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Berkeley, California

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October 19, 1966

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ABSTRACT

It is shown that if the spectrum of physical particle rest masses contains neither accumulation points or the zero point, then the number of different positive-α Landau surfaces that enter any bounded portion of the physical region of any multiple-particle scattering process is finite. This implies that if the physical-region singularities of scattering functions are confined to the closure of the set of points lying on positive-α Landau surfaces, then the scattering functions are analytic at almost all points of the physical region. The proof is made by proving an equivalent property of systems of classical point particles scattering via point interactions.

I. INTRODUCTION

There are a number of reasons for believing that the physical-region singularities of scattering functions are confined to the closure $\mathcal{M}^+$ of the set of points lying on positive-α Landau surfaces. This restriction holds for the terms of the perturbation expansion in field theory.[1] It follows also directly from a macroscopic causality condition on the mass-shell S-matrix.[2] And recent works have shown how, in simple cases, the singularities on positive-α Landau surfaces are precisely the ones that emerge from the assumption that the only physical region singularities of scattering functions are those generated by the unitarity equations.[3]

The supposition that the physical-region singularities of scattering functions are confined to $\mathcal{M}^+$ does not immediately ensure that the scattering functions are anywhere analytic; the conceivable alternative is that the positive-α Landau surfaces are everywhere dense in the physical region.

For the simplest case of the scattering of two initial particles into two final particles, each positive-α Landau surface is a normal threshold manifold, which is a manifold lying at a value of the total center-of-mass energy E that equals the sum of the rest masses of a set of the physical particles. If the spectrum of the physical-particle rest masses does not include the value zero and has no accumulation points, then the

number of these manifolds entering any bounded portion of the physical region is finite. This ensures that the set of points not lying on the closure of the set of points lying on positive-α surfaces is everywhere dense; almost every physical-region point has a neighborhood that contains no point lying on any positive-α Landau surface.

The object of the present work is to show that this result carries over to reactions of arbitrary numbers of particles. It will be shown that if the spectrum of physical-particle rest masses contains neither the value zero nor accumulation points, then the number of different positive-α Landau surfaces entering any bounded portion of the physical region is finite. Since the complement of the closure of any single positive-α Landau surface is everywhere dense in the physical region,[2] the same is true of any finite sum of such surfaces. Thus the assumption that the physical-region singularities are confined to the closure of the set of points lying on positive-α Landau surfaces entails that each scattering function be analytic at almost every point of the physical region, provided the spectrum of physical-particle rest masses contains neither zero nor accumulation points.

The result just stated was used in a recent work of the author on the crossing properties of the S-matrix.[4] It has also been a tacit assumption in many other works in analytic S-matrix theory.

Coleman and Norton[5] have recently emphasized that the set of physical-region points lying on the positive-α Landau surface corresponding to a Landau diagram D is precisely the set of points such that the classical point-particle multiple-scattering process pictured by D is dynamically possible. By definition, each point on a positive-α Landau surface of a given process is a point (in the space of the external energy-momentum vectors of this process) such that the Landau equations[1] associated with a corresponding Landau diagram D are satisfied. But the Landau loop equations are precisely the requirement that the Landau diagram D can be converted into a space-time diagram D of the same structure by simply replacing each line L_i of D by the "space time" vector $\alpha_i q_i$, where q_i and α_i are the momentum-energy vector and parameter α associated with line L_i. The parameter α_i is considered to be a proper time divided by the mass of particle i, and the positive-α condition corresponds to the requirement that the particles move forward in time.[5] The other two Landau equations ensure that the energy-momentum conservation laws are satisfied at each vertex and that each particle has the correct mass value. The space-time locations of the classical particles are not restricted, but those of the external particles are in fact determined by the gradient to the Landau surface.[2]

By virtue of this Coleman-Norton correspondence, the number of Landau surfaces that enter a given portion of the physical region is the same as the number of classical point-

particle multiple-particle scattering processes that are dynamically possible in this portion of the physical region. However, it may happen that several different multiple-scattering processes give Landau surfaces that exactly coincide with one another. Such Landau surfaces are, in our terminology, are not "different" and need be counted only once.

