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CORRECT UNITARIZATION OF "EFFECTIVE POTENTIALS"

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ABSTRACT

The two-particle Feynman graph, usually described as a function of energy and momentum-transfer, is correctly used as a potential in Schrödinger's equation after the graph is described as a function of momentum and a radial parameter.

A current procedure in calculating the S matrix for elastic scattering of various elementary particles is to unitarize the Feynman amplitude for the scattering process via Schrödinger's equation. This procedure is meaningful only if this unitarization is done correctly.

The amplitude for the scattering process is written as $A(s,t)$, where t is the momentum transfer and s the center-of-mass energy defined by

$$\sqrt{s} = (p^2 + m_1^2)^{1/2} + (p^2 + m_2^2)^{1/2} . \tag{1}$$

Here m_1 and m_2 are the masses and p^2 is the momentum-square in the elastic-scattering process. By means of Fourier (or Laplace) transforms with respect to t, one obtains an amplitude as a function of r, the radial parameter:

$$V(s,r) = \mathcal{F}_{rt}[A(s,t)] . \tag{2}$$

Treating s as a parameter, one solves the Schrödinger equation

$$\left[-\frac{1}{\mu} \nabla_r^2 + V(s,r)\right] \psi = \frac{K^2}{\mu} \psi , \tag{3}$$

where μ is the reduced mass, and K^2 is the energy eigenvalue. A partial-wave reduction can be made and the familiar set of partial-wave amplitudes calculated.

This is clearly incorrect: For example, a nonrelativistic reduction of equations arising from minimization of the Action of the free Lagrangian for the two scattering particles would yield the homogenous Schrödinger equation,

$$\frac{1}{\mu}(\nabla^2 + K^2)\psi = 0 . \tag{4}$$

If one includes interaction with the virtual particle fields, Eq. (3) is obtained. In Eq. (1) p^2 then "means"

$$\sqrt{s} \sim (-\nabla^2 + m_1^2)^{1/2} + (-\nabla^2 + m_2^2)^{1/2} . \tag{5}$$

The $\sim$ sign is used, because the operator ∇^2 is not yet defined; however s is clearly not a parameter, but a function of the momentum of the scattering particle.

Such a problem has recently been solved for the nucleon-nucleon interaction mediated by one-meson exchanges.[1] Detailed calculation of nucleon-nucleon scattering over a range of 0 to 310-MeV incident laboratory energy, as well as detailed calculations of the deuteron and of nuclear matter were entirely successful. Besides giving encouragement that an understanding of nuclear forces is in sight, these calculations also illustrate the importance of exact treatment of momentum-dependence (arising from the nonrelativistic reduction of Feynman amplitudes) and the incorrectness of treating such potentials in an energy-dependent manner.

The nonrelativistic procedure is straightforward. One collects all polynomials of p from spinor factors of the scattering particles, and expands all energy factors:

$$E_1 = (K^2 + m_1^2)^{\frac{1}{2}} \simeq 1 + \frac{K^2}{2m_1^2} + \cdots \tag{6a}$$

and

$$E_2 = (K^2 + m_2^2)^{\frac{1}{2}} \simeq 1 + \frac{K^2}{2m_2^2} + \cdots \quad . \tag{6b}$$

Then all powers of K higher than K^2 are dropped. These results are clearly only valid for $K/m_< < 1$, where $m_<$ is the lesser of m_1 and m_2. Even though this is a severe restriction for such processes as $\pi K \to \pi K$, at least the problem will be done correctly. If one admits some rather questionable mathematics, higher terms of the expansion (6) might be used in the following treatment.

Now one solves

$$\left[-\frac{\nabla^2}{\mu} + V(r,\nabla)\right]\psi = \frac{K^2}{\mu}\psi \ . \tag{7}$$

The potential term is made Hermitian; V is symmetrized in particles 1 and 2, and ∇^2 is symmetrically treated:

$$\nabla^2 W \Longrightarrow \frac{1}{2}(\nabla^2 W + W\nabla^2) \ . \tag{8}$$

There is no problem in solving the differential Eq. (8). However, if V is to be useful, it should be applicable to other physics problems (such as three-particle scattering), most of which are understood in integral-equation formulations:

$$\psi(Kr) = \phi(Kr) + \int d\underset{\sim}{r}' \, G(K, r, r') \, \overrightarrow{V(r',\nabla_{r'})} \, \psi(Kr'); \tag{9}$$

ϕ is the solution to the homogenous equation, G the Green's function solution, and the arrow implies that $\nabla_{r'}$ operates on $\psi(Kr')$.

