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OFF-ENERGY-SHELL PARTIAL-WAVE AMPLITUDES

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ABSTRACT

Representations of the two-body partial-wave transition amplitudes of the form

$$t_\ell(p,p';k^2) = F_\ell(p,p';k^2)\, t_\ell(k^2) ,$$

are displayed for several approaches to the calculation of these amplitudes. These representations separate the off-energy-shell behavior from the better known and more experimentally accessible on-shell behavior and they allow the insertion (for $k^2 > 0$) of the experimental on-shell t-matrix, expressed in terms of the phase shift, into calculations involving off-shell transition amplitudes. Such representations permit us to compare the off-shell behavior generated by two different approaches to the calculation of off-shell amplitudes, both of which yield the same on-shell amplitude.

Dispersion representations for the full off-shell amplitude $t_\ell(p,p';k^2)$ are obtained independent of any dynamical assumptions. Use of the off-shell unitarity relation in these dispersion representations allows us to determine the full off-shell amplitude from the half-off-shell amplitudes $t_\ell(p,k;k^2)$ and thus from the experimental quasiphase parameters of Sobel. This approach leads to a new approximation to the off-shell amplitude useful when the scattering is assumed to be dominated by the contribution of a single partial wave.

I. INTRODUCTION

A useful representation or approximation for the off-shell two-body partial-wave transition amplitude $t_\ell(p,p';k^2)$ should satisfy at least the following general non-dynamical requirements:

(a) Reduction: It must reduce to the known form of $t_\ell(k^2)$ on the energy shell;

(b) Unitarity: It must satisfy two-particle elastic unitarity on and off the energy shell,[1] that is

$$\mathrm{Im}\, t_\ell(p,p';k^2) = -\frac{\pi}{2} k\, t_\ell(p,k;k^2)\, t_\ell^*(k,p';k^2) \; ; \qquad (1)$$

(c) Analyticity: It should be an analytic function of k^2, with the only singularities in k^2 being the right-hand unitarity cut and bound-state or resonance poles. It should be analytic in $p(p')$ for $p^2 > 0$ $(p'^2 > 0)$ and it should have singularities for $p^2 < 0$ and $p'^2 < 0$ which become the left-hand or driving singularities of the on-shell amplitude[2] when $p^2 = p'^2 = k^2$;

(d) Time-Reversal: It should satisfy time-reversal invariance, that is $t_\ell(p,p';k^2) = t_\ell(p',p;k^2)$.

In addition, the limitations of available electronic computers make it desirable to have a separable representation

or approximation for the off-energy-shell transition amplitude for use in multiparticle scattering calculations. That is, we would like to have

$$t_\ell(p,p';k^2) = \sum_i \alpha_i(p;k^2)\ \gamma_i(k^2)\ \alpha_i(p';k^2)\ ,$$

so that the off-shell amplitude would be separable in incident and outgoing relative momenta.

We must stress that very little is known about the off-energy-shell behavior of transition amplitudes. Perhaps the best experimental test of the off-shell behavior of two-body amplitudes is p-p bremsstrahlung, but the experimental data is sparse and rather difficult to interpret.

Throughout this paper we use the conventions of Goldberger and Watson,[3] and units in which $\hbar = 2m = 1$, so that $k^2 = s$, the c.m. energy. For simplicity, we consider only the non-relativistic single-channel case.

We commence, in Sec. II, with a review and some comments on the usual methods of calculating off-shell amplitudes, in order to provide background and motivation for our work. In Sec. III we investigate representations of the form $t_\ell(p,p';k^2) = F_\ell(p,p';k^2)\ t_\ell(k^2)$ for several previous approaches to the calculation of off-shell amplitudes.[4] Finally, in Sec. IV, we present a method of calculating the full off-shell amplitude from a Cauchy formula incorporating the off-shell unitarity

relation. We shall see that this allows the full off-shell amplitude to be determined (except, perhaps, for a subtraction constant) in terms of the half-off-shell amplitudes and the bound-state wave functions. Thus, in the absence of bound states, the full off-shell amplitude is determined almost completely by experimental quantities, i.e., the phase shifts and quasiphase parameters. We also obtain an approximation for the off-shell amplitude useful when one partial wave dominates the scattering.

II. EARLIER APPROACHES TO CALCULATION OF OFF-SHELL AMPLITUDES

In the potential theory approach to the calculation of off-shell transition amplitudes, unitarity is enforced by putting a potential representing the force between the interacting particles into the Schrödinger equation or, equivalently, into the Lippmann-Schwinger equation for the transition amplitude. Time-reversal invariance is ensured by demanding that the potential have the symmetry $V_\ell(p,p') = V_\ell(p',p)$. Usually, a local potential form or a non-local separable potential form is chosen to parameterize the on-shell data. This chosen form is fitted numerically to the on-shell amplitude, which can be expressed in terms of the experimental phase shifts. Then the potential form is inserted into the Lippmann-Schwinger equation, which defines a unique continuation of the transition amplitude off the energy shell. Thus, the potential form <u>completely determines</u> the off-shell behavior of the resulting amplitude, and assumption of a potential form to fit the on-shell data implies assumption of the off-shell behavior of the amplitude. This is obviously a highly arbitrary procedure since many potential forms (some of which may lead to very different off-shell behaviors) can be used to fit the on-shell data. All that can be done is to try various potential forms which fit the on-shell data in, for example, nuclear-matter or multiparticle-scattering calculations and see

which form gives an off-shell behavior that coincides most closely with experiment.

