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A STUDY OF MULTI-CHANNEL DYNAMICS IN THE NEW STRIP APPROXIMATION

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Naren F. Bali and Shu-yuan Chu

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Naren F. Bali*

and

Shu-yuan Chu

Lawrence Radiation Laboratory
University of California
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ABSTRACT

A systematic study of the dynamics of scattering with several coupled two-particle channels is made using the new strip approximation. The existence of a unique solution to the ND^{-1} equations is established from maximal analyticity of the second degree. The method used is an extension of Jones' proof in the single-channel case, making use of an explicit expression for the determinant of D constructed by Gross. The general method is then applied to the particular case of a $\pi\pi$-$K\bar{K}$ two-channel calculation of the ρ-meson, first with fixed-spin particle and then with Reggeized particle exchange as the generalized potential. A detailed discussion of the numerical result is presented with the conclusion that the effect of the inelastic channel $(K\bar{K})$ is not important in the present approximation scheme.

I. INTRODUCTION

The strip concept regarding the four line connected parts can be stated most easily in terms of the usual invariants s, t and u. In the physical region of the s-channel, we have resonance peaks in the low energy region (s smaller than a few GeV^2) and for high s we have diffraction peaks in the foreward and backward direction with width less than a few GeV^2 in the momentum transfer variables (t or u). Otherwise the four line connected part is small. Similar statements can be made for the physical region of the t- and u-channels. On a Mandelstam diagram the regions where the four line connected part is large will be strip-like regions. If we assume the same strip structure for the unphysical regions we will require the four line connected part to be small unless the magnitude of at least one of the invariants is small. The new strip approximation[1] is based on the observation that this strip structure can be achieved very naturally if we approximate the four line connected parts as the sum of direct and crossed channel Regge poles satisfying the Mandelstam Representation. A typical s-channel Regge-trajectory $\alpha(s)$ that can reach the right half angular-momentum-plane is shown in Fig. 1. The strip region will be the finite interval of s for which $\mathrm{Re}\alpha(s) > 0$ with $\mathrm{Im}\alpha(s) \leq 1$ where the Regge pole contribution of the general form:

$$-\pi[2\alpha(s) + 1]\, \beta(s)\, \frac{P_{\alpha(s)}(-z_s)}{\sin \pi\alpha(s)}$$

will dominate the amplitude:

(1) With $s > 0$, we have resonance poles when $\mathrm{Re}\,\alpha(s)$ takes integral value $(\frac{d\alpha(s)}{ds} > 0)$. If $\mathrm{Im}\alpha(s) \geq 1$ the resonance becomes too broad to be recognizable.

(2) With $s < 0$, we are in the physical region of the crossed channels; since $\mathrm{Re}\,\alpha(s) > 0$ the Regge pole will dominate the high t (or u) behavior.

Furthermore if we require $\beta(s)$ to fall off rapidly for large s we have the strip struction in the unphysical region. Similar arguments can be applied to t- and u-channel Regge poles.

Assuming (multi-channel) two-particle unitarity in the direct channel across the strip, a reasonable approximation for a strip width of a few Gev if we include channels with unstable external particles, we can write down a set of coupled ND^{-1} integral equations, with the exchange of crossed channel Regge poles as generalized potentials analytic in the angular momentum variable. From these equations we can calculate the Regge poles in the direct channel.

Teplitz and Collins[2a,2b] have made a very extensive study in the single-channel case of $\pi\pi$ scattering. In this report we study the effect of additional inelastic channels. We derive in Section II the multi-channel ND^{-1} equations and prove the existence of a unique solution from maximal analyticity of the second degree.[3] In Section III, after justifying the numerical method used in solving the ND^{-1} equations, we make a detailed calculation of the $\pi\pi$-$K\bar{K}$ model of the ρ-meson. In the concluding section, we discuss the unsatisfactory aspects of our scheme and how they may be improved.

II. THE EXISTENCE OF A UNIQUE SOLUTION TO THE MULTI-CHANNEL STRIP EQUATION

In a recent paper Jones[4] gives a rather complete discussion of the single-channel N/D equations when the D-function has only a finite cut. In particular he shows that a unique solution exists by assuming maximum analyticity of the second degree (MASD), or analyticity in angular momentum.

We generalize his results to the Multi-Channel ND^{-1} equations in the new strip approximation. The ambiguity in the solution can be removed in the same way as in the single-channel case by using an explicit form for the determinant of D, constructed by Gross.[5]

The logic of the problem is as follows. We want to calculate the partial-wave scattering amplitude B_ℓ (B_ℓ is a matrix), from an input B_ℓ^P which carries all the left hand cut of B_ℓ and the right hand cut above $s = \sigma$, where σ is the strip boundary common to all channels, and the requirement that B_ℓ satisfy unitarity from $s = s_1$ to $s = \sigma$, where s_1 is the threshold of the lowest channel. We try to solve this problem by writing $B_\ell = ND^{-1}$, with the matrix D carrying the right hand cut of B_ℓ from $s = s_1$ to $s = \sigma$ and the matrix N carrying all the other cuts. The justification for this decomposition can be accomplished in two ways. We can show that for any given B_ℓ we can construct D explicitly (for example through the Omnes formula in the single channel case); we then get N from $N = B_\ell D$. Or we assume that B_ℓ can be written as ND^{-1}. We then

derive the integral equations satisfied by N and D, and justify our assumption a posteriori by proving that solutions to the integral equations for N or D actually exist. In the multi-channel case we have to rely on the second approach. Mandelstam[6] has shown the existence of D in the case when $\sigma \to \infty$, and the eigenphase shifts satisfy $\delta(\infty) - \delta(s_1) = 0$. For the case when σ is finite the latter conditions about the eigenphase shifts are not satisfied unless $\ell \to \infty$; however, we can prove the existence of N at least for large ℓ. The matrix ND^{-1} will then have the correct cut structure. To establish the uniqueness of our solution we have to remove the CDD ambiguity, that is, the possibility of adding arbitrary poles into the dispersion relation for D and N. This is accomplished for large ℓ by using the Gross formula. The continuation to lower values of ℓ can be done in exactly the same fashion as for the single-channel case.

A. The Derivation of the Strip Equation

The partial-wave scattering amplitude from the ith channel to the jth channel is defined by the following equations:

$$\left[A(s,t)\right]_{ij} = \sum_{\ell} (2\ell+1)\,(A_\ell)_{ij}\,P_\ell(z)$$

$$(B_\ell)_{ij} = \frac{(A_\ell)_{ij}}{q_i^\ell\, q_j^\ell}$$

In the following we will use matrix notation and suppress the

index ℓ except when we discuss properties concerning the angular momentum.

Our basic problem is to calculate B, assuming that we are given B^p which carries the left hand cut of B and the right hand cut above $s = \sigma$,

$$B(s) = B^p(s) + \frac{1}{\pi} \int_{s_1}^{\sigma} \frac{\mathrm{Im}B(s')}{s' - s} ds' \qquad \text{(II-A:1)}$$

and that B satisfies the multi-channel two-body unitarity from $s = s_1$ to $s = \sigma$:

$$\mathrm{Im}\, B^{-1} = -\rho\, \theta \qquad s_1 < s < \sigma \qquad \text{(II-A:2)}$$

where ρ and θ are diagonal matrices (we restrict ourselves to spinless particles):

$$\rho_{ij} = \frac{2q_i^{2\ell+1}}{\sqrt{s}} \delta_{ij}$$

$$\theta_{ij} = \theta(s - s_i)\, \delta_{ij} .$$

We proceed by writing

$$B = ND^{-1} , \qquad \text{(II-A:3)}$$

where D carries the right hand cut from $s = s_1$ to $s = \sigma$, and N carries all the other cuts.

We then have, on the right hand cut

$$\mathrm{Im}\, D = \mathrm{Im}(B^{-1}N) = (\mathrm{Im}\, B^{-1})N = -\rho\, \theta\, N \quad s_1 < s < \sigma \qquad \text{(II-A:4)}$$

and on rest of the cuts of B

$$\mathrm{Im}\, N = \mathrm{Im}(BD) = (\mathrm{Im}\, B)D = (\mathrm{Im}\, B^p)D \qquad \text{(II-A:5)}$$

If we normalize D to the unit matrix at infinity, we can write (we defer the discussion of CDD poles to Section II-C),

$$D(s) = 1 - \frac{1}{\pi}\int_{s_1}^{\sigma} ds' \, \frac{\rho(s')\, \theta\, (s')\, N(s')}{s' - s} \qquad \text{(II-A:6)}$$

and

$$N(s) = B(s)\, D(s)$$

$$= B^p(s)\, D(s) + \frac{D(s)}{\pi}\int_{s_1}^{\sigma} ds' \, \frac{\mathrm{Im}\, B(s')}{s' - s}$$

$N(s)$ is real by assumption for $s_1 < s < \sigma$, and since the second term on the right vanishes at infinity as $\frac{1}{s}$, we must have

$$\frac{D(s)}{\pi}\int_{s_1}^{\sigma} ds' \frac{\mathrm{Im}\, B(s')}{s'-s} = -\frac{1}{\pi}\int_{s_1}^{\sigma} ds' \frac{B^p(s')\ \mathrm{Im}\, D(s')}{s'-s}$$

Thus we have, using (II-A:6)

$$N(s) = B^p(s) + \frac{1}{\pi}\int_{s_1}^{\sigma} ds' \frac{B^p(s') - B^p(s)}{s'-s}\ \rho(s')\ \theta(s')\ N(s') \qquad \text{(II-A:7)}$$

We now proceed to show that (II-A:7) is soluble at least for large ℓ and defines a unique $N(s)$.

B. The Existence of N

For the present discussion[7] is is more convenient to rewrite the equations for N to display the channel indices explicitly.

$$N_{ij}(s) = B^p_{ij}(s) + \frac{1}{\pi}\int_{s_\mu}^{\sigma} \frac{B^p_{i\mu}(s') - B^p_{i\mu}(s)}{s'-s}\ \rho_\mu(s')\ N_{\mu j}(s') \qquad \text{(II-A:7)}'$$

with the convention that repeated Greek indices are summed over all channels.

As explained in Reference 1, $B_{i\mu}{}^p(s)$ will have a logarithmic singularity near $s = \sigma$:

$$B_{i\mu}{}^p(s) \xrightarrow[s \to \sigma]{} -\frac{1}{\pi}\ \mathrm{Im}\, B_{i\mu}(\sigma)\ \ln(\sigma - s) \qquad \text{(II-B:1)}$$

Preparatory to removing the singular part, we define new functions N'_{ij} by

$$N'_{ij}(s) = \rho_i^{1/2}(s)\, N_{ij}(s) \qquad \text{(II-B:2)}$$

The reason for doing this will become clear in the following:

We have for N'_{ij}

$$N'_{ij}(s) = B'^{p}_{ij}(s) + \frac{1}{\pi}\int_{s_\mu}^{\sigma} ds' \, \frac{\rho_i^{1/2}(s)\left[B^{p}_{i\mu}(s') - B^{p}_{i\mu}(s)\right]\rho_\mu^{1/2}(s')}{s' - s}\, N'_{\mu j}(s') \qquad \text{(II-B:3)}$$

where

$$B'^{p}_{ij}(s) = \rho_i^{1/2}(s)\, B_{ij}^{p}(s)$$

Separating out the singular part down to the highest threshold s_M, we have:

$$N'_{ij}(s) = B'^{p}_{ij}(s) + \int_{s_\mu}^{s_M} ds'\, U_{i\mu}(s,s')N'_{\mu j}(s') + \int_{s_M}^{\sigma} ds'\, K_{i\mu}(s,s')\, N'_{\mu j}(s')$$
$$- \frac{\lambda_{i\mu}}{\pi^2}\int_{s_M}^{\sigma} ds'\, k(s,s')\, N'_{\mu j}(s') \qquad \text{(II-B:4)}$$

where

$$U_{i\mu}(s,s') = \frac{1}{\pi} \frac{\rho_i^{1/2}(s)\left[B_{i\mu}{}^{p}(s') - B_{i\mu}{}^{p}(s)\right] \rho_\mu^{1/2}(s')}{s' - s}$$

$$K_{i\mu}(s,s') = U_{i\mu}(s,s') + \frac{\lambda_{i\mu}}{\pi^2} k(s,s')$$

$$k(s,s') = \frac{\ln(\sigma-s') - \ln(\sigma-s)}{s' - s}$$

$$\lambda_{i\mu} = \rho_i^{1/2}(\sigma) \left[\mathrm{Im}\, B_{i\mu}{}^{p}(\sigma)\right] \rho_\mu^{1/2}(\sigma)$$

Note that now $\lambda_{i\mu} = \lambda_{\mu i}$, a relation not true if we use N_{ij} instead of N'_{ij}.[8]

In the single-channel case, we can use the Wiener-Hopf method to get a unique solution if the phase shift $\delta(\sigma) < \pi$.[4] In the multi-channel case, we expect the eigenphase shifts to play a similar role. To facilitate future discussion, we specify the eigenphase shifts in the following way.