As an example, suppose the sum of the masses of a set of physical particles S_1 is equal to the sum of the masses of a set of physical particles S_2. If the particles of the set S_1 are all relatively at rest, then this set can convert into the set of particles S_2, all relatively at rest. An unlimited number of conversions back and forth between these two sets of relatively-at-rest particles can evidently take place without affecting the kinematical situation. This permits an unlimited number of different Landau diagrams to be compatible with certain fixed points in momentum space. However, the Landau surfaces corresponding to these different diagrams all lie exactly on top of one another, and hence are not different. As it is the number of different Landau surfaces that must be shown finite, various Landau diagrams (or their corresponding physical processes) that differ only by conversions of this type can be considered equivalent, and collisions effecting such conversions (called trivial collisions) can be disregarded. This fact is used continually, without explicit mention, in the following proofs.

The possibility of selection rules can be ignored, as they would only decrease the number of possible processes. Each particle can therefore be identified by its mass alone; an interchange of the identities of particles of the same mass would not alter a Landau surface.

II. PROOF FOR A ONE-DIMENSIONAL WORLD

A proof is given first for a world of one space and one time dimension. Let the space and time coordinates be called x and t and consider a plot of the particle trajectories on a two-dimensional x-t diagram. The trajectory lying at the most positive value of x will be called the first trajectory and its slope will be called the velocity of the first particle. This trajectory will generally have a number of straight-line segments joined at points called the collisions of the first particle. The trajectory lying at the second highest value of x will be called the second trajectory, and its slope will be called the velocity of the second particle. There may of course be several particles that trace out a segment of the first trajectory. Their velocities are all equal to the velocity called the "velocity of the first particle", etc.

At each collision of the first particle the velocity of the first particle increases. This fact is obvious in the collision center-of-mass frame, and is carried to the general frame by a Lorentz transformation.

Let $S(E)$ be the set of all multiple-scattering processes (of point particles with point interactions) possible with a total center-of-mass energy less than E, and let $N(E)$ be the least upper bound on the number of collisions of the first particle for processes in $S(E)$. Our main problem will be to show that $N(E)$ is finite for finite E; the remainder of the proof is then easy.

Let $m > 0$ be the mass of smallest-mass particle. For $2m < E < 3m$ only two-particle processes are contained in $S(E)$. In this case we have $N(E) = 1$; a system consisting of just two particles can evidently have at most one collision of the first particle. Let E_1 be the least upper bound on the values of E such that $N(E)$ is finite. Then the requirement that $N(E)$ be finite for finite E is equivalent to the requirement that E_1 be infinite.

Suppose E_1 is finite. Then for any positive integer n there must be a reaction R_N at center-of-mass energy $E < E_1 + (m/n)$ with at least $N = n^2$ collisions of the first particle. In this reaction the total change in v_1, the velocity of the first particle, is not more than $\Delta V = 2\left[(E_1 + m)^2 - 4m^2\right]^{\frac{1}{2}}/(E_1 + m)$, which is the change it would have if both an initial and final energy of $E_1 + m$ were divided between two minimal mass particles. Pick out the n collisions of the first particle of R_N that give the greatest change in

v_1 , or, more generally, such that the smallest change δv_1 in these n collisions is not smaller than the largest change δv_1 in the remaining collisions. These n collisions divide the first trajectory of R_N into $n + 1$ segments at least one of which has at least $n - 1$ collisions. The change δv_1 at each collision of the first particle occurring in the interior of any of these $n + 1$ segments must be less than $\Delta V/n$, since otherwise the total variation of v_1 would be more than ΔV . Thus we have shown that for any positive integer n there is in $S[E_1 + (m/n)]$ a reaction R_{n-1} with $n - 1$ collisions of the first particle such that the δv_1 at each of these collisions is less than $\Delta V/n$.

