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NOTE ON THE STABILITY OF LINEAR FEEDBACK SYSTEMS

E. T. Jaynes

February 27, 1952

Berkeley, California

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E. T. Jaynes*

Microwave Laboratory, Stanford University

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ABSTRACT

The Nyquist Criterion of stability in linear systems with feedback is extended to give predictions of the ringing time of a system for which the point (1,0) lies close to but outside the Nyquist diagram. Since simple derivations of Nyquist's theorem are hard to find, one is also given.

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Representation of Network Operators

The most general time-independent linear operator α which operates on the past of a function f(t) may be represented in the integral form

$$\alpha f(t) = \int_0^\infty a(t')\, f(t-t')\, dt'. \tag{1}$$

If $f(t) = \exp(st)$ and $s = \sigma + j\omega$, we get the spectrum A(s):

$$\alpha e^{st} = e^{st} \int_0^\infty a(t')\, e^{-st'}\, dt' = e^{st}\, A(s). \tag{2}$$

Thus, the eigenfunctions of the operator α are the exponential functions exp (st), and the corresponding eigenvalues are the Laplace transforms A(s) of a(t). This is a consequence of the fact that the operator α is itself not a function of t. Any linear network may be represented by some operator α which transforms an input signal f(t) into an output signal $\alpha f(t)$. If the response to a signal of finite duration dies out so that

$$\int_0^\infty \left| a(t) \right|^2 dt < \infty \tag{3}$$

then A(s) is a bounded holomorphic function in the right half plane $\sigma > 0$, for from the Schwarz inequality

$$\left| \int f_1\, f_2\, dx \right|^2 \leq \int \left| f_1 \right|^2 dx \cdot \int \left| f_2 \right|^2 dx \tag{4}$$

we get

$$\left|A(s)\right|^2 \leq \int_0^\infty |a|^2\, dt \cdot \int_0^\infty e^{-2\sigma t}\, dt = (1/2\sigma)\int_0^\infty |a|^2\, dt. \quad (5)$$

Therefore, if we put $A(s) = x + jy$, the half-plane $\sigma > 0$ is mapped onto some <u>finite</u> region in the A-plane, as in Fig. 1.

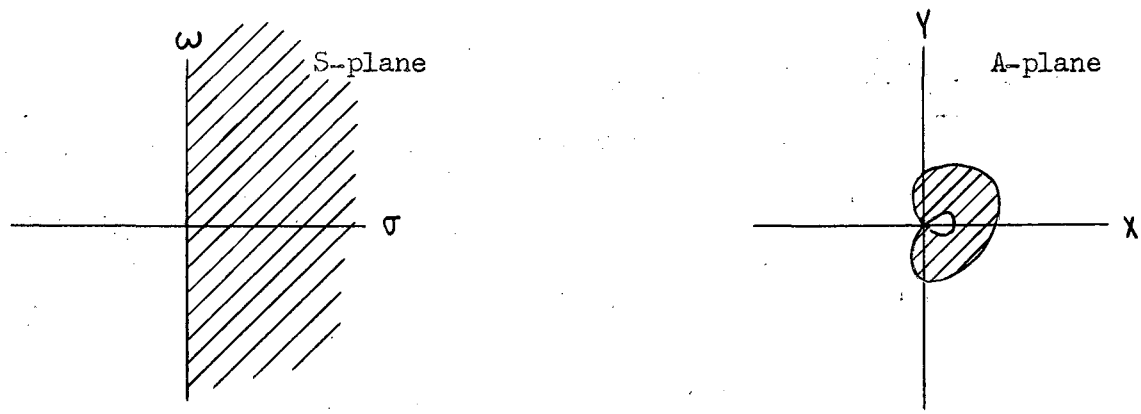

Fig. 1

The effect of the operator α on the unit pulse $f(t) = \delta(t)$ shows clearly the meaning of $A(s)$:

$$\alpha\delta(t) = \int_0^\infty a(t')\, \delta(t-t')\, dt' = a(t) = (1/2\pi j)\int_C A(s)\, e^{st} ds \quad (6)$$

where the path of integration C goes to the right of all poles of $A(s)$, and extends from $(-j\infty)$ to $(+j\infty)$, as shown in Fig. 2.

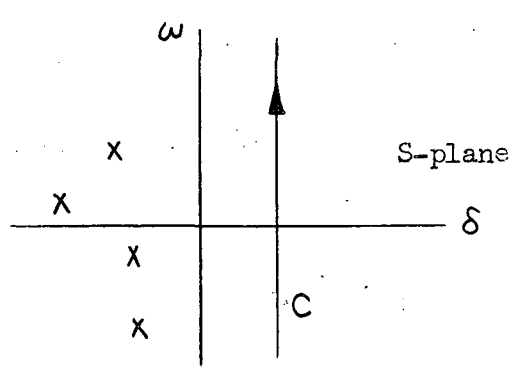

Fig. 2

Thus the output is the sum of the residues of A(s) exp(st) at each of the poles of A(s). If these poles are at s_1, s_2, ... s_n, we have an output of the form

$$a(t) = \alpha\delta(t) = \sum_{i=1}^{n} a_i e^{s_i t} , \tag{7}$$

so that the poles of A(s) represent the natural modes of the network. If any of these poles should lie in the right half plane $\sigma > 0$, we would have and exponentially increasing output, leading to a catastrophe. Therefore a passive network can never generate an operator α with poles of A(s) for which $\sigma > 0$.