The matrix

$$S = 1 + 2i\, \rho^{1/2}\theta\, B\theta\, \rho^{1/2} ,$$

which is unitary and symmetric for $s_1 < s < \sigma$, can be diagonalized by an orthogonal matrix $T(s)$. For $s_i < s < s_{i+1}$ (with $s_{M+1} = \sigma$, M is the number of channels). The first i eigenvalues can be written as

$$e_j(s) = e^{2i\,\delta_j^{(i)}(s)} \qquad j = 1,\cdots, i$$

where $\delta_j^{(i)}$ are the eigenphase shifts of the interval $s_i < s < s_{i+1}$. Because of the step functions θ, the eigenphase shifts of different intervals are not analytic continuations of one another. But also because of the step function, we have

$$S_{ij}(s_k) = \delta_{ij} \quad \text{if } i \text{ or } j > k ,$$

which implies

$$\left.\begin{aligned} \delta_i^{(i)}(s_i) &= n_i\,\pi \\ \delta_j^{(i)}(s_i) &= \delta_j^{(i+1)}(s_i) \quad j = 1,\cdots, i. \end{aligned}\right\} \quad i = 1,\cdots, M$$

Thus we can write all the eigenphase shifts in terms of M real continuous, piecewise analytic functions:

$$\delta_i(s) = \delta_i^{(j)}(s) \qquad s_j < s < s_{j+1}$$

$$j = i,\cdots, M \qquad\qquad i = 1,\cdots, M$$

At $s = \sigma$, the orthogonal transformation $T = T(\sigma)$ that diagonalizes S also diagonalizes the matrix Λ, whose elements are the $\lambda_{i\mu}$'s. We have

$$TST^{-1} = 1 + 2i\, T\left[\rho^{1/2}(\mathrm{Re}B)\,\rho^{1/2}\right]T^{-1} - 2T\Lambda T^{-1}$$

taking the real part of both sides, we get

$$\lambda_i = \mathrm{Re}\left[\frac{e^{2i\delta_i(\sigma)}-1}{2}\right]$$

or

$$\lambda_i = \left[\sin\delta_i(\sigma)\right]^2 \quad i = 1,\cdots, M$$

where λ_i is the ith eigenvalue of Λ.

After introducing the matrix T, as explained in Reference 7, we can use the Wiener-Hopf method to get the following coupled integral equations:

$$N'_{ij}(s) = B'^{p}_{ij}(s) + \int_{s_\mu}^{s_M} ds'\, U_{i\mu}(s,s')\, N'_{\mu j}(s')$$

$$+ T^{-1}_{\mu\nu}\int_{s_M}^{\sigma} ds'\, (U_{i\mu}O_\nu)(s,s')\, \overline{N}^{O}_{\nu j}(s')$$

for $s < s_M$

$$\overline{N}^{O}_{ij}(s) = \overline{B}^{p}_{i\mu}(s) + T_{i\mu}\int_{s_\mu}^{s_M} ds'\, U_{\mu\nu}(s,s')\, N'_{\nu j}(s')$$

$$+ T_{i\mu}\, T^{-1}_{\nu\rho}\int_{s_M}^{\sigma} ds'\, (K_{\mu\nu}O_\rho)(s,s')\, \overline{N}^{O}_{\rho j}(s')$$

for $s_M < s$ (II-B:5)

where: (i) the $\overline{N}^{0}_{ij}$'s are related to N'_{ij}'s by the Wiener-Hopf equations

$$\overline{N}_{ij} = \overline{N}^{0}_{ij} - \lambda_i \int_{s_M}^{\sigma} ds' \, k(s,s') \, \overline{N}_{ij}(s') \qquad \text{(II-B:6)}$$

with

$$\overline{N}_{ij} = T_{i\mu} \, N'_{\mu j} \, . \qquad \text{(II-B:7)}$$

Just as in the single-channel case we can invert (II-B:6) to give

$$\overline{N}_{ij}(s) = \int_{s_M}^{\sigma} ds' \, O_i(s,s') \, \overline{N}^{0}_{ij}(s')$$

where $O_i(s,s')$ is the Wiener-Hopf resolvent kernel.[9]

(ii) $\overline{B}_{ij}{}^{p} = T_{i\mu} \, B'^{p}_{\mu j}$

$$(U_{i\mu} O_\nu)(s,s') = \int_{s_M}^{\sigma} ds'' \, U_{i\mu}(s,s'') \, O_\nu(s'',s')$$

$$(K_{\mu\nu} O_\rho)(s,s') = \int_{s_M}^{\sigma} ds'' \, K_{\mu\nu}(s,s'') \, O_\rho(s'',s')$$

As discussed in Reference 4, the function $\overline{N}_{ij}(s)$ will have the following behavior near $s = \sigma$:

$$\overline{N}_{ij}(s) \longrightarrow (\sigma - s)^{-1+a_i} \qquad \text{(II-B:8)}$$

with $a_i = \frac{1}{\pi}\,|\delta_i(\sigma) \pm m_i\pi| < \frac{1}{2}$, unless we fix the arbitrary constant in the general solution of (II-B:6) to a particular value, then

$$\overline{N}_{ij}(s) \xrightarrow[s \to \sigma]{} (\sigma - s)^{-a_i} \qquad \text{(II-B:9)}$$

To find the correct behavior near $s = \sigma$, we make use of the explicit expression for the determinant of D. In the next section, we will show that for large ℓ, when there are no bound states

$$\det D(s) = \exp\left\{-\frac{1}{\pi}\sum_{\mu=1}^{M}\int_{s_\mu}^{\sigma} ds' \,\frac{\delta_\mu(s')}{s' - s}\right\}$$

with $\delta_i(s_i) = 0$

and $\delta_i(\sigma) \to 0$ as $\ell \to \infty$.

It follows that the determinant of D has the following behavior near the strip boundary

$$\det D(s) \xrightarrow[s \to \sigma]{} (\sigma - s)^{-\frac{1}{\pi}\sum_i \delta_\mu(\sigma)}$$

Because of Eq. (II-A:6), we have

$$\det N(s) \xrightarrow[s \to \sigma]{} (\sigma - s)^{-\frac{1}{\pi}\sum \delta_\mu(\sigma)}$$

But N_{ij} is related to $\overline{N}_{ij}$ through (II-B:2) and (II-B:7), so we have:

$$\det \overline{N} = \det(TN') = (\det T)\det(\rho^{1/2})\det N \qquad \text{(II-B:10)}$$

so

$$\det \overline{N}(s) \xrightarrow[s \to \sigma]{} (\sigma - s)^{-\frac{1}{\pi}\sum_{\mu=1}^{M} \delta_\mu(\sigma)}$$

Since each term of the determinant of $\overline{N}$ involves one of the elements of each column of $\overline{N}$, by comparing Eqs. (II-B:8), (II-B:9) and (II-B:10), bearing in mind that $\delta_i(\sigma) \to 0$ as $\ell \to \infty$, we conclude that Eq. (II-B:9) is the correct behavior for each i, and we can make the identification

$$a_i = \frac{\delta_i(\sigma)}{\pi} \qquad \text{(II-B:11)}$$

Thus all the arbitrary constants in the solution are fixed. Once the identification (II-B:11) is made and analyticity in ℓ is assumed, we have a unique solution to (II-A:7) for all ℓ as long as all eigenphase shifts at $s = \sigma$ are less than π, because (i) if $\delta_i(\sigma) < \frac{\pi}{2}$ Eqs. (II-B:5) are Fredholm.[7] (ii) if $\frac{\pi}{2} < \delta_i(\sigma) < \pi$, we may define

$$\overline{N}^0_{ij}(s) = \overline{N}^0_{ij}(s)(\sigma - s)^{-\left[\frac{\delta_i(\sigma)}{\pi} - \frac{1}{2} + \varepsilon\right]}$$

with

$$1 - \frac{\delta_i(\sigma)}{\pi} > \varepsilon > 0$$

and the integral equations for $\overline{N}^{0}_{ij}(s)$ are Fredholm.[4]

In the next section we will discuss the CDD ambiguities and continuation to lower ℓ values.

C. CDD Ambiguities and Maximal Analyticity of the Second Degree

Just as in the single channel case[10] we might have written for D_ℓ and N_ℓ

$$D_\ell(s) = 1 - \frac{1}{\pi}\int_{s_1}^{\sigma} ds' \frac{\rho_\ell(s')\,\Theta(s')\,N_\ell(s')}{s'-s} + \sum_{i=1}^{n} \frac{\Gamma_i}{s-\beta_i} \qquad \text{(II-A:6)'}$$

$$N_\ell(s) = B_\ell^p\left[1 + \sum_{i=1}^{n}\frac{\Gamma_i}{s-\beta_i}\right] + \frac{1}{\pi}\int_{s_1}^{\sigma} ds' \frac{B_\ell^{\,p}(s') - B_\ell^{\,p}(s')}{s'-s}\,\rho_\ell(s')\Theta(s')N_\ell(s') \qquad \text{(II-A:7)'}$$

if Γ_i (the residue matrix) and β_i are suitably chosen so that $D_\ell(s)$ is real outside the interval $s_1 < s < \sigma$. It is clear that (II-A:7)' will have a solution, at least if all the β_i are outside the interval $s_1 < s < \sigma$. These are the well known CDD ambiguities.

We choose our convention for the eigenphase shifts as

$$\delta_i(s_i) = 0 \qquad \text{for } i = 1,\cdots, n\,.$$

The most general expression for the determinant of D is[4]

$$\det D(s) = \frac{\prod_{i=1}^{m}(s-\mu_i)}{\prod_{i=1}^{n}(s-\beta_i)} (s-\sigma)^{n-m} \exp\left\{ -\frac{1}{\pi}\sum_{\mu=1}^{M} \int_{s_\mu}^{\sigma} ds' \frac{\delta_\mu(s')}{s'-s} \right\}$$

For any physically reasonable B_ℓ^P we expect $B_\ell^P(s) \to 0$ for $s_1 < s < \sigma$ as $\ell \to \infty$ (e.g. through Froissart-Gribov transformation as in the single channel case).[4] With our convention we then have $\delta_i(\sigma) \to 0$ for $i = 1, \ldots, M$. In this limiting case, we expect

$$D_\ell \xrightarrow[\ell \to \infty]{} 1 + \sum_{i=1}^{n} \frac{\Gamma_i}{s-\beta_i}$$

$$N_\ell \xrightarrow[\ell \to \infty]{} B_\ell^P \left[1 + \sum_{i=1}^{n} \frac{\Gamma_i}{s-\beta_i} \right].$$

So there shouldn't be any singularity at $s = \sigma$, thus $n = m$. Furthermore there are no bound states for large ℓ, so $m = 0$, which implies $n = 0$. Therefore we conclude that for large ℓ we have a unique solution with no CDD poles in D.

Now we will establish that N_ℓ and D_ℓ are analytic in ℓ, then we can infer from MASD that we have a unique solution for lower values of ℓ:

From the discussion in Reference 4, we are assured of ℓ analyticity for $N_\ell(s)$ when $s_1 < s < s_M$ and for $\overline{N}_\ell^0(s)$ when

$s_M < s < \sigma$, at least for ℓ values such that $\mathrm{Re}\,\ell > -1$ and $\delta_\ell(\sigma) < \pi$ (we suppress the channel indices). The $\overline{N}_\ell$ defined by

$$\overline{N}_\ell(s) = \int_{s_M}^{\sigma} ds'\, O_\ell(s,s')\, \overline{N}_\ell^{\,0}(s') \qquad s_M < s < \sigma$$

are analytic functions of ℓ as long as the integrals exist, e.g. if $\delta_\ell(\sigma) < \pi$. Similarly N_ℓ for $s_M < s < \sigma$ defined by

$$N_\ell(s) = \rho_\ell^{-1/2}(\sigma)\, T_\ell^{-1}(\sigma)\, \overline{N}_\ell(s)$$

are analytic in ℓ for $\delta_\ell(\sigma) < \pi$. Finally the functions D_ℓ , defined by

$$D_\ell = 1 - \frac{1}{\pi} \int_{s_1}^{\sigma} ds' \frac{\rho_\ell(s')\, \theta(s') N_\ell(s')}{s' - s}$$

are analytic in ℓ when the N_ℓ are analytic, provided the integrals exist. The integrals certainly exist for $\delta_\ell(\sigma) < \pi$, therefore it is obvious that D_ℓ cannot develop any singularity (or poles) as long as $\delta_\ell(\sigma) < \pi$ and will always maintain the same normalization at infinity. The question of what happens, if $\delta_\ell(\sigma) > \pi$ as ℓ decreases and if $\delta_\ell(\sigma) < \pi$ once again when we continue to decrease ℓ , has been discussed in Reference 4. The conclusion is that the solution to our problem, whenever $\delta_\ell(\sigma) < \pi$, is correctly given by the unique solution to the integral equations satisfied by N_ℓ and D_ℓ with no CDD poles.