Any given reaction may have a portion in which the particles can be divided into groups such that the particles in each group collide only with each other. For any reaction in $S(E_1 + m)$ the number of collisions of the first particle in any such portion must be bounded, since otherwise the least upper bound E_1 could be lowered. Thus for sufficiently large n it is not possible that the particles of the reaction R_{n-1} just constructed are divided into groups of particles that interact only among themselves; all the particles of R_{n-1} are connected to one another by collisions, for sufficiently large n .

By virtue of the above arguments, if E_1 is finite, there must be, for every positive integer n , a reaction R_{n-1} at

a center of mass energy $E < E_1 + (m/n)$ having $n - 1$ collisions of the first particle, and such that at each of these $n - 1$ collisions the change δv_1 of v_1 is less than $\Delta V/n$. In this reaction R_{n-1}, let δ_i be the maximum magnitude of the difference between the velocity of the first particle and the velocity of the second particle at the ith collision of the first particle. And let δ be the greatest of the δ_i. Because δv_1 is less than $\Delta V/n$ it follows that δ must be less than $b\,\Delta V/n$, where b is $(E_1 + m)/m$. This limit comes from the optimal case in which the second particle has the least possible mass, m, and the first particle has mass E_1, which is an upper bound.

Between collisions with the first particle the velocity of the second particle must increase monotonically. It follows from this that the difference in velocities of the first and second particles must always be less than δ, and hence also less than $b\Delta V/n$.

In order that the velocity of the second particle always differ by less than $b\Delta V/n$ from the velocity of the first particle, the maximum change of velocity of the second particle in a collision with a third particle must be less than $2b\,\Delta V/n$. This means, in turn, that the velocity of the third particle can differ from that of the second by at most $2b^2\,\Delta V/n$. [In the special case where the third and second particles can simultaneously collide with the first particle, the condition on the velocity of the third particle is more stringent].

Reapplication of the same arguments shows that the difference between the velocities of the third and fourth particles is always less than $4b^3\ \Delta V/n$, and so on. Since the total number of particles in the reaction must be less than $b = (E_1 + m)/m$, it follows that the velocities of all the particles of R_{n-1} must, in the center of mass frame, be less than $C\ \Delta V/n$, where $C = \frac{1}{2}(2b)^b$ is a constant determined by E_1/m. From this limit on the center-of-mass velocities one obtains as an upper bound on the center-of-mass kinetic energy the value[6] $\frac{1}{2}(E_1 + m)\ (C\ \Delta V/n)^2$.

According to the above result, the kinetic energy of the particles of R_{n-1} approaches zero as n approaches infinity. This requires that E_1 be equal to the sum of rest masses of some set of physical particles, and that for n larger than some finite value L, the sum of the rest masses of the particles of R_{n-1} be equal to E_1; otherwise the kinetic energy could not approach zero as E approaches E_1. The limit L is the greater of the two values L_1 and L_2 defined by $[\bar{E}_1 - E_1 - (m/L_1)] = 0$, where $\bar{E}_1$ is the smallest sum of rest masses that is greater than E_1, and by $E(L_2^{\ 2}) - \tilde{E}_1 = \frac{1}{2}(E_1 + m)\ (C\Delta V/L_2)^2$, where $\tilde{E}_1$ is the greatest sum of particle rest masses that is less than E_1, and $E(N)$ is the largest value of E' such that $N(E')$ is less than or equal to N.

Take n greater than L. Then the kinetic energy is $\epsilon = E - E_1$. But then the total variation of the velocity of the first particle in R_{n-1} is bounded by the value Δv defined by[6] $\frac{1}{2}m\ (\Delta v/2)^2 = \epsilon$.

We now repeat the arguments given before, but with $R_{n-1} = R_{r^2}$ in place of R_N and with Δv in place of ΔV. In place of the earlier bound $\frac{1}{2}(E_1 + m)\ (C\ \Delta V/n)^2$ on the kinetic energy, we now get $\frac{1}{2}(E_1 + m)\ (C\Delta v/r)^2$. That is, we have, for n larger than L,

$$\epsilon < \tfrac{1}{2}(E_1 + m)\ \left(\frac{C\Delta v}{r}\right)^2 = \frac{4(E_1 + m)\ C^2 \epsilon}{m \qquad (n - 1)} .$$

This gives $n - 1 < (2b)^{2b+1}$, which requires n, and hence $N(E)$, to be finite for finite E.