Potential theory is really only a parameterization of the scattering amplitude, because we have only a posteriori knowledge of the potential. That is, the correct potential is the one which leads to the experimentally measured transition amplitude. Therefore, at this stage in our knowledge, there is no reason for preferring a local potential parameterization to a separable potential parameterization when the separable potential approach may be more convenient.

Also, within the usual potential theory approach to the calculation of off-shell amplitudes, there is no convenient way of isolating the different off-shell behavior given by two different potential forms, both of which fit the same on-shell data.

The other main approach to the calculation of off-shell transition amplitudes is the single-pole dominance model wherein we assume that the ℓth partial-wave is dominated by a single bound state or resonance pole. We say single-pole dominance because the introduction of more than one pole term in a given partial wave leads to a non-unitary transition amplitude in this approach.

Lovelace shows that the residues of bound-state or resonance poles in the partial-wave transition amplitudes factor

in the incident and outgoing relative momenta.[5] Thus, in the vicinity of a bound-state or resonance pole, we may write the transition amplitude in a separable form. When a single pole dominates a given partial wave, it is frequently assumed that the resulting separable form describes the transition amplitude for all energies in that partial-wave, although it is strictly true only for energies near the bound state or resonance energy. Therefore, in the single-pole dominance model we write

$$t_\ell(p,p';k^2) = \frac{a_\ell(p)\, a_\ell(p')}{D_\ell(k^2)} ,$$

where $a_\ell(p)$ is the bound-state or resonance form factor. Also, $D_\ell(k^2)$ is zero at the bound-state or resonance pole and is chosen such that $t_\ell(p,p';k^2)$ satisfies off-shell unitarity. In the case of a bound state, the form factor $a_\ell(p)$ can be related to the bound-state wave function in momentum space.

In particular, Lovelace[5] notes that the on-shell transition amplitude can be written in the (unitary) N/D form,

$$t_\ell(k^2) = N_\ell(k^2)/D_\ell(k^2) ,$$

and

$$D_\ell(k^2) = 1 - \int_0^\infty \frac{dq\, q^2\, N_\ell(q^2)}{k^2 - q^2 + i\epsilon} . \qquad (2)$$

Here $N_\ell(k^2)$ has only left-hand singularities and $D_\ell(k^2)$ has only right-hand singularities. Lovelace next points out that the off-shell amplitude has only a right-hand cut in k^2 and that the bound-state and resonance form factors $a_\ell(p)$ satisfy dispersion relations in p^2 with only a left-hand cut. He then suggests that the off-shell transition amplitude can be written as

$$t_\ell(p,p';k^2) = \frac{\left[N_\ell(p^2)\, N_\ell(p'^2)\right]^{\frac{1}{2}}}{D_\ell(k^2)}, \tag{3}$$

where $D_\ell(k^2)$ is given by Eq. (2), <u>provided</u> that the partial wave is dominated by a bound state or resonance.

We obtain an identical form for the off-shell amplitude by introducing a <u>single term</u> separable potential of the form $V_\ell(p,p') = g_\ell(p)\, g_\ell(p')$ into the Lippmann-Schwinger equation, so that the introduction of a single term separable potential and Lovelace's form of the single-pole dominance model are completely equivalent. Parenthetically, it can be shown that if we choose a single separable potential form $g_\ell(k)$ such that $g_\ell^{\,2}(k) = B_\ell(k^2)$ where $B_\ell(k^2)$ is the Born term in the integral equation for $N_\ell(k^2)$, then the Lippmann-Schwinger equation gives a form for $t_\ell(k^2)$ identical to that obtained in the determinantal approximation to the N/D equations. However, if we choose $g_\ell^{\,2}(k) = N_\ell(k^2)$ as in Lovelace's approach, and then fit the

resulting form for $t_\ell(k^2)$ to the on-shell experimental data, we will achieve a parameterization of $N_\ell(k^2)$ in the <u>exact</u> on-shell partial-wave scattering amplitude.

This approach has the drawback that $N_\ell(k^2)$ must be of definite sign for $k^2 > 0$ [$g_\ell(k)$ must be real] to avoid the introduction of spurious cuts into the amplitude. Further, when we demand that $N_\ell(k^2)$ be of definite sign for $k^2 > 0$, we find that we cannot reproduce a phase shift that changes sign. It has been suggested that a sign-changing phase shift might be reproduced by a single term separable potential with an explicit energy dependence. However, it is readily shown that separable potentials of the form

$$V_\ell(p,p';k^2) = \lambda(k^2)\, g_\ell(p)\, g_\ell(p')$$

<u>do not</u> lead to transition amplitudes that satisfy the requirements of analyticity and unitarity.