III. A $\pi\pi$-$K\bar{K}$ MODEL OF THE ρ-MESON

The dynamic calculation of the ρ-meson, considered as a resonance in channels consisting of two pseudoscalar mesons with vector meson exchange as the main binding force, has been made by various authors.[12a,12b] The inherent difficulty in all models using the N/D method,[12a] is that the output ρ-width is too large. (The calculations by Balázs and by Finkelstein,[12b] using an equivalent potential approach which perhaps is equivalent to a partial Mandelstam iteration, seems to show substantial improvement in this aspect.) As it turns out, our method suffers the same difficulty.

First we justify the numerical method by applying a result obtained by Jones and Tiktopoulos.[13] Then as a preliminary step before the fully Reggeized calculation, we consider the case when fixed-spin particles are being exchanged. We thus develop a basis of comparison against which we can test the effect of Reggeization. Next we construct the generalized potential from Regge-pole exchange. There is some complication because the π- and K-meson masses are different and we have to make an approximation. The numerical results are discussed in detail. The unsatisfactory aspects of our scheme and how they may be improved will be discussed in the concluding section.

A. Justification of the Numerical Method

The theorem proved by Jones and Tiktopoulos[13] can be stated as follows. The integral equation

$$\psi(s) = \phi(s) + \int_a^b ds' \mathcal{K}(s,s')\psi(s')$$

is given where $\phi(\chi)$ is in $L^2(a,b)$, i.e. $\int_a^b ds|\phi(s)|^2 < \infty$ and the resolvent $(1 - \mathcal{K})^{-1}$ of the integral operator K exists. Suppose further that $\mathcal{K} = W + C$ where the norm of W is less than 1 and C is square integrable $(\int_a^b |C(s,s')|^2 dsds' < \infty)$. Then the method of matrix inversion can be used to invert the integral equation (for this we need K to be piecewise continuous) provided we choose the mesh points carefully near the singular point.

The multi-channel equations for N can be easily shown to satisfy the above conditions. Diagonalizing Eq. (II-B:4), we have

$$\overline{N}_{ij}(s) = \overline{B}_{ij}(s) + \int_{s_\mu}^{s_M} ds' \, \overline{U}_{i\mu}(s,s')\overline{N}_{\mu j}(s') + \int_{s_M}^{\sigma} ds' \, \overline{K}_{i\mu}(s,s')\overline{N}_{\mu j}(s')$$

$$- \frac{\lambda_i}{\pi^2} \int_{s_M}^{\sigma} ds' \, k(s,s')\overline{N}_{\mu j}(s') \qquad i,j = 1,\cdots, M$$

where

$$\overline{N}_{ij}(s) = T_{i\mu}N'_{\mu j}(s)$$

$$\overline{U}_{i\mu}(s,s') = T_{i\nu}U_{\nu\rho}(s,s')T^{-1}_{\rho\mu}$$

$$\overline{K}_{i\mu}(s,s') = T_{i\nu}K_{\nu\rho}(s,s')T^{-1}_{\rho\mu}$$

We can rewrite the above equations for each fixed column index j as a single integral equation

$$\psi_j(s) = \phi_j(s) + \int_{s_1}^{M\sigma - (M-1)s_1} \mathcal{K}(s,s')\psi_j(s') \qquad j = 1,\cdots, M$$

where

$$\psi_j(s) = \overline{N}_{ij}[s - (i-1)(\sigma-s_1)]$$

$$\phi_j(s) = \overline{B}_{ij}[s - (i-1)(\sigma-s_1)]$$

$$\begin{aligned}\mathcal{K}(s,s') = {} & \overline{U}_{i\mu}[s - (i-1)(\sigma-s_1), s' - (\mu-1)(\sigma-s_1)]\theta[s_M - (\mu-1)(\sigma-s_1) - s'] \\ & + \overline{K}_{i\mu}[s - (i-1)(\sigma-s_1), s' - (\mu-1)(\sigma-s_1)]\theta[s' - s_M + (\mu-1)(\sigma-s_1)] \\ & - \frac{\lambda_i}{\pi^2} k[s - (i-1)(\sigma-s_1), s' - (\mu-1)(\sigma-s_1)]\theta[s' - s_M + (\mu-1)(\sigma-s_1)]\end{aligned}$$

for

$$s_1 + (i-1)(\sigma-s_1) < s < s_1 + i(\sigma-s_1)$$

$$s_1 + (\mu-1)(\sigma-s_1) < s' < s_1 + \mu(\sigma-s_1)$$

and

$$\theta(s) = \begin{cases} 1 & \text{if } s > 0 \\ 0 & \text{if } s < 0 \end{cases}$$

Now the existence of the resolvent $(1 - \mathcal{K})^{-1}$ as well as the square-integrability of the $\overline{U}_{i\mu}$ and $\overline{K}_{i\mu}$'s has been established in Section II of this report. Since $\lambda_i = \sin^2 \delta_i(\sigma)$ are less than 1 for all i, the operator $\mathcal{K}$ is the sum of two operators: one with norm less than 1; the other, square integrable. Therefore the theorem of Jones and Tiktopoulos applies.

B. Fixed-spin Particle Exchange

1. Generalized Potential

Near a t-channel pole of definite spin ℓ_t, isospin I_t and mass m_R, the scattering amplitude

$$A_{ij}(s,t) \sim (2\ell_t + 1) \frac{R_{ij}(t)}{m_R^2 - t} P_{\ell_t}(z_t)$$

By crossing, the generalized potential in the s-channel for angular momentum ℓ_s and isospin I_s from exchange of this particle in the t-channel in the zero width approximation is given by

$$\left[B_{\ell_s}{}^{p}(s)\right]_{ij}^{I_s} = \beta_{I_s I_t}(2\ell_t+1)R_{ij}(t=m_R^2)P_{\ell_t}\left[z_t(t=m_R^2)\right]Q_{\ell_s}\left[z_s(t=m_R^2)\right]\frac{1}{2(q_i q_j)^{\ell_s+1}}$$

where $\beta_{I_s I_t}$ is the isospin crossing matrix element. The explicit form of the crossing matrices used is as follows:

for $\pi\pi \to \pi\pi$

$$\beta^{\pi\pi}_{I_s I_t} = \begin{array}{c|ccc} I_s \backslash I_t & 0 & 1 & 2 \\ \hline 0 & \frac{1}{3} & 1 & \frac{5}{3} \\ 1 & \frac{1}{3} & \frac{1}{2} & -\frac{5}{6} \\ 2 & \frac{1}{3} & -\frac{1}{2} & \frac{1}{6} \end{array}$$

$\pi\pi \to K\bar{K}$

$$\beta^{\pi K}_{I_s I_t} = \begin{array}{c|cc} I_s \backslash I_t & \frac{1}{2} & \frac{3}{2} \\ \hline 0 & \frac{\sqrt{6}}{3} & \frac{2\sqrt{6}}{3} \\ 1 & \frac{2}{3} & -\frac{2}{3} \end{array}$$

$K\bar{K} \to K\bar{K}$

$$\beta^{KK}_{I_s I_t} = \begin{array}{c|cc} I_s \backslash I_t & 0 & 1 \\ \hline 0 & \frac{1}{2} & \frac{3}{2} \\ 1 & \frac{1}{2} & -\frac{1}{2} \end{array}$$

We list below the generalized potential arising from the exchange of particles shown in Fig. 2. (Channel indices 1 for $\pi\pi$ channel, 2 for $K\bar{K}$ channel.)

$$\left[B^{p}_{\ell_s}(s)\right]^{I_s}_{11} = 2 \times \beta^{\pi\pi}_{I_s 1} \times 3 \times \left\{\gamma_{11}\right\} \times \left(\frac{m_\rho^{\,2}-4m_\pi^{\,2}}{4}\right)\left(1+\frac{2s}{m_\rho^{\,2}-4m_\pi^{\,2}}\right) Q_{\ell_s}\left(1+\frac{2m_\rho^{\,2}}{s-4m_\pi^{\,2}}\right)$$

$$\times \frac{1}{2(q_1)^{2\ell_s+2}}$$

where

$$\gamma_{11} = \Gamma_\rho\, m_\rho^{\,2} \frac{1}{\sqrt{m_\rho^{\,2}-4m_\pi^{\,2}}} \Bigg/ \frac{m_\rho^{\,2}-4m_\pi^{\,2}}{4}$$ is the reduced residue.

$$q_1^{\,2} = \frac{s-4m_\pi^{\,2}}{4}$$

The factor of 2 comes from the u-channel contribution which Bose-statistics requires to be the same as that of the t-channel contribution for allowed ℓ_s, I_s combination.

$$\left[B^{p}_{\ell_s}(s)\right]^{I_s}_{12} = 2 \times \beta^{\pi K}_{I_s \frac{1}{2}} \times 3 \times \left\{\frac{3}{4} \times \gamma_{11} \times \frac{m_{K^*}^{\,2} - (m_\pi^{\,2}+m_K^{\,2})^2}{4}\right\} \times$$

$$\times \left(1 + \frac{2m_{K^*}^{\,2}\, s}{[m_{K^*}^{\,2}-(m_\pi+m_K)^2][m_{K^*}^{\,2}-(m_\pi-m_K)^2]}\right) Q_{\ell_s}\left(\frac{m_{K^*}^{\,2} - \frac{s}{2} - m_\pi^{\,2} - m_K^{\,2}}{2q_1q_2}\right) \frac{1}{2(q_1q_2)^{\ell_s+1}}$$

where

$$q_2^{\,2} = \frac{s-4m_K^{\,2}}{4}$$

The factor $\frac{3}{4}$ is the ratio of reduced residues given by exact SU_3 symmetry.[15]

$$\left[B^{p}_{\ell_s}(s)\right]^{I_s}_{22} = \left[\beta^{KK}_{I_s 1} \times 3 \times \left\{\frac{1}{2} \times \gamma_{11} \times \frac{m_\rho^{\,2}-4m_K^{\,2}}{4}\right\} \left(1 + \frac{2s}{m_\rho^{\,2}-4m_K^{\,2}}\right) Q_{\ell_s}\left(1 + \frac{2m_\rho^{\,2}}{s-4m_K^{\,2}}\right)\right.$$

$$\left. + \beta^{KK}_{I_s 0} \times 3 \times \left\{\frac{3}{2} \times \gamma_{11} \times \frac{m_\phi^{\,2}-4m_K^{\,2}}{4}\right\} \left(1 + \frac{2s}{m_\phi^{\,2}-4m_K^{\,2}}\right) Q_{\ell_s}\left(1 + \frac{2m_\phi^{\,2}}{s-4m_K^{\,2}}\right)\right]$$

$$\times \frac{1}{2(q_2)^{2\ell_s+2}}$$

The factors $\frac{1}{2}$ and $\frac{3}{2}$ in the curly brackets are again the SU_3 ratios of reduced residues. Notice that we do not have a factor of 2 here since ρ- and ϕ-mesons cannot be exchanged in the u-channel. We treat the ϕ-meson purely as a member of the same octet which contains the ρ-meson, and the ω-meson as a singlet entirely decoupled to the octet. The actual ω-ϕ mixing will make little difference to the output as will become apparent from the discussion in the following section.

2. Results

The numerical results comparing the single-channel and two-channel cases for various values of the strip width σ, the input ρ-width Γ_ρ^{in}, the K^*-width $\Gamma_{K^*}^{in}$ and the ϕ-width Γ_ϕ^{in} are summarized in Table 1 and Figs. 3-5.

We conclude that the one-channel and two-channel cases are qualitatively very much the same. In order to get a resonance peak in the p-wave cross section at the experimental ρ-mass,[14] we need an input ρ-width several times larger than the experimental value or a strip width much larger than one might expect from the fact that resonance peaks die out when s is larger than a few GeV^2. The output ρ-width is always several times larger than the experimental value (see, however, discussion in (e) below). The $I = 1$ trajectory is too flat and the intercept too high, while the $I = 0$ trajectory violates the Froissart limit slightly. Both kinds of trajectories turn over too soon, if we take values given by the zero of the real part of the determinant of D (see Footnote 14). These qualitative features of the output are not sensitive to the changes in the input parameters.

The inclusion of the inelastic channel does give some additional binding and, as expected, will narrow the output ρ-width, though not by a large amount. The details are as follows.

(a) The spacing between the $I = 0$ and $I = 1$ trajectories is smaller in the two-channel case, thus the Froissart limit is violated no worse than in the one-channel case, although the corresponding $I = 1$ trajectory intercept is larger.

(b) Keeping Γ_ρ^{in} at the experimental value, in order to get the ρ-peak at the right position $(\sqrt{s} = 5.3\ m_\pi)$ we need a much higher σ for the one-channel case $(1280\ m_\pi^2)$ than for the two-channel case $(600\ m_\pi^2)$; also the output ρ-width is narrower and the trajectory steeper in the latter case.

(c) If we allow both σ and Γ_ρ^{in} to vary, keeping the correct output ρ-peak position, we notice that for larger σ we get a narrower output ρ-width but flatter trajectories.

