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AN ANALYTICAL AND NUMERICAL STUDY OF THE BOUND-STATE SOLUTIONS TO THE PION-NUCLEON BETHE-SALPETER EQUATION

Klaus D. Rothe

(Ph. D. Thesis)

July 24, 1967

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UNIVERSITY OF CALIFORNIA

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ABSTRACT

The Bethe-Salpeter equation describing the interaction of pseudoscalar mesons and nucleons via pseudoscalar coupling is solved numerically for energies below the elastic threshold by use of variational techniques. We consider only the "ladder" approximation with a local potential corresponding to the exchange of an elementary nucleon. Simple generalizations of this form of the interaction are considered as well. In the absence of a cutoff this leads to a marginally singular integral equation. We examine in detail the boundary conditions to be imposed on the solutions in order to lead to a discrete eigenvalue spectrum. The study of this problem is considerably simplified at zero total c.m. energy, where the (Wick rotated) equation is invariant under four dimensional rotations. In order to take full advantage of this symmetry, we construct a new set of spinor spherical harmonics belonging to the representations $\left(\frac{n \pm 1}{2}, \frac{n}{2}\right)$ and $\left(\frac{n}{2}, \frac{n \pm 1}{2}\right)$ of the four-dimensional rotation group. The discussion is then extended to the

general case, in which we examine briefly the formal structure of the $E \neq 0$ solutions. Attention is given to the role of the "relative time" in a relativistic theory, and the closely related problem of arriving at a time-independent approximation of the BS equation. In particular, we consider the static and nonrelativistic limit of the equation, and discuss the validity of the approximations involved. The eigenvalue spectrum obtained in the two limits is compared to that of the original equation. Finally, a brief discussion is given of the $N\theta$ and $V\theta$ sectors of the Lee model, designed to illustrate still further our observations on the role of the "relative time" in a relativistic theory.

I. INTRODUCTION

In recent years there has been renewed interest in the relativistic two-body equations of Salpeter and Bethe.[1] In the absence of a theory of the strong interactions, these off-shell equations provide at least a means for performing dynamical calculations within a manifestly covariant framework. A complete treatment of the field-theoretical problem would require us to represent the kernel of the integral equation as the sum of all irreducible Feynman graphs contributing to the elastic two-body amplitude. The solution of the complete problem is evidently hopeless at the present time. In fact, even in the "ladder" approximation, in which we retain only the lowest order term in the expansion of the kernel in powers of G^2 (the square of the coupling constant), the equation has for some time been considered intractable, the difficulties being largeley due to the presence of a degree of freedom in the equation, the "relative time," which has no analog in nonrelativistic quantum mechanics. The numerical program initiated by C. Schwartz[2] demonstrated however that the (Wick rotated) BS equation, in the ladder approximation, could be solved accurately by conventional numerical techniques. This led to renewed interest in the BS (Bethe-Salpeter) equation as a computational tool, a number of calculations having extended since then the bound state calculation by C. Schwartz to the elastic scattering region[3] and as far as the second inelastic threshold.[4]

In this paper we continue this numerical program by making a quantitative investigation of the "pion-nucleon"[5] bound states in conventional pseudoscalar meson theory, the dynamical framework being provided

by the BS equation in the ladder approximation (see Fig. 1). This problem has received little attention within this particular framework, although it has been extensively studied by the dispersion theoretic techniques of S-matrix theory, in which it has served as a prototype of so-called "bootstrap" calculations. In the present calculation we consider the 3 - 3 resonance (which we refer to as the N^*) and the nucleon as dynamical states of the πN system. Since we restrict ourselves to the bound state problem, we are left with the nucleon as the only dynamical state of immediate physical interest. If the usual arguments given within a dispersion theoretic framework[6] should serve as an indication, we expect the N^* to contribute here the dominant force. However, the inclusion of the N^* exchange would force us to introduce a cutoff right from the start. Since it was the original aim of this paper, if possible, to introduce no arbitrary parameters into the calculation, we have considered here only the nucleon exchange force.[7] This choice of interaction still leads to a marginally singular integral equation. Unfortunately standard mathematics has little to say about the existence of solutions and the nature of the eigenvalue spectrum of such singular equations. This has led to repeated speculation that these equations may either have no solutions at all, or that their eigenvalue spectrum may be continuous. An explicit example is provided by the nucleon-nucleon BS equation,[8] whose eigenvalue spectrum was found to be continuous unless the integral equation was supplemented with additional boundary conditions not already contained in it. In this paper we are

faced with a similar situation. It is for this reason that we denote a substantial part of this paper to a detailed study of the boundary conditions to be imposed on the solutions. We find that the behavior of the BS wave function near the light cone is critically dependent on the strength of the potential, and that a proper treatment of these boundary conditions is imperative for our numerical calculation to be successful. The problem of giving a proper treatment of the equation near the light cone has also been encountered in connection with the nucleon-nucleon problem, and has been studied at great length in a ϕ^4 field theory.[9] In our case the discussion of this problem is however considerably more complicated because of the presence of spin.

Because of the difficulty of solving the BS equation even in its simplest form, the ladder approximation, early efforts have been directed toward obtaining approximate solutions by introducing an instantaneous interaction into the originally covariant equation. This procedure led to a time-independent, three dimensional equation. However, the validity of these approximations is particularly questionable if we are dealing with a marginally singular equation, since they lead quite often to a supersingular (as in the present case) or even a nonsingular equation (as in the N - N problem). In either case, the original character of the BS equation has been sufficiently changed so that no correspondence need be expected between the eigenvalue spectrum of the original equation and its approximation. In this paper we consider two such time-independent approximations as applied to the pion-nucleon BS equation and discuss in particular the "static" and "nonrelativistic" limits. We examine in this connection the role of the "relative time" in the BS equation, or

for that matter, in a relativistic theory in general. In particular we will illustrate our observations by presenting a brief discussion of the Lee model from a somewhat unconventional point of view. It is hoped that the point of view taken in this paper will help clarify some of these important questions.

The subject material has been arranged as follows: In Sec. II we motivate field-theoretically the precise form of the BS equation of interest, defining all the relevant quantities. Section III is devoted to a study of the asymptotic behavior of the solutions to the Wick rotated integral equation. In Sec. IV we present some of the mathematical results which will be needed later on. In particular we construct a new set of spinor spherical harmonics belonging to the irreducible representations $\left(\frac{n \pm 1}{2}, \frac{n}{2}\right)$ and $\left(\frac{n}{2}, \frac{n \pm 1}{2}\right)$ of the four-dimensional rotation group. The main properties of these functions are discussed and a number of useful formulae are derived. In Sec. V we discuss in detail the behavior of the solutions near the origin of the (Wick rotated) Euclidean space, by studying the BS equation in its differential form. No attempt is made in this section to select among the "regular" and "irregular" solutions. If a physical interpretation of the BS amplitude is to be avoided, the criterion for such a selection must come directly from the integral equation itself. This is discussed in Sec. VI. The numerical techniques and results of our calculation are presented in Secs. VII and VIII. Finally, in Sec. IX we turn our attention to some additional topics of particular interest. We discuss the role of the "relative time"

in a relativistic theory and consider in this connection the "static" and "nonrelativistic" limit of the equation, as well as presenting some numerical results. A number of additional topics are left for the Appendices, where we present, in addition to some mathematical details relating to the conventional spinor spherical harmonics, a proof of elastic unitarity; the group theoretical results which we use in Sec. IV; some details on the formal structure of the $E = 0$ solutions; a discussion of the connection between the covariant and old-fashioned perturbation theory; and a discussion of the $N\theta$ and $V\theta$ sectors of the Lee model from a somewhat unconventional point of view, but in line with our work in this paper.

We follow in general the notation of Ref. 3. Three-vectors are represented by bold-face type, and natural units $(\hbar = c = 1)$ are used throughout this work.

II. THE BETHE-SALPETER EQUATION

We begin by motivating the precise form of the BS equation from field-theoretic considerations. Although there exists no real need to make this explicit connection with field theory, it will allow us to use the language of Feynman diagrams whenever this seems desirable, in addition to providing us with an unambiguous definition of all the relevant quantities. Since we are concerned only with the bound-state problem, it will be particularly convenient to work with the relativistic two-body wave function for the "pion" and "nucleon"[5] in a state $|\beta\rangle$, as defined[10] by

$$\Psi^i(x_1, x_2) = \langle \underset{\sim}{0}| \, T\big(\psi_H(x_1)\, \phi_H{}^i(x_2)\big)|\beta\rangle \, .$$

Here $\phi_H{}^i(x)$ and $\psi_H(x)$ are the Heisenberg fields of the pion and the nucleon respectively, $|\underset{\sim}{0}\rangle$ is the "physical" vacuum, and T is the time ordering operator. $\psi_H(x)$ is an eight-component object in the combined spin and isotopic spin space of the nucleon. Denoting by $\phi^i(x)$ and $\psi(x)$ the pion and nucleon fields in the "interaction representation," we have the equivalent definition[11]

$$\Psi^i(x_1, x_2) = \langle 0| \, T\big(\psi(x_1)\, \phi^i(x_2)\, S\big)|\, \underset{\sim}{k}_1, \underset{\sim}{k}_2\rangle \qquad (2.1)$$

for the relativistic two-body wave function in the scattering region. Here S is the scattering operator, $|0\rangle$ is the "bare" vacuum state, and $|\underset{\sim}{k}_1, \underset{\sim}{k}_2\rangle$ is the state of a free nucleon and pion of momentum $\underset{\sim}{k}_1$

and $\underset{\sim}{k}_2$ respectively.[12] (We have suppressed all spin and isotopic spin quantum numbers.) We take the interaction Hamiltonian density to have the conventional form

$$\mathcal{H}(x) = \sum_j i \frac{G}{2} [\bar{\psi}(x), \gamma_5 \tau^j \psi(x)] \phi^j(x) , \tag{2.2}$$

where τ^i are the usual Pauli matrices, and G is referred to as the "pseudoscalar coupling constant." Experimentally $G^2/4\pi \approx 14$. By inserting the familiar Dyson expansion of the S-matrix[13] into the definition (2.1) we arrive in a straightforward manner at an integral equation for $\Psi(x_1, x_2)$ in the scattering region

$$\begin{aligned} \Psi^i(x_1,x_2) &= \Psi_0{}^i(x_1,x_2) \\ &+ \frac{G^2}{(4\pi)^2} \sum_j \int dx_1' \, dx_2' \, G_1(x_1 - x_1') \, G_2(x_2 - x_2') \, I(x_1' - x_2') \\ &\times [2P^{ij}(\tfrac{3}{2}) - P^{ij}(\tfrac{1}{2})] \, \Psi^j(x_2', x_1'), \quad x_1 = (\underset{\sim}{r}_1, t_1), \quad x_2 = (\underset{\sim}{r}_2, t_2), \end{aligned} \tag{2.3}$$

where we have restricted ourselves to the "ladder" approximation (see Fig. 1). Here $G_1(x)$ and $G_2(x)$ are the one-particle Green's functions for the nucleon and pion respectively

$$\begin{aligned} G_1(x) &= [m_1 + i \, \gamma\cdot\partial] \, \Delta(x;m_1), \quad G_2(x) = \Delta(x;m_2) \\ \Delta(x;m) &= -i \int \frac{d^4p}{(2\pi)^4} \frac{e^{ip\cdot x}}{p^2 + m^2 - i\epsilon} = \left(\frac{m}{2\pi}\right)^2 \frac{K_1(m|x|)}{m|x|} \end{aligned} \tag{2.4}$$

where $|x| = (\underset{\sim}{r}^2 - t^2)^{\frac{1}{2}}$ and $K_1(z)$ is the first order modified Bessel function.[14] $I(x)$ is the nucleon exchange potential corresponding to the Born diagram of Fig. 2 and is given by

$$I(x) = 4M^2 \, [M - i\,\gamma\cdot\partial] \, \frac{K_1(M|x|)}{M|x|} \; ; \tag{2.5}$$

$\Psi_0(x_1,x_2)$ is the wave function for the free pion and nucleon

$$\Psi_0(x_1,x_2) = U(\underset{\sim}{k}_1)\, e^{ik_1\cdot x_1} \; e^{ik_2\cdot x_2} \tag{2.6}$$

where

$$k_i = (\underset{\sim}{k}_i, \omega_i), \; \omega_i = (\underset{\sim}{k}_i^{\,2} + m_i^{\,2})^{\frac{1}{2}}$$

and $U_{\alpha\sigma}(\underset{\sim}{k}) \equiv U_\alpha(\underset{\sim}{k},\sigma)$ is the nucleon Dirac spinor with the covariant normalization $\overline{U}(\underset{\sim}{k})\, U(\underset{\sim}{k}) = 1$ $\left(\overline{U}(\underset{\sim}{k},\sigma) \equiv U^\dagger(\underset{\sim}{k},\sigma)\,\gamma^0\right)$. The matrices $P(I)_{\alpha\beta}^{\;\;ij}$ are the usual isotopic spin projection operators

$$P(\tfrac{1}{2}) = \frac{1 - \vec{t}\cdot\vec{\tau}}{3} \; ,$$

$$P(\tfrac{3}{2}) = \frac{2 + \vec{t}\cdot\vec{\tau}}{3} \; ,$$

where $\vec{\tau}$ and $\vec{t}$ are the isotopic spin operators of the nucleon and meson respectively. (In the Cartesian basis, $t_{jk}^{\;\;i} = -i\epsilon_{ijk}$). Using the fact

that $P(I)\,P(I') = P(I)\,\delta_{II'}$ we arrive immediately at the integral equation for the two-body wave function in a state of definite total isotopic spin I:

$$\Psi^I(x_1,x_2) = \Psi_0{}^I(x_1,x_2) + \lambda^I \int dx_1'\, dx_2'\, G_1(x_1 - x_1')\, G_2(x_2 - x_2')\, I(x_1' - x_2')\, \Psi^I(x_2',x_1'). \tag{2.7}$$

Here

$$\lambda^I = \eta(I)\,\frac{G^2}{(4\pi)^2}\,, \quad \eta(\tfrac{1}{2}) = -1, \quad \eta(\tfrac{3}{2}) = 2\,. \tag{2.8}$$

We have omitted all other isotopic spin labels, $\Psi^I(x_1,x_2)$ being only a function of the total isotopic spin I. In the future we omit this label as well. The sign preceding λ^I in (2.7) is of course prescribed by our "derivation." Moreover, the requirement of elastic unitarity provides here an additional check, as we point out in Appendix C.

We proceed now to eliminate one of the integration variables in Eq. (2.7) by making use of the translational invariance of the theory. To this end we consider the canonical transformations

$$\begin{aligned} P &= p_1 + p_2 \quad ; \quad X = \mu_1 x_1 + \mu_2 x_2, \\ p &= \mu_2 p_1 - \mu_1 p_2 \equiv (\underset{\sim}{p}, p^0) \; ; \quad x = x_1 - x_2 \equiv (\underset{\sim}{r}, t), \end{aligned} \tag{2.9}$$

where μ_1 and μ_2 are constants subject to the condition

$$\mu_1 + \mu_2 = 1 . \tag{2.10}$$

Since we are dealing with an exchange potential it turns out to be convenient to make the choice $\mu_1 = \mu_2 = \frac{1}{2}$.[15] This is the choice made throughout this work, unless explicitly stated otherwise.

Translational invariance allows us to write

$$\Psi(x_1, x_2) = \psi(x)\, e^{iK\cdot X} .$$

Substitution into (2.7) leads then to the following integral equation for $\psi(x)$ (we now label the wave function by the momentum of the incident wave):

$$\psi_{\underset{\sim}{k}}(x) = U(\underset{\sim}{k})\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{r} - i\omega_{12}t/2} - \lambda \int d^4x' G(x - x')\, I(x')\, \psi_{\underset{\sim}{k}}(-x'), \tag{2.11}$$

where $\omega_{12} = \omega_1 - \omega_2$ and $G(x)$ is the two-particle Green's function,

$$G(x) = - \int d^4Z\, G_1(\tfrac{1}{2}x - Z)\, G_2(\tfrac{1}{2}x + Z)\, e^{iK\cdot Z} .$$

Performing the integration we obtain

$$G(x) = [m_1 + i\,\gamma\cdot\partial + \gamma^0 \tfrac{E}{2}]\, G_0(x) \tag{2.12}$$

in the c.m. system of the pion and nucleon, where $E = \omega_1 + \omega_2$ and

$$G_0(x) = \int \frac{d^4q}{(2\pi)^4} \frac{e^{iq\cdot x}}{[\underset{\sim}{q}^2 - (q^0 + \frac{E}{2})^2 + m_1^2 - i\epsilon][\underset{\sim}{q}^2 - (q^0 - \frac{E}{2})^2 + m_2^2 - i\epsilon]} \tag{2.13}$$

is of course just the two-particle Green's function for two spinless particles. Because of the poles of the integrand it is desirable for the purpose of later discussions to cast the integral (2.13) into a different form. Making use of the familiar Feynman parameterization technique we arrive at

$$G_0(x) = \int_{-\omega_1}^{\omega_2} \frac{d\beta}{E} \; e^{-i(\beta + \frac{1}{2}\omega_{12})t} \int \frac{d^4p}{(2\pi)^4} \frac{e^{ip\cdot x}}{[p^2 + \beta^2 - \underset{\sim}{k}^2 - i\epsilon]} ,$$

where $\underset{\sim}{k}^2$, ω_1 and ω_2 are related to the total c.m. energy E as follows:

$$\begin{aligned} \underset{\sim}{k}^2 &= \frac{1}{4E^2} [E^2 - (m_1 + m_2)^2] [E^2 - (m_1 - m_2)^2] \\ \omega_1 &= \frac{E^2 + (m_1^2 - m_2^2)}{2E} , \quad \omega_2 = \frac{E^2 - (m_1^2 - m_2^2)}{2E} \end{aligned} \tag{2.14}$$

and

$$\omega_{12} = \omega_1 - \omega_2 = \frac{m_1^2 - m_2^2}{E} . \tag{2.15}$$

The four-dimensional integral in momentum space has been evaluated in Ref. 3 to be proportional to the zeroth order modified Bessel function K_0 . Hence,

$$G_0(x - x') = \frac{i}{8\pi^2} \int_{-\omega_1}^{\omega_2} \frac{d\beta}{E} e^{-i(\beta+\frac{1}{2}\omega_{12})(t-t')} K_0[(\beta^2 - \underset{\sim}{k}^2)^{\frac{1}{2}} |x - x'|] \tag{2.16}$$

where $-\frac{1}{2}\pi \leqslant \arg [(\beta^2 - \underset{\sim}{k}^2)^{\frac{1}{2}} |x - x'|] \leqslant \frac{1}{2}\pi$, as explained in Ref. 3. With the aid of

$$[m_1 - i\,\gamma\cdot\partial - \gamma^0 \frac{E}{2}] [-\Box^2 + i\,E\,\partial_0 + m_2^2 - \frac{E^2}{4}] G(x) = \delta^4(x)$$

$$\Box^2 = \underset{\sim}{\nabla}^2 - \partial_0^2 , \quad \gamma\cdot\partial = \underset{\sim}{\gamma}\cdot\underset{\sim}{\nabla} + \gamma^0\partial_0$$

we may rewrite Eq. (2.11) as the differential equation

$$[m_1 - i\,\gamma\cdot\partial - \gamma^0 \frac{E}{2}] [\Box^2 - i\,E\,\partial_0 + \frac{E^2}{4} - m_2^2] \psi(x) = \lambda\, I(x)\, \psi(-x), \tag{2.17}$$

supplemented with suitable boundary conditions. It is this equation which we solve in the calculational part of this paper.

Finally we would like to remark that unless additional information is required which is not already contained in the integral equation (2.11), we do not need to give a physical interpretation to the two-body wave function (2.1), but may regard it merely as one possible way of formulating the given mathematical problem. Thus, with the definition of the scattering amplitude,

$$\langle \underset{\sim}{k}_1' \sigma'; \underset{\sim}{k}_2' | T | \underset{\sim}{k}_1 \sigma, \underset{\sim}{k}_2 \rangle = \delta^4(K' - K) \frac{E}{2\pi\, \omega_1 \omega_2} f_{\sigma'\sigma}(\underset{\sim}{k}', \underset{\sim}{k}) ,$$

where T is related to the S-matrix in the usual way

$$S = 1 - i\,T\,,$$

we easily establish the connection between $f(\underset{\sim}{k}', \underset{\sim}{k})$ and the solution $\psi_{\underset{\sim}{k}}(x)$ to the integral equation (2.11):

$$f(\underset{\sim}{k}',\underset{\sim}{k}) = \frac{i\,\lambda\,m_1}{4\pi\,E}\,\bar{U}(\underset{\sim}{k}')\int d^4x\quad e^{-ik'\cdot x}\;\;I(x)\,\psi_{\underset{\sim}{k}}(-x), \tag{2.18}$$

the numerical factors being fixed by simply looking at the Born term. For later reference we give here also the momentum space version of (2.11) and (2.18). Thus, denoting by $\tilde{\psi}(p)$, $\tilde{G}(p)$ and $\tilde{I}(p)$ the Fourier transforms[16] of $\psi(x)$, $G(x)$ and $I(x)$, we obtain, from (2.11),

$$\tilde{\psi}_{\underset{\sim}{k}}(p) = (2\pi)^4\,\delta^4(p - k)\,U(\underset{\sim}{k}) - \lambda\int\frac{d^4p'}{(2\pi)^4}\;\tilde{G}(p)\,\tilde{I}(p + p')\,\tilde{\psi}_{\underset{\sim}{k}}(p'). \tag{2.19}$$

The scattering amplitude (2.18) is now related to $\tilde{\psi}_{\underset{\sim}{k}}(p)$ by

$$f(\underset{\sim}{k}',\underset{\sim}{k}) = \frac{i\,\lambda\,m_1}{4\pi\,E}\;\bar{U}(\underset{\sim}{k}')\int\frac{d^4p}{(2\pi)^4}\;\;\tilde{I}(k'+ p)\,\tilde{\psi}_{\underset{\sim}{k}}(p). \tag{2.20}$$

III. ASYMPTOTIC BOUNDARY CONDITIONS

Since it is very difficult to solve numerically the eigenvalue problem in the form of Eq. (2.17), we should like to consider instead the corresponding eigenvalue problem for $\varphi(\underset{\sim}{r}, \tau)$, the analytic continuation of $\psi(\underset{\sim}{r},t)$ to the imaginary time axis: $\varphi(\underset{\sim}{r},\tau) = \psi(\underset{\sim}{r}, -i\tau)$. Assuming the solution to be an analytic function of t, we arrive at the differential equation

$$[m_1 - i\,\gamma\cdot\partial - \gamma^o \frac{E}{2}]\,[\Box^2 + E\,\partial_4 + \frac{E}{4}^2 - m_2^{\;2}]\,\varphi(x) = \lambda\, V(\underset{\sim}{x})\,\varphi(-x) \tag{3.1}$$

for $\varphi(r,\tau)$, where

$$V(x) = I(\underset{\sim}{r}, -i\tau) = \frac{4M^2}{R}\,[K_1(MR) + i\,\gamma\cdot\frac{x}{R}\,K_2(MR)]\,. \tag{3.2}$$

Here $R = (\underset{\sim}{r}^2 + \tau^2)^{\frac{1}{2}}$, and $K_n(MR)$ is the $\underline{n}$th order modified Bessel function with the asymptotic property[14]

$$K_n(z) \to (\frac{\pi}{2z})^{\frac{1}{2}}\; e^{-z} \quad \text{as} \quad |z| \to \infty\,. \tag{3.3}$$

We have introduced the notation

$$(x_\alpha) = (\underset{\sim}{r},\tau),\; (\partial_\alpha) = (\underset{\sim}{\nabla},\partial_4),\; \partial_4 = \frac{\partial}{\partial\tau},\quad \gamma_4 = i\,\gamma^o, \tag{3.4}$$

scalar products being now defined in terms of an Euclidean metric:

$$\gamma\cdot\partial = \underset{\sim}{\gamma}\cdot\underset{\sim}{\nabla} + \gamma_4\,\partial_4, \quad \Box^2 = \underset{\sim}{\nabla}^2 + \partial_4^{\,2}, \quad x^2 = \underset{\sim}{r}^2 + \tau^2 . \tag{3.5}$$

The corresponding analytic continuation of the integral equation (2.11) has already been discussed in the literature.[3,17] We do not repeat here the arguments except to state, that for suitable restrictions (to be given at the end of this section) on the c.m. energy E, we may carry out this analytic continuation by "rotating" simultaneously[18] the t and t' variables in Eq. (2.11) through an angle $\frac{1}{2}\pi$ in the 2nd and 4th quadrants of the complex t and t' planes: $t = \tau\exp(-i\phi)$, $t' = \tau'\exp(-i\phi)$, $0 \leqslant \phi \leqslant \frac{1}{2}\pi$. Denoting the analytically continued solution by $\phi_{\underset{\sim}{k}}(x)$, we arrive thus at the integral equation

$$\phi_{\underset{\sim}{k}}(x) = U(\underset{\sim}{k})\; e^{i\underset{\sim}{k}\cdot\underset{\sim}{r} - \frac{1}{2}\omega_{12}\tau} - \lambda\int d^4x'\; H(x - x')\; V(x')\; \phi_{\underset{\sim}{k}}(-x') \tag{3.6}$$

where $d^4x = d^3\underset{\sim}{r}\; d\tau$, and

$$\begin{aligned} H(x) &= [m_1 + i\,\gamma\cdot\partial + \gamma^0\,\frac{E}{2}]\; H_0(x) \\ H_0(\underset{\sim}{r},\tau) &= -\,i\,G_0(\underset{\sim}{r}, -\,i\,\tau) . \end{aligned} \tag{3.7}$$

An explicit expression for $H_0(\underset{\sim}{r},\tau)$ may be obtained directly from (2.16):

$$H_0(x) = \frac{1}{8\pi^2}\int_{-\omega_1}^{\omega_2}\frac{d\beta}{E}\; e^{-(\beta+\frac{1}{2}\omega_{12})\tau}\; K_0[(\beta^2 - \underset{\sim}{k}^2)^{\frac{1}{2}}\,R] . \tag{3.8}$$

In the bound-state region $\phi(x)$ will have poles at (one hopes) a discrete set of values of E (for a given value of λ). At these values of E, the residue of $\phi_{\underset{\sim}{k}}(x)$ at the pole, $\varphi(x)$, satisfies the homogeneous integral equation

$$\varphi(x) = -\lambda \int d^4x' \, H(x - x') \, V(x') \, \varphi(-x') \, . \qquad (3.9)$$

The asymptotic boundary conditions to be imposed on the solutions to (3.1) are those of $\phi_{\underset{\sim}{k}}(x)$ (in the scattering region) and $\varphi(x)$ (in the bound state region) as deduced from the integral equations (3.6) and (3.9). Although the asymptotic properties of Bethe-Salpeter amplitudes have already been discussed in the literature,[3,17] we review here the results in somewhat greater detail with particular emphasis on the unequal mass kinematics of the problem.

