline{K}'}{\partial G} \frac{\partial G}{\partial K} , \qquad (A.3)$$

as the well-defined derivative.

Because $\mathcal{R}(K)$ is bounded it is also compact in the induced topology, in which the neighborhoods in $\mathcal{R}(K)$ are defined as the intersection of $\mathcal{R}(K)$ with neighborhoods in the imbedding K space. This result is well known.[32]

The basic neighborhoods in $\mathcal{R}(K)$ will be taken small enough so that each one is contained with its closure in one of the $\mathcal{R}_{\overline{K}}(K)$. This is possible because of the analytic character of $\overline{K}'_{\overline{K}}(X,G)$ and its inverse. In particular, given a point $\overline{K}$ of $\mathcal{R}(K)$ one can find a sufficiently small neighborhood $\Delta G_{\overline{K}}(K)$ of $G = 0$ such that $\overline{K}'_{\overline{K}}(X,G)$ is analytic with an analytic inverse over $U_{\overline{K}}(K) \otimes \Delta G_{\overline{K}}(K)$.[30] Thus by taking the basic neighborhoods in $\overline{K}$ space small enough so each is contained with its closure in the image of one of the $U_{\overline{K}}(K) \otimes \Delta G_{\overline{K}}(K)$,

we ensure that the closure of the restriction to $\mathcal{R}(K)$ of each of these neighborhoods is in one of the $\mathcal{R}_{\bar{K}}(K)$.

The basic neighborhoods in the space of real points $\bar{K}$ can be defined as the open sets bounded by surfaces at rational constant values of the $\bar{k}_i$. This provides also a set of neighborhoods in $\mathcal{R}(K)$. For the basic neighborhoods in $\mathcal{R}(K)$ a subset of these will be chosen. In particular, since for points on $\mathcal{R}(K)$ the g_j are analytic functions of the $\bar{k}_i$ with nonzero the gradients, the basic neighborhoods in $\mathcal{R}(K)$ can be taken small enough so that the gradients of the surfaces $g_j = 0$ are almost constant over any basic neighborhood in $\mathcal{R}(K)$.[33] Then the set of basic neighborhoods in $\mathcal{R}(K)$ is further restricted by the requirement that none of these neighborhoods be bounded by a surface corresponding to a certain constant $\bar{k}_i$ if the gradient to this constant $\bar{k}_i$ surface is "nearly parallel" to any linear combination of the gradients to the surfaces $g_j = 0$, at any point $\bar{K}$ of the neighborhood in question.

[The point here is first that one can certainly find $n - m$ constant-$\bar{k}_i$ coordinate surfaces whose gradients are not "near" the subspace spanned by the gradients at $\bar{K}$ to the surfaces $g_j = 0$. For let $\{V_i\}$ be the orthonormal set of normalized gradients to the coordinate surfaces lying at constant $\bar{k}_i$, and let $\{W_i\}$ be an orthonormal set of vectors such that the first m of them span the space $\mathcal{W}(\bar{K})$ spanned by the m gradients ∇g_j at point $\bar{K}$. Suppose $m + 1$ of the V_i lie "near" the space $\mathcal{W}(\bar{K})$, in the sense that, with a suitable ordering of the V_i, the quantity

$$\sum_{i=1}^{m+1} \sum_{j=m+1}^{n} (V_i \cdot W_j)^2 \equiv \delta$$

is smaller than unity. Then the sum over the complementary set in i is

$$\sum_{i=m+2}^{n} \sum_{j=m+1}^{n} (V_i \cdot W_j)^2 = n - m - \delta$$

$$\leqslant \sum_{i=m+2}^{n} \sum_{j=1}^{n} (V_i \cdot W_j)^2 = n - m - 1 \quad .$$

This gives $\delta \geqslant 1$, contradicting the assumption that $m + 1$ of the V_i lie near the space $\mathcal{W}(\overline{K})$. Thus at most m of the V_i can lie near $\mathcal{W}(\overline{K})$, and one can find a set of at least $n - m$ vectors V_i none of which is nearly parallel to any linear combination of the gradients to the surfaces $g_j = 0$. (By nearly parallel we can mean, specifically, that the lines make an angle of less than n^{-1} degrees). One can choose any one of these vectors as one of the x_i. Working up by induction in m one sees that one can complete the set of x_i by choosing from among the set of $\overline{k}_i$ whose gradients are not nearly parallel to the vectors of $\mathcal{W}(\overline{K})$. Thus one can find arbitrarily small neighborhoods of $\overline{K}$ in $\mathcal{R}(\overline{K})$ that are bounded only by manifolds corresponding to $\overline{k}_i$ whose gradients are not nearly parallel to any vectors of $\mathcal{W}(\overline{K})$. And since $\mathcal{W}(\overline{K})$ depends continuously on $\overline{K}$ the condition can be maintained for all $\overline{K}$ in sufficiently small neighborhoods.]

By virtue of the compactness of $\mathcal{R}(K)$ there is a finite covering of $\mathcal{R}(K)$ by neighborhoods of the type specified above. Since intersections of finite numbers of these neighborhoods are also neighborhoods of this same type, one can find a finite set $\{\mathcal{R}_\alpha(K)\}$ of these neighborhoods whose sum is just $\mathcal{R}(K)$, apart from the set of measure zero coming from the boundary points of the various $\mathcal{R}_\alpha(K)$.

By virtue of the conditions imposed on the basic neighborhoods of $\mathcal{R}(K)$, each $\mathcal{R}_\alpha(K)$ is contained in one of the $\mathcal{R}_{\overline{K}}(K)$. The inverse image of $\mathcal{R}_\alpha(K)$ under the corresponding $\overline{K}'_{\overline{K}}(X)$ will be called $U_\alpha(K)$. Let $\overline{K}_\alpha(X;K)$ be the $\overline{K}'_{\overline{K}}(X,0;K) = \overline{K}'_{\overline{K}}(X)$ that maps $U_\alpha(K)$ onto $\mathcal{R}_\alpha(K)$. Then one can write

$$\begin{aligned} F(K) &= \sum_\alpha \int_{U_\alpha(K)} f_\alpha(X;K)\, J_\alpha(X;K)\, dX \\ &\equiv \sum_\alpha F_\alpha(K) , \end{aligned} \tag{A.4}$$

where

$$f_\alpha(X;K) = f\big(\overline{K}_\alpha(X;K);K\big) \tag{A.5}$$

and

$$J_\alpha(X;K) = \frac{\partial \overline{K}_\alpha(X;K)}{\partial(X,G)} . \tag{A.6}$$

Some straightforward formal manipulations give for the derivative of $F_\alpha(K)$ with respect to K the expression

$$\begin{aligned} \frac{\partial F_\alpha(K)}{\partial K} &= \int_{U_\alpha(K)} \left(\frac{\partial f_\alpha\, J_\alpha}{\partial K}\right) dX \\ &- \sum_\beta \int_{\partial U_\alpha(K)} f_\alpha\, J_\alpha\, \delta(h_{\beta\alpha})\, \frac{\partial h_{\beta\alpha}}{\partial K}\, dX , \end{aligned} \tag{A.7}$$

where

$$h_{\beta\alpha}(X;K) = \bar{k}_{j\beta\alpha}(X;K) - c_{\beta\alpha} = 0 \qquad (A.8)$$

are the equations for the surfaces in X bounding $U_\alpha(K)$. The function $\bar{k}_{j\beta\alpha}(X;K)$ is the component of $\bar{K}_\alpha(X;K)$ associated with the boundary surface $\beta\alpha$ of $U_\alpha(K)$.