Since $N(E)$ is finite, the finite number of collisions of the first particle divide any reaction into a finite number of subreactions. In each of these subreactions the first trajectory is separated from the other trajectories. Thus the previous argument can be applied to the subreaction, but with the second trajectory in place of the first, etc. One concludes, then, that the number of collisions of the second particle is bounded by $[N(E) + 1]^2$.

Proceeding in this way one concludes that the total number of collisions is bounded, if the center-of-mass energy E is bounded. Since also the number of different types of particles that can enter into reactions in a bounded center-of-mass energy region is bounded, the total number of different types of reactions that can occur in such a region is bounded, in the one-dimensional case.

III. EXTENSION OF ONE-DIMENSION PROOF TO THE THREE-DIMENSIONAL CASE

The foregoing proof for the one-dimensional case can be easily generalized to the three-dimensional case. To do this we consider simultaneously the first x trajectory, the first y trajectory, and the first z trajectory. These are the trajectories lying at the largest values of x, y, and z, respectively.

A "triple" is a sequence of collisions (in their natural order) that includes at least one collision of the first x particle, at least, one collision of the first y particle, and at least one collision of the first z particle. A "sequence of triples" is a sequence of collisions (in their natural order) that are separated into an ordered set of triples such that the final collision of any triple is earlier than the earliest collision of the next triple.

Let $N(E)$ be the least upper bound on the number of triples in sequences of triples for reactions in $S(E)$. Let E_1 be the least upper bound on the values of E such that $N(E)$ is finite. Suppose (contrary to fact) that E_1 is finite. Then for any positive integer n there must be a reaction R_{3N} in $S[E_1 + (m/n)]$ with a sequence of at least $3N = 3n^2$ triples. Pick out those n triples that contain the collisions of the first x particle with the n largest values of δv_{ix}. Here δv_{ix} is the change in the x component of velocity of the first x particle. Pick out also those n triples that contain the n collisions of the

the first y particle with the largest δv_{iy}. Do the same also for z. These $3n$ triples separate the reaction R_{3N} into at most $3n + 1$ (sub)reactions, at least one of which, called R_{n-1}, must contain at least $n-1$ triples.

In the reaction R_{n-1} the maximum possible change of δv_{ix} at any collision of the first x particle is less than $\Delta V/n$, for exactly the same reasons as before. This bound holds also for the changes δv_{iy} and δv_{iz}. Thus the same arguments as in the one-dimensional case now give $(3/2)\ (E_1 + m)\ (C\Delta V/n)^2$ as an upper bound on the center-of-mass kinetic energy of R_{n-1}, where C is again $(1/2)\ (2b)^b$.

Continuing as before, one concludes that E_1 must be equal to the sum of rest masses of some set of physical particles, and that for sufficiently large n, the sum of the rest masses of the particles of R_{n-1}, must be precisely E_1. Thus the center-of-mass kinetic energy for R_{n-1} is again $\epsilon = E - E_1$, for sufficiently large n.

If the kinetic energy is ϵ, then the total variation of v_{ix} in R_{n-1} is no more than Δv defined $(m/2)\ (\Delta v/2)^2 = \epsilon$. From this one concludes, by the same argument as before, that $N(E)$ cannot become infinite at finite E.

From the fact that $N(E)$, the maximum number of triples in $S(E)$, is finite, it follows that the maximum number of collisions of reactions in $S(E)$ is finite. To show this we proceed as follows: For any reaction in $S(E)$ let t_1 be the greatest time

such that the subreaction consisting of the portion of the reaction occurring at $t < t_1$ contains no triple. Then let t_2 be the largest time such that the portion of the reaction occurring in the interval $t_1 < t < t_2$ contains no triple. Let $t_3, t_4, \cdots t_n$ be defined in the analogous way. This sequence of times must terminate at a time t_n with $n < N(E)$, where $N(E)$ is the maximum number of triples for reactions in $S(E)$.