Since the asymptotic form of the solution is determined by the Green's function, we begin by considering the asymptotic properties of the integral (3.8). This form is particularly suited for the intended discussion in the bound state region where $(\beta^2 - \underset{\sim}{k}^2) > 0$ for β within the integration interval (even in the unequal mass case!). In the scattering region it is however preferable to consider the alternative expression

$$H_0(x) = \frac{1}{8\pi E} \frac{e^{ikr}}{r} e^{-\frac{1}{2}\omega_{12}\tau} - \frac{1}{8\pi^2} \left(\int_{-\infty}^{-\omega_1} + \int_{\omega_2}^{\infty} \right) \frac{d\beta}{E} e^{-(\beta + \frac{1}{2}\omega_{12})\tau} \times K_0[(\beta^2 - \underset{\sim}{k}^2)^{\frac{1}{2}} R] \qquad (3.10)$$

as derived in Ref. 3. Making use of the asymptotic property (3.3) of the modified Bessel functions, we see that we will be interested in studying in general the asymptotic form of the integral

$$I_a^{\,b}(x) = \int_a^b d\beta \left(\frac{\pi}{2(\beta^2 - \underset{\sim}{k}^2)^{\frac{1}{2}} R} \right)^{\frac{1}{2}} e^{-g(\beta)R} \tag{3.11}$$

where by assumption $(\beta^2 - \underset{\sim}{k}^2) > 0$ for $a \leqslant \beta \leqslant b$, and

$$g(\beta) = \beta \cos\theta + (\beta^2 - \underset{\sim}{k}^2)^{\frac{1}{2}} , \quad \cos\theta = \tau/R . \tag{3.12}$$

The asymptotic properties of integrals of the form (3.11) may be discussed by the "method of steepest descent."[19] Let β_0 be the value of β in the interval (a,b) for which $g(\beta)$ takes on its smallest value (not necessarily a minimum). We distinguish three possibilities:

(A) $\beta_0 = a$, and $g'(a) \neq 0$; in this case $g'(a) > 0$ and

$$I_a^{\,b}(x) \sim \left(\frac{\pi}{2(a^2 - \underset{\sim}{k}^2)^{\frac{1}{2}} R} \right)^{\frac{1}{2}} \frac{e^{-g(a)R}}{g'(a)\, R} .$$

In particular:

$$\frac{1}{8\pi^2 E} e^{-\frac{1}{2}\omega_{12}\tau} I_{-\omega_1}^{\,b}(x) \sim - \frac{1}{8\pi E} \left(\frac{m_1}{2\pi R}\right)^{\frac{1}{2}} \frac{e^{\left(\frac{E}{2}\cos\theta - m_1\right)R}}{\left(\frac{E^2 + (m_1^{\,2} - m_2^{\,2})}{2E} - m_1 \cos\theta \right) R} . \tag{3.13}$$

(B) $\beta_0 = b$, and $g'(b) \neq 0$; in that case $g'(b) < 0$ and

$$I_a^{\,b}(x) \sim -\left(\frac{\pi}{2(b^2-\underset{\sim}{k}^2)^{\frac{1}{2}}\,R}\right)^{\frac{1}{2}} \frac{e^{-g(b)R}}{g'(b)R} \; .$$

In particular

$$\frac{1}{8\pi^2 E}\; e^{-\frac{1}{2}\omega_{12}\tau}\, I_a^{\,\omega_2}(x) \sim -\frac{1}{8\pi E}\left(\frac{m_2}{2\pi R}\right)^{\frac{1}{2}} \frac{e^{-\left(\frac{E}{2}\cos\theta + m_2\right)R}}{\left(\frac{E^2-(m_1^{\,2}-m_2^{\,2})}{2E} + m_2\cos\theta\right)R} \; . \tag{3.14}$$

(C) $g(\beta)$ has a true minimum for $\beta = \beta_0$ within the integration interval (a,b), including the endpoints; in this case $g'(\beta_0) = 0$, and

$$\beta_0 = -\sqrt{-\underset{\sim}{k}^2}\;\operatorname{ctn}\theta \; . \tag{3.15}$$

We arrive this time at the following asymptotic form for $I_a^{\,b}(x)$:

$$\begin{aligned} I_a^{\,b}(x) &\to \left(\frac{\pi^2}{(\beta_0^{\,2}-\underset{\sim}{k}^2)^{\frac{1}{2}}R}\right)^{\frac{1}{2}} \frac{e^{-g(\beta_0)R}}{\sqrt{g''(\beta_0)\,R}} \\ &= \pi\,\frac{e^{-\sqrt{-\underset{\sim}{k}^2}\,R\sin\theta}}{R\sin\theta} \end{aligned} \tag{3.16}$$

the result being independent of the endpoints!

Having discussed the asymptotic properties of $I_a^{\,b}(x)$, we turn now briefly to the problem of interest. Because of the unequal mass kinematics of the problem, it is helpful to distinguish four separate regions in the c.m. energy E (we take $m_1 \geqslant m_2$):

(1) $\underline{0 \leqslant E \leqslant m_1 - m_2}$

Here (see (2.14)) $\omega_1 > 0$, $\omega_2 < 0$, $\underset{\sim}{k}^2 > 0$; From (3.8) we conclude that $\beta_0 = \omega_2$, independent of the value of $\cos\theta$, so that $H_0(x)$ has the asymptotic behavior (3.14) for $-1 \leqslant \cos\theta \leqslant 1$.

(2) $\underline{m_1 - m_2 \leqslant E \leqslant (m_1^2 - m_2^2)^{\frac{1}{2}}}$

Here $\omega_1 > 0$, $\omega_2 < 0$, $\underset{\sim}{k}^2 < 0$; Considering again the integral (3.8) we easily see that $g(\beta)$ does have a true minimum within the integration interval for some values of $\cos\theta$. In fact, all three cases (A), (B) and (C) are here relevant, so that we may distinguish three regions in the $(E, \cos\theta)$ plane, the boundaries being determined by the conditions $g'(-\omega_1) = 0$ and $g'(\omega_2) = 0$:

(2a) $\omega_1/m_1 \leqslant \cos\theta \leqslant 1$; $H_0(x)$ has the asymptotic behavior (3.13);

(2b) $-1 \leqslant \cos\theta \leqslant -\omega_2/m_2$; $H_0(x)$ has the asymptotic behavior (3.14);

(2c) $-\omega_2/m_2 \leqslant \cos\theta \leqslant \omega_1/m_1$; $8\pi^2 e^{\frac{1}{2}\omega_{12}\tau} H_0(x)$ has the asymptotic behavior (3.16);

(3) $\underline{m_1^2 - m_2^2 \leqslant E \leqslant m_1 + m_2}$

Here $\omega_1 > 0$, $\omega_2 > 0$, $\underset{\sim}{k}^2 < 0$; The same conclusions as for the case (2) hold here.

(4) $\underline{E \geqslant m_1 + m_2}$

Here $\omega_1 > 0$, $\omega_2 > 0$, $\underset{\sim}{k}^2 > 0$; from (3.10) we easily deduce with the aid of the results (3.13), (3.14) and (3.16) the asymptotic form of $H_0(x)$:

$$H_0(x) = \frac{1}{8\pi E}\left[\frac{e^{ikr-\frac{1}{2}\omega_{12}\tau}}{r} - \left(\frac{m_1}{2\pi R}\right)^{\frac{1}{2}} \frac{e^{(\frac{E}{2}\cos\theta - m_1)R}}{(\omega_1 - m_1\cos\theta)R} - \left(\frac{m_2}{2\pi R}\right)^{\frac{1}{2}} \frac{e^{-(\frac{E}{2}\cos\theta + m_2)R}}{(\omega_2 + m_2\cos\theta)R}\right].$$

This asymptotic form was already obtained in Ref. 3. Again we may distinguish three domains in the $\cos\theta$ variable, whose boundaries may be read off directly from the above asymptotic expression. They evidently correspond to a) $m_1/\omega_1 \leqslant \cos\theta \leqslant 1$; b) $-1 \leqslant \cos\theta \leqslant -m_2/\omega_2$; and c) $-m_2/\omega_2 \leqslant \cos\theta \leqslant m_1/\omega_1$.

The above results are summarized in Fig. 3, the asymptotic form of the solutions being the same as that of $H_0(x)$. (We have exhibited only the exponential dependence of these asymptotic forms. To be specific we chose $m_1 \geqslant m_2$, corresponding to the physical case of interest.) Following the same type of reasoning as given in Ref. 3, we find then, with the aid of Fig. 3, that the integration in Eqs. (3.6) and (3.9) does indeed converge at infinity, provided that $E < 2m_2 + M$ and $E < 2m_1 + M$; that is, for the c.m. energy E below the first inelastic threshold. For $E \geqslant m_1 + m_2$, these restrictions imply of course the stability conditions $m_1 < m_2 + M$ for the "nucleon," and $m_2 < m_1 + M$ for the "pion."

Before concluding this discussion we should like to comment briefly on the negative energy spectrum of Eq. (3.1). Ignoring for the moment the question of boundary conditions, we readily verify that

if $\varphi(x)$ is a solution to Eq. (3.1) for $E = E_0$ ($E_0 > 0$ to be specific), then $\gamma_5 \varphi(-x)$ is also a solution for $E = E_0$, at the same value of the coupling constant. In fact, as we will see later on, the same statement can be made about the eigenvalue problem. This shows that the equation admits a positive and a negative energy spectrum whenever the equation has solutions at all. As will become evident later on, when we restate these observation for the partial-wave projections of $\varphi(x)$, this is precisely the statement of the familiar "MacDowell symmetry."[20] It arises here as a consequence of the transformation properties of the equation under space-time inversion, but can be shown to follow also from more general principles.[21] Note that as a function of x, the solutions for $E = -E_0$ have the same asymptotic behavior as the solutions $\varphi(-x)$ for $E = E_0$, and are again acceptable solutions to the integral equation (3.9) with E_0 subject to the above stated restrictions.

IV. MATHEMATICAL PRELIMINARIES

A. Some Useful Formulae

We derive here some useful formulae relating to the four-dimensional spinor spherical harmonics defined in (A6). Specifically we are concerned with products of the form $Q\, Y_n^{\ell \pm M}(\hat{R})$, where Q is any of the operators τ, ∂_4, $(\underset{\sim}{\sigma}\cdot\underset{\sim}{r})$, and $(\underset{\sim}{\sigma}\cdot\underset{\sim}{\nabla})$. The evaluation of $\tau\, Y^{\ell \pm M}$ and $\partial_4\, Y_n^{\ell \pm M}$ has already been given in Ref. 2. We state here the result:[22]

$$\tau\, Y_n^{\ell \pm}(\hat{R}) = R\left[A_{n+1}^{\ell}\; Y_{n+1}^{\ell \pm}(\hat{R}) + A_n^{\ell}\; Y_{n-1}^{\ell \pm}(\hat{R})\right], \tag{4.1}$$

$$\partial_4\, Y_n^{\ell \pm}(\hat{R}) = A_{n+1}^{\ell}\, Y_{n+1}^{\ell \pm}(\hat{R})\left(\frac{d}{dR} - \frac{n}{R}\right) + A_n^{\ell}\, Y_{n-1}^{\ell \pm}(\hat{R})\left(\frac{d}{dR} + \frac{n+2}{R}\right), \tag{4.2}$$

where

$$A_n^{\ell} = \left[\frac{(n-\ell)\,(n+\ell+1)}{4n(n+1)}\right]^{\frac{1}{2}}. \tag{4.3}$$

The evaluation of $(\underset{\sim}{\sigma}\cdot\underset{\sim}{r})\, Y_n^{\ell \pm M}$ is also straightforward. With the aid of formulae (A2) and the recursion relations for the Gegenbauer polynomials $C_n^{\ell}(t)$,[23]

$$C_{n-\ell}^{\ell+1}(t) = \frac{\ell+1}{n+1}\left[C_{n-\ell}^{\ell+2}(t) - C_{n-\ell-2}^{\ell+2}(t)\right]$$

$$(1-t^2)\, C_{n-\ell}^{\ell+1}(t) = \frac{1}{4\ell(n+1)}\left[(n+\ell)(n+\ell+1)\, C_{n-\ell}^{\ell}(t) - (n-\ell+1)(n-\ell+2)\, C_{n-\ell+2}^{\ell}(t)\right]$$

we obtain, after some algebra,

$$-i(\underset{\sim}{\sigma}\cdot\underset{\sim}{r})\, Y_n^{\ell_\pm}(\hat{R}) = R\left[B_n^{\ell_\pm}\, Y_{n+1}^{(\ell\pm1)}{}_{\mp}(\hat{R}) - C_n^{\ell_\pm}\, Y_{n-1}^{(\ell\pm1)}{}_{\mp}(\hat{R})\right], \tag{4.4}$$

where

$$B_n^{\ell_+} = \left[\frac{(n+\ell+2)\,(n+\ell+3)}{4(n+1)\,(n+2)}\right]^{\frac{1}{2}},$$

$$B^{\ell_-} = \left[\frac{(n-\ell+1)\,(n-\ell+2)}{4(n+1)\,(n+2)}\right]^{\frac{1}{2}}, \tag{4.5a}$$

and

$$C_{n+1}^{(\ell\pm1)}{}_{\mp} = B_n^{\ell_\pm}. \tag{4.5b}$$

To evaluate $(\underset{\sim}{\sigma}\cdot\underset{\sim}{\nabla})\, Y_n^{\ell_\pm}$ we consider first the Fourier transform of this product using (4.4), and recover the desired result by taking the inverse Fourier transform. Thus, expanding a general spinor field in terms of the spherical harmonics (A6)

$$\varphi(x) = \sum_{\substack{\ell_\pm, M\\ n}} F_n^{\ell_\pm M}(R)\; Y_n^{\ell_\pm M}(\hat{R})$$

and making use of the plane wave expansion ($p\cdot x = \underset{\sim}{p}\cdot\underset{\sim}{r} + p_4\tau$)

$$\delta_{\alpha\beta}\, e^{ip\cdot x} = (2\pi)^2 \sum_{\substack{\ell_\pm, M\\ n}} i^n\, \frac{J_{n+1}(PR)}{PR}\; Y_n^{\ell_\pm M}(\hat{P})^*_\alpha\, Y_n^{\ell_\pm M}(\hat{R})_\beta\,,$$

we obtain with the aid of (4.4). (We omit the magnetic quantum number M labelling $F_n^{\ell\pm M}(R)$ and $Y_n^{\ell\pm M}(\hat{R})$.)

$$(\underset{\sim}{\sigma}\cdot\underset{\sim}{\nabla})\,\varphi(x) = (2\pi)^2 \sum_{\ell\pm M n} i^{n-1} \left[B_n^{\ell\pm} \int P^3 dP\, \tilde{F}_n^{\ell\pm}(P)\, \frac{J_{n+2}(PR)}{R}\, Y_{n+1}^{(\ell\pm 1)\mp}(\hat{R}) \right.$$

$$\left. + C_n^{\ell\pm} \int P^3 dP\, \tilde{F}_n^{\ell\pm}(P)\, \frac{J_n(PR)}{R}\, Y_{n-1}^{(\ell\pm 1)\mp}(\hat{R}) \right]$$

where $\tilde{F}_n^{\ell\pm}(P)$ is the Fourier transform of $F_n^{\ell\pm}(R)$:

$$F_n^{\ell\pm}(R) = (2\pi)^2\, i^n \int P^3 dP\, \tilde{F}_n^{\ell\pm}(P)\, \frac{J_{n+1}(PR)}{PR} .$$

Finally, making use of the recursion relations for the Bessel functions,

$$\frac{J_{n+2}(PR)}{R} = -\left(\frac{d}{dR} - \frac{n}{R}\right) \frac{J_{n+1}(PR)}{PR} ,$$

$$\frac{J_n(PR)}{R} = \left(\frac{d}{dR} + \frac{n+2}{R}\right) \frac{J_{n+1}(PR)}{PR} .$$

we arrive at the desired result:

$$-i(\underset{\sim}{\sigma}\cdot\underset{\sim}{\nabla})\, Y_n^{\ell\pm}(\hat{R}) = B_n^{\ell\pm}\, Y_{n+1}^{(\ell\pm 1)\mp}(\hat{R}) \left(\frac{d}{dR} - \frac{n}{R}\right) - C_n^{\ell\pm}\, Y_{n-1}^{(\ell\pm 1)\mp}(\hat{R}) \left(\frac{d}{dR} + \frac{n+2}{R}\right) \tag{4.6}$$

where the coefficients $B_n^{\ell\pm}$ and $C_n^{\ell\pm}$ have already been defined in (4.5). We note that we could have arrived of course in exactly the same manner at formula (4.2).

B. Basis Functions for the Representations $(\frac{n\pm1}{2}, \frac{n}{2})$ and $(\frac{n}{2}, \frac{n\pm1}{2})$ of O_4

Construction of Basis Functions

In this section we make use of the group theoretical results of Appendix B in order to construct explicitly the basis functions for the irreducible representations $(\frac{n\pm1}{2}, \frac{n}{2})$ and $(\frac{n}{2}, \frac{n\pm1}{2})$ of O_4, the four-dimensional rotation group. These functions are simple combinations of the four-dimensional spherical harmonics defined in (A6) and are particularly suited for our discussion of Eq. (3.1) at $E = 0$. As we indicate in Appendix B, we may obtain these basis functions by simply projecting out, with the aid of the projection operators defined in (B2), the desired components from the spherical harmonics (A6). These functions are thus labelled by the total three-dimensional angular momentum J, the magnetic quantum number M, and the index $n_\pm$ labelling the irreducible spaces. We will distinguish between the basis functions for the representations $(\frac{n\pm1}{2}, \frac{n}{2})$ and $(\frac{n}{2}, \frac{n\pm1}{2})$ by the presence and absence of a "tilde," respectively. Specifically,

$$\begin{aligned} \mathcal{Y}_{n_\pm}^{JM} &= N \; \mathcal{P}_{n_\pm} \, Y_n^{J\ell M} , \\ \tilde{\mathcal{Y}}_{n_\pm}^{JM} &= \tilde{N} \; \tilde{\mathcal{P}}_{n_\pm} \, Y_n^{J\ell M} , \end{aligned} \tag{4.7}$$

where $\mathcal{P}_{n_\pm}$ and $\tilde{\mathcal{P}}_{n_\pm}$ are the projection operators (B2) for the irreducible spaces of the representations $(\frac{n\pm1}{2}, \frac{n}{2})$ and $(\frac{n}{2}, \frac{n\pm1}{2})$ respectively, and N, $\tilde{N}$ are normalization constants. One may readily verify that it is immaterial whether we apply the projection operators

to $Y_n^{\ell\pm}$ or $Y_n^{(\ell+1)-}$ (same J). After proper normalization we end up with identical results, as it has to be. To be specific we apply the projection operators to $Y_n^{\ell+}$. Making use of

$$(\vec{\sigma}\cdot\vec{L})\, Y_n^{\ell+} = \ell\, Y_n^{\ell+} ,$$

$$(\vec{\sigma}\cdot\vec{L})\, Y_n^{\ell-} = -(\ell + 1)\, Y_n^{\ell-} ,$$

where the operators L_i are the usual generators of infinitesimal rotations in three-dimensional space, and noting the following properties of the coefficients A_n^{ℓ}, $B_n^{\ell\pm}$ defined in (4.3) and (4.5)

$$A_n^{\ell}\, B_{n-1}^{\ell+} + A_n^{\ell+1}\, C_n^{\ell+} = A_{n+1}^{\ell}\, C_{n+1}^{\ell+1} + A_{n+1}^{\ell+1}\, B_n^{\ell+}$$

$$= \frac{1}{2(n+1)}\, [(n - \ell)\,(n + \ell + 2)]^{\frac{1}{2}}$$

$$A_n^{\ell}\, C_{n-1}^{\ell+} = A_{n-1}^{\ell+1}\, C_n^{\ell+} , \quad A_{n+1}^{\ell}\, B_{n+1}^{\ell+} = A_{n+2}^{\ell+1}\, B_n^{\ell+}$$

we obtain, with the aid of the results of the preceding section,

$$(\underset{\sim}{\sigma}\cdot\underset{\sim}{A})\, Y_n^{\ell+} = \frac{1}{2}\left(\ell\, Y_n^{\ell+} + [(n - \ell)\,(n + \ell + 2)]^{\frac{1}{2}}\, Y_n^{(\ell+1)-}\right),$$

$$(\underset{\sim}{\sigma}\cdot\underset{\sim}{B})\, Y_n^{\ell+} = \frac{1}{2}\left(\ell\, Y_n^{\ell+} - [(n - \ell)\,(n + \ell + 2)]^{\frac{1}{2}}\, Y_n^{(\ell+1)-}\right),$$

where the operators A_i and B_i have been defined in (B.1). Making use of these results and the definitions (4.7) we obtain then finally

$$\mathcal{Y}_{n_+}^{JM} = \frac{i}{\sqrt{2(n+1)}} \left[(n+\ell+2)^{\frac{1}{2}} Y_n^{\ell+} + (n-\ell)^{\frac{1}{2}} Y_n^{(\ell+1)-} \right]$$

$$\mathcal{Y}_{n_-}^{JM} = \frac{1}{\sqrt{2(n+1)}} \left[(n-\ell)^{\frac{1}{2}} Y_n^{\ell+} - (n+\ell+2)^{\frac{1}{2}} Y_n^{(\ell+1)-} \right] \tag{4.8a}$$

as basis functions for the $(\frac{n\pm1}{2}, \frac{n}{2})$ representation, and

$$\tilde{\mathcal{Y}}_{n_+}^{JM} = \frac{1}{\sqrt{2(n+1)}} \left[(n+\ell+2)^{\frac{1}{2}} Y_n^{\ell+} - (n-\ell)^{\frac{1}{2}} Y_n^{(\ell+1)-} \right]$$

$$\tilde{\mathcal{Y}}_{n_-}^{JM} = \frac{1}{\sqrt{2(n+1)}} \left[(n-\ell)^{\frac{1}{2}} Y_n^{\ell+} + (n+\ell+2)^{\frac{1}{2}} Y_n^{(\ell+1)-} \right] \tag{4.8b}$$

as basis functions for the $(\frac{n}{2}, \frac{n\pm1}{2})$ representation. The quantum numbers J and M labelling $\mathcal{Y}_{n_\pm}^{JM}$ and $\tilde{\mathcal{Y}}_{n_\pm}^{JM}$ take on the values $1/2, 3/2, \cdots (n\pm\frac{1}{2})$ and $M = -J, -J+1, \cdots, J$; we are thus left with $(n+1)[(n\pm1)+1]$ basis functions for each of the irreducible spaces $(\frac{n\pm1}{2}, \frac{n}{2})$ and $(\frac{n}{2}, \frac{n\pm1}{2})$, in agreement with the dimensionality of these representations as given in Appendix B. That the functions (4.8) do indeed span an invariant manifold under the transformations of O_4 can be readily verified with the aid of the commutation relations

$$[\mathcal{A}_i, \mathcal{P}_{n_\pm}] = 0, \quad [\mathcal{B}_i \ \mathcal{P}_{n_\pm}] = 0 ;$$

$$[\tilde{\mathcal{A}}_i, \tilde{\mathcal{P}}_{n_\pm}] = 0, \quad [\tilde{\mathcal{B}}_i, \tilde{\mathcal{P}}_{n_\pm}] = 0 ;$$