The derivative with respect to K is to be interpreted, always, as the derivative on any variable upon which the k_i of K depend analytically. In particular, if the function is defined only over a restricted set, then the derivative is with respect to any appropriate local coordinate, in the sense of Def. (2.1). With this understanding the derivative $\partial f_\alpha/\partial K$ exists over the closure of $\bar{U}_\alpha(K)$ by virtue of our original assumption. It is therefore also uniformly bounded over $U_\alpha(K)$. The derivative of J_α exists in the usual sense, hence also in the sense of Def. 2.1. It is therefore also uniformly bounded over $U_\alpha(K)$. The set $U_\alpha(K)$ is bounded and hence the first term on the right in (A.7) is finite and, in the sense that it is independent of the phase of the variation dK, well-defined.

The second term on the right in (A.7) is also finite and well defined. The function $h_{\beta\alpha}$ is analytic in K in the usual sense, hence also in the sense of Def. 2.1. For any particular term β one can transform to a set of variables in which the $\bar{k}_i$ corresponding to the surface $\beta\alpha$ is one of the x_j. Then the $\delta(h_{\beta\alpha})$ just eliminates this one of the dx_j in dX. The remaining integral is finite because $U_\alpha(K)$, and hence $\partial U_\alpha(K)$, is of bounded extent. By virtue of the method of construction the number of sides β is finite and hence so is the

second term in (A.7). The formal derivative (A.7) is therefore finite and well-defined.

Since the formal derivative $\partial F/\partial K$ given by (A.7) is well-defined, the function $F(K)$ is analytic in K to the extent that the formal expression actually represents the limit of $\Delta F/\Delta K$. For real ΔK this is true. However, for complex K the meaning of $F(K)$ is not yet defined.

For complex K near a real $K_0 \in \mathcal{S}$ one can take $F(K)$ to be defined by (A.4). The many-variable version of the Cauchy Theorem[34] then permits the contour for $F_\alpha(K_0 + \Delta K)$ to be taken to consist of a central part $U_\alpha(K_0)$ plus a boundary strip running between $\partial U_\alpha(K_0)$ and $\partial U_\alpha(K_0 + \Delta K)$. Using this form for the contour one obtains (A.7) as the limit of $\Delta F/\Delta K$ for all complex ΔK. Thus the function $F(K)$ defined by (A.4) is analytic at points K_0 in $\mathcal{S}$.

For real K the various possible ways of choosing the local coordinates and the $U_\alpha(K)$ all lead, via (A.4), to the same function $F(K)$, by virtue of the factors J_α. Since the extension to complex K via any one of these choices gives an analytic function, the extension must be independent of the particular choice used in (A.4).

<u>Definition.</u> A <u>local coordinate patch</u> in $\mathcal{R}(K)$ will mean the image in $\mathcal{R}(K)$, under an analytic one-to-one mapping $\bar{K}(X)$, of a bounded open set in the space of points $X = \{x_1, \cdots, x_{n-m}\}$. The set X is the set of local coordinates corresponding to the local coordinate patch. We further specify that the x_j be a subset of the set of $\bar{k}_j$. That this is possible follows from arguments given in Theorem A'.

Theorem A. Let $\mathcal{R}(K)$ be the set of real $\overline{K}$ described in Theorem A', and let $\{\overline{K}:G(\overline{K};K) = 0\}$ be the set of all complex $\overline{K}$ satisfying the same conditions, $g_j(\overline{K};K) = 0$ for all j. Suppose $\mathcal{R}'(K) \subset \{\overline{K}:G(\overline{K};K) = 0\}$ is the image of $\mathcal{R}(K)$ under a mapping $\overline{K} \to \overline{K}' = \overline{K} + i\,T'(\overline{K};K)$, where $T'(\overline{K};K)$ is real and continuous over $\mathcal{R}(K) \otimes \mathcal{S}$ with continuous [hence uniformly bounded] first derivatives with respect to the local coordinates corresponding to some [hence every] finite covering of $\mathcal{R}(K)$ by local coordinate patches. Suppose $\mathcal{R}'(K)$ is close to $\mathcal{R}(K)$ in the sense that the image $U'_\alpha(K)$ of $U_\alpha(K)$ is within the region where $\overline{K}_\alpha(X,G)$ is analytic with analytic inverse, the relevant minors of $\partial g_j/\partial \overline{k}_i$ still being nonzero. Suppose $f(\overline{K};K)$ is analytic at points $(\overline{K};K)$ of $\{(\overline{K};K):K\in \mathcal{S},\ \overline{K}\in R'(K)\}$. Then $F(K)$ defined by (A.4), but with $U'_\alpha(K)$ in place of $U_\alpha(K)$ is analytic at K in $\mathcal{S}$, provided $\mathcal{R}'(K)$ is sufficiently close to $\mathcal{R}(K)$. [This final condition of closeness means that for some $\varepsilon > 0$ we have, using the metric in $\overline{K}$ space, $|T(\overline{K};K)| < \varepsilon$ for all $\overline{K}$ in $\mathcal{R}(K)$, for any fixed point K of $\mathcal{S}$. Though this condition is used in the following proof, it probably is not necessary for the validity of the theorem. The explicit definition of $F(K)$ is given by (A.9)].

Outline of Proof. The boundaries of the sets $U_\alpha(K)$ map into surfaces H_γ in $\mathcal{R}(K)$, the surface H_γ being, by construction, the intersection of $\mathcal{R}(K)$ with a portion of the manifold $\overline{k}_{i\gamma} = c_\gamma$. Here c_γ is a (real) rational number and $i\gamma$ means i_γ. The image of H_γ in $\mathcal{R}'(K)$ is H'_γ, on which $\overline{k}_{i\gamma}$ is given by $\overline{k}_{i\gamma} = c_\gamma + i\,t'_{i\gamma}(\overline{K}/H_\gamma\,;K)$. The symbol $\overline{K}/H_\gamma$ means that $\overline{K}$ is considered restricted to H_γ.