(d) To see the effect of $\Gamma_{K^*}^{in}$, Γ_ϕ^{in} and the ρ-contribution in $K\bar{K} \to K\bar{K}$, we allow them to vary independently instead of using the values given by SU_3. Because their contributions are small compared to the ρ-contribution in $\pi\pi \to \pi\pi$, unless we assign them values drastically different from those given by SU_3 there will be no appreciable changes. Increasing Γ_ϕ^{in}, $\Gamma_{K^*}^{in}$ will narrow the output ρ-width and flatten the output trajectory. Changing Γ_ϕ^{in} is more effective.

(e). If we use Γ_ϕ^{in} 7 times and $\Gamma_{K^*}^{in}$ 0.46 times the value given by SU_3 we could get an output ρ-width of $0.9\ m_\pi$, although the intercept of the ρ-trajectory is very large (0.9). This confirms that one way of achieving the experimental ρ-width is to get the ρ-meson primarily as a bound state of an inelastic channel weakly coupled to the $\pi\pi$ channel, though it is not possible to do so with reasonable input in our scheme. Notice in Fig. 4 that we have two $I = 1$ trajectories in this latter case. The leading trajectory is primarily a bound state of the $K\bar{K}$ channel due to the unusually large

attraction of the ϕ-exchange; the secondary one, presumably due to ρ-exchange in the $\pi\pi$ channel, is mainly a resonance of the $\pi\pi$ channel.

C. Reggeized Particle Exchange

1. The Generalized Potentials

The potentials were obtained following the Chew-Jones prescription.[1] We assume that the double spectral functions are large only in the strip like regions indicated with i , j , k in Fig. 6. In the case of an equal mass channel the contribution to the amplitude in the s-channel from a Regge pole in the j-region can be written (omitting signature and isotopic spin complication):

$$R_j^{\sigma}(s,t) = \frac{1}{2}\Gamma_j(t)\int_{\sigma}^{\infty} \frac{P_{\alpha_j(t)}\left(-1 - \frac{2s}{t-4}\right)}{s' - s}\, ds' \qquad \text{(III-C:1)}$$

where

$$\Gamma_j(t) = [2\alpha_j(t) + 1]\gamma_j(t)(-q_t^{\,2})^{\alpha_j(t)} \qquad \text{(III-C:2)}$$

$\alpha_j(t)$ and $\gamma_j(t)$ are the trajectory and reduced residue functions of the Regge pole under consideration. The integral in Eq. (III-C:1) is defined by analytic continuation when it does not converge.

The full amplitude can then be written as the sum of Regge poles in all strip regions:

$$A(s,t) = \sum_i [R_i^\sigma(s,t) + \xi_i R_i^\sigma(s,u)]$$

$$+ \sum_j [R_j^\sigma(t,s) + \xi_j R_j^\sigma(t,u)]$$

$$+ \sum_k [R_k^\sigma(u,s) + \xi_k R_k^\sigma(u,t)] \qquad \text{(III-C:3)}$$

where the ξ's are the signature factors of the Regge poles and σ is the strip width, taken to be the same for all channels. The isotopic spin crossing matrix elements have been omitted. The amplitude thus constructed will have a double spectral function different from zero in the shaded regions of Fig. 6. The Reggeized potential for an equal mass channel can now be obtained directly from this amplitude by separating the appropriate terms from the partial wave projection of the amplitude. Using the Wong projection formulae we can write the contribution of a single Regge pole in the $\pi\pi \to \pi\pi$ case as:[2a] (m_π taken to be 1)

$$\left[B_\ell^P(s)\right]_{11}^I = \sum_j \frac{1}{2\pi q_1^{2\ell+2}} \int_{-\infty}^{0} dt \left[\mathrm{Im}Q_\ell\left(1 + \frac{t}{2q_1^2}\right)\right] \times$$

$$\times \left\{ \beta_{II'}^{\pi\pi}\Gamma(t) \int_{-4q_t^2}^{\sigma} dt\, P_\alpha(t) \left(-1 - \frac{u'}{2q_t^2}\right) \left[\frac{1}{u'-s} + \frac{1}{u'-u}\right] \right.$$

$$+ (-1)^{I'} \beta_{II'}^{\pi\pi} \int_{\sigma}^{\infty} du'\, \Gamma(t')P_\alpha(t') \left(-1 - \frac{u'}{2q_t^2}\right) \left[\frac{1}{u'-u} - (-1)^I \frac{1}{u'-t}\right]$$

either

$$+ \frac{\pi\Gamma(t)\beta^{\pi\pi}_{II'}}{\sin \pi\alpha(t)} \left[(-1)^{I'} P_{\alpha(t)}\left(-1 - \frac{s}{2q_t^2}\right) + P_{\alpha(t)}\left(1 + \frac{s}{2q_t^2}\right)\right] \quad \text{if} \quad \left(-1 - \frac{s}{2q_t^2}\right) < 1$$

or

$$+ \Gamma(t)\beta^{\pi\pi}_{II'}\left[\pi P_{\alpha(t)}\left(-1 - \frac{s}{2q_t^2}\right)\begin{Bmatrix} \cot \frac{\pi\alpha(t)}{2} ; & I' = 0,2 \\ -\tan \frac{\pi\alpha(t)}{2} ; & I' = 1 \end{Bmatrix} - 2Q_{\alpha(t)}\left(-1 - \frac{s}{2q_t^2}\right)\right] \quad \text{if} \quad \left(-1 - \frac{s}{2q_t^2}\right) > 1 \Bigg\}$$

$$- \frac{1}{4} \int_{-\infty}^{4-\sigma} ds' \frac{\Gamma(s')\delta_{II'}}{(s'-s)(-q_{s'}^2)^{\ell+1}} \int_{4-s'}^{\sigma} dt' \, P_{\alpha(s')}\left(-1 - \frac{t'}{2q_{s'}^2}\right) P_{\ell}\left(-1 - \frac{t'}{2q_{s'}^2}\right)$$

(III-C:4)

where

$$s + t + u = s' + t' + u' = 4$$

A similar expression can be written for the case $K\bar{K} \to K\bar{K}$.

In the case of $\pi\pi \to K\bar{K}$, the external particles have different masses. The more complicated functional relation between q_t^2 and t means that an expression of the type Eq. (III-C:1) will fail to have the correct analytic properties in s and t because the double spectral function will be non-zero outside the shaded regions shown in Fig. 6. The procedure adopted to avoid this difficulty is to

approximate q_t^2 by

$$(q_t^*)^2 = \frac{[t - (m_\pi + m_K)^2][(M^*)^2 - (m_\pi - m_K)^2]}{4(M^*)^2}$$

where M^* is the mass of the resonance being exchanged in the t-channel. In the same spirit, we define the cosine of the scattering angle in the t-channel as

$$z_t^* = 1 + \frac{s}{2(q_t^*)^2}$$

With the above approximation we can rewrite Eq. (III-C:1) to read (suppress the subscript j)

$$R^{*\sigma}(s,t) = \frac{1}{2}\Gamma^*(t) \int_\sigma^\infty \frac{P_{\alpha(t)}\left(-1 - \frac{s}{2(q_t^*)^2}\right)}{s' - s} ds' \qquad \text{(III-C:5)}$$

where now

$$\Gamma^*(t) = [2\alpha(t) + 1]\gamma(t)[-(q_t^*)^2]^{\alpha(t)}$$

It is readily seen that this expression has the correct analytic properties, and when $t \to (M^*)^2$ it reduces to Eq. (III-C:1). As we expect the main contribution to the potential should come from values of t close to $(M^*)^2$, this seems to be a reasonable approximation. With this new form for $R_j^{*\sigma}(s,t)$, the same steps as led to Eq.

(III-C:4) can be carried out and an expression for the potential in the $\pi\pi \to K\bar{K}$ case can thus be derived.

The potential due to exchange of the Pomeranchuk trajectory, which is present in both the $\pi\pi$ and $K\bar{K}$ channels, is very repulsive in our scheme. As pointed out by Collins[2a] a combination of an attractive and a repulsive potential can give rise to poles on the physical sheet in the present method. One way to avoid this difficulty is to "renormalize" the potential[16] by subtracting from the potential its value at $s = 0$ and adding back the same quantity now computed from a partial wave projection of the amplitude. In the $\pi\pi \to \pi\pi$ case, we have

$$\left[B_\ell^P(s)\right]_{11}^I = \frac{1}{2\pi q_s^{2\ell+1}} \int_{-\infty}^{0} dt \;\; \mathrm{Im}Q_\ell\left(1 + \frac{t}{2q_s^2}\right)\left[V^P(s,t) - V^0(s,t)\right]$$

$$+ \frac{\beta_{I0}^{\pi\pi}}{2\pi q_s^{2\ell+1}} \int_4^{\sigma} dt \; Q_\ell\left(1 + \frac{t}{2q_s^2}\right) \sum_{\ell' \text{ even}} (2\ell' + 1) \; \mathrm{Im}A_{\ell'}(t)$$

where $V^P(s,t)$ is the contribution from exchange of the Pomeranchuk trajectory to the potential in the $\pi\pi \to \pi\pi$ case. It has been shown by Collins[2b] that the second term is negligible, and the "renormalization" can be carried out using only the first term. He also pointed out that this renormalizing technique can lead to difficulties, and is not entirely justifiable. We shall use it in the following because it is the only way to prevent the potential from being overwhelmed by P-exchange.

2. Results

In the one-channel calculation[2a,2b] it has been noted that we cannot get enough attraction from either a fixed-spin particle exchange or a Regge-pole exchange with input ρ-width equal to the experimental value. Thus in order to get the peak in the cross section at the correct ρ-mass, we have to use an input ρ-width several times larger than the experimental value. In the fixed-spin case this leads to an $I = 0$ trajectory with intercept larger than 1, violating the Froissart bound. In the Reggeized case we have an additional restriction, i.e. the imaginary part of the potential at the strip boundary is required by unitarity to satisfy the relation $0 \leq \rho_\ell(\sigma)\ \mathrm{Im}B^P(\sigma) \leq 1$, which will be violated usually before we violate the Froissart bound, especially in the lower ℓ values. the other difficulties are large output ρ-width and flat output trajectory. From the two-channel fixed-spin particle exchange case discussed above we find that the inelastic channel $K\bar{K}$, through $\Gamma^{in}_{K^*}$ and Γ^{in}_{ϕ}, provides additional attraction and narrows the output ρ-width a little. We now investigate whether we can get the same kind of effect with various types of parametrization of the input trajectories and residues in the Reggeized case.

(a) Reproducing the fixed-spin particle exchange case: we use extremely flat linear trajectories for ρ, K^* and ϕ passing through 1 at the corresponding resonance energies. The residues, taken to be constant, are adjusted to give respectively the experimental ρ-width, the K^*-width and the ϕ-width with ratios to the ρ-width fixed by

SU_3 symmetry. We get very much the same potentials as from the fixed-spin particle exchange; in particular there are very small imaginary parts.

(b) Various parametrizations of the input trajectories and residue functions: we use a one-pole formula for the various trajectories[2b]

$$\alpha(t) = J_R - a\frac{t_B}{t_R} - a\frac{1 - \frac{t_B}{t_R}}{1 - \frac{t}{t_B}}$$

where $J_R - a$ gives the intercept at $t = 0$, t_R is the square of the mass of the resonance, J_R the spin of the resonance and t_B is the pole-position. Two forms of the residue functions have been used: the Chew-Teplitz formula[18]

$$\gamma(t) = C\alpha'(t)[\bar{t} - t]Q_{\alpha(t)}\left(1 + \frac{56}{\bar{t} - 4}\right)\Big/\left(\frac{\bar{t} - 4}{4}\right)^{\alpha(t)+1}$$

which has been used in Ref. 2b to get a self-consistent solution, and the one-pole formula

$$\gamma(t) = \frac{A}{1 - \frac{t}{B}}$$

which can give a steeply-falling residue function as suggested by the results of fitting the high energy scattering data.[19]

To start with, we use the same parametrization for α_ρ and α_p and the residue functions for $\gamma_{\rho\pi\pi}$ and $\gamma_{p\pi\pi}$ which gives a

self-consistent solution in the one-channel calculation as explained in Ref. 2b, and assume the P and ρ contribution in $K\bar{K}$ to be of the same form as that in $\pi\pi$, with P coupled equally strongly to the $K\bar{K}$- and the $\pi\pi$-channels but with $\gamma_{\rho K\bar{K}}/\gamma_{\rho\pi\pi} = 0.5$, the SU_3 ratio for all t. As for the trajectories α_ϕ, α_{K^*} we take the pole position t_B to be the same as for α_ρ but adjust the intercept at $t = 0$ so that they run more or less parallel to the ρ-trajectory. For the residue functions $\gamma_{\phi K\bar{K}}$ and $\gamma_{K^*\pi K}$ we again use the same form as for $\gamma_{\rho\pi\pi}$ with the magnitude fixed by SU_3 ratios. The resulting output as compared to the one-channel result is shown in Figs. 7-9. We notice that the output is very similar to the one-channel case and we still get approximately self-consistent ρ and P in the $\pi\pi$ channel. This result presumably is due to the fact that the K^* and ϕ trajectories, lying lower than the ρ-trajectory, are dominated by the ρ-contribution in the $\pi\pi$-channel. The ρ- and P-contribution in the $K\bar{K}$ channel is small as compared to that in the $\pi\pi$ channel because the $K\bar{K}$ threshold is much higher and we cannot exchange ρ and P in the u-channel. The output in this case gives $\gamma_{\rho K\bar{K}}/\gamma_{\rho\pi\pi} \sim 0.15$ and $\gamma_{PK\bar{K}}/\gamma_{P\pi\pi} \sim 0.09$, far smaller than the SU_3 ratios of 0.5 and 1 respectively. Although these ratios are not directly subject to experimental test, we know that the K^*-width and ϕ-width calculated from SU_3 ratios agree reasonably well with the ρ-width. These departures from the SU_3 ratios therefore seem to indicate a defect in our calculation scheme. However, if we increase the ϕ-contribution in the $K\bar{K}$

channel by a factor of 6 , we find $\gamma_{\rho K\bar{K}}/\gamma_{\rho\pi\pi} \sim 0.2$ and $\gamma_{PK\bar{K}}/\gamma_{P\pi\pi} \sim 0.1$. This slight improvement suggests that, if in a better dynamic model (see discussion in Section IV below) the attraction in the $K\bar{K}$ channel can be greatly enhanced, which we know will improve the output ρ-width, the above defect may also be cured. The output also suffers the same difficulties as the one-channel case: large output ρ-width, flat trajectories, violation of unitarity for lower ℓ values. Since these qualitative features apparently are not sensitive to the presence of the $K\bar{K}$ channel, a completely self-consistent solution is considered as not significant.