Notice that the single-pole dominance model is also only a parameterization of the on- and off-shell behavior of the transition amplitude, because of the arbitrary assumption that the form of the amplitude valid near a pole is valid at all energies. In addition, the choice of the form factors $a_\ell(p)$, which determine the off-shell behavior, is quite arbitrary at our present state of knowledge. For example, the bound-state

form factors depend on the poorly known momentum-space wave functions of the bound states.

The point of this review is to show that the available methods for calculating off-shell transition amplitudes lead, in practice, merely to bases for the parameterization of the off-shell amplitudes consistent with the four general requirements we set down earlier. The theme of our work, then, is that approaches which allow us to reduce the uncertainty in our knowledge of off-shell scattering amplitudes by directly incorporating a maximum of experimental information into the methods for calculating or representing these off-shell amplitudes will be valuable supplements to the present trial-and-error approaches.

III. SEPARATION OF OFF-SHELL BEHAVIOR

It seems worthwhile to have representations or parameterizations for the off-shell partial-wave transition amplitude of the form $t_\ell(p,p';k^2) = F_\ell(p,p';k^2)\, t_\ell(k^2)$, where $F_\ell(k,k;k^2) = 1$ and time-reversal invariance demands that $F_\ell(p,p';k^2) = F_\ell(p',p;k^2)$.

The value of such representations is threefold:

(a) They enable us to separate the off-shell behavior of the transition amplitude from the on-shell behavior, which is better known and more accessible to experiment. That is, they permit us to study the off-shell behavior of amplitudes independently, to some extent, of their on-shell behavior.

(b) They allow us to include the experimental data from on-shell two-body scattering in calculations involving off-shell transition amplitudes by writing

$$t_\ell(k^2) = -\frac{2}{\pi k} \exp[i\,\delta_\ell(k^2)] \sin \delta_\ell(k^2) , \qquad (4)$$

for $k^2 > 0$. Thus, we can take on-shell two-body scattering into account <u>exactly</u> in such calculations, and the only approximation will lie in the treatment of off-shell effects and the region $k^2 < 0$.

(c) They permit us to compare the off-energy-shell behaviors

produced by two different approaches to the calculation of off-energy-shell amplitudes, both of which yield the same on-shell amplitude. This would be useful in illuminating, for example, the differences between the various potential forms which have been used to fit the nucleon-nucleon scattering data.

The separations are useful for $k^2 > 0$, where $t_\ell(k^2)$ is determined by the experimental phase shifts. However, if we assume that the on-shell t-matrix is known from some other theory, they are good for all k^2.

Note that, since one of our main aims is to isolate the different off-shell behaviors given by two methods that yield the same on-shell amplitude, the explicit appearance of the on-shell amplitude in some of our expressions for the "separations" is not an important drawback.

It is clear that the off-shell factor $F_\ell(p,p';k^2)$ has poles as a function of k^2 at the zeros of the on-shell amplitude $t_\ell(k^2)$. This follows from the facts that if $t_\ell(k_0{}^2) = 0$, $t_\ell(k^2)$ can be represented as

$$t_\ell(k^2) = t_\ell'(k_0{}^2)\,(k^2 - k_0{}^2)\,,$$

for $k^2 \approx k_0{}^2$, and

$$F_\ell(p,p';k^2) = t_\ell(p,p';k^2)/t_\ell(k^2) . \tag{5}$$

In potential theory, the transition amplitude is obtained from the Lippmann-Schwinger equation

$$t_\ell(p,p';k^2) = V_\ell(p,p') + \int_0^\infty dq \frac{q^2}{k^2 - q^2 + i\epsilon} V_\ell(p,q)\, t_\ell(q,p';k^2) . \tag{6}$$

Now, suppose we know that $t_\ell(k^2)$ has n zeros at values of k^2 denoted by k_i^2. Let us write the off-energy-shell factor $F_\ell(p,p';k^2)$ in the form

$$F_\ell(p,p';k^2) = X_\ell(p,p';k^2) \Big/ \prod_{i=1}^{n} (k^2 - k_i^2) . \tag{7}$$

We obtain the following integral equation for $X_\ell(p,p';k^2)$ directly from the Lippmann-Schwinger equation:

$$X_\ell(p,p';k^2) = \frac{V_\ell(p,p')}{t_\ell(k^2)} \prod_{i=1}^{n} (k^2 - k_i^2) + \int_0^\infty dq \frac{q^2}{k^2 - q^2 + i\epsilon} V_\ell(pq)\, X_\ell(q,p';k^2) . \tag{8}$$