Each of the real functions $t'_{i\gamma}(\bar{K}/H_\gamma\ ;\ k)$ can, by virtue of the Weierstrass approximation theorem,[35] be approximated over H_γ, for fixed K, to arbitrary pointwise precision by a real polynomial $t''_{i\gamma}(\bar{K};K)$ in the variables $\bar{k}_i$ of $\bar{K}$. Indeed all the $t'_{i\gamma}(\bar{K}/H_\gamma\ ;\ K)$ having the same index i can be approximated by one single polynomial $t''_i(\bar{K};K)$. We assume this is done so that the index $i\gamma$ on $t''_{i\gamma}(\bar{K};K)$ can be interpreted as an i_γ. Sometimes, as in (A.7), $\beta\alpha$ is used in place of γ to identify a boundary surface.

The $U_\alpha(K)$ can be selected so that for each point $\bar{K}$ of $\mathcal{R}(K)$ there is, in $\mathcal{R}(K)$, a coordinate patch $N(\bar{K})$ containing $\bar{K}$, such that the $\bar{k}_{i\gamma}$ associated with each H_γ that intersects $N(\bar{K})$ is a member of the set of local coordinates X corresponding $N(\bar{K})$. If the original $U_\alpha(K)$ do not satisfy this condition then the c_γ can be slightly shifted so that the condition is satisfied. [Suppose, for example, that a surface $h_\gamma(X) \equiv \bar{k}_{i\gamma}(X) - c_\gamma = 0$ intersects the intersection I_a of a set of coordinate surfaces $x_j = 0$, where j runs over the set $\{1, \cdots, a \leqslant n - m\}$. And suppose $\partial h_\gamma / \partial x_j = 0$, for $j = a + 1, \cdots, n - m$, at some point of $I_a \cap \{h_\gamma = 0\}$. This is a typical case where the gradient of h_γ is not independent of the gradients of some subset of the x_j and hence h_γ cannot be taken as one of the x_j. There may be a connected set of points in I_a for which this condition on the gradient remains satisfied, but all points of this set must lie at $\bar{k}_{i\gamma} = c_\gamma$. Thus a slight shift of c_γ will move this entire set of points on $\{h_\gamma = 0\}$ for which the grandient condition is satisfied out of the set I_a. A finite number of applications of this argument will give the required result.]

By the Heine-Borel covering theorem the compact $\mathcal{R}(K)$ can be covered by a finite number of coordinate patches of the type specified above. These real neighborhoods $N(\overline{K})$ in $\mathcal{R}(K)$ can be extended to complex neighborhoods $N^*(\overline{K})$ in $\{\overline{K}:G(\overline{K};K) = 0\}$ such that the mapping $\overline{K}(X)$ associated with $N(\overline{K})$ remains analytic and single-valued over $N^*(\overline{K})$. We shall require that for some finite covering of $\mathcal{R}(K)$ by these local coordinate patches $N(\overline{K})$ the image in $\mathcal{R}'(K)$ of each $N(\overline{K})$ lies in $N^*(\overline{K})$. This requirement certainly can be satisfied if $\mathcal{R}'(K)$ lies sufficiently close to $\mathcal{R}(K)$. This condition is far from necessary, however.

The complexification of the neighborhoods $N(\overline{K})$ leads to a complexification $x_j \to z_j$ of the corresponding local coordinates. Because the z_j are independent variables over the corresponding $N^*(\overline{K})$, the requirement just imposed ensures that the values of the $\overline{k}'_{i\gamma}$ on H'_γ are independent variables. In particular, the $\overline{k}'_{i\gamma}$ associated with the various H'_γ can be simultaneously shifted by sufficiently small amounts without moving off the surface $\{\overline{K}:G(\overline{K};K) = 0\}$.

Because the $\overline{k}'_{i\gamma}$ on H'_γ are independent variables, in this sense, a surface $\mathcal{R}''(K) \subset \{\overline{K}:G(\overline{K};K) = 0\}$ can be defined by the mapping $\overline{K} \to \overline{K}'' = \overline{K} + i\,T''(\overline{K};K)$, where the i_γ component of $T''(\overline{K}/H_\gamma\,;K)$ is a polynomial $t''_{i\gamma}(\overline{K}/H_\gamma\,;K)$ of the type discussed earlier, and where all components of $T''(\overline{K};K)$ are, for fixed K, continuous in $\overline{K}$ over $\mathcal{R}(K)$ with continuous [hence uniformly bounded] first derivatives in the local coordinates of any fixed finite covering of $\mathcal{R}(K)$.

The surface $\mathcal{R}''(K)$ can be made to lie arbitrarily close to $\mathcal{R}'(K)$. Thus by virtue of the many-variable Cauchy theorem[34] the contour can be taken to run over $\mathcal{R}''(K)$ instead of $\mathcal{R}'(K)$, without changing the value of the integral.

The construction described above is carried out for the original real $K = K_0$ in $\mathcal{D}$. For nearby real K the boundaries of the real $U_\alpha(K)$ are taken to be defined by the same equations $h_\gamma = 0$ that are used at $K = K_0$. The boundaries of the images $U''_\alpha(K)$ of $U_\alpha(K)$, under $\overline{K} \to \overline{K}''$, are defined by taking the $t''_{i\gamma}(\overline{K};K)$ to be independent of K. This can be done because they are independent variables, in the sense discussed above.

The function $F(K)$ is defined by (A.4), but with the X in $\overline{K}_\alpha(X;K)$ replaced by $Z''_\alpha(X;K)$, which is the function that maps $U_\alpha(K)$ onto $U''_\alpha(K)$. In particular, we have

$$\begin{aligned} F_\alpha(K) &= \int_{U''_\alpha(K)} f_\alpha(Z'';K) \frac{\partial \overline{K}_\alpha(Z'';K)}{\partial Z} dZ'' \\ &= \int_{U_\alpha(K)} f_\alpha\left(Z''_\alpha(X;K);K\right) \frac{\partial \overline{K}_\alpha\left(Z''_\alpha(X;K);K\right)}{\partial Z} \frac{\partial Z''}{\partial X} dX , \end{aligned} \tag{A.9}$$

where

$$f_\alpha(Z'';K) \equiv f\left(\overline{K}_\alpha(Z'';K);K\right) .$$

As the real K varies from its original value K_0 certain of the boundaries of the $U_\alpha(K)$ may move. The integral $F_\alpha(K)$ can be considered to be composed of a central part lying over the fixed $U_\alpha(K_0)$ plus a boundary part that is the strip connecting $\partial U''_\alpha(K_0)$ to $\partial U''_\alpha(K)$. By virtue of Cauchy's theorem,[34] applied to the first form in (A.9), the exact shape of the interior of the contour $U''_\alpha(K)$ is not important; it can be slightly shifted without changing $F_\alpha(K)$.