Properties

We list here a number of properties of the functions (4.8) which we will need in our later work. To begin with we note that these functions are orthonormal:

$$\sum_{\alpha} \int d\Omega_R \; \mathcal{Y}_{n_\pm}^{JM}(\hat{R})^*_\alpha \; \mathcal{Y}_{n'_\pm}^{J'M'}(\hat{R})_\alpha = \delta_{JJ'}\,\delta_{MM'}\,\delta_{n_\pm n'_\pm}$$

$$\sum_{\alpha} \int d\Omega_R \; \tilde{\mathcal{Y}}_{n_\pm}^{JM}(\hat{R})^*_\alpha \; \tilde{\mathcal{Y}}_{n'_\pm}^{J'M'}(\hat{R})_\alpha = \delta_{JJ'}\,\delta_{MM'}\,\delta_{n_\pm n'_\pm} \tag{4.9}$$

where $d\Omega_R$ is the infinitesimal solid angle defined in (A.5). The addition theorem and completeness relation take the form

$$\sum_{\substack{JM\\ \pm}} \mathcal{Y}_{n_\pm}^{JM}(\hat{R})^*_\alpha \; \mathcal{Y}_{n_\pm}^{JM}(\hat{R}')_\beta = \frac{2(n+1)}{4\pi^2}\; C_n^{\,1}(\cos\gamma)\,\delta_{\alpha\beta}\;, \tag{4.10}$$

$$\sum_{\substack{JM\\ n_\pm}} \mathcal{Y}_{n_\pm}^{JM}(\hat{R})^*_\alpha \; \mathcal{Y}_{n_\pm}^{JM}(\hat{R}')_\beta = \delta_{\alpha\beta}\,\delta(\Omega_{R'} - \Omega_R)\;, \tag{4.11}$$

the functions $\tilde{\mathcal{Y}}_{n_\pm}^{JM}$ satisfying exactly the same relations. $C_n^{\,\ell}(\cos\gamma)$ are of course the Gegenbauer polynomials and $\cos\gamma$ is defined in (A11). Under space-time reflection:

$$\mathcal{Y}_{n_\pm}^{JM}(-\hat{R}) = (-)^n\; \mathcal{Y}_{n_\pm}^{JM}(\hat{R}), \quad \tilde{\mathcal{Y}}_{n_\pm}^{JM}(-\hat{R}) = (-)^n\; \tilde{\mathcal{Y}}_{n_\pm}^{JM}(\hat{R})\;. \tag{4.12}$$

From (4.8) we see that these functions are not all independent. In fact, $\mathcal{Y}_{n_\pm}^{JM}$ and $\tilde{\mathcal{Y}}_{n_\pm}^{JM}$ are related by space inversion

$$\tilde{\mathcal{Y}}^{JM}_{n_\pm}(\hat{R}) = (-)^{J-\frac{1}{2}} \; \mathcal{Y}^{JM}_{n_\pm}(\hat{R}_p), \quad \hat{R}_p = (-\underset{\sim}{x}, \tau)|R \; . \tag{4.13}$$

By construction we have, of course, (we list only a few representative examples)

$$\underset{\sim}{\mathcal{A}}^2 \mathcal{Y}^{JM}_{n_\pm} = \frac{(n \pm 1)}{2}\left[\frac{(n \pm 1)}{2} + 1\right] \mathcal{Y}^{JM}_{n_\pm}, \quad \underset{\sim}{\mathcal{B}}^2 \mathcal{Y}^{JM}_{n_\pm} = \frac{n}{2}\left(\frac{n}{2} + 1\right) \mathcal{Y}^{JM}_{n_\pm}$$

$$\underset{\sim}{\tilde{\mathcal{A}}}^2 \tilde{\mathcal{Y}}^{JM}_{n_\pm} = \frac{n}{2}\left(\frac{n}{2} + 1\right) \tilde{\mathcal{Y}}^{JM}_{n_\pm}, \quad \underset{\sim}{\tilde{\mathcal{B}}}^2 \tilde{\mathcal{Y}}^{JM}_{n_\pm} = \frac{(n \pm 1)}{2}\left[\frac{(n \pm 1)}{2} + 1\right] \tilde{\mathcal{Y}}^{JM}_{n_\pm}$$

the notation being the same as in Appendix B. Denoting by C the charge conjugation matrix

$$C_{\alpha\beta} = (-)^{\alpha+\frac{1}{2}} \; \delta_{\beta,-\alpha} = \exp(-i\pi\sigma_2)_{\alpha\beta} \tag{4.14}$$

with the property $C \, \underset{\sim}{\sigma} \, C^{-1} = - \underset{\sim}{\sigma}^*$, we have

$$C \, \mathcal{Y}^{JM}_{n_\pm}(\hat{R}) = (-)^{J-M} \; \mathcal{Y}^{J,-M}_{n_\pm}(\hat{R})^*, \quad C \, \tilde{\mathcal{Y}}^{JM}_{n_\pm}(\hat{R}) = (-)^{J-M} \; \tilde{\mathcal{Y}}^{J,-M}_{n_\pm}(\hat{R})^* \tag{4.15}$$

In those cases in which the quantum numbers J and M are of no relevance (as in the case of O_4 symmetry, for example) it is convenient to define the functions

$$X_{\hat{R}}^{n\pm}(\hat{R}')_{\alpha\beta} = 4\pi^2 \sum_{JM} \mathcal{Y}_{n_\pm}^{JM}(\hat{R}')_\alpha \, \mathcal{Y}_{n_\pm}^{JM}(\hat{R})^*_\beta \, ,$$

$$\tilde{X}_{\hat{R}}^{n\pm}(\hat{R}')_{\alpha\beta} = 4\pi^2 \sum_{JM} \tilde{\mathcal{Y}}_{n_\pm}^{JM}(\hat{R}')_\alpha \, \tilde{\mathcal{Y}}_{n_\pm}^{JM}(\hat{R})^*_\beta \, , \tag{4.16}$$

which play the same role in the four-dimensional case, as the functions $Z_{\hat{r}}^{\ell\pm}(\hat{r}')$ defined in (A3) do in the three-dimensional case. Since the functions $X_{\hat{R}}^{n\pm}(\hat{R}')$ and $\tilde{X}_{\hat{R}}^{n\pm}(\hat{R}')$ share a number of properties, it is useful to introduce the function $W_{\hat{R}}^{n\pm}(\hat{R}')$ to represent either one of them. We have then evidently

$$W_{\hat{R}}^{n\pm}(\hat{R}')^\dagger = W_{\hat{R}'}^{n\pm}(\hat{R}) \, .$$

From (4.9) we obtain the orthonormality relations

$$\int \frac{d\Omega_{R''}}{4\pi^2} \, W_{\hat{R}'}^{n\pm}(\hat{R}'')^\dagger \, W_{\hat{R}}^{n\pm}(\hat{R}'') = \delta_{n_\pm n'_\pm} \, W_{\hat{R}}^{n\pm}(\hat{R}') \tag{4.17}$$

and the addition theorem (4.10) can now be stated as follows:

$$\sum_\pm W_{\hat{R}}^{n\pm}(\hat{R}')_{\alpha\beta} = 2(n+1) \, C_n^{\,1}(\cos\gamma) \, \delta_{\alpha\beta} \, . \tag{4.18}$$

In the remaining part of this section we give a list of formulae which will be particularly useful in our later work. These formulae are the analog of those listed in (A2) for the three dimensional spherical

harmonics, and are readily derived with the aid of the results of Sec. IV-A. To this end we introduce the notation

$$(\sigma_\alpha) = (\underset{\sim}{\sigma}, \sigma_4), \quad (\tilde{\sigma}_\alpha) = (\underset{\sim}{\sigma}, -\sigma_4) \; ; \; \sigma_4 = i, \tag{4.19}$$

so that

$$\sigma \cdot x = \underset{\sim}{\sigma} \cdot \underset{\sim}{r} + i\tau, \quad \tilde{\sigma} \cdot x = \underset{\sim}{\sigma} \cdot \underset{\sim}{r} - i\tau \, , \tag{4.20}$$

for example. By considering an infinitesimal four-dimensional rotation we easily prove the transformation laws[24]

$$\begin{aligned} D^{(\frac{1}{2}0)}(R)\, \tilde{\sigma}_\alpha \; D^{(0\frac{1}{2})}(R^{-1}) &= \mathcal{R}_{\alpha\beta}^{-1}\, \tilde{\sigma}_\beta \\ D^{(0\frac{1}{2})}(R)\, \sigma_\alpha \; D^{(\frac{1}{2}0)}(R^{-1}) &= \mathcal{R}_{\alpha\beta}^{-1}\, \sigma_\beta \end{aligned} \tag{4.21}$$

where $D^{(\frac{1}{2}0)}(R)$ and $D^{(0\frac{1}{2})}(R)$ are the unitarity, irreducible representations of O_4^+ and $\mathcal{R} \sim D^{(\frac{1}{2}\frac{1}{2})}(R)$. The transformation properties (4.21) suggest that the explicit expressions for the products of the form $Q\, \mathcal{Y}_{n_\pm}^{JM}$ and $Q\, \tilde{\mathcal{Y}}_{n_\pm}^{JM}$ with Q representing such "rotationally invariant" operators as (4.20) may take a very simple form. This is indeed the case. We summarize our results below: (We suppress the quantum numbers J and M since they remain unaffected by the operations of Q; we use the notation $\hat{x} = (x_\alpha / R)$.)

$$
\begin{aligned}
-i(\sigma\cdot\hat{x})\ \mathcal{Y}_{n_\pm} &= \tilde{\mathcal{Y}}_{(n\pm1)_\mp} \\
i(\tilde{\sigma}\cdot\hat{x})\ \tilde{\mathcal{Y}}_{n_\pm} &= \mathcal{Y}_{(n\pm1)_\mp} \\
-i(\sigma\cdot\partial)\ \mathcal{Y}_{n_+} &= \tilde{\mathcal{Y}}_{(n+1)_-}\left(\frac{d}{dR} - \frac{n}{R}\right) \\
-i(\sigma\cdot\partial)\ \mathcal{Y}_{n_-} &= \tilde{\mathcal{Y}}_{(n-1)_+}\left(\frac{d}{dR} + \frac{n+2}{R}\right) \\
i(\tilde{\sigma}\cdot\partial)\ \tilde{\mathcal{Y}}_{n_+} &= \mathcal{Y}_{(n+1)_-}\left(\frac{d}{dR} - \frac{n}{R}\right) \\
i(\tilde{\sigma}\cdot\partial)\ \tilde{\mathcal{Y}}_{n_-} &= \mathcal{Y}_{(n-1)_+}\left(\frac{d}{dR} + \frac{n+2}{R}\right) .
\end{aligned}
\tag{4.22}
$$

Combining the above results we obtain the additional formulae

$$
\begin{aligned}
(\tilde{\sigma}\cdot\partial)\ (\sigma\cdot\hat{x})\ \mathcal{Y}_{n_+} &= \mathcal{Y}_{n_+}\left(\frac{d}{dR} + \frac{n+3}{R}\right) \\
(\tilde{\sigma}\cdot\partial)\ (\sigma\cdot\hat{x})\ \mathcal{Y}_{n_-} &= \mathcal{Y}_{n_-}\left(\frac{d}{dR} - \frac{n-1}{R}\right) \\
(\tilde{\sigma}\cdot\hat{x})\ (\sigma\cdot\partial)\ \mathcal{Y}_{n_+} &= \mathcal{Y}_{n_+}\left(\frac{d}{dR} - \frac{n}{R}\right) \\
(\tilde{\sigma}\cdot\hat{x})\ (\sigma\cdot\partial)\ \mathcal{Y}_{n_-} &= \mathcal{Y}_{n_-}\left(\frac{d}{dR} + \frac{n+2}{R}\right)
\end{aligned}
\tag{4.23}
$$

the corresponding formulae for $\tilde{\mathcal{Y}}_{n_\pm}{}^{JM}$ being obtained directly from here by the simple replacements $\sigma_\alpha \to \tilde{\sigma}_\alpha$ and $\mathcal{Y}_{n_\pm}{}^{JM} \to \tilde{\mathcal{Y}}_{n_\pm}{}^{JM}$ in (4.23).

With the aid of these results we may write down immediately the corresponding formulae for the spinor functions (4.16). We will not do so except for noting in particular the formula

$$(\sigma.\hat{x}')\ X_{\hat{R}}^{n_\pm}(\hat{R}')\ (\tilde{\sigma}.\hat{x}) = \tilde{X}_{\hat{R}}^{(n\pm 1)}{}_{\mp}(\hat{R}')\ , \qquad (4.24)$$

which we will need to make use of later on.

V. BOUNDARY CONDITIONS AT THE ORIGIN

In this section we deduce the behavior of the solutions to (3.1) near the origin of the four-dimensional Euclidean space. The results will include the so-called "regular" and "irregular" solutions. No attempt is made in this section to justify the selection of one or the other. Aside from physical considerations which we do not go into in this paper, the criterion for such a selection must come from a study of the integral equation (3.9) itself. This is left for Sec. VI where we find in fact, that even the integral equation (3.9) does not necessarily exclude the irregular solutions.

The study of the behavior of the solutions to (3.1) at $R = 0$ is rather involved, if we consider the general case in which $E \neq 0$. A great deal of insight into this question may be obtained, however, by considering first the problem at zero total c.m. energy E, where the equation simplifies considerably. If the behavior of the solutions at $R = 0$ should turn out to be independent of E, as one might expect on the basis of some familiar examples, it would be sufficient to consider this special case, although the generalization of the results to the case in which $E \neq 0$ will not be immediate, as we shall see.

A. $E = 0$

For $E = 0$ Eq. (3.1) is invariant with respect to the transformations of the group O_4. In order to make this symmetry most explicit it is convenient to work in the representation where

$$\gamma_\alpha = \begin{pmatrix} & -\tilde{\sigma}_\alpha \\ \sigma_\alpha & \end{pmatrix} , \quad \gamma_5 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

the matrices σ_α and $\tilde{\sigma}_\alpha$ having been defined already in (4.19). We call this the "Weyl representation." γ_α and γ_5 are matrices in the direct sum space of the representations $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ of O_4. The reason for working in this direct sum space is of course the required invariance of the theory under space inversion. Indeed, as (4.13) demonstrates explicitly, the operation of space inversion takes the representation $(\frac{n \pm 1}{2}, \frac{n}{2})$ into $(\frac{n}{2}, \frac{n \pm 1}{2})$, and vice versa. Denoting by $D^{\frac{1}{2}}(R)$ the representation matrices in the direct sum space $[(\frac{1}{2},0) \oplus (0,\frac{1}{2})]$, we have the following transformation law for the γ-matrices (see (4.21)):

$$D^{\frac{1}{2}}(R)\ \gamma_\alpha\ D^{\frac{1}{2}}(R^{-1}) = \mathcal{R}_{\alpha\beta}{}^{-1}\ \gamma_\beta .$$

It follows that for $E = 0$ Eq. (3.1) is invariant under four-dimensional rotations, so that its solutions may be classified by the eigenvalues of the complete set of commuting generators of O_4. We write the $E = 0$ equation in the form

$$\mathcal{D}_0(\partial;\lambda)\ \varphi(x) = 0 \tag{5.1}$$

where

$$\mathcal{D}_E(\partial;\lambda) = [m_1 - i\gamma\cdot\partial - \gamma^0 \tfrac{E}{2}]\ [\Box^2 + E\ \partial_4 + \tfrac{E^2}{4} - m_2{}^2] - \lambda\ V(x)\mathcal{P}(x) . \tag{5.2}$$

Here $V(x)$ is the "potential" (3.2) and $\mathcal{P}(x)$ is the operator defined by

$$\mathcal{P}(x) : \ (\underset{\sim}{r}, \ \tau) \to (-\underset{\sim}{r}, \ -\tau) \ .$$

Introducing

$$\mathcal{P}(\underset{\sim}{r}) : \ (\underset{\sim}{r}, \ \tau) \to (-\underset{\sim}{r}, \ \tau)$$

we define the "parity" operators in three- and four-space:

$$\Pi_3 \ = \ - \ i \ \gamma_4 \ \mathcal{P}(\underset{\sim}{r}), \quad \Pi_4 \ = \ \gamma_5 \ \mathcal{P}(x). \tag{5.3}$$

We note the following commutation relations:

$$[\Pi_{3,4}, \ \mathcal{D}_0(\partial;\lambda)] \ = \ 0, \quad [\Pi_3, \ \Pi_4] \ \neq \ 0 \ . \tag{5.4}$$

Because of these commutation relations we may choose the solutions of Eq. (3.1) to be also eigenfunctions of Π_3 or Π_4, but not of both. In particular, the eigenfunctions of Π_4 will have to be of the form

$$\varphi_{n_\pm}^{JM}(x) \ = \ \begin{pmatrix} F_{n_\pm}(R) \ \mathcal{Y}_{n_\pm}^{JM}(\hat{R}) \\ G_{n_\pm}(R) \ \tilde{\mathcal{Y}}_{(n\pm 1)_\mp}^{JM}(\hat{R}) \end{pmatrix} \tag{5.5}$$

with space time parity $(-)^n$. Substituting (5.5) into (5.1) and using the results of Sec. IV, we arrive at the following set of coupled

differential equations for the "radial" functions $F_{n_+}(R)$ and $G_{n_+}(R)$:[25]

$$m_1 K^n F_{n_+}(R) + \left(\frac{d}{dR} + \frac{n+3}{R}\right) K^{n+1} G_{n_+}(R) = (-)^n \lambda \frac{4M^2}{R} [K_1(MR) F_{n_+}(R) + K_2(MR) G_{n_+}(R)]$$

$$m_1 K^{n+1} G_{n_+}(R) + \left(\frac{d}{dR} - \frac{n}{R}\right) K^n F_{n_+}(R) = (-)^{n+1} \lambda \frac{4M^2}{R} [K_1(MR) G_{n_+}(R) + K_2(MR) F_{n_+}(R)] \tag{5.6}$$

where

$$K^n \equiv \frac{d^2}{dR^2} + \frac{3}{R}\frac{d}{dR} - \frac{n(n+2)}{R^2} - m_2^{\,2} . \tag{5.7}$$

Equations (5.6) exhibit explicitly the degeneracy of the solutions with respect to the quantum numbers J and M. In exactly the same manner we arrive at a set of coupled differential equations for $F_{n_-}(R)$ and $G_{n_-}(R)$. It is, however, simpler to note that if $\varphi_{n_+}^{JM}(x)$ is a solution of (5.1) with eigenvalue λ, then so is $\varphi_{(n+1)_-}^{JM} \propto \Pi_3 \varphi_{n_+}^{JM}$, on account of the commutation relations (5.4). That is, we may obtain the differential equations for $F_{n_-}(R)$ and $G_{n_-}(R)$ by simply noting that (we make a convenient choice of proportionality constant)

$$F_{(n\pm1)_\mp} = G_{n_\pm} . \tag{5.8}$$

We conclude therefore that for a given value of n, the combined set of solutions $\varphi_{n_+}^{JM}$ and $\varphi_{(n+1)_-}^{JM}$ [with J and M taking on the values $J = 1/2, 3/2, \cdots, (n+\frac{1}{2})$, $M = -J, -J+1, \cdots, J$) is $(n+1)\times$

$\times(n + 2)$ - fold degenerate. For eigenfunctions of Π_3 the additional twofold degeneracy expressed by (5.8) would correspond to a degeneracy with respect to the two values of the orbital angular momentum coupling to a given total angular momentum J .

We now turn our attention to a study of the behavior of $F_{n_\pm}(R)$ and $G_{n_\pm}(R)$ as $R \to 0$. Although the potential (3.2) has a logarithmic singularity at $R = 0$, the leading constribution exhibits an integral power law

$$\langle n_\pm' \, J'M' | V | \, n_\pm \, JM \rangle \to \delta_{n_\pm' n_\pm} \, \delta_{J'J} \, \delta_{M'M} \, \frac{8(-)^n}{R^3} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \quad \text{as} \quad R \to 0. \tag{5.9}$$

Substituting (5.9) for $V(x)$ in (5.6), we arrive at two coupled differential equations of the Fuchsian type, with $R = 0$ a regular, singular point. We conclude therefore that the six independent solutions (for a given value of λ , J, M and $n_\pm$) to Eq. (5.6) exhibit the behavior

$$F_{n_\pm}{}^i(R) \sim R^{\alpha_i^{n_\pm}}, \quad G_{n_\pm}{}^i(R) \sim R^{\beta_i^{n_\pm}} \quad (\text{with } i = 1,6) \tag{5.10}$$

as $R \to 0$, provided that $\alpha_i - \alpha_j \neq 0$, and $\beta_i - \beta_j \neq 0$, for $i \neq j$. We must have, however, $\alpha_i - \beta_i = m$ (integer). In fact, a more detailed examination of the equations shows that m only can have the values $m = \pm 1$. The α_i's and β_i's are then found to satisfy the indicial equations

$$[\alpha^{n_+} - n]\,[\alpha^{n_+} - (n+2)]\,[\alpha^{n_+} + (n+2)] = (-)^{n+1}\,8\lambda,\quad \beta^{n_+} = \alpha^{n_+} + 1,$$

$$[\tilde{\alpha}^{n_+} + n]\,[\tilde{\alpha}^{n_+} - (n+2)]\,[\tilde{\alpha}^{n_+} + (n+2)] = (-)^{n}\,8\lambda,\quad \tilde{\beta}^{n_+} = \tilde{\alpha}^{n_+} - 1 \tag{5.11a}$$

and

$$[\alpha^{n_-} - n]\,[\alpha^{n_-} + n]\,[\alpha^{n_-} - (n+2)] = (-)^{n}\,8\lambda,\quad \beta^{n_-} = \alpha^{n_-} - 1,$$

$$[\tilde{\alpha}^{n_-} - n]\,[\tilde{\alpha}^{n_-} + n]\,[\tilde{\alpha}^{n_-} + (n+2)] = (-)^{n+1}\,8\lambda,\quad \tilde{\beta}^{n_-} = \tilde{\alpha}^{n_-} + 1. \tag{5.11b}$$

We note the following useful relations:

$$\alpha^{n_\pm} = \alpha^{(n\pm 1)_\mp} \mp 1\,,$$

$$\tilde{\alpha}^{n_\pm} = \tilde{\alpha}^{(n\pm 1)_\mp} \pm 1\,. \tag{5.12}$$

For the purpose of later discussions it is also convenient to distinguish among the six independent solutions to Eqs. (5.11) by a subscript m, where m denotes the integral value which the solution in question approaches as $\lambda \to 0$. Thus

$$\alpha_m^{n_\pm} \to m \quad \text{as} \quad \lambda \to 0\,. \tag{5.13}$$

Note that $\tilde{\alpha}_{-m}^{n_\pm} = -\alpha_m^{n_\pm}$; finally we note that except for some isolated values of λ, including $\lambda = 0$, the solutions to Eqs. (5.11)

do indeed satisfy the above stated conditions for the validity of the ansatz (5.10). This can also be seen from Fig. 4 for the special case $n = 1$, where we present a plot of $\alpha^{1\pm}$ and $\tilde{\alpha}^{1\pm}$ as a function of the coupling strength λ.

B. $E \neq 0$

We now turn our attention to the actual case of interest, that is, when $E \neq 0$. In this case it is still desirable to expand the solution in terms of the four-dimensional spinor spherical harmonics, although this leads now to an infinite set of coupled third-order ordinary differential equations. The situation is therefore much more complicated and the discussion of the boundary conditions at the origin is correspondingly more involved.