We next try to see whether we can improve the output by changing the input drastically. As it turns out, mainly because unitarity at the strip boundary puts such a stringent restriction on the imaginary part of the potential, we are unable to improve our result in any satisfactory way. We explain the details as follows:

(i) The strip width: In contrast to the fixed-spin particle exchange case, the potential, now depending on the strip width, decreases when we increase the latter. The additional attraction we can get by increasing σ is thus very limited.

(ii) The trajectory: A steeper input trajectory will decrease the potential and flatten the output trajectory; the approximately self-consistent ρ-trajectory used is about the best compromise we can manage.

(iii) The residues: A steeply-falling residue function gives steep output trajectory and satisfies unitarity better for all values

of ℓ , but the output residue function is never steeply-falling and, if we try to keep the ρ-peak in the cross section at the right position, the trajectory develops a large imaginary part so fast that we are unable to follow it above $\ell = 0.4$. If we allow the ρ-peak to appear at much lower energy we can get steeper trajectories (Fig. 10).

(iv) The effect of K^* and ϕ : In the fixed-spin particle exchange case we found that we could narrow the output ρ-width by changing the contribution of K^*- and ϕ-exchange, if we made the ρ-meson as a bound state of the $K\bar{K}$ channel weakly coupled to the $\pi\pi$ channel. There we had to use a $\gamma_{\phi K\bar{K}}$ too large to be acceptable; in the present case we find that unitarity at the strip boundary rules out such a possibility. In Fig. 11 we plot two illustrative cases where we increase the attraction in the $K\bar{K}$ channel by very large factors. Even though unitarity is violated severely already at $\ell = 1$, we still cannot get the experimental ρ-width.

IV. CONCLUSION

Our two-channel calculation in Section III-B of the output ρ-width for fixed-spin particle exchange agrees with earlier results[12a] based on the same N/D model (Regge trajectories were not considered in the earlier publications). From the discussion of Section III-C it is evident that Reggeization, because of the restriction imposed by unitarity at the strip boundary, does not improve the result in any appreciable way in our approximation scheme.

Three immediate possible improvements in the dynamics come to mind: (a) Iterate the potential to get additional strength without violating unitarity. (b) Include additional inelasticity through a complex generalized potential within the strip. This, furthermore, will allow a smooth transition across the strip boundary, avoiding the artificial singularity. (c) Treat properly the long range Pomeranchuk repulsion[20] in order to narrow the output ρ-width. All these aspects can be handled very naturally using the Mandelstam iteration scheme. In view of Finkelstein's results[12b], using the equivalent potential method by Balázs, we know that point (a) is significant. The importance of (c) is established in Ref. 20. Thus we need not be too discouraged by our results here. Adding one more channel has not in itself turned out to cure the ills of a particular strong interaction model, but we have shown that it is computationally feasible to include multi-channel effects in a Reggeized system. The qualitative results reported here should be of assistance in the construction of better models.

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FOOTNOTES AND REFERENCES

* Fellow of the Consejo Nacional de Investigaciones Científicas y Tecnicas, Argentina. On leave of absence from the F.C.G.N., Universidad de Buenos Aires.

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6. S. Mandelstam, Phys. Rev., 140, B375 (1965); the relevant discussion is contained in the Appendix.

7. S. Y. Chu, Phys. Rev., 139, AB3 (1965). The detailed derivation of Eq. (II-B:5) is given there.

8. Reference 7 overlooked this complication, but once we can diagonalize the matrix Λ the arguments remain unchanged.

9. For explicit expression of $O_i(s,s')$ see Reference 4 and V. L. Teplitz (Reference 2a).

10. The discussion in this section is exactly parallel to the single channel case discussed in Section VII of Reference 4.

11. See Reference 4 for complete detail.

12a. See for example
   F. Zachariasen and C. Zemach, Phys. Rev., 128, 849 (1962).

L. P. Balázs, Phys. Rev., 137, B168 (1965).

J. R. Fulco, G. L. Shaw, and D. Y. Wong, Phys. Rev., 137, B1242 (1965).

12b. L. P. Balázs, Phys. Rev., 137, B1510 (1965).

J. Finkelstein, Equivalent-Potential Calculation of $\pi\pi$ and πK Scattering, UCRL-16537, November, 1965 (to be published in Phys. Rev.).

13. C. E. Jones and G. Tiktopoulos, Numerical Solution of Non-Fredholm (Singular) Integral Equations by Matrix Inversion, Princeton University preprint.

14. We use this criterion rather than that $\mathrm{Re}\alpha_\rho$ should cross 1 at the ρ-mass because the determinant of D develops a large imaginary part very quickly above the $\pi\pi$ threshold; thus the zero of the real part of the determinant of D will not give us the correct value of $\mathrm{Re}\alpha_\rho$.

15. R. H. Capps, Phys. Rev. Letters, 10, 312 (1963).

16. G. F. Chew and V. L. Teplitz, Phys. Rev., 137, B139 (1965).

17. A. H. Rosenfeld et al., Data on Elementary Particles and Resonant States, UCRL-8030, August, 1965 (unpublished).

18. G. F. Chew and V. Teplitz, Phys. Rev., 136, B1154 (1964).

19. See for example,

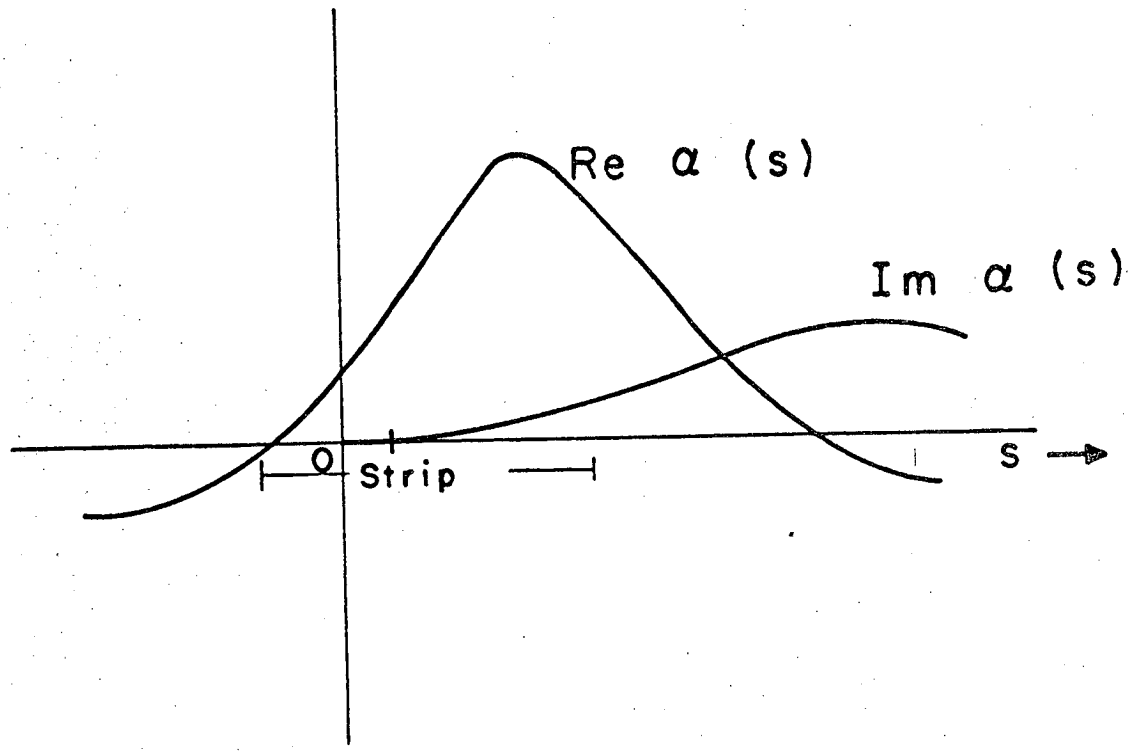
R. Philips and W. Rarita, Phys. Rev., 139, B1336 (1965).

20. G. F. Chew, Phys. Rev., 140, B1427 (1965).

Table I. Results of the cases with fixed-spin particle exchange as the generalized potential.[a]

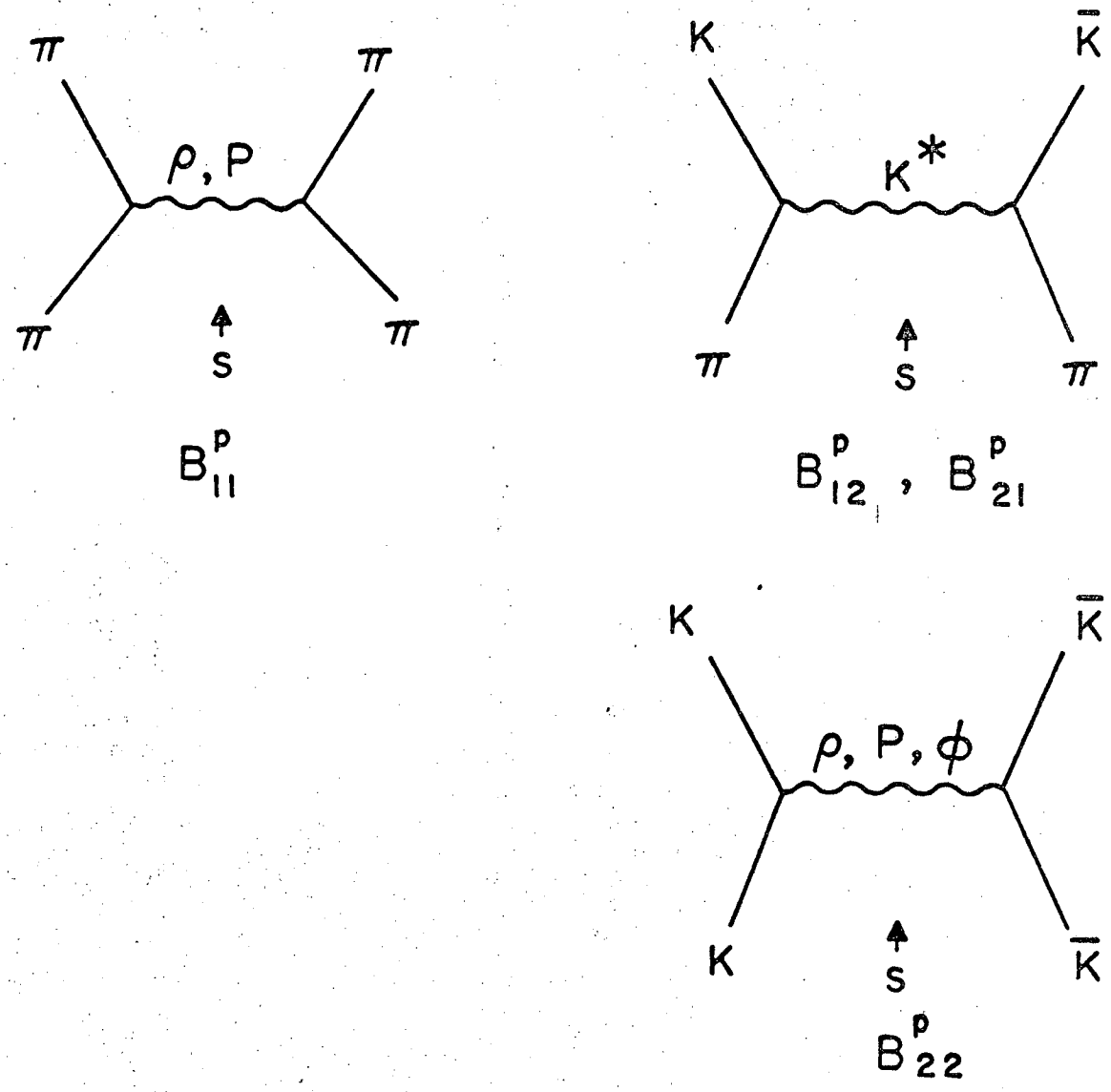
Case No.	Cases			Input				Output			
				$\Gamma_\rho^{in}(m_\pi)$	$\Gamma_{K^*}^{in}(m_\pi)$	$\Gamma_\phi^{in}(m_\pi)$	$\sigma(m_\pi^2)$	$\Gamma_\rho^{out}(m_\pi)$	σ_{max}(mb)	$\alpha_\rho(0)$	$\alpha_P(0)$
(A)1	One-Channel			0.9	-	-	1280	4.0	101	0.88	1.19
(A)2				1.26	-	-	500	4.2	95	0.81	1.15
(A)3				1.58	-	-	300	4.3	92	0.78	1.1
(A)4				1.9	-	-	200	4.45	83	0.72	1.06
(B)11	Two-Channel	Without ϕ	Use SU_3 Ratios	0.9	SU_3	-	600	3.75	106	0.87	1.13
(B)12				1.1	SU_3	-	400	3.9	103	0.83	1.1
(B)13				1.65	SU_3	-	200	4.25	91	0.77	1.04
(B)21			Effect of K*	1.0	0.7	-	300	3.8	100	0.82	1.05
(B)22				1.0	1.2	-	200	3.6	105	0.80	0.99
(C)11		With ϕ	Use SU_3 Ratios	1.07	SU_3	SU_3	400	3.8	105	0.84	1.09
(C)12				1.65	SU_3	SU_3	200	4.15	96	0.78	1.04
(C)21			Effect of ϕ	1.03	SU_3	$2\times SU_3$	400	3.55	106	0.85	1.09
(C)22				0.94	SU_3	$4\times SU_3$	400	3.0	113	0.87	1.08
(C)31			Effect of ϕ,K*	0.725	$2\times SU_3$	$2\times SU_3$	400	3.25	111	0.85	1.06
(C)32				0.9	$0.46\times SU_3$	$7\times SU_3$	400	0.9	121	0.9	1.04