Because of this freedom in the choice of contour, the location of central part of the contour, lying over $U(K_0)$, can be prescribed by taking the function $Z''_\alpha(X;K)$ to be independent of K. This function is then analytic in K and, consequently, so is the part of the integrand in (A.9) lying over $U(K_0)$. Thus this central contribution to $F_\alpha(K)$, which corresponds to the first term in (A.7), is analytic in K.

For the calculation of the contribution to $F_\alpha(K)$ coming from the boundary strip near $H_{\beta\alpha} = H_\gamma$ we choose a set of local coordinates X_γ which has an element $x_{j\gamma}$ that is $\bar{k}_{i\gamma}$. (We may need several such coordinate systems to cover H_γ, but a finite number will certainly suffice.) In this coordinate system γ the equation for H''_γ is

$$z''_{i\gamma} \equiv \bar{k}''_{i\gamma} = c_\gamma + i\,\bar{t}''_{i\gamma}\left(\bar{K}_\gamma(X_\gamma\,;\,K)/H_\gamma\right)$$

$$\equiv z''_{j\gamma}(X_\gamma/H_\gamma\,;\,K)\,,$$

which, by the theorem on compositions of analytic functions,[36] is analytic in K. The meaning of X_γ/H_γ is evident.

The other edge of this boundary strip lies on $\partial U''(K_0)$, hence on the image in γ space of $Z''_\alpha\left(X_\alpha/H_\gamma(K_0);K_0\right)$, which is

$$\hat{Z}''_\gamma(X_\gamma/H_\gamma\,;\,K)$$

$$\equiv \hat{Z}_\gamma\left[\bar{K}_\alpha\left(Z''_\alpha\{X_\alpha[\bar{K}_\gamma(X_\gamma/H_\gamma;K_0)];K_0\};K\right)\right] \quad .$$

The i_γ component of this equation is

$$\hat{z}''_{i\gamma}(X_\gamma/H_\gamma\ ;\ K) = \bar{k}_{i\beta\alpha}\left(Z''_\alpha\{X_\gamma[\bar{K}_\gamma(X_\gamma/H_\gamma\ ;\ K_0)];K_0\};K\right) ,$$

where $\bar{k}_{i\beta\alpha}(Z''_\alpha;K)$ is the $\beta\alpha = \gamma$ component of the function $\bar{K}_\alpha(Z''_\alpha;K)$. This function $\hat{z}''_{j\gamma}(X_\gamma/H_\gamma\ ;\ K)$ is analytic in K, and hence so is

$$\Delta\, z''_{j\gamma}(X_\gamma/H_\gamma\ ;\ K) \equiv z''_{i'}(X_\gamma/H_\gamma\ ;\ K) - \hat{z}''_{j\gamma}(X_\gamma/H_\gamma\ ;\ K) .$$

Let $\bar{Z}''_\gamma$ represent the set of coordinates other than $z''_{j\gamma} = \bar{k}''_{i\gamma}$ in the set Z''_λ, and let $\bar{X}_\gamma$ be defined similarly. The value of $\bar{Z}''_\gamma$ on H''_γ has not been specified so far. The point is that the contour in $\bar{Z}''_\gamma$ can be slightly shifted, keeping $z''_{i\gamma}(X_\gamma/H_\gamma\ ;\ K)$ fixed, without altering the value of $F_\alpha(K)$. This is because the contribution to $F_\alpha(K)$ from a piece of the contour confined to H''_γ vanishes, because of the vanishing of $dz''_{i\gamma}$. This result is familiar in simple cases, where the shifting of the contour in $\bar{Z}_\gamma$ space is justified by the Cauchy theorem in $\bar{Z}_\gamma$ space.

Since the exact value of $\bar{Z}''_\gamma(X_\gamma/H_\gamma\ ;\ K)$ is not important, we shall leave it unspecified, except to require that the surface $\mathcal{R}''(K)$ be smooth (i.e., continuous with continuous first derivatives with respect to the variables of some local coordinate system).

The contribution to $F_\alpha(K)$ from the boundary strip near H''_γ is then given (up to a sign perhaps) by

$$\int_{H_\gamma} f_\gamma\left(Z''_\gamma(X_\gamma / H_\gamma \; ; \; K_0);K_0\right) \quad \frac{\partial \, \overline{K}_\gamma\left(Z''_\gamma(X_\gamma / H_\gamma \; ; \; K_0);K_0\right)}{\partial \, Z''_\gamma}$$

$$\times \quad \frac{\partial \, \overline{Z}''_\gamma(X_\gamma / H_\gamma \; ; \; K_0)}{\partial \overline{X}_\gamma} \quad \times \; \Delta \, z''_{j\gamma}(X_\gamma / H_\gamma \; ; \; K) \; d\overline{X}_\gamma$$

plus higher-order terms in ΔK. The dependence on K is through the analytic function $\Delta z''_{j\gamma}(X_\gamma / H_\gamma \; ; \; K)$. Thus the limit $\Delta z''_{i\gamma} / \Delta K$ will be well-defined (i.e., independent of the phase of ΔK). The analyticity of $F_\alpha(K)$ then follows by the same arguments that were used in Theorem A'.

APPENDIX B. The Phase Factor in the Pole-Factorization Theorem

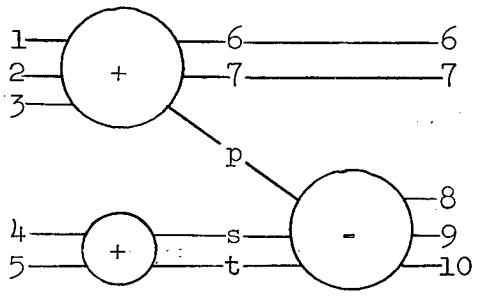

Fig. B1

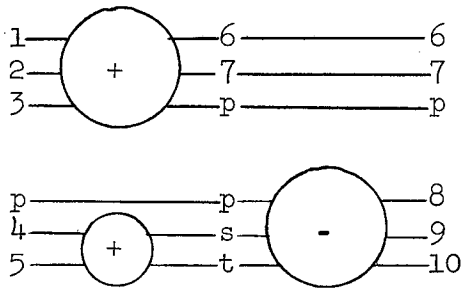

Fig. B1'