After a partial-wave reduction, one has

$$\psi_\ell(Kr) = j_\ell(Kr) + \int \underset{\sim}{dr'}\, G_\ell(K,r,r')\, \overrightarrow{V_\ell(r',\nabla_{r'})}\, \psi_\ell(Kr'). \tag{10}$$

For a scattering problem one can write

$$G_\ell(K,r,r') = -j_\ell(Kr_<)\, n_\ell(Kr_>)\ , \tag{11}$$

where respectively j_ℓ and n_ℓ are Bessel and Neumann functions, and $r_<$ and $r_>$ are the lesser and greater of r and r'. Integration by parts now allows V_ℓ to operate on G_ℓ.

The phase shift for the scattering problem, corresponding to standing-wave conditions, is

$$\tan \delta_\ell = -\frac{\mu}{K} \int \underset{\sim}{dr}\, j_\ell(Kr)\, \overleftarrow{V_\ell(Kr)}\, \psi_\ell(Kr), \tag{12}$$

where δ_ℓ is the same δ_ℓ in $e^{i\delta_\ell} \sin \delta_\ell$, which is the matrix element obtained for the outgoing wave solution. Notice that for

$$V_\ell = X_\ell + Y_\ell \frac{d}{dr} + Z_\ell \frac{d^2}{dr^2}\ , \tag{13}$$

Eq. (10) becomes

$$(1 + Z_\ell)\psi_\ell = j_\ell + \int G_\ell \overleftarrow{V_\ell}\, \psi_\ell\ . \tag{14}$$

If possible, as was

done in the nucleon-nucleon case, it is most desirable to first project the invariant helicity amplitudes of the Feynman graph onto a complete set of eigenoperators of the eigenfunctions ψ_ℓ.

Cutoffs should be made to insure that $V(r,\nabla)$ is an analytic function over all coordinate space. Then there is no problem in integrating by parts in Eq. (10). As G_ℓ is composed of polynomials, sines and cosines for the scattering problem and polynomials and exponentials for the bound-state problem, the resulting equation is no harder to solve, numerically, than one with a potential given as a simple function of the radial parameter, r. For the nucleon-nucleon case, it was found convenient to solve the radial form of Eq. (14) by matrix inversion (the diagonal elements of the matrix to be inverted do not equal zero, so that a 'pivot' element need not be searched for) using three 10-point Gaussian quadrature meshes. Coupled equations, (as arise from tensor forces for nucleons) were similarly treated.

As was mentioned, proper treatment of momentum-dependence is essential to solve the eigenvalue equation (10); solutions of the integral and differential equations agree only if the momentum-dependence is treated correctly.

The potential as defined is nonlocal:

$$\frac{1}{G(r,r')} \; V_\ell(r',\nabla_{r'}) \; G_\ell(r,r') \;=\; V_{eff}(r',r), \qquad (15)$$

where V_{eff} is a function of just r and r' (involving sines, cosines, or exponentials). Only for two-body problems, for

$$F(r',\nabla_{r'}) = W \frac{d^2}{dr^2} + \frac{d^2}{dr^2} W , \tag{16}$$

is this not true. Here one obtains

$$\frac{1}{G} F G = \left\{ \frac{d^2 W}{dr^2} + \left[\frac{L^2}{r^2} \begin{pmatrix} -K^2 \\ \\ +\alpha^2 \end{pmatrix} \right] W \right\} , \tag{17}$$

which gives a local potential. In Eq. (17), one has $L^2 = \ell(\ell + 1)$, and $-K^2$ is taken for the scattering problem, and $+\alpha^2$ for the bound-state problem.

FOOTNOTE AND REFERENCE

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1. Lester Ingber, Realistic Nuclear Forces, presented at the International Nucleon-Nucleon Conference, Gainesville, Florida, 1967 (proceedings to be published).

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