The times t_i divide the original reaction into a set of no more than $[N(E) + 1]$ subreactions each of which contains no triple. But a subreaction that contains no triple must have a first x, y, or z trajectory that is disjoint from the other x, y, or z trajectories, respectively. This implies that the particles of each of the various subreactions must separate into groups such that the particles of each group interact only with each other. And at least one of these groups must be confined to a single x, y, or z trajectory. The number of these independent groups into which a subreaction divides is evidently no greater than E/m, which is an upper bound on the number of particles in reactions in $S(E)$.

The above analysis takes the original reaction into no more than $(E/m)\,[N(E) + 1]$ independent (i.e., self-interacting) new reactions. The same analysis is next applied to each of these independent new reactions. For reactions that are confined to a single x, y, or z trajectory, one use however "doubles" instead of "triples". Doubles are the two-dimensional analog of triples;

one eliminates the x, y, or z coordinate if the reaction is confined to an x, y, or z trajectory, respectively. The number of doubles (and analogously, of singles) in reactions in $S(E)$ are shown to be bounded by essentially the same argument as was just given for triples. In fact, the bound $N(E)$ obtained previously for the number of triples is also a bound on the number of doubles (and also on the number of singles).

The original reaction is separated at the first stage of the analysis into no more than $(E/m)\,[N(E) + 1]$ independent new reactions. Each of these is separated at the second stage into no more than $(E/m)\,[N(E) + 1]$ new independent reactions. One continues in this way until the stage at which no further decomposition is obtained. This must occur after no more than E/m iterations, since the energy of a part that decomposes at a given stage must have been reduced at every earlier stage by at least m, due to the separation into independent parts. Thus the total number of independent reactions that are picked out altogether is no more than $\lambda(E) \equiv \{(E/m)\,[N(E) + 1]\}^{(E/m)} + 1$. Also, the total number of times t_i singled out in the entire course of the analysis is bounded by $\lambda(E)$.

Each (nontrivial) collision of the original reaction occurs at one of the times t_i singled out in the above analysis. For, on the one hand, the only collisions that are eliminated at any stage of the analysis are those occurring at one of the singled-out times. On the other hand, the analysis does not terminate as long as any nontrivial collision remains. In the first place

any independent part that is not confined to a single x, y, or z trajectory must contain a triple, since otherwise a trajectory can be separated out. Thus the analysis cannot terminate so long as there are still independent parts not confined to an x, y, or z trajectory. Similarly, all parts confined to a single x, y, or z trajectory must be reduced, before the analysis terminates, to parts lying on at least two trajectories. These parts lying on two trajectories are confined to one-dimensional subspaces. For such parts the analysis proceeds until the first trajectory becomes the same as the last trajectory, since otherwise the first trajectory of an independent part must have a collision. One is left, finally, with only trivial collisions.

The conclusion from the above arguments is that the collisions of any reaction in $S(E)$ are confined to a set of times t_i, the number of which is no more than $\lambda(E)$. Furthermore, the number of different types of particles that can participate in reactions in $S(E)$ is also finite, due to the spectral conditions on the particle rest masses. (Particles of the same mass can be identified, as mentioned in the introduction). But a finite number of different particles colliding only at a bounded number of different times can give only a finite number of different types of reactions. In particular, if $q(E)$ is the number of different kinds of particles with rest energy less than E, then the number of different possible collisions at a single given time t_i is no more than $[2q(E)]^{2E/m}$. Thus the total number of

different types of multiple-scattering reactions involving collisions at no more than $\lambda(E)$ different instants of time is no more than $(2q)^{2E\lambda/m}$. This upper bound on the number of different types of collisions possible in the portion of the physical region lying at center-of-mass energy less than E could be lowered with but little extra effort, should the need arise.

FOOTNOTES AND REFERENCES

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