[a]The input ρ-width of 0.9 m_π corresponds to the experimental value of 124 MeV[17], the K*-width given by SU_3 is then 46 MeV (0.34 m_π) and the ϕ-width is 4.9 MeV (0.037 m_π).



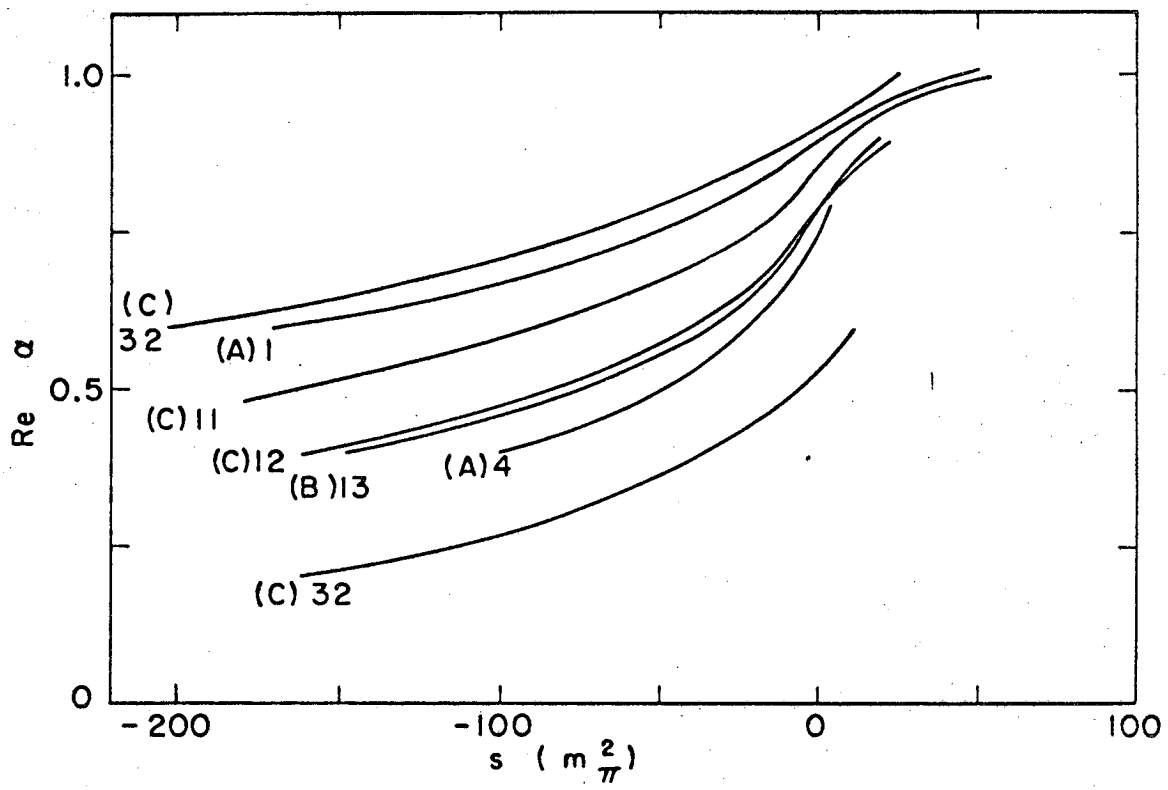
MUB-11186

Fig. 1 A typical Regge trajectory that reaches the right-half angular-momentum plane.



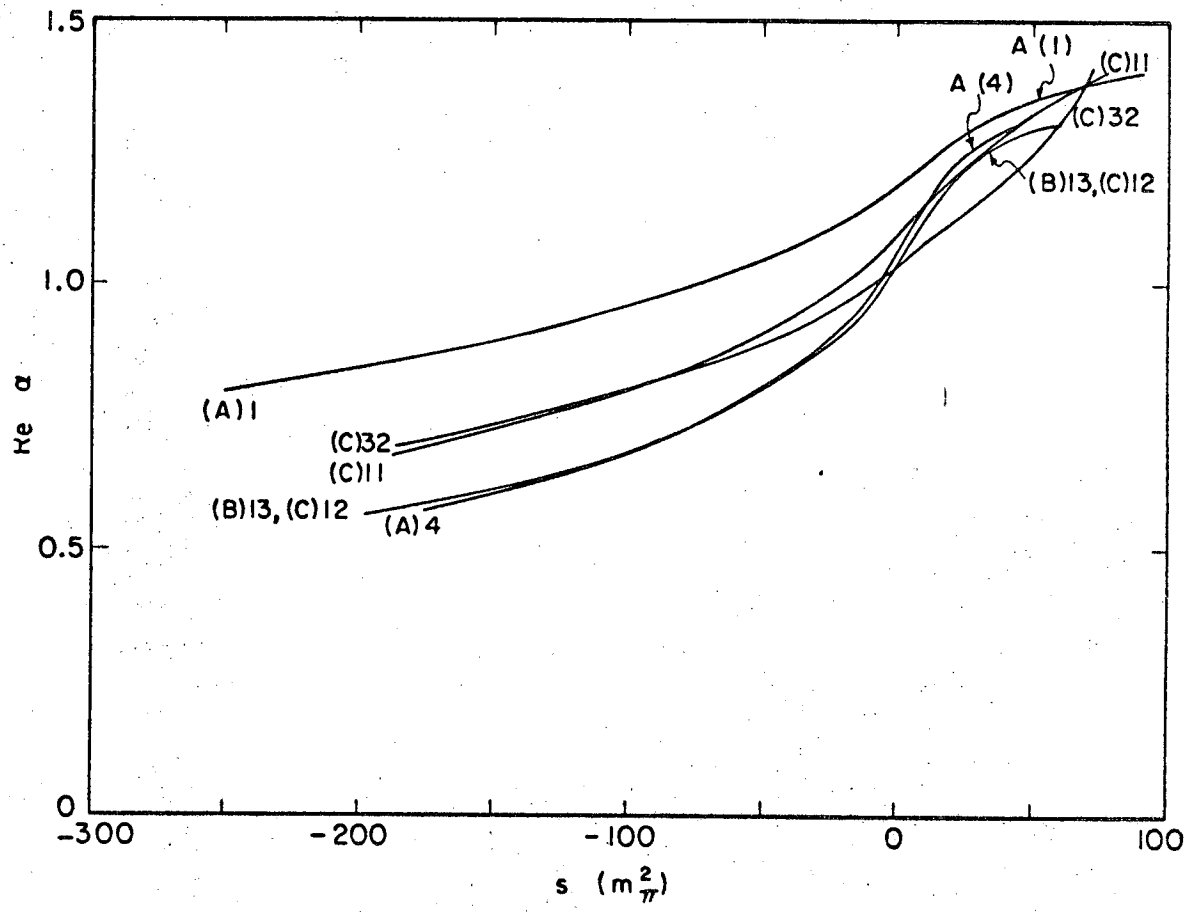
MUB-11187

Fig. 2 The particles being exchanged that give rise to the generalized potential.



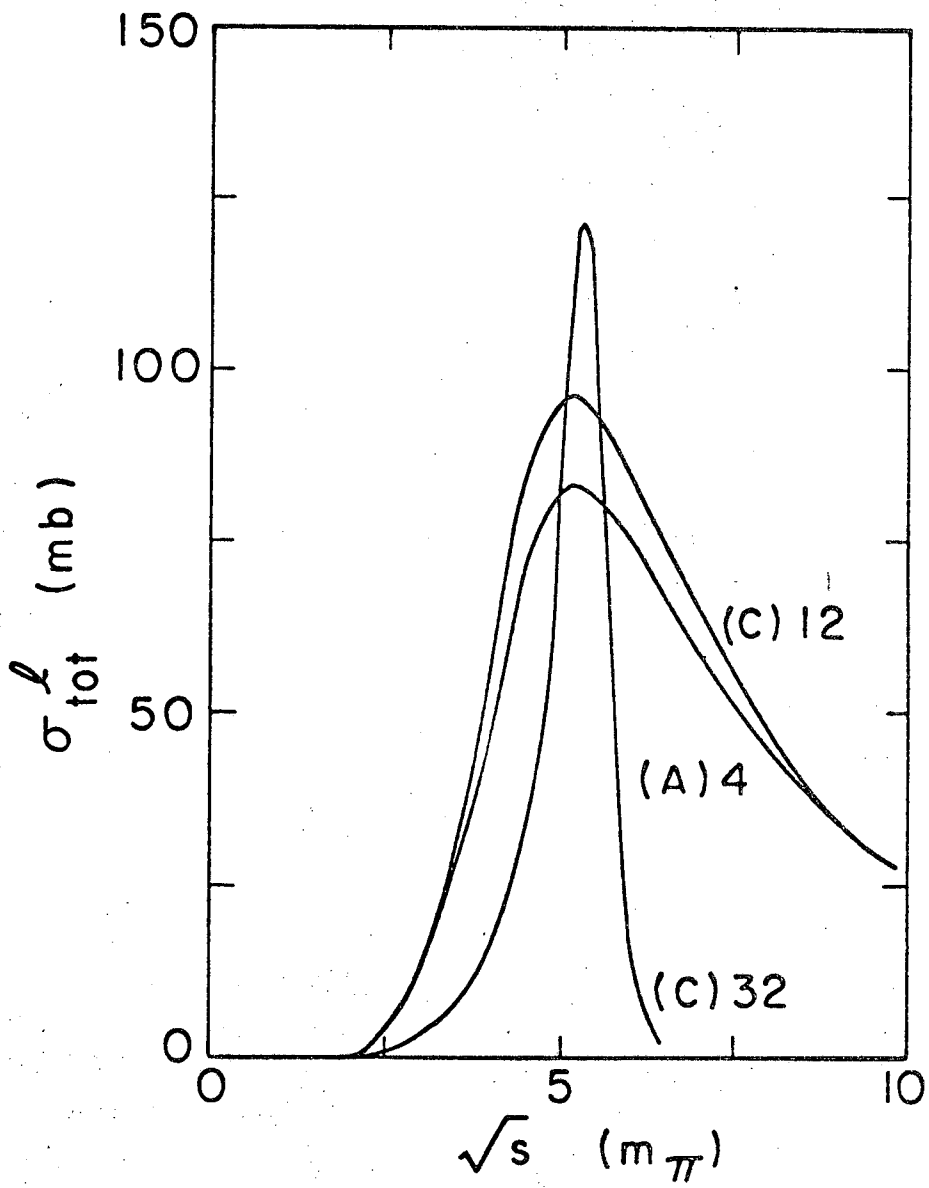
MUB-11188

Fig. 3 Some I = 1 output trajectories from fixed-spin particle exchange, labeled by the case numbers as listed in Table 1.