Feedback Networks

A feedback device operates as shown in Fig. 3.

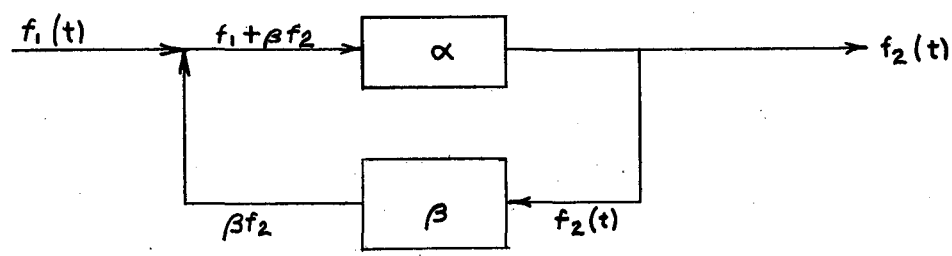

Fig. 3.

$$f_2(t) = \alpha\left[f_1(t) + \beta f_2(t)\right] \tag{8}$$

$$f_2(t) = \frac{\beta}{1 - \alpha\beta} f_1(t)$$

The feedback network β converts the network operator α into $\alpha/(1 - \alpha\beta)$. If they contain power sources, it is possible that

$$\frac{A(s)}{1 - A(s)\ B(s)} \tag{9}$$

may have a pole s_1 with $\sigma_1 > 0$, in which case the system can break into oscillation with an output $\sim \exp(s_1 t)$. Since by hypothesis A(s) has no

such pole, this can happen only if $A(s)\ B(s) = 1$ has a root in the half-plane $\sigma > 0$. This applies strictly only to response to shock excitation, but small disturbances, thermal or otherwise, are always present in physical systems. Therefore if a feedback device is not to oscillate spontaneously, there must be no s for which $Re(s) > 0$ and for which $A(s)\ B(s) = 1$. Thus, if we map the right half of the s-plane onto the plane of the complex variable $AB = x + jy$, the finite region obtained must not contain the point (1,0). An equivalent statement is that the curve which represents the locus of AB for real frequencies (which is the mapping of the imaginary axis of the s-plane onto the AB-plane) must not enclose the point (1,0). See Fig. 4.

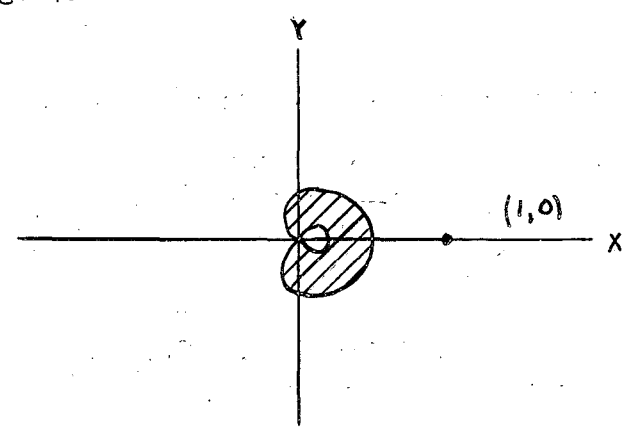

Fig. 4.

The criterion of stability just derived is recognized as Nyquist's theorem. Its intuitive content is this: one might at first expect that the criterion for oscillation is the existence of some frequency for which AB is real and greater than unity; i.e., if for some frequency the gain around the feedback loop is greater than unity and the total phase shift is zero one might expect oscillations of the frequency to start. Nyquist's theorem, however shows that this is not quite correct; the essential condition is not only the existence of a certain frequency ω, but also a

certain build-up time constant $1/\sigma$ such that when the signal exp (st) = exp $(\sigma + jw)t$ is applied, the gain around the feedback loop is exactly unity. Once one has this mathematical result it is easy to see physically why it is correct: the physical conditions imposed on the signal is, of course, that in the absence of any input excitation, the gain around the feedback loop must be exactly unity, and the condition AB = 1 determines both the real and imaginary part of s. That the wrong criterion can be satisfied without having the correct one fulfilled is shown by the example of an amplifier with an AB plot as shown in Fig. 5. There are two frequencies ω_1, ω_2 which satisfy the incomplete criterion, but nevertheless the amplifier is stable. This has been verified experimentally.