At values of k^2 near the zero of $t_\ell(k^2)$ located at $k^2 = k_j^2$, where $t_\ell(k^2)$ can be represented as $t_\ell(k^2) = t_\ell'(k_j^2)\,(k^2 - k_j^2)$,

the inhomogeneous term in the integral equation (8) becomes

$$\frac{V_\ell(p,p')}{t_\ell'(k_j^2)} \prod_{\substack{i=1\\ i\neq j}}^{n} (k^2 - k_i^2) ,$$

while the full off-shell amplitude is

$$t_\ell(p,p';k^2) = \frac{X_\ell(p,p';k^2)}{\prod_{\substack{i=1\\ i\neq j}}^{n} (k^2 - k_i^2)} \; t_\ell'(k_j^2) \; , \text{ for } \; k^2 \approx k_j^2 \; .$$

Furthermore, Kowalski[6] has shown that the transition amplitude is the solution to a non-singular integral equation obtained by Fredholm reduction from the Lippmann-Schwinger equation. This equation is

$$\begin{aligned} t_\ell(p,p';k^2) &= \frac{V_\ell(p,k)}{V_\ell(k,k)} \; t_\ell(p',k;k^2) \\ &+ \left[V_\ell(p,p') - \frac{V_\ell(p,k)\; V_\ell(k,p')}{V_\ell(k,k)} \right] \\ &+ \int_0^\infty dq \; \Lambda_\ell(p,q;k^2) \; t_\ell(q,p';k^2) \; , \end{aligned} \tag{9}$$

where the non-singular kernel is

$$\Lambda_\ell(p,q;k^2) = \frac{q^2}{k^2 - q^2} \left[V_\ell(p,q) - \frac{V_\ell(p,k)\; V_\ell(k,q)}{V_\ell(k,k)} \right] . \tag{10}$$

Kowalski also shows that the half-off-shell amplitude is given <u>exactly</u> by $t_\ell(p,k;k^2) = f_\ell(p,k)\, t_\ell(k^2)$, where $f_\ell(p,k)$ satisfies a non-singular integral equation with the kernel (10), namely,

$$f_\ell(p,k) = \frac{V_\ell(p,k)}{V_\ell(k,k)} + \int_0^\infty dq\, \Lambda_\ell(p,q;k^2)\, f_\ell(q,k) .$$

These equations are time-reversal invariant if $V_\ell(p,p') = V_\ell(p',p)$. Noyes[7] has shown that the zeros of $V_\ell(k,k)$ cause no essential problems, since k enters these equations only as a parameter and not as a variable.

Let us write the half-off-shell amplitude $t_\ell(p,k;k^2)$ as

$$t_\ell(p,k;k^2) = H_\ell(p;k^2)\, t_\ell(k^2) , \tag{11}$$

where $F_\ell(p,k;k^2) \equiv H_\ell(p;k^2)$ and $H_\ell(k;k^2) = 1$. If we cast the half-off-shell factor $H_\ell(p;k^2)$ in the form

$$H_\ell(p;k^2) = h_\ell(p;k^2) \Big/ \prod_{i=1}^{n} (k^2 - k_i^{\,2}) , \tag{12}$$

we obtain an integral equation for $h_\ell(p;k^2)$:

$$h_\ell(p;k^2) = \frac{V_\ell(p,k)}{V_\ell(k,k)} \prod_{i=1}^{n} (k^2 - k_i^{\,2}) + \int_0^\infty dq\, \Lambda_\ell(p,q;k^2)\, h_\ell(q;k^2) ,$$

immediately from Kowalski's work. Also, if we insert Eqs. (11) and (12) in Eq. (9) and use Eq. (5) and Eq. (7) we find the following equation for $X_\ell(p,p';k^2)$:

$$X_\ell(p,p';k^2) = \frac{V_\ell(p,k)}{V_\ell(k,k)} h_\ell(p';k^2)$$

$$+ \frac{\prod_{i=1}^{n} (k^2 - k_i^2)}{t_\ell(k^2)} \left[V_\ell(p,p') - \frac{V_\ell(p,k)\, V_\ell(k,p')}{V_\ell(k,k)} \right]$$

$$+ \int_0^\infty dq\, \Lambda_\ell(p,q;k^2)\, X_\ell(q,p';k^2) .$$

We see that $X_\ell(p,p';k^2)$ and $h_\ell(p;k^2)$ satisfy non-singular integral equations with the same kernel. Less convenient separations can be made by starting with Kowalski's[6] Eqs. (6) and (7).

Finally, we present a way of determining $X_\ell(p,p';k^2)$ that does not explicitly involve the on-shell t-matrix. Suppose we start with the Lippmann-Schwinger equation (6) for the off-shell amplitude. We also know that

$$t_\ell(k,p';k^2) = V_\ell(k,p') + \int_0^\infty dq \frac{q^2}{k^2 - q^2 + i\epsilon} V_\ell(k,q)\, t_\ell(q,p';k^2) ; \tag{13}$$

so, if we multiply Eq. (13) by $V_\ell(p,p')/V_\ell(k,p')$, subtract the result from Eq. (6), and use Eqs. (5), (7), (11) and (12), we find

$$X_\ell(p,p';k^2) = \frac{V_\ell(p,p')}{V_\ell(k,p')} h_\ell(p';k^2) + \int_0^\infty dq \frac{q^2}{k^2 - q^2 + i\epsilon}\left[V_\ell(p,q) - \frac{V_\ell(p,p')\, V_\ell(k,q)}{V_\ell(k,p')}\right] X_\ell(q,p';k^2) .$$

This equation is valid for $V_\ell(k,p') \neq 0$ but, again, k and p' enter this equation only as parameters and not as variables, so the zeros of $V_\ell(k,p')$ should not cause any essential problems.