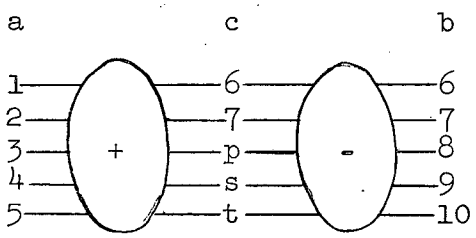

Fig. B2

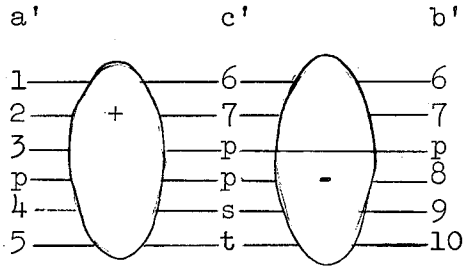

Fig. B2'

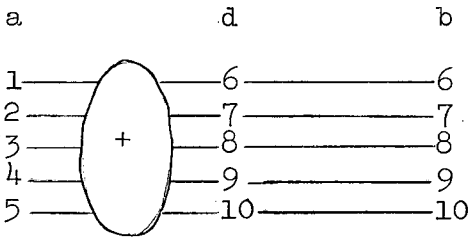

Fig. B3

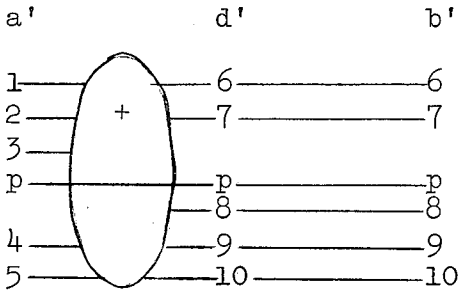

Fig. B3'

Fig. B1 - A Contribution to Fig. 3.
Fig. B1' - The corresponding contribution to Fig. 6.
Fig. B2 - The connected part corresponding to Fig. B1.
Fig. B2' - An analogous contribution corresponding to Fig. B1'.
Fig. B3 - A contribution corresponding to Fig. B2, but with a different intermediate set of particles.
Fig. B3' - An analogous contribution corresponding to Fig. B2'.

The phase factors in Fig. B1 are α_1 and α_2 whereas those in Fig. B1' are α_1' and α_2'. In order to convert Fig. 3 to Fig. 6 one must multiply by $\alpha_1' \alpha_2'/\alpha_1\alpha_2$. By virtue of E2 this is equal to $\alpha'_{ac}\, \alpha'^{*}_{bc}/\alpha_{ac}\, \alpha^{*}_{bc}$, where a, b, c, and d label the sets shown in Fig. B2, etc. But by virtue of E3 and E2 this is in turn equal to σ_p times

$$\alpha'_{ad}\, \alpha'^{*}_{bd}/\alpha_{ad}\, \alpha^{*}_{bd} = \alpha'_{ad}/\alpha_{ad} = \alpha_b$$

where α_b is the α_b of Eq. (4.10). That the other corresponding terms in Figs. 3 and 6 gave this same ratio α_b is assured by E2 and E3. The factor σ_p comes from the interchange of order of the two variables p in c' that is required to bring Fig. B2' into the form where E3 is applicable; after this interchange c' can be replaced by d'. Then E2 is used to obtain Figs. B3 and B3'.

APPENDIX C

In Ref. 13 $M_c(K)$ is shown to consist of a sum of terms $M_c^{\,i}(K)$ each of which is a limit to physical points of a function analytic in a region containing physical points as boundary points. In particular we have

$$M_c^{\,i}[\varphi] = \int M_c^{\,i}(K)\ \varphi(K)\ dK$$

$$= \lim_{\epsilon \to 0^+} \int M_c^{\,i}\big(K + i\ \epsilon\ \Delta^i(K)\big)\ \varphi(K)\ dK$$

where K is here considered a point in a real vector space of dimension $(3N - 4)$, and $\Delta^i(K)$ is a point in the same space. (i.e. local oinear coordinates are introduced). Certain properties of such functions will now be derived.

<u>Lemma C1.</u> Let $\mathfrak{D}_R$ be the space of C^∞ test functions with support confined to R, the closure of the bounded open set R' in the space R_n of n real numbers. Let $f[\varphi]$ be a functional of φ such that for any φ in $\mathfrak{D}_R$

$$f[\varphi] = \lim_{\epsilon \to 0^+} \int f\big(x + i\ \epsilon\ \Delta(x)\big)\ \varphi(x)\ d\,x\ , \qquad \text{(C1)}$$

where x and $\Delta(x)$ are elements of R_n, the components of $\Delta(z)$ are entire functions of $z = x + i\,y$, and $f(z)$ is analytic in the strip

$$S = \{z : z = x + i\ \epsilon\ \Delta(x),\ x \in R,\ 0 < \epsilon < \eta > 0\}\ .$$

If $\varphi(x)$ in $\mathcal{D}_R$ is, at points x of R' the restriction to R' of a function $\varphi(z)$ analytic and uniformly bounded in

$$S'' = \{z : \mathrm{Re}\, z \in R',\ |\mathrm{Im}\, z| < \rho(x)\},$$

where $\rho(x)$ is continuous and positive for x in R', then

$$f[\varphi] = \lim_{\epsilon \to 0^+} \int_{R-R''} f\big(x + i\, y(x) + i\, \epsilon\, \Delta(z)\big)\, \varphi\big(x + i\, y(x)\big)\, d\big(x + i\, y(x)\big) + \int_{R''} f\big(x + i\, y(x)\big)\, \varphi\big(x + i\, y(x)\big)\, d\big(x + i\, y(x)\big), \tag{C.2}$$

where the closure of R'' is in R' and $y(x)$ is any continuous real function that is zero for x not in R', and for x in R' gives a $z = x + i\, y(x)$ that is in $S \cap S'$. The set of points $x + i\, y(x)$ can be considered a contour C lying over the real points x. Then (C2) can be written in the more compact form

$$f[\varphi] = \lim_{\epsilon \to 0^+} \int_{C(R - R'')} f\big(z + i\, \epsilon\, \Delta(z)\big)\, \varphi(z)\, dz + \int_{C(R'')} f(z)\, \varphi(z)\, dz, \tag{C.2'}$$

where $C(T)$ is the part of C lying over T.