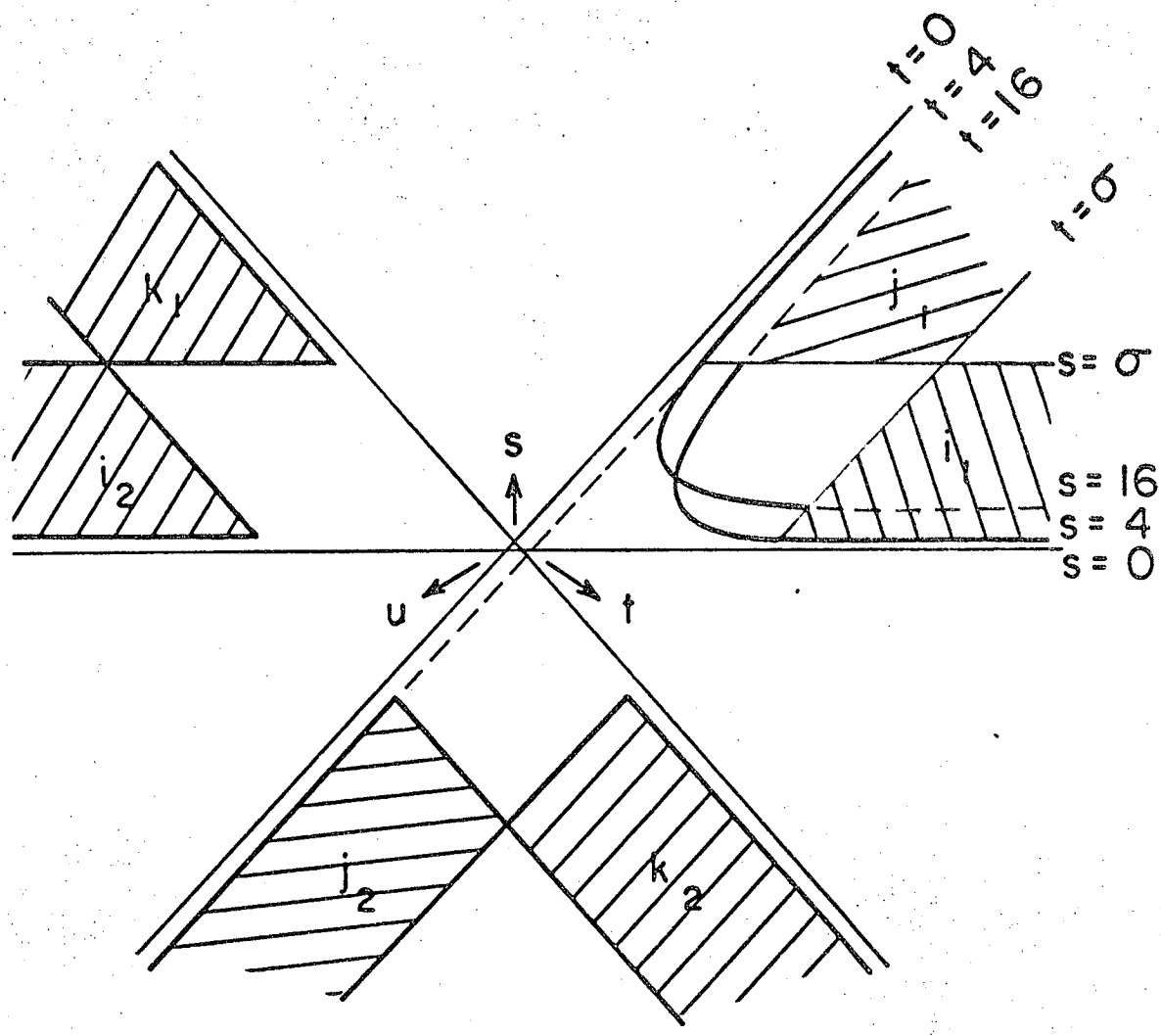
MUB-11180

Fig. 4 Some I = 0 output trajectories from fixed-spin particle exchange, labeled by the case numbers as listed in Table 1.



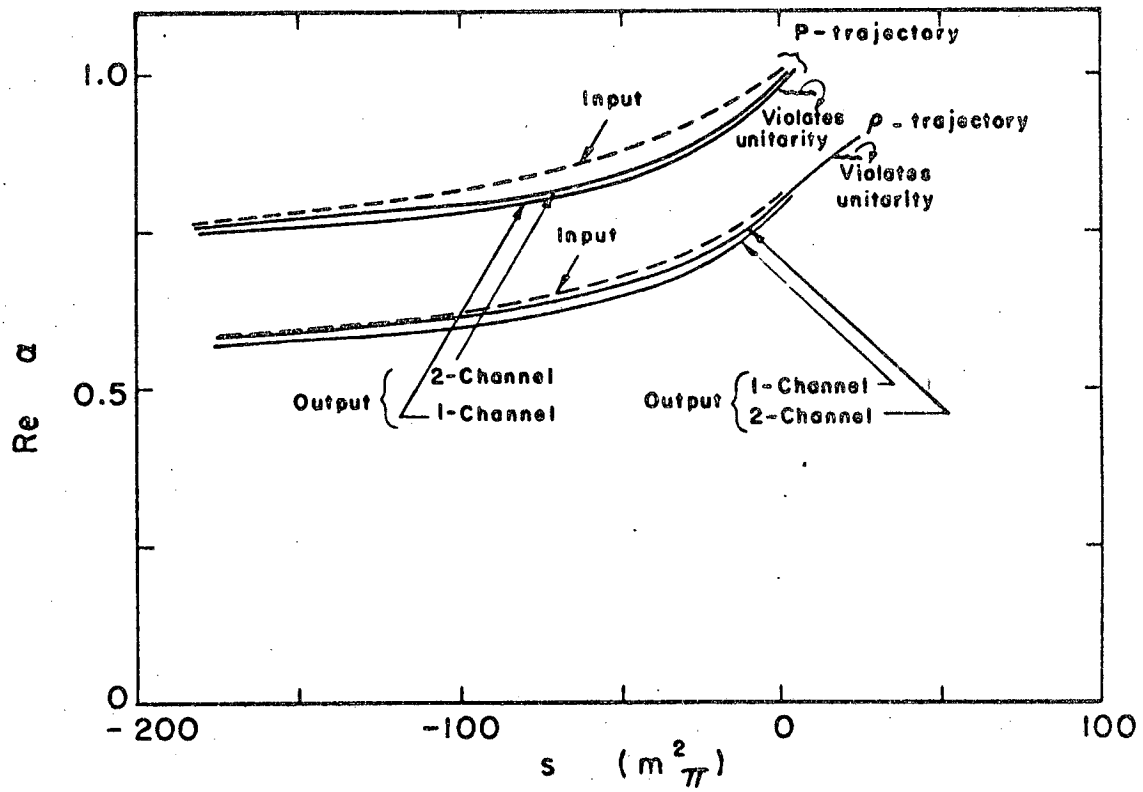
MUB-11190

Fig. 5 Some $I = 1$, $\ell = 1$ cross sections from fixed-spin particle exchange, labeled by the case numbers as listed in Table 1.



MUB-11181

Fig. 6 The Mandelstam diagram for the new strip approximation, showing the six strip $i_{1,2}$, $j_{1,2}$, and $k_{1,2}$. Also the boundaries of the double spectral function.

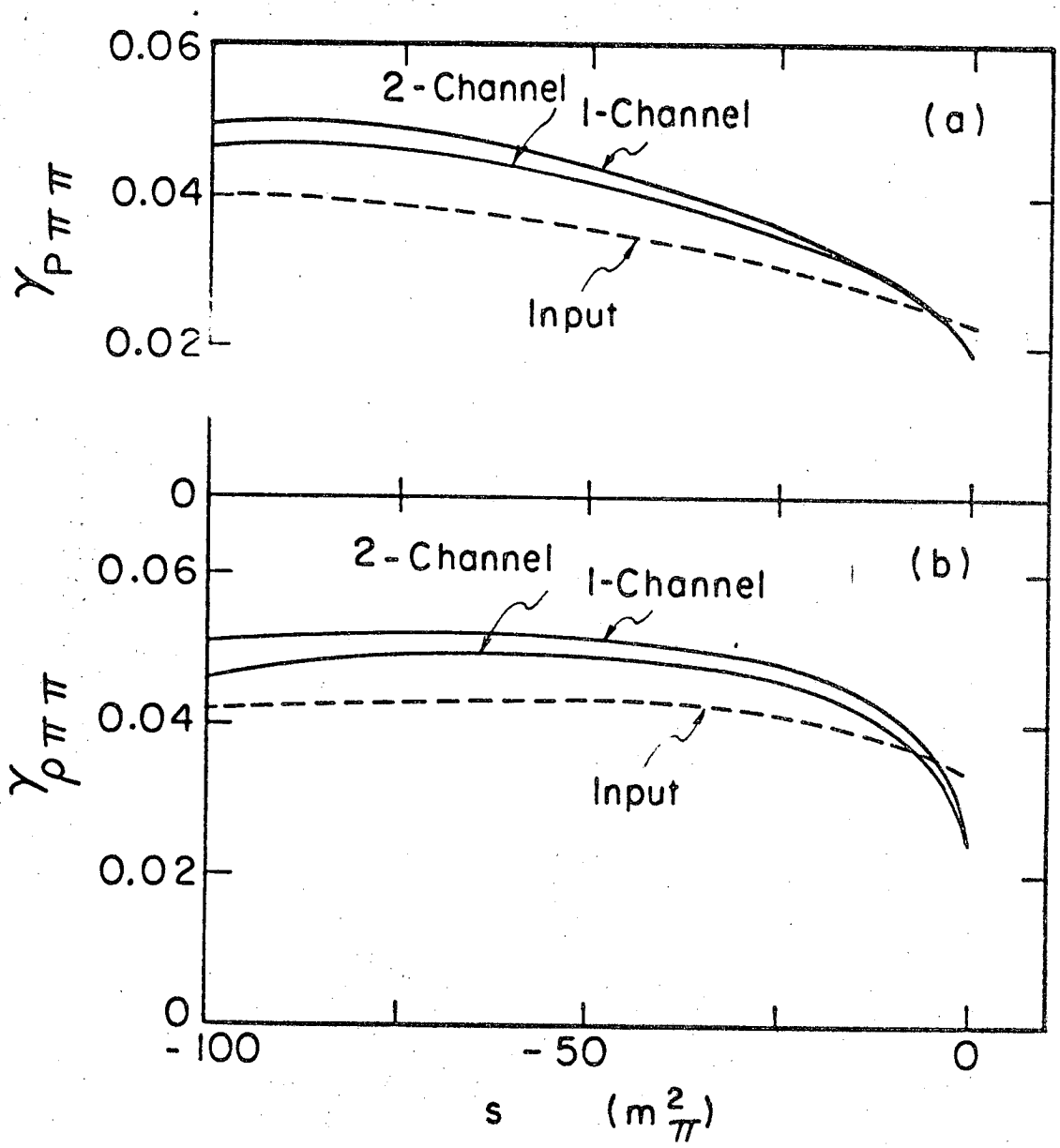


MUB-11185

Fig. 7 The approximately self-consistent ρ- and P-trajectories (parameters taken from Ref. 2b). (These are the input trajectories we used in all cases below.) ($\sigma = 100\ m_\pi^2$)