To make the separations between experimental and theoretical quantities complete in the expressions we have obtained thus far from potential theory, we must use $t_{\ell_M}(k^2)$ (obtained by solution of the Lippmann-Schwinger equation with the model potential) in the expressions for the off-shell factors, rather than the experimental model-independent $t_\ell(k^2)$ obtained from the phase shifts.

Although we have dealt in potential theory, with the partial-wave t-matrix, analogous derivations can be made for the partial-wave reaction or K-matrix that satisfies the equation

$$K_\ell(p,p';k^2) = V_\ell(p,p') + P \int_0^\infty dq \frac{q^2}{k^2 - q^2} V_\ell(p,q) K_\ell(q,p';k^2) ,$$

where P indicates a principal value integral.

In a single-pole dominance model, or when a single term separable potential is inserted into the Lippmann-Schwinger equation, the off-shell amplitude has the form of Eq. (3) and the separation is made simply by writing

$$t_\ell(p,p';k^2) = \frac{\left[N_\ell(p^2)\, N_\ell(p'^2)\right]^{\frac{1}{2}}}{N_\ell(k^2)} t_\ell(k^2)$$

or

$$t_\ell(p,p';k^2) = \frac{g_\ell(p)}{g_\ell(k)} t_\ell(k^2) \frac{g_\ell(p')}{g_\ell(k)} .$$

In this case, the relation between the off-shell behavior and the assumed separable potential form is particularly transparent. It can easily be shown that <u>any</u> separation of the form

$$t_\ell(p,p';k^2) = F_\ell(p,p';k^2)\, t_\ell(k^2) ,$$

where $F_\ell(p,p';k^2)$ is real, is constrained by unitarity to be of the form

$$F_\ell(p,p';k^2) = H_\ell(p;k^2)\, H_\ell(p';k^2) ,$$

where $H_\ell(p;k^2)$ is the multiplicative half-off-shell factor.

Watson[8] has shown that the off-shell amplitude $t_\ell(p,p';k^2)$ can always be written in potential theory as

$$t_\ell(p,p';k^2) = N_\ell(p,p';k^2)/D_\ell(k^2) , \tag{14}$$

for p^2 and p'^2 positive. Here $N_\ell(p,p';k^2)$ and $D_\ell(k^2)$ are, respectively, operator analytic and analytic for k^2 in the complex plane cut from zero to infinity. In addition, all the poles of $t_\ell(p,p';k^2)$ in the variable k^2 (for positive values of p^2 and p'^2) must appear as zeros of $D_\ell(k^2)$, and $N_\ell(p,p';k^2)$ will have cut singularities for negative p^2 and negative p'^2. Wong and Zambotti[9] have expanded $N_\ell(p,p';k^2)$ as a sum of separable terms which preserve the correct analyticity structure for $N_\ell(p,p';k^2)$ and have shown that the resulting separable approximate amplitude gives results reasonably close to the corresponding local potential amplitude. Also, it is known that if the potential is assumed to be separable, the Lippmann-Schwinger equation always reduces to a set of algebraic equations, and the solution can be displayed in closed form. If we consider a two-term non-local separable potential of the form

$$V_\ell(p,p') = \lambda_1 g_\ell(p) g_\ell(p') - \lambda_2 h_\ell(p) h_\ell(p') ,$$

where λ_1 and λ_2 are constant and $V_\ell(p,p')$ has the symmetry

required for time-reversal invariance, it can be shown that the off-shell t-matrix has the form (14) with the same general analyticity properties. Such a separable potential is needed in a partial-wave where the phase shift changes sign.

The obvious separation to be made for an amplitude of the form (14) is

$$t_\ell(p,p';k^2) = \frac{N_\ell(p,p';k^2)}{N_\ell(k,k;k^2)} \; t_\ell(k^2) \; . \tag{15}$$

The single-pole dominance and single separable potential approaches are special cases of this more general approach. This separation is no more complicated for computational use than the usual approaches in which an off-shell amplitude in the general N/D form is used. We simply use the form (15) when $k^2 > 0$ and $t_\ell(k^2) \neq 0$, with $t_\ell(k^2)$ obtained from the experimental phase shifts. Then, when $k^2 < 0$ or $t_\ell(k^2) = 0$ we revert to the usual form (14).