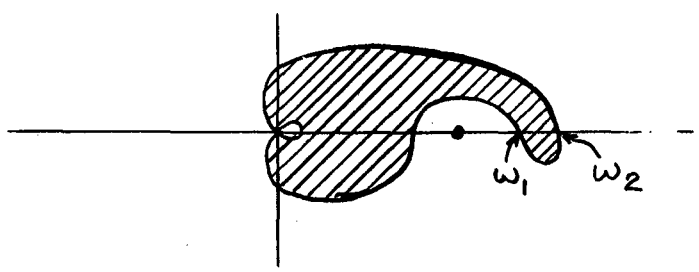

Fig. 5.

According to the above linear relations, an unstable system would generate a signal that increases exponentially with time. In actual systems this will be true only when the signal amplitude is low. For sufficiently high-level signals the network operators α, β will always become nonlinear due to such things as driving vacuum tubes beyond the linear range of their characteristics, saturating the core in an output transformer, sparking across condensers, heating of resistors, etc. In general the effect of these nonlinearities is to decrease the gain of a network. The final stable state of oscillation is again determined by the condition

$\alpha\beta = 1$, where α, β are now nonlinear operators, so that not only an amplitude and frequency, but also a wave shape (harmonic content) may be found from the condition that the gain around the feedback loop be exactly unity. Thus, the requirement $\alpha\beta = 1$ determines the behavior of an unstable system throughout its buildup process.

Regarding practical applications, it is clear that our original criterion that $A(s)\ B(s) = 1$ shall have no roots for which $\sigma > 0$ is the better one to use when the analytical expressions for the functions $A(s)$, $B(s)$ are known, while Nyquist's formulation is more convenient when the information available consists of experimental measurements of A and B at various frequencies.

Prediction of Ringing-Time from a Nyquist Diagram

Although the Nyquist diagram gives a definite and unambiguous answer to the question whether a system is stable, there remains a question of practical interest as to the behavior of systems whose stability is marginal; i.e., for which the point (1,0) lies very near to the AB plot, but outside it. All such systems act like high-Q resonant circuits, and will "ring" with a long time constant in response to any disturbance. In practice, this may be as objectionable as true instability. To see this we note that since the Nyquist diagram is a conformal map of the imaginary axis of the s-plane, the response function will then have a pole at a position $s = \sigma + j\omega$ very close to the imaginary axis, but to the left of it: $\sigma < 0$, $|\sigma| \ll \omega$. From the general definition of Q (see Fig. 6)

$$s^2 - s_o^2 = js\,\frac{s_o}{Q}$$

$$s_o^2 + \sigma^2 + \omega^2 = 0 \qquad (10)$$

Regarded as an equation in the unknown s, this has roots at $s = \sigma \pm j\omega$, as required, and we have $Q = -\omega/2\sigma$.

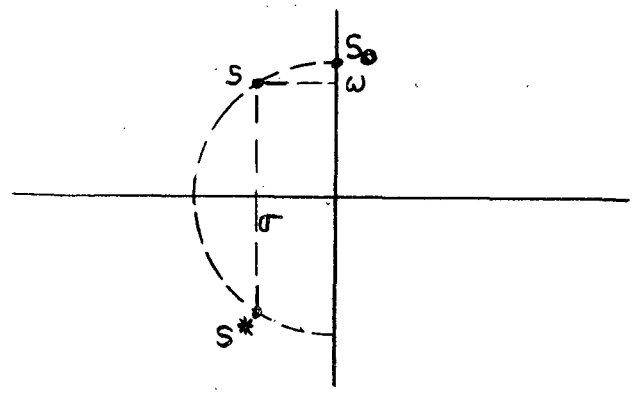

Fig. 6.

The ringing corresponding to such a pole is given by

$$\exp(st) = \exp(\sigma + j\omega)t = \exp(-t/t_o)\exp(j\omega t), \qquad (11)$$

where the ringing time-constant is

$$t_o = -1/\sigma > 0. \qquad (12)$$

Because of the well known property of conformal transformations that in small regions they consist merely of uniform expansion or contraction with preservation of angles, we can find the ringing time-constant as easily from the Nyquist diagram as from the location of the pole in the s-plane. This time-constant is proportional to the reciprocal of the distance of the pole from the imaginary axis in the s-plane, and therefore to the reciprocal of the distance of the point (1,0) from the nearest part of the Nyquist plot. The proportionality factor is found by writing σ in a form that is independent of the scale of the diagram. Since the magnification factor in the Nyquist diagram is determined by the rate at which we move along the curve as the frequency is varied, we see that the correct formula is

$$1/t_o = -\sigma = a \left| \frac{d\omega}{dZ} \right| \tag{13}$$

where a is the distance of the point (1,0) from the nearest point on the Nyquist diagram, and $|dz/d\omega|$ is the rate at which we move along the curve as the frequency is changed. The Q of the ringing is then given by $t_o = 2Q/\omega$, or

$$Q = \frac{\omega}{2a} \left| \frac{dz}{d\omega} \right| \quad . \tag{14}$$

Conclusion

The response of a system that is close to the point of instability is that of a high-Q resonant circuit. The frequency and Q of the ringing may be predicted from the Nyquist plot of the system.