We begin by restating the results of part A in a somewhat different form, the reason for this being twofold:

(a) In the $E \neq 0$ case it turns out to be more convenient to work in the "Dirac representation" related to the "Weyl representation" by a unitary transformation. In this representation

$$\gamma^0 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \quad \underset{\sim}{\gamma} = \begin{pmatrix} & \underset{\sim}{\sigma} \\ -\underset{\sim}{\sigma} & \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$$

(b) If $E \neq 0$ the space-time parity is no longer a good quantum number, whereas Π_3 still commutes with the operator $\mathcal{D}_E(\partial;\lambda)$. It is therefore desirable to restate our earlier conclusions in terms of the simultaneous eigenfunctions of $\mathcal{D}_E(\partial;\lambda)$ and Π_3.

We may obtain the $E = 0$ eigenfunctions of Π_3 by simply applying the projection operator

$$P_{\pm} = \frac{1}{2}\left[1 \pm \left(-i\,\gamma_4\,\mathcal{P}(\underset{\sim}{r})\right)\right], \quad \text{(three-parity} \pm 1)$$

to the $E = 0$ solutions (5.5). Denoting the resulting functions by $\tilde{\chi}_{n_\pm}^{JLM}$, where $L = J \mp \frac{1}{2}$ specifies the three-parity, we have

$$\tilde{\chi}_{n_\pm}^{JLM}(x) = \begin{pmatrix} F_{n_\pm}(R)\ \mathcal{Y}_{n_\pm}^{JM} + (-)^L\, G_{n_\pm}(R)\ \mathcal{Y}_{(n\pm1)_\mp}^{JM} \\ (-)^L\, F_{n_\pm}(R)\ \tilde{\mathcal{Y}}_{n_\pm}^{JM} + G_{n_\pm}(R)\ \tilde{\mathcal{Y}}_{(n\pm1)_\mp}^{JM} \end{pmatrix} \tag{5.14}$$

with three-parity $(-)^{L-J+\frac{1}{2}}$. The functions $F_{n_\pm}(R)$ and $G_{n_\pm}(R)$ appearing in (5.14) are of course identical to the ones appearing in (5.5). We note that not all of the solutions (5.14) are independent:

$$\tilde{\chi}_{n_\pm}^{JLM} \sim \tilde{\chi}_{(n\pm1)_\mp}^{JLM}$$

$$F_{n_\pm} \sim G_{(n\pm1)_\mp} \ .$$

This property had already been noted in (5.8). To be specific we choose $\tilde{\chi}_{n_+}^{JLM}(x)$ (with $L = J \mp \frac{1}{2}$) as the independent set of solutions to replace the original set of solutions $\varphi_{n_+}^{JM}$ and $\varphi_{n_-}^{JM}$ (same quantum numbers J, M and n). Going now to the Dirac representation and making use

of the definitions (4.8) we arrive at the following expressions for the $E = 0$ eigenfunctions of Π_3:[22]

$$\chi_n^{\ell_+}(x) = \begin{pmatrix} \left[\frac{2(n+\ell+2)}{n+1}\right]^{\frac{1}{2}} F(R)\, Y_n^{\ell_+} + \left[\frac{2(n-\ell+1)}{n+2}\right]^{\frac{1}{2}} G(R)\, Y_{n+1}^{\ell_+} \\ \left[\frac{2(n-\ell)}{n+1}\right]^{\frac{1}{2}} F(R)\, Y_n^{(\ell+1)_-} - \left[\frac{2(n+\ell+3)}{n+2}\right]^{\frac{1}{2}} G(R)\, Y_{n+1}^{(\ell+1)_-} \end{pmatrix} \tag{5.15a}$$

with three-parity $(-)^{\ell}$, and

$$\chi_n^{(\ell+1)_-}(x) \begin{pmatrix} \left[\frac{2(n-\ell)}{n+1}\right]^{\frac{1}{2}} F(R)\, Y_n^{(\ell+1)_-} + \left[\frac{2(n+\ell+3)}{n+2}\right]^{\frac{1}{2}} G(R)\, Y_{n+1}^{(\ell+1)_-} \\ \left[\frac{2(n+\ell+2)}{n+1}\right]^{\frac{1}{2}} F(R)\, Y_n^{\ell_+} - \left[\frac{2(n-\ell+1)}{n+2}\right]^{\frac{1}{2}} G(R)\, Y_{n+1}^{\ell_+} \end{pmatrix} \tag{5.15b}$$

with three-parity $(-)^{\ell+1}$. Here $F(R) \equiv F_{n_+}(R)$ and $G(R) \equiv G_{n_+}(R)$, so that

$$\begin{aligned} F(R) &\sim R^{\alpha^{n_+}}, \quad G(R) \sim R^{\alpha^{n_+}+1} \\ F(R) &\sim R^{\tilde{\alpha}^{n_+}}, \quad G(R) \sim R^{\tilde{\alpha}^{n_+}-1} \end{aligned} \tag{5.16}$$

as $R \to 0$, where α^{n_+} are the solutions to the indicial equations (5.11a). Comparing (5.15a) and (5.15b) we see that

$$\chi_n^{(\ell+1)_-} = (-)^n \Pi_4\, \chi_n^{\ell_+} . \tag{5.17}$$

Equation (5.17) together with the commutation relations (5.4) shows that $\chi_n^{\ell_+}$ and $\chi_n^{(\ell+1)_-}$ are solutions to Eq. (5.1) for the same eigenvalue λ. This corresponds to the twofold degeneracy already noted in connection with (5.8). With the additional degeneracy in ℓ and the magnetic quantum number M we have again a set of $2(n+1)(n+2)$ degenerate solutions.

We could have arrived at the results (5.15) also in a less direct way by substituting the expansion

$$\chi^{\ell\pm}(x) = \sum_{k=\ell}^{\infty} \begin{pmatrix} F_k(R)\, Y_k^{\ell\pm} \\ G_k(R)\, Y_{k\pm1}^{(\ell\pm1)_\mp} \end{pmatrix} \tag{5.18}$$

into (5.1). This would lead at first to an infinite set of differential equations coupled in the quantum number k. Our earlier group theoretical considerations have shown however that there exist solutions which actually decouple this infinite set of equations. By studying this question somewhat more closely we find that it is precisely the choice of coefficients in (5.15) which leads to such a decoupling via a number of cancellations. We leave the details of this study for Appendix D and turn now our attention to the case where $E \neq 0$.

If $E \neq 0$, the substitution of the expansion (5.18) into Eq. (3.1) again leads to an infinite set of coupled differential equations which can no longer be decoupled. This complicates the study of the boundary conditions at the origin considerably. We will show,

however, that if we develop the solution as a power series in E then,[26] to a given order in E, only a finite number of terms will contribute in the expansion (5.18). Indeed, consider the solutions $\phi_{\underset{\sim}{k}}(x)$ to Eq. (3.6). For E below the elastic threshold and at a given value of λ, $\phi_{\underset{\sim}{k}}(x)$ is expected to have a discrete set of poles in E, corresponding to the eigenvalues E_m of Eq. (3.9). Considering in particular one such pole at $E = E_0$, we write

$$\phi_{\underset{\sim}{k}}(x) = \frac{\tilde{\phi}_E(x)}{E - E_0} .$$

The bound state wave function $\varphi(x)$ was then defined as $\tilde{\phi}_E(x)$ evaluated at the pole: $\varphi(x) \equiv \tilde{\phi}_{E_0}(x)$; assuming that $\phi_E(x)$ can be developed into a power series in E with radius of convergence $r_0 \geqslant |E_0|$, and noting that in the limit $E \to 0$ we must recover the results (5.15), we conclude that to first order in E_0

$$\varphi_n^{\ell\pm}(x) = \chi_n^{\ell\pm}(x) + E_0\, \psi_n^{\ell\pm}(x) , \tag{5.19}$$

where $\chi_n^{\ell\pm}(x)$ are the $E = 0$ solutions (5.15) and $\psi_n^{\ell\pm}(x)$ is independent of E_0. The general form of $\psi_n^{\ell\pm}(x)$ will be determined below. The $E \neq 0$ solutions are of course no longer degenerate with respect to J and ℓ. Note, however, that both $\varphi_n^{\ell\pm}$ and $\chi_n^{\ell\pm}$ are labeled by the same quantum numbers, implying a one-to-one correspondence between the $E = 0$ and $E \neq 0$ solutions.

In order to determine the form of $\psi_n^{\ell\pm}(x)$ we substitute (5.19) into (3.1), keeping only the first order terms in E_0. We arrive then at the following set of coupled equations for $\chi_n^{\ell\pm}$ and $\psi_n^{\ell\pm}$:

$$\mathcal{D}_0(\partial;\lambda)\ \chi_n^{\ell\pm}(x) \ = \ 0\ , \tag{5.20a}$$

$$\mathcal{D}_0(\partial;\lambda)\ \psi_n^{\ell\pm}(x) \ = \ \left(\frac{\gamma^0}{2}\ [\Box^2 - m_2{}^2] - [m_1 - i\ \gamma.\partial]\ \partial_4\right)\chi_n^{\ell\pm}(x). \tag{5.20b}$$

Equation (5.20a) is of course just the $E = 0$ equation. Equation (5.20b) is of the form of an "inhomogeneous" $E = 0$ equation, the inhomogeneous term being already known from the solution of (5.20a). This suggests that we seek a solution of such form as to lead to only a finite set of coupled differential equations. An examination of the inhomogeneous term in (5.20b) shows that $\psi_n^{\ell\pm}(x)$ must in fact be of the following specific form

$$\psi_n^{\ell+}(x) \ = \ \begin{pmatrix} \left[\frac{2(n+\ell+1)}{n}\right]^{\frac{1}{2}} F_1(R)\ Y_{n-1}^{\ell+} + G_1(R)\ Y_n^{\ell+} \\ \left[\frac{2(n-\ell-1)}{n}\right]^{\frac{1}{2}} F_1(R)\ Y_{n-1}^{(\ell+1)-} + \tilde{G}_1(R)\ Y_n^{(\ell+1)-} \end{pmatrix}$$
$$+ \begin{pmatrix} F_2(R)\ Y_{n+1}^{\ell+} + \left[\frac{2(n-\ell+2)}{n+3}\right]^{\frac{1}{2}}\ G_2(R)\ Y_{n+2}^{\ell+} \\ \tilde{F}_2(R)\ Y_{n+1}^{(\ell+1)-} - \left[\frac{2(n+\ell+4)}{n+3}\right]^{\frac{1}{2}} G_2(R)\ Y_{n+2}^{(\ell+1)-} \end{pmatrix}, \tag{5.21}$$

the <u>formal</u> structure of $\psi_n^{(\ell+1)-}(x)$ being the same as that of $\Pi_4\,\psi_n^{\ell+}$. The coefficients of Y_{n-1} and Y_{n+2} have been chosen so that when the operator $\mathcal{D}_0(\partial;\lambda)$ is applied to $\psi_n^{\ell\pm}(x)$ in (5.20b), the coefficients of Y_{n-2} and Y_{n+3} in the resulting expressions vanish identically. Our discussion in Appendix D shows that the above choice of coefficients does indeed accomplish this. In general a truncation of the expansion (5.18) as represented in (5.21) cannot be achieved without running into a contradiction. Thus, for the case in question, the substitution of (5.21) into (5.20b) leads to a set of eight coupled equations for the six radial function in (5.21), so that we appear to be left with an overdetermined problem. It is, however, the structure of the equations (5.21) as considered from the group theoretical point of view which assures us that only six out of the eight equations can in fact be independent. This is indeed found to be the case.

Although we have demonstrated it only to first order in E_0, our results suggest that, subject to the assumed analyticity properties of $\tilde{\phi}_E(x)$, we may obtain the $E \neq 0$ solution $\varphi_n^{\ell\pm}$ perturbatively by starting from the $E = 0$ solution $\chi_n^{\ell\pm}(x)$, with only a finite number of terms contributing to its partial wave expansion (such as (5.18)) to a given order in E_0. As for the radius of convergence r_0 of the expansion of $\tilde{\varphi}_E(x)$ in powers of E, we note that certainly $r_0 < E_1$, where E_1 is the first excited state in the bound state spectrum of Eq. (3.9).

Having determined the structure of the $E \neq 0$ solutions to first order in E, we now examine the behavior of the radial functions

in (5.21) at $R = 0$. This is a straightforward, although tedious task, since we are dealing now with a finite set of coupled equations. We will not present the details which are purely algebraic in character. The results of this study are listed in Table I. Introducing the expansion

$$\varphi_n^{\ell\pm}(x) = \sum_{k=\ell}^{\infty} \begin{pmatrix} F_{nk}^{\ell\pm}(R)\, Y_k^{\ell\pm} \\ F_{nk}^{\ell\pm}(R)\, Y_{k\pm1}^{(\ell\pm1)\mp} \end{pmatrix} , \tag{5.22}$$

where we have followed the notation (5.19), we may summarize the contents of this table by the statements

$$F_{nk}^{\ell\pm} \sim R^{\alpha_m^{n_+} + |k-n|} , \quad G_{nk}^{\ell\pm} \sim R^{\alpha_m^{n_+} + |(k\pm1)-n|} \tag{5.23a}$$

$$F_{nk}^{\ell\pm} \sim R^{\tilde{\alpha}_m^{n_+}-1+|k-(n+1)|} , \quad G_{nk}^{\ell\pm} \sim R^{\tilde{\alpha}_m^{n_+}-1+|(k\pm1)-(n+1)|} \tag{5.23b}$$

as $R \to 0$, where $\alpha_m^{n_+}$ and $\tilde{\alpha}_m^{n_+}$ are the six solutions to the indicial equations (5.11a). Although we have shown only the first-order solutions (5.19) to exhibit the behavior (5.23) at $R = 0$, we note that these boundary conditions are independent of J and M, and are in fact determined by the $E = 0$ solution itself. We therefore assume (5.23) to be correct independent of the value of E.

VI. THE INTEGRAL EQUATION AND BOUNDARY CONDITIONS AT THE ORIGIN

In order to complete our discussion of boundary conditions, we need to examine which of the boundary conditions (5.23) are actually contained in the integral equation (3.9). Our objective in this section is actually twofold: In addition to establishing which of the six independent solutions (5.23) to Eq. (3.1) (for given values of J, ℓ, M and n) are also prospective solutions of the integral equation (3.9), it seems of considerable interest to learn how the boundary conditions (5.23) may be obtained directly from the integral equation itself. We restrict our discussion to the case in which $E = 0$, since, as we have seen in Sec. V, this is entirely sufficient for our purpose. In this case the expression (3.8) for the "Wick rotated" Green's function simplifies considerably, the result of taking the limit $E \to 0$ being

$$H_0(x) = \frac{1}{8\pi^2} \int_0^1 d\beta \, K_0[(m_2^2 + \beta(m_1^2 - m_2^2)^{\frac{1}{2}} R]. \tag{6.1}$$

For simplicity we will take $m_1 = m_2 = m$. In that case we obtain upon substituting (6.1) into (3.7),

$$H(x) = \frac{m}{8\pi^2} [K_0(m|x|) - i\gamma\cdot\hat{x} K_1(m|x|)]. \tag{6.2}$$

As in Sec. V-A it is convenient to work in the "Weyl representation." The $E = 0$ solutions are then of the form (5.5). Introducing the operator

$$Q(\hat{x}) = \begin{pmatrix} 1 & \\ & -i(\sigma\cdot\hat{x}) \end{pmatrix} \tag{6.3}$$

we write the $E = 0$ solutions (5.5) in the form

$$\varphi^{JM}_{n_\pm}(x) = \begin{pmatrix} F_{n_\pm}(R)\mathcal{Y}^{JM}_{n_\pm} \\ -i(\sigma\cdot\hat{x})\, G_{n_\pm}(R)\mathcal{Y}^{JM}_{n_\pm} \end{pmatrix} = Q(\hat{x})\, \tilde{\varphi}^{JM}_{n_\pm} \ . \tag{6.4}$$

With the definitions

$$\tilde{H}(x,x') = Q(\hat{x})^{-1}\, H(x - x')\, Q(\hat{x}'), \quad \tilde{V}(x) = Q(\hat{x})^{-1}\, V(x)\, Q(\hat{x}) \tag{6.5}$$

we may rewrite Eq. (3.9) as an integral equation for $\tilde{\varphi}(x)$:

$$\tilde{\varphi}(x) = -\lambda \int d^4x'\, \tilde{H}(x,x')\, \tilde{V}(x')\, \tilde{\varphi}(-x') \ . \tag{6.6}$$

We have

$$\tilde{V}(x) = \frac{4M^2}{R} \begin{pmatrix} K_1(MR) & -K_2(MR) \\ -K_2(MR) & K_1(MR) \end{pmatrix} \tag{6.7}$$

and

$$\tilde{H}_{11}(x,x') = \frac{m}{8\pi^2} K_0(m|x - x'|)$$

$$\tilde{H}_{22}(x,x') = \frac{m}{8\pi^2} (\tilde{\sigma}\cdot\hat{x}) K_0(m|x - x'|) (\sigma\cdot\hat{x}') \tag{6.8}$$

$$\tilde{H}_{12}(x,x') = \frac{m^2}{8\pi^2} \left[R(\tilde{\sigma}\cdot\hat{x}) \frac{K_1(m|x - x'|)}{m|x - x'|} (\sigma\cdot\hat{x}') - R' \frac{K_1(m|x - x'|)}{m|x - x'|}\right]$$

$$\tilde{H}_{21}(x,x') = \frac{m^2}{8\pi^2} \left[R \frac{K_1(m|x - x'|)}{m|x - x'|} - R'(\tilde{\sigma}\cdot\hat{x}) \frac{K_1(m|x - x'|)}{m|x - x'|} (\sigma\cdot\hat{x}')\right] .$$

Making use of the addition theorem (4.18) we have the expansions

$$\delta_{\alpha\beta} K_0(m|x-x'|) = \sum_{n_\pm} \frac{1}{4(n+1)} [\epsilon_n I_n(mR_<)K_n(mR_>) - 2 I_{n+2}(MR_<)K_{n+2}(mR_>)] W^{n\pm}_{\hat{R}'}(\hat{R})_{\alpha\beta} .$$

$$\delta_{\alpha\beta} \frac{K_1(m|x - x'|)}{m|x - x'|} = \sum_{n_\pm} \frac{I_{n+1}(mR_<) K_{n+1}(mR_>)}{(mR)(mR')} W^{n\pm}_{\hat{R}'}(\hat{R})_{\alpha\beta} \tag{6.9}$$

where $R = (\underset{\sim}{r}^2 + \tau^2)^{\frac{1}{2}}$, $R_> = \max\{R,R'\}$, etc., and $\epsilon_n = 2$ for all positive integers n except for $n = 0$ where $\epsilon_0 = 1$; $I_n(x)$ and $K_n(x)$ are modified Bessel functions with the properties

$$\left.\begin{aligned} I_n(x) &\sim \Gamma(n+1) \left(\frac{x}{2}\right)^n \\ K_n(x) &\sim \frac{1}{2} \Gamma(n) \left(\frac{x}{2}\right)^{-n} \qquad (n \neq 0) \\ K_0(x) &\sim -\ln x , \end{aligned}\right\} \quad \text{as } x \to 0 \tag{6.10}$$

With the aid of the expansions (6.9) and (4.24) we obtain the following set of coupled integral equations for the radial functions $F_{n_\pm}(R)$ and $G_{n_\pm}(R)$ in (6.4):

$$F_{n_\pm}(R) = 2M^2\lambda(-)^{n+1}\int R'^2 dR' \Big\{ \frac{m}{4(n+1)} [\epsilon_n I_n(mR_<)K_n(mR_>) - 2I_{n+2}(mR_<)K_{n+2}(mR_>)]$$

$$\times [K_1(MR')\, F_{n_\pm}(R') + K_2(MR')\, G_{n_\pm}(R')]$$

$$-[\frac{1}{R'} I_{\bar{n}+1}(mR_<)K_{\bar{n}+1}(mR_>) - \frac{1}{R} I_{n+1}(mR_<)K_{n+1}(mR_>)][K_1(MR')G_{n_\pm}(R') + K_2(MR')F_{n_\pm}(R')] \Big\}$$

$$G_{n_\pm}(R) = 2M^2\lambda(-)^{n+1}\int R'^2 dR' \Big\{ [\frac{1}{R'} I_{n+1}(mR_<)K_{n+1}(mR_>) - \frac{1}{R} I_{\bar{n}+1}(mR_<)K_{\bar{n}+1}(mR_>)]$$

$$\times [K_1(MR')\, F_{n_\pm}(R') + K_2(MR')\, G_{n_\pm}(R')]$$

$$- \frac{m}{4(\bar{n}+1)} [\epsilon_{\bar{n}} I_{\bar{n}}(mR_<)K_{\bar{n}}(mR_>) - 2I_{\bar{n}+2}(mR_<)K_{\bar{n}+2}(mR_>)][K_1(MR')G_{n_\pm}(R') + K_2(MR')F_{n_\pm}(R')] \Big\} \tag{6.11}$$

where $\bar{n} = (n \pm 1)$. In order to deduce the behavior of the radial functions at $R \approx 0$ we make again the ansatz that in this limit $F_{n_\pm}(R) \sim R^\alpha$, and $G_{n_\pm}(R) \sim R^\alpha$; Breaking up the integration in (6.11),

$$\int_0^\infty dR' = \int_0^R dR' + \int_R^\infty dR'$$

we may then evaluate explicitly the integral $\int_0^R dR'$, by making use of (6.10). Moreover, in the limit $R \to 0$ the dominant contribution to the

integral $\int_R^\infty dR'$ comes from the neighborhood of $R' = R$, so that the leading term in this integral may be obtained by simply substituting for $F_{n_\pm}(R)$ and $G_{n_\pm}(R)$ their values at $R \approx 0$. Performing the indicated integrations we arrive at the following indicial equations for α and β :

$$4\lambda(-)^{n+1}\left\{\frac{1}{[\alpha + \bar{n} + 1][\alpha - \bar{n} - 1]} - \frac{1}{[\alpha + n + 2][\alpha - n]}\right\} = 1 \quad (6.12a)$$

with $\beta = \alpha + 1$, and

$$4\lambda(-)^{n+1}\left\{\frac{1}{[\beta + \bar{n} + 2][\beta - \bar{n}]} - \frac{1}{[\beta + n + 1][\beta - n - 1]}\right\} = 1 \quad (6.12b)$$

with $\alpha = \beta + 1$; Comparing Eqs. (6.12) with the indicial equations (5.11) obtained earlier, we see that they are in fact identical. This then completes our first objective, that is, to learn how the boundary conditions (5.23) may be deduced directly from the integral equation itself. It remains now to be seen for which of the six solutions (5.23) the integral equation (6.11) remains well defined. An examination of the equations leading to (6.12) (we do not present the details here) shows that we need to observe the inequalities

$$\begin{aligned} \alpha^{n_+} &> -(n+2), \quad & \tilde{\alpha}^{n_+} &> -n \\ \alpha^{n_-} &> -n, \quad & \tilde{\alpha}^{n_-} &> -n \end{aligned} \quad (6.13)$$

if the integration in (6.11) is not to diverge at $R' = 0$. [Compare (6.13) with the $\lambda = 0$ solutions to the indicial equations!]

At this point it is convenient to define what we will refer to as the "regular" and "irregular" solutions. We separate the solutions to the indicial equations (5.11) into two groups, depending on whether they take on positive (including zero) or negative values for $\lambda = 0$. We will refer to these as the "regular" and "irregular" solutions, respectively. Thus, to take an example, the solutions $\alpha_n^{n_+}$, $\alpha_{n+2}^{n_+}$, and $\tilde{\alpha}_{n+2}^{n_+}$ to Eqs. (5.11a) belong to the first class, whereas $\alpha_{-(n+2)}^{n_+}$, $\tilde{\alpha}_{-(n+2)}^{n_+}$, and $\tilde{\alpha}_{-n}^{n_+}$ belong to the second class. We make the corresponding classification of the six independent solutions (for given values of J, M, ℓ and n) to Eq. (3.1). Consider then a typical case such as represented in Fig. 4, where we have plotted $\alpha^{1_\pm}$ and $\tilde{\alpha}^{1_\pm}$ as a function of the coupling constant λ. The intersection of a given line $\lambda = \lambda_1$ with the curves gives the six (possibly pairwise complex) solutions to the indicial equations. We see that the restrictions (6.13) do not necessarily exclude the irregular solutions. Thus, if we take the 1_+ solutions in Fig. 4a as an example, the conditions (6.13) are seen to exclude only one of the irregular solutions for $\lambda > 0$, whereas they exclude all three for $\lambda \leqslant 0$. By merely counting the number of free parameters, we would expect therefore Eq. (3.9) to have a continuous eigenvalue spectrum for $\lambda > 0$, unless we supplement the integral equation with additional boundary conditions. To this end we require the solutions to be regular

at $R = 0$. This choice of boundary conditions has the virtue that we treat the solutions for positive and negative values of the coupling constant in a like fashion. Another reason for favoring this choice is that for a non-singular force the integral equation would include without question only the regular solutions.