Multiparticle scattering calculations are frequently done by fitting the on-shell data with a potential theory or a single-pole dominance model. Then the resulting off-shell amplitude is inserted into the multiparticle scattering equations. In the range $k^2 < 0$, where the on-shell amplitude is no more accessible to experiment than the off-shell amplitude, one must use the method originally chosen. But, for $k^2 > 0$, we suggest

the use of the separation proposed herein, so that the experimental on-shell two-body data in the form of phase shifts can be included in the calculations, thus taking on-shell two-body scattering effects into account exactly. It is also possible that one might insert the known on-shell scattering data and some convenient form for the parameterization of the off-shell effects into a calculation involving off-shell amplitudes. We might then fit the result to experiment and thereby obtain a parameterization of the full off-shell amplitude in a given partial wave. We further suggest the use of these separations to expose the differences in off-energy-shell behavior produced by various methods for calculation of the transition amplitude which have been constructed so that they yield the same on-shell results. Such a comparison may be made by plotting the half-off-shell factors

$$H_\ell(p;k^2) \qquad \text{or} \qquad h_\ell(p;k^2)$$

vs p and the symmetrically off-shell factors

$$F_\ell(p,p;k^2) \qquad \text{or} \qquad X_\ell(p,p;k^2)$$

vs p at given values of k^2. Alternatively, these factors can be studied as functions of $\rho = p^2 - k^2$, where ρ measures the distance off the energy shell.

IV. DISPERSION RELATIONS FOR OFF-SHELL AMPLITUDES

The non-dynamical constraints of analyticity and unitarity, coupled with the assumption that $t_\ell(p,p';s)$ goes (at worst) to a constant as $|s| \to \infty$, allow us to derive dispersion relations in the energy variable $s = k^2$ for the full off-shell two-body partial-wave transition amplitude $t_\ell(p,p';s)$. We will show later that $t_\ell(p,p';s)$ goes to a constant as $|s| \to \infty$ within the framework of potential theory, but for the moment we will take this as an added assumption, since we do not wish to introduce any dynamical postulates at this time.

Let us consider a partial wave in which there are N bound states at energies $s = -s_n$. We know[5] that the residue $R_n(p,p')$ of the nth bound-state pole can be factored as $R_n(p,p') = a_n(p)\, a_n(p')$, where the bound-state form factor satisfies $a_n(p) = -(s_n + p^2)\,\psi_n(p)$ and $\psi_n(p)$ is the bound-state wave function in momentum space. We then obtain the following once-subtracted dispersion relation for $t_\ell(p,p';s)$:

$$t_\ell(p,p';s) = t_\ell(p,p';s_0) + \sum_n \frac{a_n(p)\, a_n(p')}{s + s_n} \left(\frac{s - s_0}{-s_n - s_0}\right)$$

$$+ (s - s_0)\,\frac{1}{\pi}\int_0^\infty \frac{\mathrm{Im}\, t_\ell(p,p';\xi)}{(\xi - s)\,(\xi - s_0)}\, d\xi\,, \qquad (16)$$

where the subtraction has been made at $s = s_0$ with s_0 on the physical sheet of the complex energy Riemann surface.

If we make the subtraction at $s_0 = 0$, we find

$$t_\ell(p,p';s) = t_\ell(p,p';0) + \sum_n \frac{a_n(p)\, a_n(p')}{s + s_n} \left(\frac{-s}{s_n}\right)$$

$$+ \frac{s}{\pi} \int_0^\infty \frac{\mathrm{Im}\, t_\ell(p,p';\xi)}{(\xi - s)} \cdot \frac{d\xi}{\xi} . \tag{17}$$

The off-shell unitarity relation (1) tells us that the subtraction constant $t_\ell(p,p';0)$ is real provided the half-off-shell amplitude $t_\ell(p,s^{\frac{1}{2}};s)$ is finite for all p when $s = 0$.

If, however, we assume that $\mathrm{Im}\, t_\ell(p,p';s) \to 0$ as $s \to \infty$ along the real axis, we can make the subtraction at $s_0 = \infty$, whence

$$t_\ell(p,p';s) = t_\ell(p,p';\infty) + \sum_n \frac{a_n(p)\, a_n(p')}{s + s_n}$$

$$+ \frac{1}{\pi} \int_0^\infty \frac{\mathrm{Im}\, t_\ell(p,p';\xi)}{(\xi - s)}\, d\xi , \tag{18}$$

and $t_\ell(p,p';\infty)$ is real by assumption.

Let us consider the possibility of making the subtraction at the position of a bound-state or resonance pole at $s = s_p$. The residue of the pole will, of course, factor in the form

$R(p,p') = g(p)\, g(p')$, where $g(p)$ denotes the bound-state or resonance form factor. The resulting once-subtracted dispersion relation is

$$t_\ell(p,p';s) = \pm \frac{g(p)\, g(p')}{s - s_p} + \sum_n \frac{a_n(p)\, a_n(p')}{s + s_n} \left(\frac{s - s_p}{-s_n - s_p} \right)$$

$$+ \frac{(s - s_p)}{\pi} \int_0^\infty \frac{\mathrm{Im}\, t_\ell(p,p';\xi)}{(\xi - s)\,(\xi - s_p)}\, d\xi \,. \qquad (19)$$

The plus sign in the first term holds if s_p is chosen as a bound-state pole, in which case the sum in the second term is over the remaining bound-state poles. The minus sign in the first term is appropriate if s_p is chosen as a resonance pole. The sign difference between the bound-state and resonance cases arises because the bound-state and resonance poles are, respectively <u>inside</u> and <u>outside</u> the large circular contour (indented to exclude the right hand unitarity cut) on the physical sheet of the complex energy Riemann surface, which was used to derive the dispersion relations (16) and (19). This is because the resonance poles lie on the second sheet of the energy variable s, just below the physical region, which is reached by approaching the real s axis from above in the first, or physical, sheet.