$$\alpha_\rho(t) = 0.5 + 0.3\Big/\left(1 - \frac{t}{70}\right)$$

$$\alpha_{\underline{P}}(t) = 0.625 + 0.375\Big/\left(1 - \frac{t}{110}\right)$$



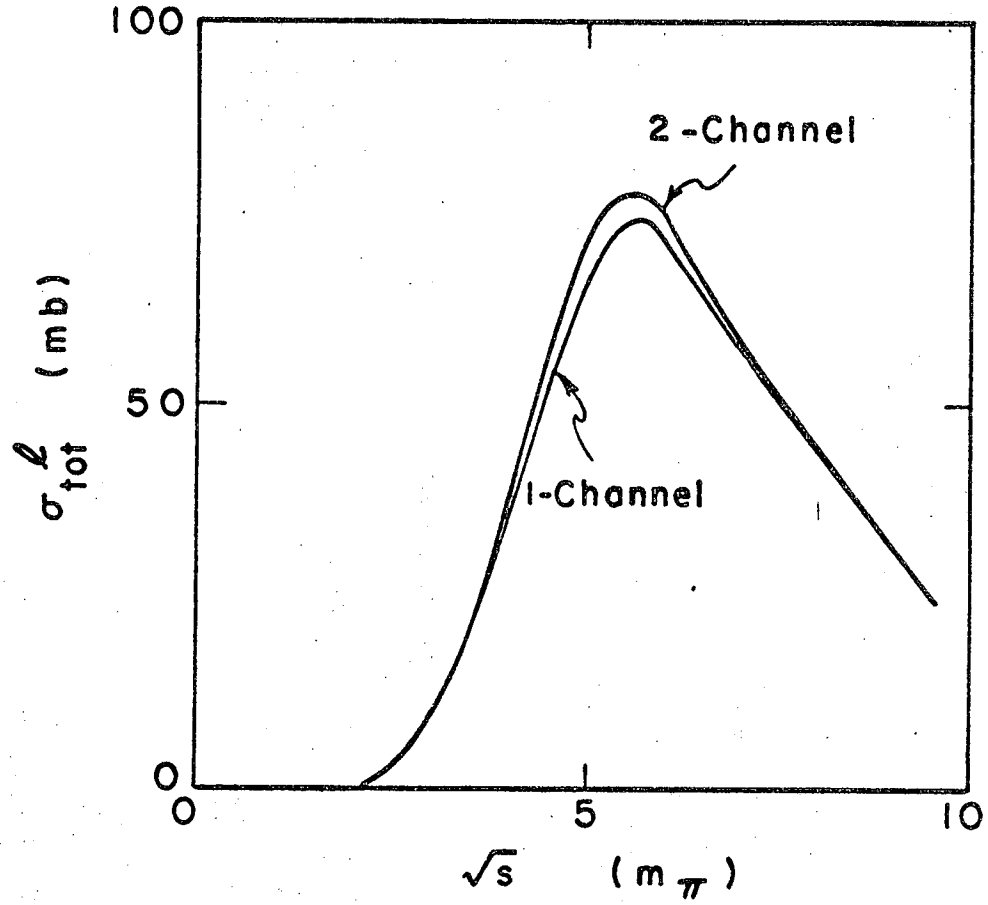
MUB-11184

Fig. 8 The approximately self-consistent residue functions $\gamma_{\rho\pi\pi}$ and $\gamma_{\underline{P}\pi\pi}$ (parameters taken from Ref. 2b). ($\sigma = 100\ m_\pi^2$)

$$\gamma_{\rho\pi\pi} = 125\ \alpha'_\rho(t)\ (40-t)\ Q_{\alpha_\rho(t)}\ (2.55)\Big/(9)^{\alpha_\rho(t)+1}$$

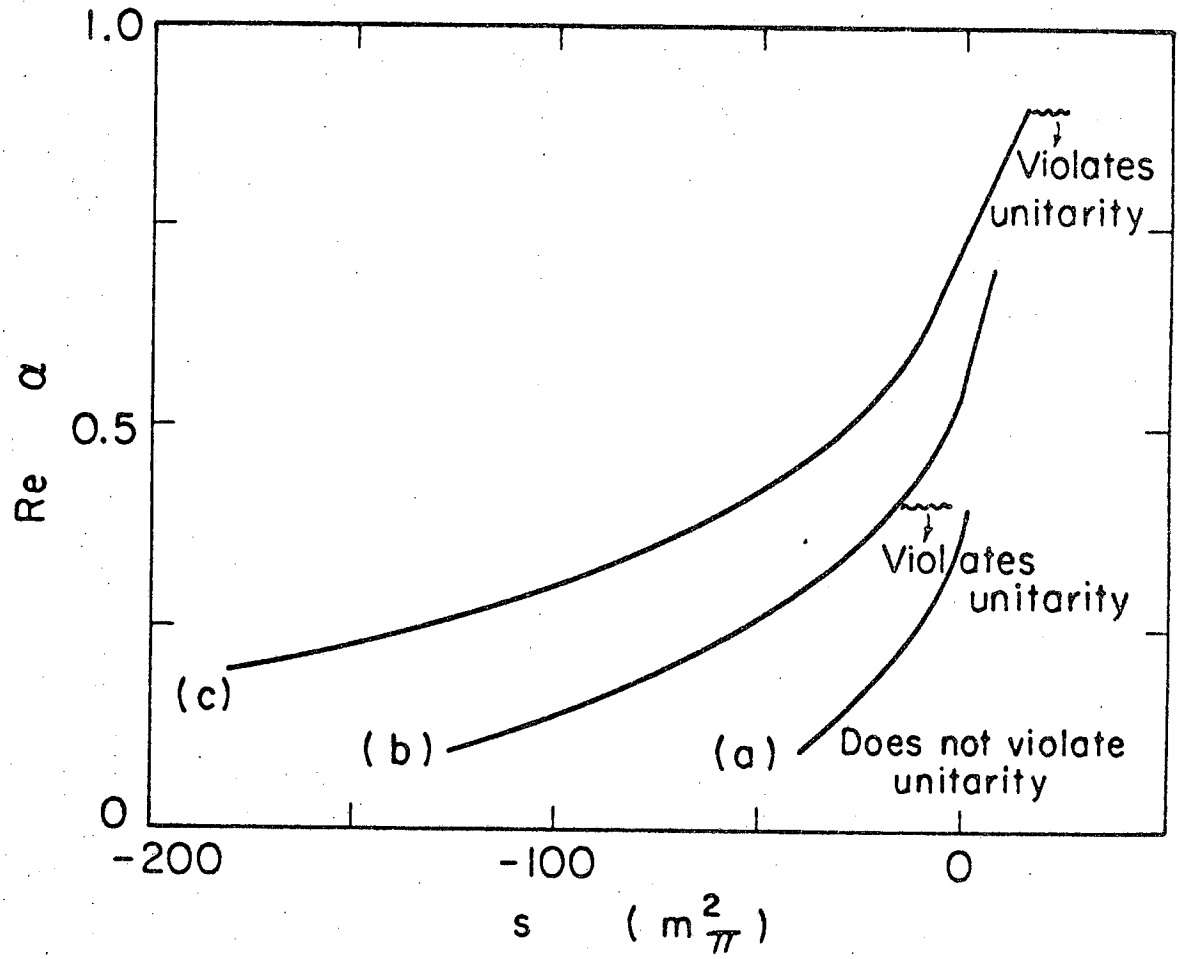
$$\gamma_{\underline{P}\pi\pi} = 230\ \alpha'_{\underline{P}}(t)\ (50-t)\ Q_{\alpha_\rho(t)}\ (2.22)\Big/(11.5)^{\alpha_{\underline{P}}(t)+1}$$

Other residues are given by SU_3.



MUB-11182

Fig. 9 The $I = 1$, $\ell = 1$ cross section for the approximately self-consistent case.



MUB-11189

Fig. 10 Output I = 1 trajectories for steep input residue functions (the trajectories are the same as the approximately self-consistent case) ($\sigma = 200\ m_\pi^2$).

(a) $\gamma_{\rho\pi\pi} = 0.0375 \Big/ \left(1 - \frac{t}{10}\right)$ $\left(\rho\text{-peak at } 28(m_\pi^2)\right)$

(b) $\gamma_{\rho\pi\pi} = 0.05 \Big/ \left(1 - \frac{t}{10}\right)$ $\left(\rho\text{-peak at } 22(m_\pi^2)\right)$

(c) $\gamma_{\rho\pi\pi} = 0.065 \Big/ \left(1 - \frac{t}{10}\right)$ $\left(\rho\text{-peak at } 15(m_\pi^2)\right)$

Other residues are fixed by SU_3 ratios.

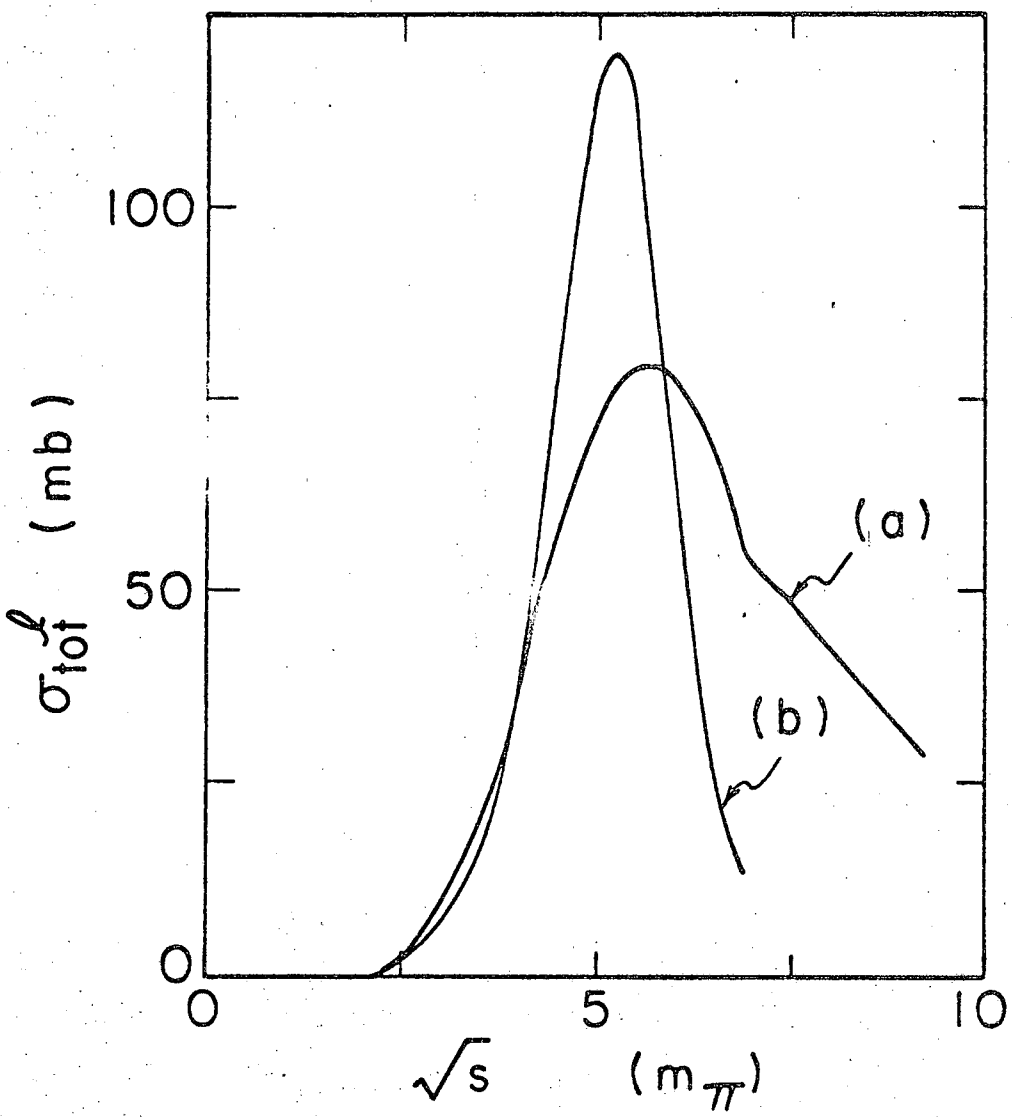


Fig. 11 The $I = 1$, $\ell = 1$ cross section.

MUB-11183

(a) $\sigma = 100\ m_\pi^2$

$B_{11}^{P}(s)$, $B_{12}^{P}(s)$ same as the approximately self-consistent case,

$B_{22}^{P}(s)$ is 15 times the approximate self-consistent case.

The kink in the cross section is due to the presence of a secondary trajectory which is mainly a resonance of the $\pi\pi$ channel. (As a measure of violation of unitarity we have $\lambda_1 = 0.61$ $\lambda_2 = 1.23$ for $\ell = 1$.)

(b) $\sigma = 400\ m_\pi^2$

$$\gamma_{\rho\pi\pi} = 0.0136\Big/\left(1 - \frac{t}{50}\right) \quad \text{and} \quad \gamma_{\underline{P}\pi\pi} = 0.0038\Big/\left(1 - \frac{t}{50}\right)$$

$\gamma_{\rho K\bar{K}}$ is 10 times the value given by SU_3. ($\lambda_1 = 0.52$ $\lambda_2 = 1.77$ for $\ell = 1$)

FIGURE CAPTIONS

Fig. 1. A typical Regge trajectory that reaches the right-half angular-momentum plane.

Fig. 2. The particles being exchanged that give rise to the generalized potential.

Fig. 3. Some $I = 1$ output trajectories from fixed-spin particle exchange, labeled by the case numbers as listed in Table 1.

Fig. 4. Some $I = 0$ output trajectories from fixed-spin particle exchange, labeled by the case numbers as listed in Table 1.

Fig. 5. Some $I = 1$, $\ell = 1$ cross sections from fixed-spin particle exchange, labeled by the case numbers as listed in Table 1.

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$\gamma_{\rho K\bar{K}}$ is 10 times the value given by SU_3. ($\lambda_1 = 0.52$ $\lambda_2 = 1.77$ for $\ell = 1$).

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