Having thus completed our study of the boundary conditions to be imposed on the solutions to Eq. (3.1), we like to comment again briefly on the MacDowell symmetry (see Sec. III) and its relation to the eigenvalue problem. Labeling the bound-state solutions by the corresponding values of the energy E and coupling constant λ, we may now restate our earlier conclusions as follows: If $\varphi_n^{\ell\pm}(x;\lambda,E_0)$ is a solution of Eq. (3.1) for $E = E_0$, then $\varphi_n^{(\ell\pm1)\mp}(x;\lambda,-E_0) \propto \Pi_4\ \varphi_n^{\ell\pm}(x;\lambda,E_0)$ is also a solution for $E = -E_0$, at the same value of λ. In fact both of these solutions are regular, if either one of them is. Combining this with our earlier remarks in Sec. III concerning their asymptotic behavior, we conclude that if $\varphi_n^{\ell\pm}(x;\lambda,E_B)$ solves the <u>eigenvalue</u> problem for $E = E_B$, then $\varphi_n^{(\ell\pm1)\mp}(x;\lambda,-E_B)$, as defined above, solves the eigenvalue problem for $E = -E_B$.[28] That is,

$$E_B(\lambda;\ \ell_\pm) = -E_B\big(\lambda;\ (\ell \pm 1)_{\mp}\big) \tag{6.14}$$

which is a more familiar way of stating the MacDowell symmetry.

VII. NUMERICAL TECHNIQUES

A. General Treatment

In the preceding sections we have prepared the ground for doing the calculation we have in mind; that is, to solve Eq. (3.1) subject to the conditions that $\varphi(x)$ exhibit the asymptotic behavior of Fig. 3 and that it be regular at $R = 0$. Writing equation (3.1) in the form

$$D_E(\partial)\,\varphi(x) = \lambda\, V(x)\,\varphi(-x) \tag{7.1}$$

we easily verify that the quotient

$$[\lambda] = \frac{\int d^4x\, \phi^\dagger(-x)\, D_E(\partial)\, \phi(x)}{\int d^4x\, \phi^\dagger(x)\, V(-x)\, \phi(x)} \tag{7.2}$$

is stationary with respect to infinitesimal variations in $\phi(x)$ about the solution to (7.1). The integrals in (7.2) converge at infinity for E below the two-body elastic threshold, i.e. $E < m_1 + m_2$. This can be easily verified with the aid of the results of Sec. III as summarized in Fig. 3. In the scattering region the integral in the numerator ceases to converge, however (the integral in the denominator continues to converge for E below the first inelastic threshold), so that the stationary expression (7.2) can be used only for calculations in the bound state region. The choice of scalar product was of course dictated by the desired stationary properties. As an extra bonus we notice that

the operators $D_E(\partial)$ and $V(x)$ are self-adjoint with respect to the scalar products chosen, so that we will have to deal only with Hermitian (actually real, symmetric) matrices in the calculation. We also note that $\lambda = \lambda^*$ as long as the denominator in (7.2) does not vanish. However, since $V(x)$ is not a positive definite operator, this is not guaranteed, and Eq. (7.1) does in fact have complex as well as real eigenvalues λ.

In the actual calculation we considered Eq. (7.1) as an eigenvalue problem in either E or λ, depending on our choice of potential. Thus, in the case of a non-singular potential, such as the potential (3.2) in the presence of a cutoff, it was clearly preferable to regard Eq. (7.1) as in eigenvalue problem in λ, since this allowed us, at least in principle, to represent correctly the asymptotic boundary conditions, as well as the boundary conditions at the origin. On the other hand, in the case of a singular potential it was preferable to regard Eq. (7.1) as an eigenvalue equation in E, since then the boundary conditions at the origin could still be incorporated correctly into the calculation, although the asymptotic boundary conditions would have to depend implicitly on the computed values of E. It was important that not the converse point of view was taken, since for an attractive potential the boundary conditions at the origin are expected to play the dominant role. In fact, our numerical calculation showed that it was absolutely crucial that the boundary conditions at the origin were treated properly, as it was to be expected, since we are dealing with a non-Fredholm equation.

In Sec. VI we argued on the basis of the integral equation (3.9) that the solutions to Eq. (3.1) should be regular at $R = 0$. The solutions to the eigenvalue problem are then in general a linear combination of the three independent regular solutions to Eq. (3.1) (for a given set of quantum numbers J, M, ℓ and n). To take proper account of all three of these solutions would require the inversion of extremely large matrices in the actual variational calculation. Hence, if the matrix sizes are to be kept at reasonable values, we need to compromise by either building into our "trial" functions the correct boundary conditions at the origin and representing the radial functions otherwise poorly; or by including only the dominant contribution at $R = 0$ and improving instead on the form of the radial trial functions. From the numerical point of view the latter choice is definitely to be preferred. An inspection of (5.23) shows that at least for a restricted range of the coupling constant the boundary conditions (5.23a) with $m = n$ can be considered as leading at $R = 0$ although this choice will not represent the dominant behavior for all the radial functions. We will thus consider in the actual calculation only the leading boundary conditions as given by

$$F_{nk}^{\ell_\pm} \sim R^{\alpha_n^{n_+}+|k-n|}, \quad G_{nk}^{\ell_\pm} \sim R^{\alpha_n^{n_+}+|(k\pm 1)-n|} \quad \text{as} \quad R \to 0 . \tag{7.3}$$

Now, it follows from our discussion in Sec. V that the specification of the angular momentum and parity of a particular

channel of interest is insufficient to single out a unique solution, since we are left with the quantum number n labeling the solutions (5.22) as an additional degree of freedom. Since this quantum number has no direct physical interpretation, it follows that, to take an example, the "nucleon" could in principle be considered as a bound state in any one of the infinite number of $J^P = \frac{1}{2}^+$, $I = \frac{1}{2}$ channels distinguished by the quantum number n. In view of the role of this quantum number in labeling the solutions, we would expect on a merely intuitive basis that higher values of n would correspond to higher excited states. The stability of the nucleon under the strong interactions would then suggest that it should be interpreted as a bound state in the channel labeled by the lowest value of n. We have thus restricted our attention to a study of the solutions $\varphi_\ell^{\ell_+}(x)$ and $\varphi_\ell^{(\ell+1)_-}(x)$ as a logical starting point for our calculation. We will refer to these solutions simply as $\varphi^{\ell_\pm}(x)$, and write

$$\varphi^{\ell_\pm}(x) = \sum_{k=\ell}^{\infty} \begin{pmatrix} F_k^{\ell_\pm}(R)\, Y_k^{\ell_\pm} \\ G_k^{\ell_\pm}(R)\, Y_{k\pm1}^{(\ell\pm1)_\mp} \end{pmatrix} . \tag{7.4}$$

Then, according to our choice of boundary conditions (7.3) we will, in the actual numerical calculation, require the radial functions to exhibit the behavior

$$F_k^{\ell_\pm}(R) \sim R^{\alpha_\ell^{\ell_\pm}+|k-\ell|} , \quad G_k^{\ell_\pm}(R) \sim R^{\alpha_\ell^{\ell_\pm}\pm1+|k-\ell|} \tag{7.5}$$

as $R \to 0$, where we have made use of (5.12) in order to put the results into a more symmetric form. Noting also that

$$C\,\mathcal{D}_E(\partial;\lambda)C^{-1} = \mathcal{D}_E^*(\partial;\lambda^*) , \tag{7.6}$$

where $\mathcal{D}_E(\partial;\lambda)$ is the operator defined in (5.2) and C is the charge conjugation matrix (4.14), we conclude with the aid of (4.15) [the spherical harmonics (A6) satisfy the same relations] that, for the bound state problem, $F_k^{\ell\pm}(R)$ and $G_k^{\ell\pm}(R)$ in (7.4) may be chosen to be real, for λ real. Specifically, we took the radial trial functions to be of the form

$$\begin{aligned} F_k^{\ell\pm}(R) &= R^{\alpha_\ell^{\ell\pm}+(k-\ell)} \sum_{m=0}^{M} a_{km} R^m e^{-\gamma R} , \\ G_k^{\ell\pm}(R) &= R^{\alpha_\ell^{\ell\pm}\pm 1+(k-\ell)} \sum_{m=0}^{M} b_{km} R^m e^{-\gamma R} , \end{aligned} \tag{7.7}$$

where $k = \ell, K$. The actual solutions have logarithmic singularities at $R = 0$ so that we could very likely have improved on the convergence of our calculation by including explicitly these logarithmic singularities in the expansion (7.7). We have not done so for the sake of simplicity. In practice only fairly low values of K and M in (7.7) were considered; thus typically $2 \leqslant K \leqslant 5$ and $0 \leqslant M \leqslant 8$. The stationary expression (7.2) was then converted to a matrix equation for the expansion coefficients (a_{km}, b_{km}) by performing all the integrations.

This left us with a set of matrices $D_E(k'm'i';kmi)_{ab}$ and $V(k'm'i';kmi)_{ab}$ corresponding to the differential operator- and potential-integrals respectively. Here m and k have the same meaning as in (7.7) and i labels the three independent regular solutions. (In the actual calculation only one value of i was considered, corresponding to the choice of boundary conditions (7.5).) The indices a, b label the upper and lower components in the expansion (7.4). In order to speed up the calculation we expressed $D_E(I,J)$ as a polynomial in E : $D_E = D_0 + ED_1 + E^2D_2 + E^3D_3$; the matrices D_j and V were real, symmetric: $D_j(k'm'i';kmi)_{ab} = D_j(kmi;k'm'i')_{ba}$, with the corresponding statement for $V(I,J)$.

In the calculations involving the singular potential (3.2) we were not free to consider arbitrary values of the coupling constant strength λ . Evidently the six solutions to the indicial equations (5.11a) are real for only a restricted range (λ_1, λ_2) of coupling-constant values. (We will refer to $\lambda_{1,2}$ as the "critical" values of the coupling constant λ .) Outside this range we have two pairs of complex conjugate solutions. In fact, as Fig. 4 illustrates, the regular and irregular solutions do, or do not mix at $\lambda = \lambda_{1,2}$, depending on the sign of λ and the particular quantum numbers involved. Still, from a purely mathematical standpoint one would expect that, at least in principle, a well-defined eigenvalue problem could be formulated even for coupling constant strengths exceding the critical value. We would thus expect that our boundary condition of

regularity would continue to lead to a discrete eigenvalue spectrum for $|\lambda| \geqslant |\lambda_{1,2}|$. In particular, if there should be no mixing between the regular and irregular solutions at the critical values of the coupling constant (such as for $\lambda < 0$ in Fig. 4a, for example) we could continue to choose these boundary conditions at the origin to be real:

$$F^{\ell\pm}_{nk}(R) \sim R^{\alpha^{n_+}+|k-n|} \ , \ F^{\ell\pm}_{nk}(R) \sim A\, R^{\alpha_R^{n_+}+|k-n|} \cos\left(\alpha_I^{n_+} \ell n\, R + B\right) ,$$

with the corresponding statement for $G^{\ell\pm}_{nk}(R)$. Here α^{n_+} and $\alpha_R^{n_+} \pm i\, \alpha_I^{n_+}$ denote the real and complex solutions to the indicial equations (5.11a), and A, B are constants to be fixed by the eigenvalue problem. (The overall normalization constant is of course irrelevant.) The solution is thus seen to oscillate infinitely fast at $R = 0$, the existence of an eigenvalue problem depending of course critically on the choice of phase B. Aside from the obvious numerical problems, it is very doubtful whether a meaningful physical interpretation can be given to these solutions.[29] We have restricted therefore our attention to the coupling constant range (λ_1, λ_2), where the values of $\lambda_{1,2}$ corresponding to the solution $\varphi_n^{\ell\pm}(x)$ are given by[30]

$$\lambda_{1,2} = (-)^n \ (A \pm \sqrt{B})/8 \tag{7.8}$$

where

$$A = (2a_2^3 - 9\, a_1\, a_2 + 27\, a_0)/27$$

$$B = -8\, [(3a_1 - a_2^2)/3]^3/54$$

and $a_0 = n(n+2)^2$, $a_1 = -(n+2)^2$, $a_2 = -n$; in Table II we have listed the calculated values of $\lambda_{1,2}$ corresponding to the solutions $\varphi^{\ell\pm}(x)$ for $\ell = 0$ through 5.

Although the present calculation was originally intended to be free of arbitrary parameters, the range (7.8) of coupling constant values considered in the singular problem was too restrictive in the $J^P = \frac{1}{2}^+$ channel (see Table II) as to give a bound state, so that we needed to modify the nucleon exchange potential (3.2) appropriately if bound-state solutions were to be obtained at all. We considered therefore the two modifications

$$\lambda\, V(x) = \frac{4M^2}{R}\, [a\, K_1(MR) + b\, i\, \gamma\cdot\hat{x}\, K_2(MR)]\ , \tag{7.9a}$$

$$V(x) = \frac{4M^2}{R}\, [K_1(MR) + i\, \gamma\cdot\hat{x}\, f(R, R_c)\, K_2(MR)]\ . \tag{7.9b}$$

Here R_c is a cutoff parameter and $f(R;R_c)$ is a cutoff function which was chosen to be of the form

$$f(R;R_c) = R/(R + R_c)\ . \tag{7.10}$$

The bound-state solutions were thus studied as a function of the three masses m_1, m_2 and M, the parameters a and b appearing in (7.9a), and the cutoff parameter R_c. We were of course free to give a any desired value, whereas the parameter b was still restricted to the range (λ_1, λ_2) as given by (7.8). For $a = b$ the potential (7.9a) reduces of course to the form (3.2) corresponding to the exchange of an elementary nucleon. We also note that for the choice of potential (7.9b) the boundary conditions (7.5) need to be replaced by

$$F_k^{\ell\pm}(R) \sim R^k , \quad G_k^{\ell\pm}(R) \sim R^{k\pm 1} \quad \text{as} \quad R \to 0 . \tag{7.11}$$

Before we proceed to a discussion of the numerical results, there are several "technical" points which seem worth mentioning. In practice it was found that the "optimum" value of the exponential parameter γ in the expansion (7.7) as determined in the course of the calculation was in general far off from the one suggested by the known asymptotic properties of the solution because of the short range character of the force. The starting point of our numerical calculation therefore consisted in searching for a "good" value of the parameter γ by studying the convergence properties of the calculation at small but increasing matrix sizes. We then improved on the accuracy of the calculated eigenvalues by enlarging systematically our space of basis functions; that is, by keeping an increasing number of terms in the expansions (7.4) and (7.7). Since we expect the terms in the expansion (7.4) to become of decreasing importance as we go to increasing values

of k, we made the index M in the sum (7.7) a function of k, observing at all times the inequality $M(k) > M(k + 1)$. In practice we did not invert matrices larger than 54 by 54. For matrices of this size an accuracy of 1% or less was common for the case in which $m_1 = m_2$, although an accuracy of $\approx$ 10% could already be obtained for much smaller matrix sizes. This is to be compared with other calculations.[3,4] When making this comparison it is to be kept in mind, of course, that the presence of spin alone in our problem doubles the matrix sizes for a given choice of trial function.

Because of the orthogonality of the spherical harmonics it was evidently crucial to determine precisely the total number of terms which needed to be included in the expansion (7.4) if the computed eigenvalues were to be accurate to some specified amount. In practice it was found that we never had to include more than the first five terms in the expansion (7.4) if an accuracy of about .1% was desired. This suggests that our method of reducing the genuine two-dimensional partial differential equation in the variables R and $\cos\theta$ to an infinite set of coupled ordinary differential equations by expanding the solution in the form (7.4) has definite advantages over solving the partial differential equation directly, since it provides us with an approximation scheme in which we actually need to solve only a small set (in our case at most five) of coupled ordinary differential equations.

Tables III and IV illustrate the observations just made. In Table III we present a typical sequence of approximations for five

different choices of the asymptotic behavior of the trial functions. The first column specifies the form of the trial function, the notation being

$$K \; ; \; M(\ell), \; M(\ell + 1), \cdots, M(K) \tag{7.12}$$

where $M(k)$ is the maximum value of the summation index m in the expansion (7.7), as a function of the four-dimensional angular momentum k labeling the radial functions, and K is the maximum value of k considered in the partial wave expansion (7.4). The table illustrates how the choice of the exponential parameter in (7.7) affected the convergence rate as well as the direction from which the computed eigenvalues approached the final value. We see that independent of the choice of γ, the numbers do eventually converge to the same final value.

Table IV illustrates our above observation that special care had to be taken to include a sufficient number of terms in the expansion (7.4). Table IVa shows the effect of improving only on the radial dependence; the numbers are seen to converge to the same final value independent of the choice of γ, as one would expect, since we may correct for a poor choice of the exponential parameter by including simply a larger number of terms in the expansion (7.7). Table IVb shows on the other hand, that by improving only on the "angular" dependence of the solutions we cannot correct a poor choice of γ, so that the numbers are seen to converge to different values for different choices

of γ . However, in all cases the numbers are seen to converge to the "wrong" value, the "correct" value being 1.649 as seen from Table III. Table IVb also shows that it was entirely sufficient for our purpose to include only the first five terms in the expansion (7.4), as we had already pointed out.

We have outlined here the general numerical procedure which was followed in the present calculation. We turn now to a disucssion of certain difficulties inherent in our choice of variational principle.

B. Weakly Bound States With $m_2/m_1 << 1$.

If we want to study the physically interesting case, that is, the bound states of the pions and nucleons with the experimentally measured masses, we run into several difficulties with the Rayleigh-Ritz variational principle (7.2). The fact that we are dealing with weakly bound states, such as the "nucleon," for example, and with a very small mass ratio, $m_2/m_1 = 0.144$, implies a very asymmetric asymptotic behavior of the solution with respect to the forward and backward light cone, so that in the bound state domain the solution is actually exponentially rising in one part of the $(\cos\theta, E)$ plane, and decaying in another, although the product $\varphi^\dagger(-x)\,\varphi(x)$ is always exponentially decaying. These circumstances made it in general rather difficult to obtain reasonably accurate solutions for the case in which $m_2/m_1 = 0.144$, although the variational principle (7.2) proved quite adequate for mass ratios $0.4 \leqslant m_2/m_1 \leqslant 1$, the lower bound being actually a function of the bound state energy considered. There have been presented, of course,

alternative ways of formulating the variational problem[3] which, in addition to being much more efficient, do overcome to some extent the above difficulties. However, instead of pursuing these alternatives, we will try to "patch up" the variational calculation as presented so far, by transferring some of the undesirable asymptotic properties of the solution to the potential. To this end we consider the transformation

$$\tilde{\varphi}(x) = e^{\frac{1}{2}\mu R\cos\theta}\,\varphi(x)\,, \tag{7.13}$$

where μ is some parameter whose value we are free to choose. If we set $\mu = (m_1 - m_2)$, then $\tilde{\varphi}(x)$ will exhibit the asymptotic behavior $\tilde{\varphi} \sim \exp[\frac{1}{2}(E - m_1 - m_2)R]$ along both the positive and negative time direction, replacing the original asymptotic behavior (see Fig. 3) $\varphi \sim \exp[(\frac{E}{2} - m_1)R]$ for $\cos\theta = 1$, and $\varphi \sim \exp[(\frac{E}{2} - m_2)R]$ for $\cos\theta = -1$. We have thus achieved a complete symmetry with respect to the forward and backward time direction in the four-dimensional Euclidean space. Moreover, the solution is now exponentially decaying for all $E < m_1 + m_2$. We also note that this particular choice of μ replaces the asymptotic expression (see again Fig. 3) $\exp[-\sqrt{-\underset{\sim}{k}^2}\, r - \frac{\omega}{2}\, 12\,\tau]$ by $\exp[-\sqrt{-\underset{\sim}{k}^2}\, r - \frac{\omega}{2}\, 12\,(1 - \eta)\tau]$, where $\eta = E/(m_1 + m_2)$. Hence for weakly bound states $(\eta \approx 1)$ we have reduced the original asymptotic expression to an essentially time independent form.[31] Another choice for the parameter μ would obviously be $\mu = \omega_{12}$, and would correspond to the choice $\mu_i = \omega_i/E$ in (2.9). The result of the transformation (7.13) is

now that $\tilde{\varphi}(x)$ is exponentially decaying everywhere in the $(\cos\theta, E)$ plane for $(m_1 - m_2) < E < m_1 + m_2$; however, the asymptotic behavior of $\tilde{\varphi}(x)$ with respect to the forward and backward time direction is still asymmetric.

From the practical point of view the transformation (7.13) merely involves the substitutions

$$m_1 \to m_1 - \mu\gamma^0 , \quad E \to E - \mu$$

$$\varphi(x) \to \tilde{\varphi}(x) , \quad V(x) \to \tilde{V}(x) = e^{\mu\tau} V(x)$$

in (7.2). As trivial as this change may appear, from a practical standpoint the situation is now considerably more complicated, since we are no longer dealing with a rotationally invariant potential. However, we still need to compute only one-dimensional integrals. Thus in order to evaluate the integral $\int d^4x\, \tilde{\phi}^\dagger(x)\, \tilde{V}(-x)\, \tilde{\phi}(x)$ in (7.2) we make use of the expansion

$$e^{-\mu R\cos\theta} = 2 \sum_{k=0}^{\infty} (-)^k (k+1) \frac{I_{k+1}(\mu R)}{\mu R} C_k^1(\cos\theta) \qquad (7.14)$$

and of the reduction of the direct product of two four-dimensional spherical harmonics. In practice we needed only the reduction coefficient $\tilde{C}^\ell(nn'N)$ as given by the integral

$$\tilde{C}^{\ell}(nn'N) = N_n^{\ell} N_{n'}^{\ell} \int_0^{\pi} d\theta \, (\sin\theta)^{2\ell+2} \, C_{n-\ell}^{\ell+1}(\cos\theta) \, C_{n'-\ell}^{\ell+1}(\cos\theta) \, C_N^{1}(\cos\theta). \tag{7.15}$$

Here $C_n^{\ell}(\cos\theta)$ are the Gegenbauer polynomials and N_n^{ℓ} is the normalization coefficient defined in (A7). Note that $\tilde{C}^{\ell}(nn'0) = \delta_{nn'}$ and $\tilde{C}^{0}(nn'N) = 1$. From the orthonormality property

$$\int_{-1}^{1} dx \, (1 - x^2)^{\ell+\frac{1}{2}} \, C_{n-\ell}^{\ell+1}(x) \, C_{n'-\ell}^{\ell+1}(x) = \delta_{nn'} \, (N_n^{\ell})^{-2}$$

we deduce immediately that

$$\tilde{C}^{\ell}(nn'N) = 0 \quad \text{unless} \quad |n - n'| \leqslant N \leqslant n + n' . \tag{7.16}$$

It also follows from (7.15) and the property $C_n^{\ell}(-t) = (-)^n C_n^{\ell}(t)$ of the Gegenbauer polynomials, that

$$\tilde{C}^{\ell}(nn'N) = (-)^{n+n'+N} \, \tilde{C}^{\ell}(nn'N) \tag{7.17}$$

so that

$$\tilde{C}^{\ell}(nn'N) = 0 \quad \text{unless} \quad n + n' + N = \text{even integer}. \tag{7.18}$$

In practice the coefficients $\tilde{C}^{\ell}(nn'N)$ were computed numerically by expressing the integral (7.15) as a quadruple sum over products of gamma functions.

Making use of the above results we may now convert the variational expression (7.2) to a matrix equation just as before. In particular, with the expansions (7.4) and (7.7) for our trial function $\tilde{\phi}(x)$ corresponding to the transformed solution (7.13) we obtain for a typical element of the matrix $[\tilde{V}(nm;n'm')_{ab}]$ replacing the integral $\int d^4x\, \tilde{\phi}^{+}(x)\, \tilde{V}(-x)\, \tilde{\phi}(x)$ in the stationary expression (7.2)

$$\tilde{V}(nm;n'm')_{11} = \sum_k (-)^k(k+1)\,\{\tilde{C}^{\ell}(n,n',k)\, U_1^{\ell}(n+n'+m+m';k) + [A^{\ell}_{n'+1}\,\tilde{C}^{\ell}(n,n'+1,k) + A^{\ell}_{n'}\,\tilde{C}^{\ell}(n,n'-1,k)]\, U_2^{\ell}(n+n'+m+m';k)\} \tag{7.19}$$

where

$$U_{\nu}^{\ell}(N;k) = \frac{8M^2}{\mu}\int dR\; e^{-2\gamma R}\,\rho(R)\, R^{2(\alpha_{\ell}^{\ell_+}-\ell)+N+1}\, K_{\nu}(MR)\, I_{k+1}(\mu R) \tag{7.20}$$

is now a typical integral that had to be computed, with $\rho(R)$ denoting a cutoff function. (For the singular problem $\rho(R) = 1$.) Only the Bessel functions $I_{K+1}(\mu R)$ and $I_K(\mu R)$, with $K = \max\{k\}$ in the sum (7.19), were computed explicitly, the rest being obtained with the aid of the recursion relations

$$I_{n-1}(x) = I_{n+1}(x) + \left(\frac{2n}{x}\right) I_n(x)\ .$$

(Note that as a consequence of (7.16) and (7.18) only a finite number

of terms need to be summed in (7.19).) The integrals were computed numerically by use of Gaussian integration techniques.[32]

The effect of introducing the modification (7.13) is illustrated in Table V where we have listed for a sequence of increasing matrix sizes the computed coupling constant values needed to give a bound-state of the actual pion nucleon system at the mass of the nucleon, $E = 1.0$. The potential was taken to be of the form (7.9b) with $R_c = 0.7$ and $M = m_1 = 1.0$. For the choice $\mu = 0$ in (7.13) the "convergence" of the calculation is seen to be <u>extremely</u> poor, whereas the choice $\mu = 0.856$ is seen to lead to a quite satisfactory convergence of the numerical results. In practice the modification (7.13) provided us with one additional variational parameter which we were free to adjust so as to optimize the convergence properties of the calculation.