Let us consider the dispersion integral term in Eqs. (16) through (19). We shall have to digress slightly to discuss

some properties of the half-off-shell amplitudes. Since $t_\ell(p,k;k^2) = t_\ell(k,p;k^2)$ by time-reversal invariance, we need only discuss the former. Kowalski[10] has shown, from time-reversal invariance and the off-shell unitarity relation that we may <u>always</u> write

$$t_\ell(p,k;k^2) = H_\ell(p;k^2)\, t_\ell(k^2)$$

with $H_\ell(p;k^2)$ real. In addition, Sobel[11] has shown that, based on the same general considerations of unitarity, time-reversal invariance, and conservation of angular momentum and parity which allow us to make the phase shift representation of the on-shell amplitude, the half-off-shell amplitude can always be represented by

$$t_\ell(p,k;k^2) = -\frac{2}{\pi k} \exp[i\, \delta_\ell(k^2)]\, \Delta_\ell(p^2;k^2) .$$

The real quasiphase parameter $\Delta_\ell(p^2;k^2)$ is such that $\Delta_\ell(k^2;k^2) = \sin \delta_\ell(k^2)$. Using the expression (4) for the on-shell amplitude, we find

$$t_\ell(p,k;k^2) = [\Delta_\ell(p^2;k^2)/\sin \delta_\ell(k^2)]\, t_\ell(k^2) ,$$

whence

$$H_\ell(p;k^2) = \Delta_\ell(p^2;k^2)/\sin\,\delta_\ell(k^2)\ .$$

From the off-shell unitarity relation (1), we have

$$\mathrm{Im}\ t_\ell(p,p';k^2) = -\ \frac{\pi}{2}\,k\,H_\ell(p;k^2)\ |t_\ell(k^2)|^2\ H_\ell(p';k^2)\ . \qquad (20)$$

If we use Eq. (4), we find

$$\mathrm{Im}\ t_\ell(p,p';k^2) = -\ \frac{2}{\pi k}\,H_\ell(p;k^2)\,\sin^2\,\delta_\ell(k^2)\ H_\ell(p';k^2) \qquad (21)$$

or

$$\mathrm{Im}\ t_\ell(p,p';k^2) = -\ \frac{2}{\pi k}\,\Delta_\ell(p^2;k^2)\ \Delta_\ell(p'^2;k^2)\ . \qquad (22)$$

Now, we insert the off-shell unitarity relation (1), or one of its variations (20) through (22), into the dispersion relations (16) through (19). Notice that, if we have a separable representation or approximation for the multiplicative half-off-shell factor $H_\ell(p;s)$ of the form $H_\ell(p;s) = \alpha_\ell(p)\ \beta_\ell(s)$, the dispersion integral term in Eqs. (16) through (19) will be separable in incident and outgoing relative momenta. In fact, Eq. (19) will provide, in this case, a representation for $t_\ell(p,p';s)$ that is a sum of separable terms.

Therefore, based only on the non-dynamical assumptions of analyticity, unitarity, and asymptotic behavior, we have shown that the full off-shell transition amplitude $t_\ell(p,p';s)$ is <u>determined</u> by the half-off-shell scattering data and bound-state and resonance form factors, at least to within a subtraction constant. This is in contrast to the earlier methods, wherein we had to assume $V_\ell(p,p')$ to obtain $t_\ell(p,p';s)$ [or else assume $t_\ell(p,p';s)$] and then try the resulting transition amplitude in calculations to see if we had the correct form for $t_\ell(p,p';s)$. Notice that this dispersion theoretic approach is valid for all values of s, including $s < 0$. Indeed, if we had the requisite half-off-shell scattering data, we might use the dispersion relations to assess the validity of approximating $t_\ell(p,p';s)$ by a bound-state or resonance pole.

Now we would like to introduce a new approximation to the off-shell transition amplitude based on this dispersion theoretic approach. First, let us note that the partial cross section is given by $\sigma_\ell = (2\ell + 1)\ \frac{4\pi}{s} \sin^2 \delta_\ell(s)$. Unfortunately, the partial cross section cannot be extracted from experiment unless a phase shift analysis is done. However, suppose the ℓth partial wave dominates the scattering so that $\sigma_E(s)$, which denotes the total elastic cross section, is given by $\sigma_E(s) \approx \sigma_\ell(s)$. Then, we find that

$$\sin^2 \delta_\ell(s) \approx \frac{s}{4\pi(2\ell + 1)} \sigma_E(s) ,$$

which, with Eq. (21), yields

$$\mathrm{Im}\, t_\ell(p,p';s) = \frac{-s^{\frac{1}{2}}}{2\pi^2(2\ell + 1)} H_\ell(p;s)\, \sigma_E(s)\, H_\ell(p';s) . \qquad (23)$$

Thus, in the case where a given partial wave dominates the scattering, we may insert Eq. (23) into our dispersion relations and find that the partial-wave amplitude in the dominant wave is determined by the total elastic cross section, the quasiphase parameters, and bound-state and resonance form factors, at least to within a subtraction constant.