<u>Proof.</u> By Cauchy's Theorem, generalized to several variables,[34] (C1) is equivalent to

$$f[\varphi] = \lim_{\epsilon \to 0^+} \int_C f\big(z + i\,\epsilon\,\Delta(z)\big)\,\varphi(z)\,dz \;, \tag{C3}$$

since for some $\epsilon' > 0$ and every fixed $0 < \epsilon < \epsilon'$ the functions are analytic in the region through which the contour is shifted. The points on the boundary of R', where the $\varphi(z)$ are not analytic, give no contribution because of the boundedness condition on $\varphi(z)$. For the part $C(R'')$ of C the limit $\epsilon \to 0$ can be taken, since the integrand is analytic, hence continuous, in ϵ at these points.

Lemma C2. If the conditions of Lemma C1 are satisfied and if $|f(z)|$ is bounded over the intersection of S with some neighborhood of $R - R'$, then

$$f[\varphi] = \int_C f(z)\,\varphi(z)\,dz \;. \tag{C2''}$$

Proof. The boundedness of $f(z)$ and $\varphi(z)$ assures that contribution from $R' - R''$ vanishes as $R'' \to R'$.

Corollary 1. If $f[\varphi]$ and φ satisfy the conditions of Lemmas C1 and C2 than $f[\varphi]$ is finite (noninfinite).

Proof. The right side of (C2") is finite.

Corollary 2. If in place of Eq. (C1) we have

$$f[\varphi] = \lim_{\epsilon \to 0^+} \int \big[f^+\big(x + i\,\epsilon\,\Delta^+(x)\big) - f^-(x - i\,\epsilon\,\Delta^-(x)\big]\,\varphi(x)\,dx$$

and if $f^{\pm}(z)$ and $\varphi(x)$ satisfy the conditions of Lemmas C1 and C2 then we have, in place of Eq. (C2"), rather

$$f[\varphi] = \int_{C^+} f^+(z)\,\varphi(z)\,dz - \int_{C^-} f^-(z)\,\varphi(z)\,dz .$$

Proof. The manipulations of the proof of Lemma C2 can be carried through for each term separately.

In the following theorems x is a single real variable (i.e. $n = 1$). The set R' will be the real set $R' = \{x : |x| < a\}$.

Theorem C1. Let $f[\varphi]$ be a functional of φ such that for any φ in $\mathcal{D}_R$

$$f[\varphi] = \lim_{\epsilon \to 0^+} \int [f^+(x + i\epsilon) - f^-(x - i\epsilon)]\,\varphi(x)\,dx ,$$

where the $f^{\pm}$ are analytic in the strips

$$S^{\pm} = \{z : \mathrm{Re}\, Z \in R,\ 0 < \pm \mathrm{Im}\, z < \eta > 0\} .$$

Suppose the $|f^{\pm}(z)|$ are bounded in $S^{\pm} \cap N$, where N is a neighborhood of $R - R'$. Then the vanishing of $f[\varphi]$ for all φ in $\mathcal{D}_R$ implies that the limits $f^{\pm}(x)$ exist, are analytic, and are equal, for all x in R'.

Proof. By virtue of the second corollary to Lemma C2 one has

$$f[\varphi] = \int_{C^+} f^+(z)\,\varphi(z)\,dz - \int_{C^-} f^-(z)\,\varphi(z)\,dz \qquad \text{(C5)}$$

for any $\varphi(z)$ satisfying the conditions of the Lemma C1, where C^+ and C^- are certain (compact) contours from $x = -a$ to $x = a$

that lie just above and below the real axis for $|x| < a$. Let $\psi(x)$ be the function $\psi(z) = \exp[-(a^2 - z^2)^{-1}]$ and let $\varphi(z\,;\,z') \equiv \psi(z)\,(z - z')^{-1}$. Then define

$$F(z') = \int_{C^+ + \overline{C}^+} f^+(z)\,\varphi(z\,;\,z')\,dz - \int_{C^- + \overline{C}^-} f^-(z)\,\varphi(z;z')\,dz, \tag{C6}$$

where $\overline{C}^+$ is a contour from a to $-a$ that lies inside S^+ and above C^+ for $|x| < a - b$; and $\overline{C}^-$ is a contour from a to $-a$ that lies inside S^- and below C^- for $|x| < a - b$. For $a \geqslant |x| \geqslant a - b > 0$ the contours $\overline{C}^{\pm}$ are taken to coincide with $-C^{\pm}$, respectively. By Cauchy's Theorem $F(z') = 2\pi i\, f^+(z')\,\psi(z')$ if z' is between C^+ and $\overline{C}^+$, and $F(z') = 2\pi i\, f^-(z')\,\psi(z')$ if z' is between C^- and $\overline{C}^-$. By virtue of the vanishing of (C5) one also has for z' in either of these two regions

$$F(z') = \int_{\overline{C}^+} \frac{f^+(z)\,\psi(z)}{(z - z')}\,dz - \int_{\overline{C}^-} \frac{f^-(z)\,\psi(z)}{(z - z')}\,dz\,. \tag{C7}$$

In view of the analyticity and boundedness conditions on $f^{\pm}$ and φ, (C7) implies that $F(z')$ is a single analytic function throughout the interior of $\overline{C}^+ - \overline{C}^-$. This implies that $f^+(z')$ and $f^-(z')$ are both equal to $F(z')/2\pi i\,\psi(z')$, and hence are analytic, inside $\overline{C}^+ - \overline{C}^-$.

<u>Theorem C2.</u> The condition of boundedness on $|f^{\pm}|$ in the statement of Theorem C1 can be replaced by the condition that $|f^{\pm}(z)|$ be bounded in $S^{\pm} \cap N$ by $C \exp B\,|\mathrm{Im}\, z|^{-m}$ for some positive values of the constants C, B and m.

Proof. One can replace in the proof of Theorem C1 the function $\psi(z) = \exp[-(a^2-z^2)^{-1}]$ by $\exp[-A(a^2 - z^2)^{-m}]$. And the curve $z(x)$ can be taken to approach the endpoints at $x = \mp a$ along the lines $\arg(z \pm a) = \pm \pi/4m$. Then for points on the contour sufficiently near $x = \pm a$ the function $|\psi(z)|$ is less than $\exp - A'|\mathrm{Im}\, z|^{-m}$ where $A' \equiv A[\sin(\pi/4m)]^m(2a)^{-m}/2$. If one chooses A so that $A' > B$ then $|f^{\pm}\varphi|$ is bounded near $x = \pm a$. The same argument also shows that $|f^{\pm}(z)\,\varphi(z;z')|$ is bounded for z near $\pm a$, if $z' \neq \pm a$. But it is the boundedness of these products, rather than of the $|f^{\pm}|$ themselves, that is actually needed both in Lemma C2 and its corollaries, and in the proof of Theorem C1 itself. Thus the arguments in that proof carry over immediately to the present case.