VIII. NUMERICAL RESULTS

In the present calculation we devote our attention exclusively to the $J = 3/2$ and $J = 1/2$ channels. For convenience we divide this discussion into several parts corresponding to the different values of the total angular momentum J, isotopic spin I and parity that were considered. To a large extent we are concerned only with the general features of the eigenvalue spectrum of Eq. (3.1), which are not expected to depend critically on the ratio m_2/m_1. Because of the technical difficulties encountered when dealing with a ratio $m_2/m_1 << 1$ (as in the pion-nucleon case), we have placed somewhat greater emphasis on the case where $m_1 = m_2$. We note that the last significant figure of the computed eigenvalues as listed in Tables VI through XIII is uncertain.

A. The $N^*(J^P = 3/2^+, I = 3/2)$ Channel.

In this channel the coupling constant λ is positive (recall that λ is related to the pseudoscalar coupling constant G as in (2.8)); the parameters a and b in (7.9a) were taken to be positive, as well. In Table VI (Fig. 5) we have summarized our results for the choice of potential (7.9a) with $m_1 = m_2 = M = 1.0$. The energy spectrum was computed for several values of a and b. We see from Fig. 5 that both the singular and nonsingular components of the potential (7.9a) "act"[33] attractively in this channel. We also note that, for the range of coupling constant values considered here, there exist no excited states in the energy spectrum.[34] We see from Table VI

that for the undamped nucleon exchange potential (3.2) we obtain a bound state of mass $M_B = 1.649$ (in our units) for $\lambda = 1.6$ ($G^2/4\pi = 10$). Finally, although we have not verified this explicitly for this particular channel, we expect the curves for E vs. a in Fig. 5 to continue smoothly into the negative energy region, since the eigenvalue problem is well defined for all energies $|E| < m_1 + m_2$, as we have noted already in Sec. VI. We will see this explicitly in our discussion of the $J^P = \frac{1}{2}^+$, $I = \frac{1}{2}$ channel.

In Table VII we have summarized the corresponding results for the correct pion-nucleon kinematics, that is, for $m_1 = M = 1.0$ and $m_2 = 0.144$, the pion mass. Again we studied the energy spectrum for several values of the parameter a in (7.9a), with $b = 1.6$. For the choice $\mu = 0$ in (7.13) the convergence of the calculation was rather poor for energies close to threshold, that is, for $1.6 \leqslant a \leqslant 1.9$. (See Table VII.) The modification (7.13) did lead however to some improvement for a suitable choice of the parameter μ. Table VII shows that for the exchange potential (3.2), that is, for $a = b$ in (7.9a), we obtain a bound state of mass $M_B = 1.04$ at a coupling constant $\lambda = 1.6$ ($G^2/4\pi = 10$). It is quite reasonable to interpret this bound state as the $N^*(1236)$, since, for the singular force (3.2), the N^* could emerge in our calculation as a weakly bound state, while a more realistic treatment of the short range part of the force may "predict" it as a resonance (for the same value of the coupling constant). The above value for the mass of the N^* should be compared with the

value calculated by Abers and Zemach,[6] who obtained the N^* in their N/D calculation as a bound state with a mass $m(N^*) = 1.13$ (in our units) at the experimentally measured value of the coupling constant, for a force input corresponding to the exchange of an elementary nucleon. The results of Table VIII give qualitative support to our observation that with the modification (7.10) of the short range part of the force and a reasonable choice of the cutoff parameter R_c, we would not obtain the N^* as a bound state at the experimental value of the coupling constant.

In Table IX we give the energy spectrum as computed for several values of the mass M of the exchanged nucleon, for $m_1 = m_2 = 1.0$ and the potential (3.2) with $\lambda = 1.6$. We find that for increasing values of M (with a corresponding decrease in the range of the force) the two-body system becomes more deeply bound, which seems at first to contradict our intuition. A glance at the potential (3.2) shows however, that the mass of the exchanged nucleon controls not only the range of the force, but also the effective coupling constant strength.

Finally, we have summarized in Table X the bound state energies as computed for several values of m_2, with $m_1 = M = 1.0$, and for the choice of potential (7.9a) with $a = 1.9$ and $b = 1.6$. The convergence of the numerical results was quite good for m_2 in the range from 0.4 to 1.0, even for the choice $\mu = 0$ in (7.13).

B. The $J^P = 1/2^+$, $I = 3/2$ Channel.

Again only positive values of the parameters a, b and λ are to be considered here. No stable particle with the above quantum

numbers has been observed to date. Because of the very restricted range of coupling constant values that could be considered here in the singular case (see Table II), we were unable to obtain any bound states unless we considered the more general forms (7.9) for the potential, with $a \neq b$ and $R_c \neq 0$. This was the case in all $J = \frac{1}{2}$ channels, independent of the isotopic spin and parity, and was of course the reason for considering at all the modifications (7.9). In Table XI (Fig. 6) we present the bound state energies as computed for several values of the parameters a and b in (7.9a), with $m_1 = m_2 = M = 1.0$. Qualitatively the situation is very similar to that represented in Fig. 5, except that now the singular component of the potential (7.9a) acts repulsively, whereas it acted attractively in the N^* channel. The nonsingular component acts attractively, as before. One might expect that the situation would be reversed if we considered negative values of the parameters a and b. In particular, since the overall force was found to be attractive for $a, b > 0$, we might expect it to be repulsive for $a, b < 0$. This is not the case, bound state solutions having been obtained for both signs of the forces, as the following discussion will show.

C. The Nucleon ($J^P = \frac{1}{2}^+$, $I = \frac{1}{2}$) Channel.

In this channel $\lambda < 0$. The parameters a and b in (7.9a) were taken to be negative as well. In Table XII we list the computed bound state energies at several values of the coupling constant λ, for the choice of potential (7.9b) with $R_c = 1.0$ and

$m_1 = m_2 = M = 1.0$. We have found it instructive to study this time explicitly both the positive and negative energy spectrum. As we have pointed out already, our eigenvalue problem is just as well defined in the negative energy domain as it is in the positive one, provided that $-(m_1 + m_2) < E < (m_1 + m_2)$. The results of Table XII as plotted in Fig. 7 indicate a somewhat "abnormal" situation, the computed binding energies bearing an inverse relationship to the magnitude of the coupling constant. However, in view of the MacDowell symmetry as stated in the form (6.14), this "abnormal" situation in the 1_- channel (we use the notation $\ell_\pm$) is a direct consequence of the existence of a "normal" situation in the 0_+ channel. This has been explicitly verified numerically. Stated more generally, given a normal situation in the $\ell_\pm$ channel, we will have a corresponding abnormal situation in the $(\ell \pm 1)_\mp$ channel. It is clearly sufficient to compute the positive and negative energy spectrum in all angular momentum channels of a given parity, since the energy spectrum in the remaining channels of opposite parity may be obtained from here with the aid of (6.14). We note that the MacDowell symmetry has been shown[21] to be a consequence of extended Lorentz invariance (invariance under complex Lorentz transformations) which is itself a consequence of the invariance of the S matrix under real Lorentz transformations. Hence, the usual requirement of Lorentz invariance provides the physical reason for the existence of the abnormal situation observed above.

Finally, Table XIII summarizes for the nucleon channel, the data complementary to that presented in Table VIII for the N^* channel. Fixing the energy of the bound state at the nucleon mass ($E = 1.0$) we have computed the eigenvalues λ as a function of the cutoff parameter R_c in the potential (7.9b), with $m_1 = M = 1.0$ and $m_2 = 0.144$, the pion mass. [Since we are dealing here with a very weakly bound state in addition to a very small mass ratio m_2/m_1, we found it imperative to consider the modification (7.13). For the choice $\mu = 0.856$ the convergence of our numerical results was quite good.] Comparing the contents of Tables VIII and XIII we note that the values of the pseudoscalar coupling constant G needed to give a bound state of the πN system at the nucleon mass is approximately three-half times larger in the N^* channel than in the nucleon channel! (Recall that an estimate based on the relative sizes of the Born term in the various channels would predict the nucleon as contributing the dominant force in the N^* channel, not in the nucleon channel!) We observe however that the situation in the N^* channel is a normal one, whereas in the nucleon channel we are dealing with an abnormal situation, so that the strength of the force (as measured in terms of the binding energy of the bound state) is here inversely related to the strength of the coupling constant.

We should like to conclude this discussion on a somewhat pessimistic note. Our results have shown that any estimate of the relative strength of the forces in the various channels as based on

the Born diagram of Fig. 2 will fail completely in the bound state problem, as of course was expected. Moreover, our calculation has also shown that, for the bound state problem, the sign of the coupling constant does not in general provide a criterion for distinguishing between attractive and repulsive forces (unless the operators involved are either positive or negative definite, as is the case in a ϕ^3 theory with equal mass particles.). This implies in particular that the study of crossing matrices alone cannot provide in general an estimate of the relative sign and strength of the forces, except when this study is made in the context of a particular dynamical model.

IX. TIME INDEPENDENT APPROXIMATIONS TO THE BS EQUATION

As we have seen, a great deal of effort was involved in the solution of Eq. (3.1). It is clear that the presence of the "relative time" in the BS equation presented the greatest difficulties in our analytical discussion as well as in the actual numerical solution. This motivates us to ask a number of questions: What is the role of the relative time variable in the BS equation? Under what conditions would a time-independent description of a physical situation be just as adequate? Is it possible to arrive under these conditions at a time independent equation (with a local or nonlocal potential) by starting from the original covariant equation? In particular, we may ask whether the static limit of Eq. (3.1) exists, and if it exists, what particular form it would take. Static models for the πN interaction have received a great deal of attention in the past, most of the discussions having been given, however, within a dispersion theoretic framework, although the pioneering work of Chew and Low[35] was presented in a field theoretic context.

In order to answer the above questions, we must begin by understanding the role of the relative time variable in a covariant theory. This is discussed in the following section, where we find that this role is intimately connected with the presence or absence of antiparticles in the theory. In Secs. B and C we will then consider two particular time-independent approximations to Eq. (3.1), which we will call the "static" and "nonrelativistic" limits. The "static limit"

involves approximations which have a simple diagrammatic interpretation and which will lead, at least formally, to the well known Chew-Low static model for the pion-nucleon interaction (in the ladder approximation). The "nonrelativistic" limit will involve the approximations already considered by Blankenbeckler and Sugar[36] and by Schwartz and Zemach.[3] In every case, however, we need to introduce a cutoff in order to make the approximations at least somewhat meaningful. In Sec. D we will then compare the solutions to the equations in the static and nonrelativistic limit with those of the original BS equation. The discussion of a related topic is left for Appendix E.

A. The Role of the "Relative Time"

We begin by considering our problem from a somewhat "old fashioned" point of view. We like to solve the eigenvalue equation

$$H_s|B\rangle = E_B|B\rangle \tag{9.1}$$

where H_s is the full, time-independent Hamiltonian in the "Schrodinger picture" and $|B\rangle$ are the stationary eigenstates of H_s. This is in general a hopeless problem since the exact treatment of Eq. (9.1) within a relativistic framework would necessarily lead to an infinite set of coupled, three-dimensional integral equations for the probability amplitudes $A(m,k_i;n,p_j;\bar{n},\bar{p}_k)$ of finding the "bound state" $|B\rangle$ in a

state of m bare mesons with momenta k_i, n bare nucleons with momenta p_j, and $\bar{n}$ bare antinucleons with momenta $\bar{p}_k$. Hence, in any practical calculation we have to truncate somehow this set of coupled equations. Thus, speaking diagrammatically, one might ignore all higher (than some given number) intermediate particle states, and include the contribution from the lower lying intermediate particle states exactly. Another less obvious possibility would consist in including some well defined contribution from all intermediate particle states. Since we are dealing with off-shell equations it is by no means clear which of these truncations would be most realistic. This is an interesting question, but has not been considered here any further in this paper. The presence of an arbitrary number of particles in the intermediate states is characteristic of any covariant treatment, since particle-antiparticle pairs must be included. On the other hand, in some field theoretical models ignoring the antiparticle contributions, only a finite number of particles in the intermediate states may contribute. A classic example is provided by the $V\theta$ sector of the Lee model,[37] corresponding to the time- ordered ladder graphs of Fig. 8. Only one- and two-particle intermediate states contribute here, corresponding to the fact that the $|V\theta\rangle$ and $|N\theta\theta\rangle$ represent a complete set of free particle states for this particular sector. Returning to our original problem, we would like to consider now the second type of truncation suggested above. Using again the diagrammatic language, we would like to truncate the infinite set of coupled, three-dimensional integral equations arrived at from Eq. (9.1) by including only the contribution

from the ladder graphs, ignoring "crossed rungs," etc. This will leave us again with an infinite set of coupled equations, which are in fact equivalent to the covariant equation (2.11). Indeed, from the point of view of perturbation theory we find that, grouping together all time-ordered diagrams of a given order N, we obtain invariably $N!$ diagrams which differ merely in the ordering of the vertices. These $N!$ non-covariant diagrams may be collected however into a single covariant Feynman diagram by introducing an additional degree of freedom corresponding to the time-variable in coordinate space. To give a specific example, we consider the box diagram of Fig. 9, as represented by the integral[38]

$$\int \frac{d^4q'}{(2\pi)^4} \frac{\bar{U}(\underset{\sim}{k}',\sigma')[M + \gamma\cdot(k'+q)][M - \gamma\cdot(\frac{K}{2}+q)][M + \gamma\cdot(k+q)]\, U(\underset{\sim}{k},\sigma)}{[(k'+q)^2 + M^2 - i\epsilon][(\frac{K}{2}+q)^2 + M^2 - i\epsilon][(k+q)^2 + M^2 - i\epsilon][(\frac{K}{2}-q)^2 + \mu^2 - i\epsilon]} \tag{9.2}$$

aside from irrelevant kinematical factors. It is convenient to separate the particle and antiparticle contributions in the integral (9.2) by writing

$$\frac{[M - \gamma\cdot p]}{p^2 + M^2 - i\epsilon} = \left(\frac{M}{E(\underset{\sim}{p})}\right)\left[\frac{U(\underset{\sim}{p})\,\bar{U}(\underset{\sim}{p})}{E(\underset{\sim}{p}) - p^0 - i\epsilon} - \frac{V(-\underset{\sim}{p})\,\bar{V}(-\underset{\sim}{p})}{E(\underset{\sim}{p}) + p^0 - i\epsilon}\right] \tag{9.3}$$

for the nucleon propagator, with the corresponding separation of the meson propagator. Here $U(\underset{\sim}{p})_{\alpha\sigma} \equiv U(\underset{\sim}{p},\sigma)_\alpha$ and $V(\underset{\sim}{p})_{\alpha\sigma} \equiv V(\underset{\sim}{p},\sigma)_\alpha = [\gamma_5\, U(\underset{\sim}{p})\, C^{-1}]_{\alpha\sigma}$ are the usual Dirac spinors for the nucleon and

antinucleon respectively.* The separation (9.3) corresponds to the diagrammatic decomposition of Fig. 10, and leads to the replacement of (9.2) by the sum of sixteen separate integrals. It is however our claim that by a careful algebraic manipulation of the individual terms in this sum, we may rewrite it as the sum of twenty-four integrals bearing a one-to-one correspondence to the time-ordered diagrams of old fashioned perturbation theory. The reason that the decomposition of Fig. 10 does not lead immediately to such a one-to-one correspondence is of course, that even for a detailed specification of the quantum numbers labeling the particles in the virtual intermediate states, the interactions at the vertices may still proceed in more than just one given time-order. Figure 11 illustrates our point. Thus, the time-ordered diagram of Fig. 11a is the only fourth order diagram contributing to the πN interaction if no antinucleons are to be present in the intermediate states. On the other hand, all four diagrams of Fig. 11b are characterized by the same quantum numbers labeling the particles in the virtual intermediate states, and merely differ in the time-ordering of the vertices. In practice the "rules of the game" as applied to the box-diagram of Fig. 9 were found to be very simple: Having made the separation (9.3) in the integral (9.2), and performing the p^o integration, a particular integral was found to correspond

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* Note that the sum over spins in (9.3) is implicit in the matrix multiplication: $[U(\underset{\sim}{p})\ \bar{U}(\underset{\sim}{p})]_{\alpha\beta} = \sum_{\sigma} U(\underset{\sim}{p},\sigma)_{\alpha}\bar{U}(\underset{\sim}{p},\sigma)_{\beta}$, etc. We will make frequent use of the shorthand notation used in (9.3).

to 1) a single time ordered diagram, as in Fig. 11a, if the integrand involved only one pole in either the upper or lower half of the p^o plane; 2) the sum of several time-ordered diagrams, such as in Fig. 11b, if the integrand exhibited more than one pole in either half of the p^o plane. It is a simple, although tedious, algebraic excercise to verify these statements explicitly for the box diagam of Fig. 9,[39] and becomes a forbidding task for higher order diagrams. The correctness of our observations can however be demonstrated in general, as well.

The above considerations have led then to a clear interpretation of the role of the relative time variable in the BS equation (or a covariant theory in general). Its introduction represents thus merely an alternative mathematical way of dealing with a difficulty intrinsic to any relativistic problem, that is, the infinite dimensionality of the Hilbert space of particle states. In particular it follows, that if for some reason (such as conservation laws) only one time-ordering of the diagrams should be permissible, the time-variable no longer plays a useful role, and we should be able to eliminate it explicitly from the theory. We will make specific use of these ideas in the following sections.

We like to conclude this discussion by commenting briefly on the importance of the antiparticle contributions in a relativistic theory. It appears that it is difficult to make a reliable statement in this respect. An examination of the time-ordered graphs of "old fashioned perturbation theory" shows (we refer here specifically to the πN problem) that individual diagrams (of a given order) involving

antinucleons in the intermediate states are supressed with respect to the diagrams (of the same order) involving no antinucleons at all, if the ratio of the meson mass to the nucleon mass is small $(\mu/M << 1)$, provided that the dominant contribution comes from intermediate particle states with momenta small compared to the respective particle masses. This presupposes the presence of a sufficiently strong damping mechanism. However, even if all these conditions should be met, it should still be remembered, that by neglecting the antinucleons in the theory we are excluding $N! - 1$ (perhaps "small") Nth order diagrams out of the total of $N!$ diagrams.

B. The Static Limit.

Starting from Eq. (2.19) we arrive, by making a number of approximations, at the "Chew Low equations" (in the ladder approximation) for the static pion-nucleon interaction. For the discussion we have in mind it is convenient to work with the equation for the scattering amplitude. Thus, introducing the Stapp M- function by

$$f(\underset{\sim}{k}',\underset{\sim}{k}) = \frac{2M}{8\pi E} \bar{U}(\underset{\sim}{k}') M(k',k) U(\underset{\sim}{k}) \tag{9.4}$$

where $f(\underset{\sim}{k}',\underset{\sim}{k})$ is the on-shell scattering amplitude defined by (2.20), we arrive from (2.19) at the following off-shell equation for $M(p,k)$:

$$M(p,k) = i\lambda \tilde{I}(p,k) - \lambda \int \frac{d^4p'}{(2\pi)^4} \tilde{I}(p+p') \tilde{G}_E(p') M(p',k). \tag{9.5}$$

For later convenience we have labeled the momentum space Green's function explicitly by the total c.m. energy E. We proceed to make a series of approximations, which, motivated by our earlier considerations, require however the presence of a properly chosen cutoff in order to have any validity. Assuming then a suitable "damping mechanism" to be present in Eq. (9.5), we proceed to neglect the antinucleon contributions, corresponding to the replacements

$$\tilde{I}(p) \to \tilde{I}'(p) = -i(4\pi)^2 \left(\frac{M}{E(\underset{\sim}{p})}\right) \frac{\gamma_5\, U(\underset{\sim}{p})\, \bar{U}(\underset{\sim}{p})\, \gamma_5}{[E(\underset{\sim}{p}) - p^o - i\epsilon]} ,$$

$$\tilde{G}_E(p) \to \tilde{G}'_E(p) = \left(\frac{M}{E(\underset{\sim}{p})}\right) \frac{U(\underset{\sim}{p})\, \bar{U}(\underset{\sim}{p})}{[E(\underset{\sim}{p}) - (p^o + \frac{E}{2}) - i\epsilon][\omega(\underset{\sim}{p})^2 - (p^o - \frac{E}{2})^2 - i\epsilon]} . \tag{9.6}$$

These approximations are most easily interpreted diagrammatically, the omition of the antinucleons leading evidently to the replacement of the sum of ladder graphs of Fig. 1 for the BS wave function by the sum of time-ordered ladder graphs of Fig. 8 for the T-matrix (with the replacement of the dressed V-quantum by the N-quantum). These considerations indicate that the p^o variable can no longer play a useful role in equation (9.5), suggesting that we should be able to perform the p^o-integration explicitly. In order to see why this should be the case, we examine the analytic structure of $M(p,k)$ in the p^o variable, as deduced from equation (9.5) by an iterative scheme. We easily verify by going through an elementary "pinch analysis," that $M(p,k)$ has only singularities in the lower half of the p^o plane, as a result of

the absence of the antinucleons. Moreover, basing again our arguments on perturbation theory, we verify that the p^0 integration converges sufficiently fast, such as to allow us to close the p^0 contour of integration in the upper half of the complex p^0 plane. Doing so, and defining the off the energy shell T-matrix by

$$T(\underset{\sim}{p},\underset{\sim}{k}) = \bar{U}(\underset{\sim}{p})\, M\left(\underset{\sim}{p}, \frac{E}{2} - \omega(\underset{\sim}{p})\ ;\ \underset{\sim}{k}, \frac{E}{2} - \omega(\underset{\sim}{k})\right) U(\underset{\sim}{k}) \tag{9.7}$$

we arrive at the following integral equation for $T(\underset{\sim}{p},\underset{\sim}{k})$:

$$T(\underset{\sim}{p},\underset{\sim}{k}) = \lambda\, V(\underset{\sim}{p},\underset{\sim}{k};E) + \lambda \int \frac{d^3\underset{\sim}{q}}{(2\pi)^3}\, V(\underset{\sim}{p},\underset{\sim}{q};E)\ \mathcal{G}_E(\underset{\sim}{q})\, T(\underset{\sim}{q},\underset{\sim}{k}) \tag{9.8}$$

where

$$V(\underset{\sim}{p},\underset{\sim}{k};E) = -(4\pi)^2 \left(\frac{M}{E(\underset{\sim}{p}+\underset{\sim}{k})}\right) \frac{\bar{U}(\underset{\sim}{p})\, \gamma_5\, U(\underset{\sim}{p}+\underset{\sim}{k})\, \bar{U}(\underset{\sim}{p}+\underset{\sim}{k})\, \gamma_5\, U(\underset{\sim}{k})}{[E - E(\underset{\sim}{p}+\underset{\sim}{k}) - \omega(\underset{\sim}{p}) - \omega(\underset{\sim}{k}) + i\epsilon]} \tag{9.9}$$

$$\mathcal{G}_E(\underset{\sim}{p}) = \left(\frac{M}{2\omega(\underset{\sim}{p})\, E(\underset{\sim}{p})}\right) \frac{1}{[E - E(\underset{\sim}{p}) - \omega(\underset{\sim}{p}) + i\epsilon]} \quad . \tag{9.10}$$

As our above remarks have anticipated, this is precisely the equation we would have arrived at by simply applying the "rules" of old fashioned perturbation theory[39] to the sum of ladder graphs in Fig. 8.