We hope to apply this approximation to the low-energy pion-nucleon interaction, which is dominated by P-wave scattering. In that case, the P-wave multiplicative half-off-shell factor $H_1(p;s)$ and the nucleon and 3-3 resonance form factors can be determined from Chew-Low theory.[5] Additionally, the Chew-Low theory tells us that $H_1(p;s)$ factors as $H_1(p;s) = \alpha(p)\, \beta(s)$. If we subtract at the nucleon bound-state pole, Eq. (19) then leads to a two-term separable approximation to the off-shell πN transition amplitude in the P_{11} wave. Similarly, subtraction at the 3-3 resonance pole in Eq. (19) leads to a two-term separable approximation to the off-shell P_{33} pion-nucleon t-matrix.

We may estimate the validity of this approximation on the energy shell by choosing an energy at which the phase shift is known and using this phase shift to obtain $t_\ell(s)$ from Eq. (4). We may then use our approximation (23), which gives

$$\mathrm{Im}\, t_\ell(s) = \frac{-s^{\frac{1}{2}}}{2\pi^2(2\ell+1)}\, \sigma_E(s) \,, \tag{24}$$

since $H_\ell(s^{\frac{1}{2}};s) \equiv 1$. Equation (24) can be inserted into the dispersion relations and the resulting value of $t_\ell(s)$ can be compared with the value known from phase shift analysis.

This single wave dominance model is slightly more general than the usual approximation in which a transition amplitude is approximated by a single pole in the dominant partial wave. It is hoped that such a scheme will be useful for approximating transition amplitudes for two-particle scattering processes for which phase shift analyses have not been carried out.

Let us now assume that we are operating within the dynamical framework of potential theory. The Lippmann-Schwinger equation (6) shows that $t_\ell(p,p';s) \to V_\ell(p,p')$, which is constant as a function of s, when $s \to \pm\infty$. In other words, the Born approximation becomes exact as we go to very high energies. Then, the Sugawara-Kanazawa[12] theorem tells us that $t_\ell(p,p';s) \to$ constant as $|s| \to \infty$ in any direction in the complex plane, justifying our assumption on the asymptotic behavior of $t_\ell(p,p';s)$ in potential theory.

We have shown that, in potential theory, $t_\ell(p,p';\infty) = V_\ell(p,p')$, which is real when $V_\ell(p,p')$ is real. Thus, when we deal with a real potential, we have $\mathrm{Im}\, t_\ell(p,p';k^2) \to 0$ as $k^2 \to \infty$ and we may use Eq. (18). Incidentally, when $\mathrm{Im}\, t_\ell(p,p';k^2) \to 0$ as $k^2 \to \infty$ we may set $p = p'$ and use the off-shell unitarity relation (1) to find

$$- \frac{\pi}{2}\, k\, |t_\ell(p,k;k^2)|^2 \to 0 \quad \text{as} \quad k^2 \to \infty .$$

In potential theory, the half-off-shell multiplicative factor $H_\ell(p;s)$ which appears in Eqs. (20) and (21) is calculated from Eq. (12). If we have reason to believe that the phase shift in the ℓth partial wave does not change sign, and the partial wave contains at most one bound state, we may hope to parameterize the partial-wave scattering amplitude by inserting a single term separable potential in the Lippmann-Schwinger equation. We may fit the separable potential to the scattering length, effective range, and bound state binding energy, if any. In this case

$$V_\ell(p,p') = \lambda\, g_\ell(p)\, g_\ell(p'), \quad H_\ell(p;s) = g_\ell(p)/g_\ell(s^{\frac{1}{2}})$$

and Eq. (18) gives a representation of $t_\ell(p,p';s)$, which is a sum of three separable terms.

V. CONCLUSION

We have attempted to reduce the uncertainty in our knowledge of off-energy-shell two-body partial-wave transition amplitudes by introducing representations for, and a method of calculating, these amplitudes that directly incorporate as much experimental data as possible into the determination of these amplitudes. It is hoped that this approach might lead to more detailed and accurate knowledge of the off-energy-shell behavior of transition amplitudes. This knowledge would be useful for constructing good approximations to the off-shell amplitudes for use in the multiparticle scattering calculations which depend on these amplitudes for input. Also if we had detailed numerical information on the off-shell amplitude $t_\ell(p,p';s)$ we could use this information as input to the Lippmann-Schwinger equation and solve for the potential $V_\ell(p,p';s)$, which gives the experimental amplitude. This <u>experimentally determined</u> potential might be compared with the relevant limit of some more fundamental description of the forces between elementary particles.

FOOTNOTES AND REFERENCES

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