Remark C1. Theorems similar to C2 have been proved by other authors,[37,38] under the more stringent assumption that $f[\varphi]$ is a distribution. This distribution assumption demands that $f(z)$ be bounded near $\mathrm{Im}\, z = 0$ by some negative power of $|\mathrm{Im}\, z|$.[39] It is not clear that we wish to impose such a strict requirement on the allowed functions. In fact, from the S-matrix view point is is natured to allow all functionals $f[\varphi]$ on $\mathfrak{D}_R$ that can be expressed as sums of limits of analytic functions. Theorem C2 is a step in this direction. The condition, required in this theorem, that f be bounded by an exponential of an inverse power, while already very weak, can be much further weakened by replacing the constant power m by $C' \exp B' |\mathrm{Im}\, z|^{-m'}$. Moreover this new m can again be replaced in the same way, and so on. Thus the bound on $f(z)$ can be made extremely weak. Whether the boundedness condition can be removed altogether is still an open question, as far as I know.

Theorem C. If a functional $f[\varphi]$ has a finite value for all infinitely differentiable functions $\varphi(x)$ of compact support $\mathcal{R}$ that are restrictions to $\mathcal{R}$ of functions analytic in the interior of R, and if for such φ the functional is given by

$$f[\varphi] = \lim_{\epsilon \to 0} \int [f^{+}(x + i\epsilon) - f^{-}(x - i\epsilon)]\, \varphi(x)\, dx \qquad (C8)$$

where the functions $f^{\pm}(z)$ are analytic in the strips

$$S^{\pm} = \{z : \mathrm{Re}\, z \in R,\ 0 < \pm \mathrm{Im}\, z < \eta > 0\} \qquad (C9)$$

and are bounded at points of $S^{\pm}$ near the boundary of R by $C \exp B\, |\mathrm{Im}\, z|^{-m}$, for some positive values of constants C, B and m, then the functions $f^{\pm}(z)$ are unique up to a common additive function that is analytic at interior points of R.

Proof. This follows from Theorem C2 by taking the $f^{\pm}$ of that theorem to be differences of possible functions $f^{\pm}$ of this theorem. The extra condition in this theorem that φ be analytic at interior points of $\mathcal{R}'$ does not alter the proofs, since only functions having this property were used.

Remark C2. The particular boundedness condition used in Theorem C can, according to the Remark C1, be greatly weakened, if the need should arise. Also, class of φ for which (C8) holds can be much further restricted, if the need should arise.

APPENDIX D

The mass-shell paths C_{ij} connecting various crossed reaction and Hermitian conjugate points are constructed by following the distortions of paths $C_{ij}(a_p)$ as the effective masses a_p increase from zero to μ_p^2. At the start, where all $a_p = 0$, the $C_{ij}(a_p)$ all reduce to a single common point. As the a_p increase the various end points of the $C_{ij}(a_p)$ move along definite singularity free paths and the interiors of the $C_{ij}(a_p)$ are distorted so as to avoid a certain set of Landau surfaces $\mathcal{D}$.

In this continuation in a_p the various $C_{ij}(a_p)$ may be distorted in such a way that it becomes possible to find a closed loop lying in the set of C_{ij}. Since no surfaces of the specified set $\mathcal{D}$ cross this loop as it develops from a single point at $a_p = 0$ to its form at $a_p = \mu_p^2$ one can say that, in a certain sense, none of these surfaces lies "within" the loop. Thus one might expect that the mass-shell loop should be able to be shrunk (staying within the mass shell) to a point, without crossing any of these surfaces. This is in fact, true, within limits, if the set of surfaces $\mathcal{D}$ is such that it is possible to construct some function singular on just this set $\mathcal{D}$.

This result is proved by an application of the continuity theorem for functions of several complex variables.[40] First it can be noted that the actual loop, as it grows from a point to its final form, can, at each stage, be approximated to arbitrary precision (pointwise) by a curve that is a boundary of a disc lying on an analytic manifold. In

particular if the equation for the loop at any particular value of the a_p is given in terms of a set of mass-shell variables z_i by $z_i = z_i(\theta)$, where θ is a cyclic variable, then these equations can be approximated to arbitrary pointwise precision[41] by the expansion

$$z_i \simeq \sum_{-N}^{N} (\exp i\,\theta)^n \; c_n$$

$$= \bar{z}_i \; (\exp i\,\theta) \quad .$$

The surface $\bar{z}_i(z)$ is a one (complex) dimensional analytic manifold.[30] The curve $\{\bar{z}_i(z); |z| = 1\}$ passes arbitrarily close to the original curve.

If the original curve is always confined to a bounded region, as we shall suppose, and remains at more than some finite minimum distance from $\mathcal{D}$ throughout the contribution, which we can suppose, then N can be held fixed over the entire journey from $a_p = 0$ to $a_p = \mu_p$. Since the boundary curve $\{z_i(z); |z| = 1\}$ crosses no singularity of $\mathcal{D}$, neither can the interior points $[z_i(z), |z| < 1]$, by virtue of the continuity theorem. Thus finally at $a_p = \mu_p$ we can shrink the curve to a point by the transformation $|z| \to 0$.

The above argument applies within the limit set by the requirement that a system of analytic local coordinates z_i can be found such that the loop lies within the coordinate patch corresponding to these coordinates. Though the question can be pursued further it is simpler to restrict the paths so that no closed loop occurs within the set of paths.

For the set of analytic functions with singularities lying on $\mathcal{S}$ (at least within the relevant region) we have in mind the truncated or renormalized perturbation theory functions. It remains to be shown, however, that the restriction imposed on $\mathcal{S}$ by the requirement that only positive-α singularities enter the physical region of the larger process actually forces $\mathcal{S}$ to be identical to the analogous set of singularity surfaces in perturbation theory.

FOOTNOTES AND REFERENCES

1. H. P. Stapp, Phys. Rev. 125, 2139 (1962).

2. G. C. Wick, A. S. Wightman, and E. Wigner, Phys. Rev. 88, 101 (1952).

3. H. P. Stapp, Appendix I of Ref. 1. There crossing was assumed to hold without any extra phase factors, but the no-scattering parts were allowed to be different from unity.

4. H. P. Stapp, The Decomposition of the S Matrix and the Connection Between Spin and Statistics, Lawrence Radiation Laboratory Report UCRL-10289, June 1962.

5. J. Gunson, Birmingham Preprint (1962) and J. Math. Phys. 6, 827 (1965).

6. D. Olive, Towards an Axiomation of S Matrix Theory, July 1963 (Cambridge preprint. This is an earlier version of Ref. 7.)