It should be noted that Eq. (9.8) is marginally singular, that is, our approximations did not destroy this particular feature of the original Eq. (9.5). However, it is clear that the sequence of approxi-

mations leading to Eq. (9.8) are certainly unjustified in the absence of a cutoff, and are in fact very questionable under any circumstances, as our qualitative discussion in Part A has indicated. We will therefore simply accept the approximations carried out so far, and proceed from here.

If we attempted to solve Eq. (9.8) as it stands, we would be faced with several difficulties. Since it is a marginally singular integral equation, the asymptotic behavior of $T(\underset{\sim}{p},\underset{\sim}{k})$ in the off-shell variable $\underset{\sim}{p}$ is critically dependent on the coupling strength. Hence any attempt to solve Eq. (9.8) numerically would require us to take explicitly care of these asymptotic boundary conditions. However, even if proper care is taken in this respect, some preliminary calculations seem to indicate that there exist basic difficulties in solving such singular integral equations by the usual meshpoint techniques. Another unrelated difficulty which we encounter, is the presence of the non-rotationally invariant quantities $E(\underset{\sim}{p} + \underset{\sim}{p}')$ in Eq. (9.8). This complicates the partial way analysis of this equation considerably. We will not try to overcome these difficulties, but will remedy these by simply introducing to begin with a cutoff in Eq. (9.5), and by making an additional approximation in Eq. (9.8) designed to simplify the partial wave analysis.

Denoting by $\underset{\sim}{q}$ the momenta of the nucleons in the virtual intermediate states of the ladder graphs of Fig. 8, we could expect, in the presence of a suitably chosen cutoff, the inequality $|\underset{\sim}{q}|/M << 1$

to hold. Under these conditions we could go to the full static limit of Eq. (9.8). It is in this full static limit that we recover the classic Chew-Low equations in the ladder approximation, which ignore the nucleon recoil altogether. However, for reasons to be discussed later, we will choose a cutoff function $g(p)$, having the asymptotic dependence $g(p) \sim p^{-2}$ as $p^2 \to \infty$. For such a slow asymptotic fall-off we cannot go to the full static limit of Eq. (9.8), but need to take into account at least some of the nucleon recoil. Keeping in mind the above mentioned difficulties of solving Eq. (9.8) directly, we proceed to make the following approximations in (9.9):

$$\left(U(\underset{\sim}{p})\,\gamma_5\,U(\underset{\sim}{p}+\underset{\sim}{p}')\,\tilde{U}(\underset{\sim}{p}+\underset{\sim}{p}')\,\gamma_5\,U(\underset{\sim}{p}')\right) \approx -\frac{1}{4M^2}(\vec{\sigma}\cdot\vec{p}')(\vec{\sigma}\cdot\vec{p}) \tag{9.11}$$

$$E(\underset{\sim}{p}+\underset{\sim}{p}')[E-\omega(\underset{\sim}{p})-\omega(\underset{\sim}{p}')-E(\underset{\sim}{p}+\underset{\sim}{p}')] \approx M\left[E-M-\omega(\underset{\sim}{p})-\omega(\underset{\sim}{p}')-\frac{(\underset{\sim}{p}+\underset{\sim}{p}')^2}{2M}\right]. \tag{9.12}$$

In making the approximation (9.11) we went to the full static limit in which $E(\underset{\sim}{p}) \approx M$. With these additional approximations, we arrive now easily at a partial wave projection of $V(\underset{\sim}{p},\underset{\sim}{k};E)$. Thus, making use of (A3) we note the following expansion:

$$\frac{1}{(\underset{\sim}{p}+\underset{\sim}{p}')^2+\mu^2} = \frac{1}{2pp'}\sum_{\ell_\pm}(-)^\ell\,Q_\ell(x_\mu)\,Z_{\hat{p}}^{\ell_\pm}(\hat{p}'),$$

where $Q_\ell(z)$ are the Legendre functions of the second kind, and

$$x_\mu = \frac{\underset{\sim}{p}^2+\underset{\sim}{p}'^2+\mu^2}{2pp'}, \quad p \equiv |\underset{\sim}{p}|.$$

Defining

$$y_M \equiv \left[E - M - \omega(\underset{\sim}{p}) - \omega(\underset{\sim}{p}') - \frac{\underset{\sim}{p}^2 + \underset{\sim}{p}'^2}{2M}\right] \frac{M}{2pp'}$$

and introducing the expansion

$$V(\underset{\sim}{p}, \underset{\sim}{p}';E) = \sum_{\ell_\pm} \mathcal{V}^{\ell_\pm}(p,p';E)\, Z^{\ell_\pm}_{\hat{p}'}(\hat{p})$$

we obtain with the aid of the formulas listed in Appendix A

$$\mathcal{V}^{\ell_\pm}(p,p';E) = \frac{4\pi^2}{M} \left[2\, y_M\, Q_\ell(y_M) - Q_{\ell\pm1}(y_M)\right] . \tag{9.13}$$

Finally, since we are interested only in the bound state problem it is convenient to rewrite Eq. (9.8) as an eigenvalue equation for the bound state wave function $\varphi(\underset{\sim}{p})$. Proceeding as usual, and denoting by $\varphi^{\ell_\pm}(p)$ the partial wave projection of $\varphi(\underset{\sim}{p})$, we arrive at the eigenvalue equation

$$\varphi^{\ell_\pm}(p) = \lambda\, \mathcal{G}_E(p) \int \frac{dp'}{2\pi^2}\, p'^2\, \mathcal{V}^{\ell_\pm}(p,p';E)\, \varphi^{\ell_\pm}(p') . \tag{9.14}$$

In continuing Eq. (9 .8) to the bound state region we treated E, $\underset{\sim}{p}$ and $\underset{\sim}{k}$ as independent variables.

C. The "Non-Relativistic" Limit.

We consider in this section another time-independent approximation to Eq. (2.11), which leads to the equation for a non-relativistic (NR) pion and nucleon moving in the field of strongly singular, central forces plus a force arising from the coupling of the nucleon-spin to the orbital motion of the system. The approximation to be considered has the disadvantage of not having a direct diagrammatic interpretation and being based instead on the intuitive notion that the dependence of the BS wave-function on the relative time is a relativistic effect. As we have tried to emphasize in the previous sections (see also Appendix E) such an intuitive notion can be misleading.

Consider Eq. (2.11). We seek a limit in which the solution[40] $\psi(x)\exp(i\frac{\omega}{2}12^t)$ is independent of the relative time. We call this the NR limit of the equation. Since the Green's function determines essentially the time-development of the solution, we expect $\bar{G}_0(x) \equiv G_0(x)\exp(i\frac{\omega}{2}12^t)$ itself to be independent of time in this limit. A glance at Fig. 3 shows that for E sufficiently close to threshold, $\bar{G}_0(x)$ will asymptotically be essentially independent of the relative time. Now, letting $m_{1,2} \to \epsilon\, m_{1,2}$ and rescaling simultaneously E so that the ratio $\eta = E/(m_1 + m_2)$ remains fixed, we have

$$G_0(x) \to G_0(\epsilon\, x) \quad \text{as} \quad m_{1,2} \to \epsilon\, m_{1,2} \quad . \tag{9.15}$$

Hence, as we let the masses $m_{1,2}$ tend simultaneously to infinity in the above manner, keeping E fixed (in units of $m_1 + m_2$), the Green's function $G_0(x)$ becomes asymptotic at decreasing values of x. This limiting process is of course equivalent to letting the mass M of the exchanged nucleon tend to zero, keeping all other parameters fixed. This was the point of view taken in Ref. 3, and shows that we must supplement the condition $|\eta| \simeq 1$ for a NR situation with the requirement that $M \to 0$, in order to arrive at a time-independent equation.

The precise asymptotic form of $\bar{G}_0(x)$ for $|\eta| \gtrsim 1$ was found in Sec. III to be given by [the same result applies here as to $H_0(x)$][41]

$$\bar{G}_0(x) \to \mathcal{G}_0(r) = \frac{i}{8\pi E} \frac{e^{ikr}}{r}, \tag{9.16}$$

where $k = i\sqrt{-\underset{\sim}{k}^2}$ in the bound state region. In the limit $m_{1,2} \to \infty$, $\eta = \text{const} \gtrsim 1$, we may thus replace equation (2.11) by

$$X(\underset{\sim}{r}) = U(\underset{\sim}{k})\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}} - \lambda \int d^3\underset{\sim}{r}'\, \mathcal{G}_0(\underset{\sim}{r} - \underset{\sim}{r}')\, V(\underset{\sim}{r}')\, X(-\underset{\sim}{r}') \tag{9.17}$$

where we have performed the time integration in (2.11) and introduced the definitions

$$\chi(\underset{\sim}{r}) \equiv [\psi(\underset{\sim}{r},t)\ \exp(i\ \tfrac{\omega}{2}\ 12^{t})]_{t=0}$$

$$\overset{0}{\chi}(\underset{\sim}{r}) = [m_1 + i\ \vec{\gamma}\cdot\vec{\nabla} + \gamma^{o}\ \omega_1]\ \overset{0}{\chi}_0(r)$$

$$V(\underset{\sim}{r}) = -\ i\ (4\pi)\ [M - i\ \vec{\gamma}\cdot\vec{\nabla} - \gamma^{0}\ \omega_{12}]\ V_0(r;M) \tag{9.18}$$

$$V_0(r;M) = \frac{1}{r}\ \exp\left[\ -\ \sqrt{M^2 - \omega_{12}^{\ 2}}\ r\ \right] .$$

We are thus left with an energy dependent potential, with an energy dependent range given by

$$r_0 = \left[M^2 - \left(\frac{m_1^{\ 2} - m_2^{\ 2}}{E}\right)\right]^{-\frac{1}{2}} . \tag{9.19}$$

By rewriting Eq. (9.17) in momentum space, we may compare the approximations involved here with those made in the case of the static limit. We will not go into a discussion of this point.

Equation (9.17) is readily converted to the differential equation

$$[m_1 - i\ \vec{\gamma}\cdot\vec{\nabla} - \gamma^{o}\ \omega_1]\ \chi(\underset{\sim}{r}) = -\ \frac{i\lambda}{2E}\ V(\underset{\sim}{r})\ \chi(-\underset{\sim}{r}) , \tag{9.20}$$

where we have made use of

$$[m_1 - i\ \vec{\gamma}\cdot\vec{\nabla} - \gamma^{o}\ \omega_1]\ \overset{0}{\chi}(\underset{\sim}{r}) = \frac{i}{2E}\ \delta^3(\underset{\sim}{r}) .$$

It should be noted that Eq. (9.20) as it stands is "supersingular" and cannot be expected to follow from the original "marginally singular" equation (2.11) by the above limiting process in the masses $m_{1,2}$. In fact, the solutions to such supersingular equations are known to exhibit essential singularities at the origin.[42] Hence if our approximations are to be meaningful under the stated limiting conditions, we will have to consider Eq. (2.11) in the presence of a cutoff, just as in the case of the static limit. This will be done in the following section. For the present discussion we will assume $V_0(r;M)$ in (9.18) to have been replaced by a sufficiently well behaved potential $U_0(r)$ such as to make the approximations leading to Eq. (9.20) meaningful.

Equation (9.20) is of the form of the Dirac equation for a relativistic spin $\frac{1}{2}$ particle moving in some fixed external, spin-dependent field, an interpretation not suggested by our approximations. We proceed therefore to cast Eq. (9.20) into a more manifest, non-relativistic form, by rewriting it as an equation for the "dominant" (upper) component $\varphi(\underset{\sim}{r})$ in $\chi(\underset{\sim}{r})$:

$$(\underset{\sim}{\nabla}^2 + \underset{\sim}{k}^2)\, \varphi(\underset{\sim}{r}) = [U(\underset{\sim}{r}) - \frac{2}{r}\, \Phi(r)\, \vec{\sigma}\cdot\vec{L}]\, \varphi(-\underset{\sim}{r}) \tag{9.21}$$

where $L = -i\vec{r} \times \vec{\nabla}$,

$$\Phi(r) = \frac{d}{dr}\, \tilde{U}_0(r)\,, \quad \tilde{U}_0(r) = \frac{2\pi}{E}\, \lambda\, U_0(r)\,,$$

$$U(r) = (m_1 + \omega_1)(M - \omega_{12})\, \tilde{U}_0(r) - \frac{d}{dr}\, \Phi(r) + \Phi(r)^2\, \rho(\underset{\sim}{r}) \tag{9.22}$$

and $\mathcal{P}(\underset{\sim}{r})$ is the operator introduced in Sec. V. In arriving at (9.21) we have assumed that effectively $|\zeta\, U_0(r)| << 1$, where $\zeta = (M - \omega_{12})/(m_1 + \omega_1)$. This condition can be met for weakly bound states and for a sufficiently weak coupling constant λ, if $U_0(r) \sim r^{\alpha}$ $(\alpha \geqslant 0)$ as $r \to 0$. In particular, for $M = 0$ and $m_1 = m_2$, Eq. (9.21) follows exactly from (9.20).

D. Numerical Results.

Although our discussion in this section has been largely academic, it is still of some interest to compare the eigenvalue spectrum of the various equations considered in this paper. We have thus solved numerically Eqs. (3.1), (9.14), and (9.20) for energies below the elastic threshold and for the choice of invariant cutoff

$$g(p) = \frac{\Lambda^2}{p^2 + \Lambda^2} \tag{9.23}$$

where p^2 is the square of the four-momentum transfer from the nucleon to the pion in the exchange graph of Fig. 2. The reason for this choice is that it still allows for a simple connection between the various equations considered in this paper. Thus, when the definition $\xi(\Lambda, M) = \Lambda^2/(\Lambda^2 - M^2)$ is introduced, the replacement $\tilde{I}(p) \to \tilde{I}(p)\, g(p)$ in Eq. (9.5) leads in coordinate space to the replacement of $I(x)$ by[43]

$$I'(x) = 4\, \xi(\Lambda, M)[M - i\, \gamma\cdot\partial] \left[M \frac{K_1(M|x|)}{|x|} - \Lambda \frac{K_1(\Lambda|x|)}{|x|} \right] \tag{9.24}$$

in Eq. (2.11), and to the replacement of $V_0(r;M)$ by

$$U_0(r) = \xi(\Lambda,M)\,[V_0(r;M) - V_0(r;\Lambda)] \tag{9.25}$$

in Eq. (9.18). Thus, for $\Lambda > M$ the introduction of the cutoff (9.23) leads to a modification of the short-range part of the force, leaving the long-range part unchanged.

For Eq. (9.14) our choice of cutoff introduces of course undesirable singularities into the kernel of the original integral Eq. (9.5). Hence, rather than trying to justify the intermediate steps, we simply proceed by analogy with the substitutions (9.24) and (9.25) and replace the potential (9.13) by

$$\mathcal{V}^{\ell\pm}(p,p';E) = \frac{4\pi^2}{M}\,\xi(\Lambda,M)\,[(2\,y_M\,Q_\ell(y_M) - Q_{\ell\pm1}(y_M)) - (M \to \Lambda)]. \tag{9.26}$$

Equation (9.20) was solved by again using variational techniques, the stationary expression considered in this connection being formally the same as in (7.2). Since the total angular momentum J and the parity are evidently good quantum numbers, we may take the solutions to be of the form

$$\varphi^{\ell\pm M}(\underset{\sim}{r}) = \begin{pmatrix} F^{\ell\pm}(r)\;\; \mathcal{Y}^{\ell\pm M}(\hat{r}) \\ G^{\ell\pm}(r)\;(-i\,\underset{\sim}{\sigma}\cdot\hat{r})\;\mathcal{Y}^{\ell\pm M}(\hat{r}) \end{pmatrix}$$

with parity $(-)^{\ell}$. We easily deduce the boundary conditions

$$F^{\ell\pm}(r) \sim r^{\ell}, \quad G^{\ell\pm}(r) \sim r^{\ell\pm 1}, \quad \text{as} \quad r \to 0,$$

and the asymptotic behavior $F^{\ell\pm}(r), \; G^{\ell\pm}(r) \sim \exp\left[-\sqrt{m_1^2 - \omega_1^2}\; r\right]$.

On the other hand, Eq. (9.14) was solved directly in momentum space by the usual meshpoint techniques. In order to maximize the order of the polynomial that could be integrated exactly for a given number of meshpoints, we used the Gaussian integration techniques.[32] Still it was found, that we needed to invert matrices of sizes comparable to those inverted in the solution of Eq. (3.1) if a comparable accuracy was to be achieved. This may be traced to the fairly complicated singularity structure of the Kernel in (9.14). Indeed, we also considered the conventional integral equation in the $V\theta$ sector of the Lee model for some suitable choice of cutoff, treating the V-quantum as bare. This corresponds to taking the extreme static limit in Eq. (9.14), and dropping the spin dependence. The analytic structure of this equation is particularly simple, and a remarkable accuracy of the computed eigenvalues was achieved already at fairly small matrix sizes. The singularity structure of the kernel is thus seen to play a very important role in the numerical solution of an integral equation. In practice we may deal with these problems by incorporating, in the variational approach, the "leading" singularities of the solution (provided they are known) into our trial functions, or by absorbing, in the meshpoint

approach, some of the singularities of the kernel into the weight function, using Gaussian integration techniques.

In Table XIV, Fig. 12 we have summarized our results for the three equations referred to above. For the choice $m_1 = m_2 = M = 1.0$ and $E/(m_1 + m_2) = 0.95$ we computed the coupling constant eigenvalues at several values of the cutoff parameter Λ. The quantum numbers chosen were those of the N^*. We have listed only the lowest, positive value of λ in the coupling constant spectrum. The sharp rise of the λ vs. Λ curves in Fig. 12 as $\Lambda \to 0$ is of course due to the choice (9.23) for the cutoff function. We have also studied the solutions for the choice $m_1 = M = 1.0$, $m_2 = 0.144$, corresponding to the actual pion-nucleon kinematics. As our discussion in Part A of this section would have anticipated, there was no qualitative difference between these results and those presented in Fig. 12.

X. SUMMARY

Our calculation has demonstrated that even marginally singular Bethe-Salpeter equations may be solved by standard numerical techniques, provided that proper care is taken of the boundary conditions at the origin (of the Wick-rotated space-time). From the computational point of view, the presence of spin in the problem did not in any fundamental way affect our ability to solve the equation, although the algebraic aspect of the problem was considerably more involved than in the absence of spin, the added complexity being principally due to the particular approach taken in this paper. Thus we could have chosen to solve the BS equation directly in the form of a two-dimensional differential or integral equation. Other[4] calculations have shown, however, that such an approach would require the inversion of typically 100-by-100-dimensional matrices in order to compute the desired numbers to an accuracy of a few percent. This is to be compared with our calculation in which twenty-seven was the maximum "effective" matrix size ever considered, the corresponding accuracy of the computed eigenvalues being frequently better than 1%. In fact, in most of the cases the inversion of a 3 by 3 matrix was sufficient to obtain an accuracy of about 10%. Our particular treatment offers additional advantages as well. Thus we recall that in practice we needed to include at most the first five terms in the expansion of the solution in terms of the four-dimensional spherical harmonics, in order to achieve a 0.1% accuracy (or better). Moreover, our approach allows us to give a proper treatment of the boundary conditions at the origin. In fact, the

critical dependence of these boundary conditions on the coupling constant strength in marginally singular equations such as considered in this paper suggests, that our approach may well be the approach to the singular problem. Indeed, attempts to solve the marginally singular integral equation for the $V\theta$ sector of the Lee model (in the full static limit and in the absence of a cutoff) by meshpoint techniques have led to difficulties even with a proper treatment of the boundary conditions at infinity. This makes it questionable whether the singular BS equation, in the form of a two-dimensional integral equation, could, be solved successfully by the usual meshpoint techniques.

As for the physical content of our calculation, we do not know the range of validity of the ladder approximation, nor is it generally believed that the forces in nature are as singular as the ones we have considered (in the absence of a cutoff). In addition, a more realistic treatment would have to include the forces arising from the exchange of higher spin particles, such as the N^* for example. We have thus strongly emphasized the mathematical rather than the physical aspect of this problem.

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APPENDIX A: THE 3- AND 4-DIMENSIONAL SPINOR SPHERICAL HARMONICS

We list here some well known properties of the conventionally used spinor spherical harmonics in three and four dimensions. The notation introduced in this appendix is used throughout this work.

Three-Dimensional Spinor Spherical Harmonics

Definition:

$$\mathcal{Y}^{\ell\pm M}(\hat{r})_\alpha = \sum_m i^\ell \, C(\ell \tfrac{1}{2} \ell \pm \tfrac{1}{2} ; m \, \alpha \, M) \, \mathcal{Y}_{\ell m}(\hat{r}) \tag{A1}$$

where $C(\ell, s, J; m, \alpha, M)$ are the Clebsch-Gordon coefficients as defined by Rose,[44] and $\mathcal{Y}_{\ell m}(\hat{r})$ are the usual, spin-zero spherical harmonics, with $\hat{r} = \underset{\sim}{r}/r$. The functions $\mathcal{Y}^{\ell\pm M}(\hat{r})_\alpha$ are orthonormal,

$$\sum_\alpha \int d\Omega_r \, \mathcal{Y}^{\ell\pm M}(\hat{r})^*_\alpha \, \mathcal{Y}^{\ell'\pm M'}(\hat{r})_\alpha = \delta_{\ell\pm, \ell'\pm} \, \delta_{MM'}$$

and satisfy the following useful relations: (we omit the magnetic quantum number M)

$$(\vec{\sigma}\cdot\hat{r}) \, \mathcal{Y}^{\ell\pm}(\hat{r}) = \pm i \, \mathcal{Y}^{(\ell\pm 1)\mp}(\hat{r})$$

$$-i(\vec{\sigma}\cdot\vec{\nabla}) \, \mathcal{Y}^{\ell+}(\hat{r}) = \mathcal{Y}^{(\ell+1)-}(\hat{r}) \left(\frac{d}{dr} - \frac{\ell}{r}\right) , \tag{A2}$$

$$i(\vec{\sigma}\cdot\vec{\nabla}) \mathcal{Y}^{\ell-}(\hat{r}) = \mathcal{Y}^{(\ell-1)+}(\hat{r}) \left(\frac{d}{dr} + \frac{\ell+1}{r}\right) .$$

In those cases in which the magnetic quantum number M is of no relevance (as is the case if we are dealing with rotationally invariant quantities, such as the T matrix, for instance) it is convenient to introduce the functions

$$Z^{\ell_\pm}_{\hat{k}}(\hat{k}')_{\alpha\beta} = 4\pi \sum_M \mathcal{Y}^{\ell_\pm M}(\hat{k}')_\alpha \; \mathcal{Y}^{\ell_\pm M}(\hat{k})^*_\beta . \qquad (A3)$$

Evidently

$$Z^{\ell_\pm}_{\hat{k}}(\hat{k}')^\dagger = Z^{\ell_\pm}_{\hat{k}'}(\hat{k}) ;$$

the additional theorem for the spherical harmonics (A1) then takes the form $\sum_\pm Z^{\ell_\pm}_{\hat{k}}(\hat{k}') = (2\ell + 1)\mathcal{P}_\ell(\hat{k}\cdot\hat{k}')$. We also have the orthonormality property

$$\int \frac{d\Omega_r}{4\pi} \; Z^{\ell_\pm}_{\hat{k}}(\hat{r})^\dagger \; Z^{\ell'_\pm}_{\hat{k}'}(\hat{r}) = \delta_{\ell_\pm, \ell'_\pm} \; Z^{\ell_\pm}_{\hat{k}'}(\hat{k}) .$$

Finally, it follows directly from (A2) that

$$(\vec{\sigma}\cdot\hat{k}') \; Z^{\ell_\pm}_{\hat{k}}(\hat{k}') \; (\vec{\sigma}\cdot\hat{k}) = Z^{(\ell\pm 1)}_{\hat{k}}{}_{\mp}(\hat{k}') .$$

Four-Dimensional Spinor Spherical Harmonics

We introduce first the notation and conventions which are followed throughout our work. A point $(x_\alpha) = (x_1, x_2, x_3, x_4) = (\underset{\sim}{r}, \tau)$ in the four dimensional Euclidean space can also be represented in terms of the length $R = (\underset{\sim}{r}^2 + \tau^2)^{\frac{1}{2}}$ of the four-vector, the polar and

azimuthal angles ϑ and ϕ in the $(x_1 x_2 x_3)$ plane, and the angle θ between the four-vector and the four-axis:

$$x = (R\sin\theta \sin\vartheta \sin\phi,\ R\sin\theta \sin\vartheta \cos\phi,\ R\sin\theta \cos\vartheta,\ R\cos\theta) . \tag{A4}$$

We use the notation $\hat{R} = (x_\alpha/R)$ to denote a four-vector of unit length. The differential element of solid angle $d\Omega_R$ and the differential volume element $d\,V$ are given by

$$\begin{aligned} d\Omega_R &= \sin^2\theta \sin\vartheta \; d\theta \; d\vartheta \; d\phi \; . \\ d\,V &= R^3 \, d\,R \, d\Omega_R \; . \end{aligned} \tag{A5}$$