7. D. Olive, Phys. Rev. 135, B745 (1964).

8. H. P. Stapp, Seminars in Problems in S-Matrix Theory, August 1963. (privately circulated). That Gunson's method might be used to justify an effective continuation in mass, which is the central idea of both this reference and of the present paper, was suggested earlier in the Proceedings of the 1963 Midwest Conference on Theoretical Physics (Notre Dame, May-June 1963). The main difficulty in achieving this is to take all possible Landau singularities into account.

9. J. B. Boyling, Nuovo Cimento 33, 1356 (1964).

10. E. Y. C. Lu and D. I. Olive, Spin and Statistics in S-Matrix Theory, Cambridge Preprint, Nov. 1965.

11. David Olive (Cambridge University), private communication, Oct. 1965. The proof of spin and statistics given in Section VII was constructed upon receiving this communication, in which Olive stated that he and Lu had obtained a proof based on his version of the crossing argument.

12. H. P. Stapp, The Analytic S-Matrix Framework, in "The Trieste Lectures," High Energy Physics and Elementary Particles (IAEA, Vienna, 1965).

13. C. Chandler and H. P. Stapp, Asymptotic Causality and the Physical Region Analyticity Properties of Scattering Functions (in preparation). The results obtained in this reference fall slightly short of the summary given in the present paper. The reader is referred to the original for an exact account.

14. J. C. Polkinghorne, Nuovo Cimento 25, 901 (1962).

15. P. V. Landshoff and D. I. Olive, Extraction of Singularities from the S-Matrix, Cambridge Preprint, April 1965.

16. David Branson, The Cross Discontinuity Condition in the S-Matrix, Lawrence Radiation Laboratory Report UCRL-16702, Feb. 1966.

17. H. P. Stapp, Antiparticles in S-Matrix Theory, in "The Trieste Lectures," High Energy Physics and Elementary Particles (IAEA, Vienna, 1965).

18. A proof of this fact is given in Ref. 8, and also in Ref. 19. A proof explicitly making use of the definition (2.1) of analyticity is given in Ref. 20.

19. Peter Minkowsky, David N. Williams, and Rudolf Seiler, in Proceedings of the Symposium of the Lorentz Group, Seventh Annual Summer Institute for Theoretical Physics, 1964 (University of Colorado, Boulder, Colorado) (Lemma A).

20. H. P. Stapp, 1963 Madras Lectures on Analytic S-Matrix Theory (Matscience, Madras, India, 1963), p. 72.

21. That the Landau surfaces can be expressed in this parametric form has been noticed also by Logunov, Todorov, and Chernikov. See 1962 International Conference on High Energy Physics at CERN, p. 695. My derivation consists of expressing the q_n of Eq. (2.38) first in terms of the internal momenta incident on V_n, using (2.36b), and then expressing the internal p_j in terms of the $\Delta_j = \alpha_j p_j = \Delta_j \omega$ of diagram $\bar{D}$.

22. S. Bochner and W. Martin, Several Complex Variables, (Princeton University Press, Princeton, New Jersey 1948) p. 39.

23. Functionals $f[\varphi]$ more general than distributions could be used here.

24. H. J. Bremmerman, Distributions, Complex Variables and Fourier Transformations (Addision-Wesley, Reading, Massachusetts, 1965), p. 48.

25. The path of continuation from P^+ to P_0 to P^- is really a collection of paths, one for each term in $A^{\pm}$. The shifting of these paths into the mass shell is to be done in such a way that they all pass through a common point $P_0(a_i)$, at which A_0 is added. This requirement may entail that some of the paths have to jump across certain cuts, as is discussed in the following paragraph. So long as only a finite number of singularity manifolds are relevant, the required distortions and continuations are possible. Natural boundary are considered to be infinite collections of singularity manifolds and are assumed to give no contributions to the residue if none of the constituent manifolds do. The set of points where the manifold property fails is very thin and the paths can be moved by them without difficulty.

26. H. P. Stapp, Phys. Rev. 128, 1863 (1962).

27. It is an earlier version of the following argument that, communicated to J. R. Taylor and converted by him to Hilbert space form, served as a key element in the proof of Lu and Olive. The formalism of Taylor (Ref. 28) is a transcription of the M-function formalism into a Hilbert space form.

28. J. R. Taylor, J. Math. Phys. 7, 181 (1966).

29. J. R. Taylor, Cluster Decomposition of S-Matrix Elements, University of Wisconsin Preprint, (1965).

30. Robert C. Gunning and Hugo Rossi, Analytic Functions of Several Complex Variables (Prentice-Hall, Englewood Cliffs, New Jersey, 1965) pp. 16-18.

31. B. A. Fuks, Introduction to the Theory of Analytic Functions of Several Complex Variables (American Mathematical Society, Providence, Rhode Island, 1963), p. 6 and p. 203.

32. N. Dunford and J. Schwartz, Linear Operators, Vol. I (Interscience Publishers Inc., New York, 1958) p. 17.

33. That this can be achieved is assured by an application of Theorem 8 of page 8 of Ref. 30. Almost constant can mean, for instance, less $(1000n)^{-1}$ percent variation.

34. See Ref. 31, 264. It is important to note that if the region of integration $V = U_\alpha(K)$ is bounded in part by a portion of a manifold defined by $z_{j\gamma} = c_\gamma$, where $j\gamma \equiv j_\gamma$, then the values of the remaining z_j on the interior of this portion of the boundary can be varied in a smooth way through a region of analyticity

of the integrand without altering the integral. This is because $dZ = dz_1 \wedge dz_2 \cdots \wedge dz_{n-m}$ is zero if any dz_j is zero, and hence the extra piece of contour gives no contribution.

35. See e.g. R. Courant, Methods of Mathematical Physics (New York University, Institute for Mathematics and Mechanics, 1950), p. 47.

36. See e.g. Ref. 30 Theorem 5, p. 6.

37. H. G. Tillmann, Math. Zeitschrift 77, 106-124 (1961).

38. See R. F. Streater and A. S. Wightman, PCT, Spin and Statistics, and All That. (W. A. Benjamin Inc., New York, 1964), Section 2-5. In both this reference and the preceding one the main interest is the many-dimensional case.

39. See Ref. 37 or Theorem 2-10 of Ref. 38.

40. See the article by A. S. Wightman in the Les Houches Lectures, Dispersion Relations and Elementary Particles, R. Omnes and C. Dewitt, ed. (Wiley and Sons Inc., New York, 1960), p. 260.

41. G. H. Hardy and W. W. Rogosinski, Fourier Series (Cambridge University Press, London, 1950) p. 21.

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