The four dimensional spinor-spherical harmonics are defined by

$$Y_n^{\ell\pm M}(\hat{R})_\alpha = N_n^{\ell} \; \mathcal{Y}^{\ell\pm M}(\hat{r})_\alpha \; \sin^\ell\theta \; C_{n-\ell}^{\ell+1}(\cos\theta) \; . \tag{A6}$$

Here $C_n^{\ell}(t)$ are the Gegenbauer polynomials[23] and N_n^{ℓ} is the normalization constant

$$N_n^{\ell} = \left[\frac{2^{2\ell+1}(n+1)(n-\ell)!\,(\ell!)^2}{\pi(n+\ell+1)!} \right]^{\frac{1}{2}} \; . \tag{A7}$$

The functions $Y_n^{\ell\pm M}$ are orthonormal,

$$\sum_\alpha \int d\Omega_R \; Y_n^{\ell\pm M}(\hat{R})^*_\alpha \; Y_{n'}^{\ell'\pm M'}(\hat{R})_\alpha = \delta_{\ell\pm\ell'\pm} \, \delta_{nn'} \, \delta_{MM'} \; , \tag{A8}$$

and satisfy the completeness relation

$$\sum_{\ell_\pm, M, n} Y_n^{\ell_\pm M}(\hat{R})^*_\alpha \; Y_n^{\ell_\pm M}(\hat{R}')_\beta = \delta_{\alpha\beta} \; \delta(\Omega_{R'} - \Omega_R) \tag{A9}$$

as well as the addition theorem

$$\sum_{\ell_\pm, M} Y_n^{\ell_\pm M}(\hat{R})^*_\alpha \; Y_n^{\ell_\pm M}(\hat{R}')_\beta = \delta_{\alpha\beta} \; \frac{n+1}{2\pi^2} \; C_n^{1}(\cos\gamma) \tag{A10}$$

where

$$\begin{aligned} \cos\gamma &= \hat{R}\cdot\hat{R}' = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\omega \\ \cos\omega &= \cos\vartheta\cos\vartheta' + \sin\vartheta\sin\vartheta'\cos(\phi - \phi'). \end{aligned} \tag{A11}$$

APPENDIX B: THE FOUR-DIMENSIONAL ROTATION GROUP*

In this appendix we construct the projection operators $\mathcal{P}_{n_\pm}$ and $\tilde{\mathcal{P}}_{n_\pm}$ for the irreducible spaces of the representations $(\frac{n \pm 1}{2}, \frac{n}{2})$ and $(\frac{n}{2}, \frac{n \pm 1}{2})$ of O_4, the four-dimensional rotation group. It is well known[45] that the group O_4 is characterized by the six Hermitian generators of infinitesimal rotations A_i and B_i $(i = 1,2,3)$ with the familiar angular momentum commutation relations

$$[A_i, A_j] = i\,\epsilon_{ijk}\,A_k, \quad [B_i, B_j] = i\,\epsilon_{ijk}\,B_k\,.$$

Moreover $[A_i, B_i] = 0$. $\underset{\sim}{A}^2$ and $\underset{\sim}{B}^2$ are the two Casimir operators of the group; in a particular irreducible representation labeled by the pair of indices (a,b), $\underset{\sim}{A}^2 = a(a+1)I$, $\underset{\sim}{B}^2 = b(b+1)I$; the representations (a,b) are unitarity and of dimensionality $(2a+1)\times(2b+1)$. To give a specific example we note that on the space of scalar fields, A_i and B_i are just the differential operators

$$\begin{aligned} A_i &= \frac{1}{2}\,[L_i - i(x_i\,\partial_4 - x_4\,\partial_i)] \\ B_i &= \frac{1}{2}\,[L_i + i(x_i\,\partial_4 - x_4\,\partial_i)] \end{aligned} \tag{B1}$$

where $\underset{\sim}{L} = -i(\underset{\sim}{r} \times \underset{\sim}{\partial})$ are the usual generators of infinitesimal rotations in three dimensions.

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* I am indepted to Professor C. Zemach for very informative discussions on the four-dimensional rotation group.

In the following discussion we denote by (A_i, B_i) the six generators in the $(\frac{n}{2}, \frac{n}{2})$ representation and by (a_i, b_i), $(\tilde{a}_i, \tilde{b}_i)$ the generators in the $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representation, respectively. We have then in the $(\frac{n}{2}, \frac{n}{2})$ representation

$$\underset{\sim}{A}^2 = \frac{1}{4} n (n+2) I , \quad \underset{\sim}{B}^2 = \frac{1}{4} n(n+2) I \quad ;$$

in the $(\frac{1}{2}, 0)$ representation,

$$a_i = \tfrac{1}{2} \sigma_i , \quad b_i = 0$$

$$\underset{\sim}{a}^2 = \frac{3}{4} \quad , \quad \underset{\sim}{b}^2 = 0 \quad ;$$

in the $(0, \frac{1}{2})$ representation,

$$\tilde{a}_i = 0 \quad , \quad \tilde{b}_i = \tfrac{1}{2} \sigma_i$$

$$\underset{\sim}{\tilde{a}}^2 = 0 \qquad \underset{\sim}{\tilde{b}}^2 = \frac{3}{4} \quad .$$

We have the following reduction of the direct product space,

$$(\tfrac{n}{2}, \tfrac{n}{2}) \otimes (\tfrac{1}{2}, 0) \sim (\tfrac{n+1}{2}, \tfrac{n}{2}) \oplus (\tfrac{n-1}{2}, \tfrac{n}{2}) ,$$

$$(\tfrac{n}{2}, \tfrac{n}{2}) \otimes (0, \tfrac{1}{2}) \sim (\tfrac{n}{2}, \tfrac{n+1}{2}) \oplus (\tfrac{n}{2}, \tfrac{n-1}{2}) .$$

The combined set of functions $Y_n^{\ell_+ M}(\hat{R})$ and $Y_n^{(\ell+1)_-, M}(\hat{R})$ defined in (A6) (with the indices ℓ and M taking on all allowed values for a given value of n) form a basis for either of these two $2(n+1)^2$-dimensional direct product spaces. In Sec. IV we make use of precisely this fact, when we construct the basis functions for the irreducible spaces $(\frac{n\pm1}{2},\frac{n}{2})$ and $(\frac{n}{2},\frac{n\pm1}{2})$ contained in the above reduction of the direct product space.

We construct next the projection operators for the irreducible spaces $(\frac{n\pm1}{2},\frac{n}{2})$ and $(\frac{n}{2},\frac{n\pm1}{2})$ which we denote by $\mathcal{P}_{n_\pm}$ and $\tilde{\mathcal{P}}_{n_\pm}$ respectively. To this end we define the generators of infinitesimal rotations in the above direct product spaces (the direct product with the identity is understood, so that $A_i = A_i\,(\frac{n}{2},\frac{n}{2}) \otimes I(\frac{1}{2},0)$, etc):

$$\mathcal{A}_i = A_i + a_i \quad , \quad \mathcal{B}_i = B_i + b_i \quad ,$$

$$\tilde{\mathcal{A}}_i = A_i + \tilde{a}_i \quad , \quad \tilde{\mathcal{B}}_i = B_i + \tilde{b}_i \quad .$$

Now, in the $(\frac{n+1}{2},\frac{n}{2})$ representation,

$$\underset{\sim}{\mathcal{A}}^2 = \frac{n^2+4n+3}{4} = \frac{n(n+2)}{4} + \frac{3}{4} + \underset{\sim}{\sigma}\cdot\underset{\sim}{A}$$

$$\underset{\sim}{\mathcal{B}}^2 = \frac{1}{4}\,n(n+2)$$

or

$$\underset{\sim}{\sigma}\cdot\underset{\sim}{A} = \frac{n}{2} \, .$$

Similarly, in the $(\frac{n-1}{2}, \frac{n}{2})$ representation,

$$\underset{\sim}{\mathcal{A}}^2 = \frac{n^2 - 1}{4} = \frac{n(n+2)}{4} + \frac{3}{4} + \underset{\sim}{\sigma}\cdot\underset{\sim}{A}$$

$$\underset{\sim}{\mathcal{B}}^2 = \frac{1}{4}\, n(n+2)$$

or

$$\underset{\sim}{\sigma}\cdot\underset{\sim}{A} = -\frac{(n+2)}{2} .$$

In exactly the same manner we conclude that

$$\underset{\sim}{\sigma}\cdot\underset{\sim}{B} = \frac{n}{2} \qquad \text{in the } \left(\frac{n}{2}, \frac{n+1}{2}\right) \text{ representation}$$

$$\underset{\sim}{\sigma}\cdot\underset{\sim}{B} = -\frac{(n+2)}{2} \text{ in the } \left(\frac{n}{2}, \frac{n-1}{2}\right) \text{ representation.}$$

The desired projection operators are thus given by

$$\begin{aligned} \mathcal{P}_{n_+} &= \frac{\frac{1}{2}(n+2) + \underset{\sim}{\sigma}\cdot\underset{\sim}{A}}{n+1}, \qquad \mathcal{P}_{n_-} = \frac{\frac{1}{2}\, n - \underset{\sim}{\sigma}\cdot\underset{\sim}{A}}{n+1}, \\ \tilde{\mathcal{P}}_{n_+} &= \frac{\frac{1}{2}(n+2) + \underset{\sim}{\sigma}\cdot\underset{\sim}{B}}{n+1}, \qquad \tilde{\mathcal{P}}_{n_-} = \frac{\frac{1}{2}\, n - \underset{\sim}{\sigma}\cdot\underset{\sim}{B}}{n+1} . \end{aligned} \tag{B2}$$

We easily verify that

$$\mathcal{P}^2_{n_\pm} = \mathcal{P}_{n_\pm} ; \quad \mathcal{P}_{n_\pm}\mathcal{P}_{n_\mp} = 0 ; \quad \mathcal{P}_{n_+} + \mathcal{P}_{n_-} = 1$$

with $\tilde{\mathcal{P}}_{n_\pm}$ satisfying the same relations.

APPENDIX C: PROOF OF ELASTIC UNITARITY

In this appendix we show that the scattering amplitude $f(\underset{\sim}{k}',\underset{\sim}{k})$ defined in (2.18) satisfies elastic unitarity. For the purpose of this discussion it is convenient to "rotate" the t-integration contour in (2.18) to the imaginary axis. The result is

$$f(\underset{\sim}{k}',\underset{\sim}{k}) = \frac{\lambda m_1}{4\pi E}\, \tilde{U}(\underset{\sim}{k}') \int d^4x\, e^{-i\underset{\sim}{k}'\cdot\underset{\sim}{r}+\frac{\omega}{2}12\tau}\, V(x)\, \phi_{\underset{\sim}{k}}(-x) \tag{C1}$$

where $\phi_{\underset{\sim}{k}}(x)$ is the solution to Eq. (3.6). We easily show that

$$\gamma^0\, \phi_{\underset{\sim}{k}}(-\underset{\sim}{r},\tau) = \phi_{-\underset{\sim}{k}}(\underset{\sim}{r},\tau)\ . \tag{C2}$$

It follows then from (C1) and (C2) that

$$f^\dagger(\underset{\sim}{k},\underset{\sim}{k}') = \frac{\lambda m_1}{4\pi E} \int d^4x\ \phi^\dagger_{-\underset{\sim}{k}'}(x)\, V(-x)\, U(\underset{\sim}{k})\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}-\frac{\omega}{2}12\tau} \tag{C3}$$

where the "dagger" denotes the Hermitian conjugate in the spin space only, and $\phi^\dagger_{-\underset{\sim}{k}}(x)$ evidently satisfies the equation

$$\phi^\dagger_{-\underset{\sim}{k}}(-x) = \bar{U}(\underset{\sim}{k})\, e^{-i\underset{\sim}{k}\cdot\underset{\sim}{r}+\frac{\omega}{2}12\tau} - \lambda \int d^4x'\ \phi^\dagger_{-\underset{\sim}{k}}(x')\, V(-x')\, H^\dagger(x'-x). \tag{C4}$$

We note here parenthetically that with the aid of

$$C\, U^*(-\underset{\sim}{k})\, C^{-1} = U(\underset{\sim}{k}), \quad C\, V^*(x)\, C^{-1} = V(x) \tag{C5}$$

where C is the usual charge conjugation matrix as defined in (4.14), we easily show that

$$f(\underset{\sim}{k}',\underset{\sim}{k}) = f(-\underset{\sim}{k}', -\underset{\sim}{k}) \tag{C6}$$

and

$$f^{\dagger}(\underset{\sim}{k},\underset{\sim}{k}') = C\, f^{*}(-\underset{\sim}{k}', -\underset{\sim}{k})\, C^{-1} . \tag{C7}$$

Statements (C6) and (C7) follow of course directly from the invariance of the theory under space inversion and time-reflection. Combining the above results we obtain

$$\begin{aligned} &\frac{4\pi E}{m_1}\left[f(\underset{\sim}{k}', \underset{\sim}{k}) - f^{\dagger}(\underset{\sim}{k}, \underset{\sim}{k}')\right] \\ &= \lambda^2 \int d^4x\, d^4x'\, \phi^{\dagger}_{-k'}(x')\, V(-x')\, [H^{\dagger}(x'-x) - H(x'-x)]V(x)\, \phi_{\underset{\sim}{k}}(-x). \end{aligned} \tag{C8}$$

By making use of

$$[m - \vec{\gamma}\cdot\vec{p} + \gamma^{o}\sqrt{\underset{\sim}{p}^2 + m^2}\,] = 2m \sum_{\sigma} U(\underset{\sim}{p},\sigma)\, \bar{U}(\underset{\sim}{p},\sigma)$$

it is straightforward to show that

$$H(x) - H^{\dagger}(x) = \frac{i\, m_1 |\underset{\sim}{k}|}{2\pi E} \int \frac{d\Omega_k}{4\pi} \sum_{\sigma} U(\underset{\sim}{k},\sigma)\, \bar{U}(\underset{\sim}{k},\sigma)\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{r} - \frac{1}{2}\omega_{12}\tau} . \tag{C9}$$

Substituting (C9) into (C8) we arrive then at the statement of elastic unitarity,

$$i[f(\underset{\sim}{k}',\underset{\sim}{k}) - f^{\dagger}(\underset{\sim}{k},\underset{\sim}{k}')] = \frac{|\underset{\sim}{k}|}{2\pi} \int d\Omega_q \quad f(\underset{\sim}{k}',\underset{\sim}{q})\, f^{\dagger}(\underset{\sim}{k},\underset{\sim}{q}). \tag{C10}$$

With introduction of the partial wave expansion

$$f(\underset{\sim}{k}',\underset{\sim}{k}) = \sum_{\ell_\pm} \quad t^{\ell_\pm}(E) \quad Z^{\ell_\pm}_{\hat{k}}(\hat{k}')$$

where $Z^{\ell_\pm}_{\hat{k}}(\hat{k}')_{\alpha\beta}$ is the complete set of functions defined in (A3), the statement of elastic unitarity (C10) takes the following form for $t^{\ell_\pm}(E)$:

$$\mathrm{Im}\ t^{\ell_\pm}(E) = -|\underset{\sim}{k}| \quad |t^{\ell_\pm}(E)|^2 ,$$

where

$$t^{\ell_\pm}(E) = \frac{e^{i\delta_{\ell_\pm}} \sin \delta_{\ell_\pm}}{|\underset{\sim}{k}|} .$$

Note that with the usual phase convention implicit in our definition (2.20) of the scattering amplitude, the sign preceding the coupling constant λ in Eq. (3.6) is uniquely fixed by the requirement of elastic unitarity (C10).

APPENDIX D: SOME DETAILS ON THE STRUCTURE OF THE $E = 0$ SOLUTIONS

It was stated in Sec. V, that we could have arrived at the explicit forms (5.15) for the $E = 0$ solutions in a different way, independent of any group theoretical considerations. However, had we been unaware of the existing four-dimensional symmetry in the problem, we would most likely not have sought solutions involving only a few terms in an expansion such as (5.18). Thus, to repeat here our earlier arguments, the substitution of the expansion (5.18) into (5.1) would leave us with an infinite set of coupled equations for the radial functions $F_k(R)$ and $G_k(R)$. To seek a solution involving only a finite number of terms in the expansion (5.18) would imply a truncation of this infinite set of equations which would lead in general to an overdetermined problem. Only for $E = 0$ our analysis in Sec. V suggests that this overdetermined set of equations actually does have a solution. We will now present the details.

By substituting the expansion (5.18) into (5.1) we arrive at an infinite set of coupled equations. However, instead of starting from this set of equations, we will simplify our discussion, by making the following ansatz (which we know to be correct):

$$X_n^{\ell\pm}(x) = \begin{pmatrix} F(R)\, Y_n^{\ell\pm} + G(R)\, Y_{n+1}^{\ell\pm} \\ \tilde{F}(R)\, Y_n^{(\ell\pm1)\mp} + \tilde{G}(R)\, Y_{n+1}^{(\ell\pm1)\mp} \end{pmatrix} \tag{D1}$$

This ansatz will not detract from the point we are trying to make, and is only designed to simplify our presentation. From here on the

discussion will follow exactly the same lines as in the more general treatment. We will only consider the $X_n^{\ell+}(x)$ solutions, the corresponding results for the $X_n^{\ell-}(x)$ solutions being obtained in exactly the same way. Moreover, we may always obtain the $X_n^{\ell-}$ solutions from $X_n^{\ell+}$ with the aid of (5.17) which is a direct consequence of the commutation relations (5.4). Substituting the expansion (D1) for $X_n^{\ell+}$ into Eq. (5.1) we arrive at a set of eight coupled equations for the four radial functions. This is in general an overdetermined problem. If a solution is to exist at all we must require that the coefficients of Y_{n-1} and Y_{n+2} in these equations vanish identically. This leads to the conditions

$$\begin{aligned} A_n^{\ell} F &= C_n^{(\ell+1)-} \tilde{F} & A_{n+2}^{\ell} G &= -B_{n+1}^{(\ell+1)-} \tilde{G} \\ C_n^{\ell+} F &= A_n^{\ell+1} \tilde{F} & B_{n+1}^{\ell+} G &= -A_{n+2}^{\ell+1} \tilde{G} \quad . \end{aligned} \tag{D2}$$

Noting that

$$\begin{aligned} C_n^{(\ell\pm1)\mp} C_n^{\ell\pm} &= A_n^{\ell} A_n^{\ell\pm1} \\ B_n^{(\ell\pm1)\mp} B_n^{\ell\pm} &= A_{n+1}^{\ell} A_{n+1}^{\ell\pm1} \end{aligned} \tag{D3}$$

we conclude that the equations (D2) do indeed have a solution. We are thus left with the following set of equations:

$$m_1 K^n F + \left(\frac{d}{dR} + \frac{n+3}{R}\right) K^{n+1} \left[A^{\ell}_{n+1} G - C^{(\ell+1)-}_{n+1} \tilde{G}\right]$$

$$= \frac{4M^2}{R} (-)^n \lambda \left[K_1(MR) F + K_2(MR) \left(A^{\ell}_{n+1} G - C^{(\ell+1)-}_{n+1} \tilde{G}\right)\right] \tag{D4a}$$

$$m_1 K^{n+1} G + \left(\frac{d}{dR} - \frac{n}{R}\right) K^n \left[A^{\ell}_{n+1} F + B^{(\ell+1)-}_{n} \tilde{F}\right]$$

$$= \frac{4M^2}{R} (-)^{n+1} \lambda \left[K_1(MR) G + K_2(MR) \left(A^{\ell}_{n+1} F + B^{(\ell+1)-}_{n} \tilde{F}\right)\right] \tag{D4b}$$

$$m_1 K^n F + \left(\frac{d}{dR} + \frac{n+3}{R}\right) K^{n+1} \left[C^{\ell+}_{n+1} G - A^{\ell+1}_{n+1} \tilde{G}\right]$$

$$= \frac{4M^2}{R} (-)^n \lambda \left[K_1(MR) \tilde{F} + K_2(MR) \left(C^{\ell+}_{n+1} G - A^{\ell+1}_{n+1} \tilde{G}\right)\right] \tag{D5a}$$

$$m_1 K^{n+1} \tilde{G} - \left(\frac{d}{dR} - \frac{n}{R}\right) K^n \left[B^{\ell+}_{n} F + A^{\ell+1}_{n+1} \tilde{F}\right]$$

$$= \frac{4M^2}{R} (-)^{n+1} \lambda \left[K_1(MR) \tilde{G} - K_2(MR) \left(B^{\ell+}_{n} F + A^{\ell+1}_{n+1} \tilde{F}\right)\right] \tag{D5b}$$

where K^n is the operator defined in (5.7). These are four equations for only two radial functions (on account of (D2)). However, multiplying Eqs. (D5a) and (D5b) through by $C^{(\ell+1)-}_{n}/A^{\ell}_{n}$ and $-B^{(\ell+1)-}_{n+1}/A^{\ell}_{n+2}$ respectively, making use of (D2) and noting that

$$C^{(\ell+1)-}_{n} C^{\ell+}_{n+1} = A^{\ell}_{n} A^{\ell}_{n+1} \quad ; \quad B^{(\ell+1)-}_{n+1} B^{\ell+}_{n} = A^{\ell}_{n+2} A^{\ell}_{n+1}$$

$$A^{\ell+1}_{n+1} C^{(\ell+1)-}_{n} = A^{\ell}_{n} C^{(\ell+1)-}_{n+1} \quad ; \quad A^{\ell+1}_{n+1} B^{(\ell+1)-}_{n+1} = A^{\ell}_{n+2} B^{(\ell+1)-}_{n}$$

we conclude that Eqs. (D5) are in fact identical to Eqs. (D4), as it had to be of course, if the problem is to have a solution at all. Finally, a little more algebra shows that, aside from an overall normalization constant, the solutions (D1) are precisely of the form (5.15). In fact, comparing equations (D4) and (5.6) we easily verify with the aid of (D2) that they are the same equations.

APPENDIX E: A TIME-DEPENDENT FORMULATION OF THE LEE MODEL

In this appendix we underline still further our remarks in Part A of Sec. IX concerning the role of the relative time variable in a covariant theory, by discussing a simple, "time independent," field theoretical model in the Bethe-Salpeter language. We are referring here to the familiar Lee model.[37] Such simple models have already served in the past to clarify some poorly understood aspects of quantized field theories. In this section we give a very brief and elementary discussion of a particular sector of the Lee model, using the perhaps more familiar language of covariant perturbation theory. This might appear as making a "simple" problem unnessessarily complicated, but will serve to underline still further the equivalence of the "time-dependent" approach and the more old-fashioned "time-independent" approach to the theory of quantized fields. At the same time it will serve to clearify the connection between the Tamm-Dancoff techniques for solving Eq. (9.1) and the covariant Feynman approach. We will limit ourselves to the presentation of only a few examples. This will suffice to underline the basic ideas and the reader may proceed from here to recover all the familiar results discussed extensively in the literature.

The basic ingredient of our approach is, as we have pointed out already in Sec. X, that we will work in the "Interaction picture," rather than the "Schrodinger picture" as it is conventionally done. In that case the Lee-model Hamiltonian density is given by

$$\mathcal{H}_I(x) = \lambda_0 \int d^4x' \, \rho(x - x') \, [V^\dagger(x) \, N(x) \, \theta(x') + N^\dagger(x) \, V(x) \, \theta^\dagger(x')] \tag{E1}$$

where $V(x)$, $N(x)$ and $\theta(x)$ are the quantized fields in the interaction picture associated with the V, N and θ particles, antiparticles being ignored here altogether. The source function $\rho(x)$ has the effect of spreading out the distribution of the N and V quanta in space but not in time

$$\rho(x) = \tilde{\rho}(|\underset{\sim}{x}|)\, \delta(t) , \qquad \text{(E2)}$$

so that we may still seek our solutions to be stationary eigenstates of H_s (see (9.1)). From the Hamiltonian (E1) we generate in the usual fashion the full, unitary S-matrix from the Dyson expansion. Substituting this expansion into the definition (2.1) of the Bethe-Salpeter amplitude, we easily arrive at an integral equation in the $N\theta$ and $V\theta$ sectors of the Lee model. Let us consider first the rather trivial $N\theta$ sector. Making use of the translational invariance of the theory we arrive as in Sec. II at the integral equation

$$\psi_E(x) = e^{i\underset{\sim}{k}\cdot\underset{\sim}{r} - i\omega_{12}t/2} - \lambda_0^2 \int d^4y\, d^4y'\, G_E(x-y)\rho(y)V_E(y-y')\rho(y')\psi_E(y') \qquad \text{(E3)}$$

for the wave function $\psi_E(x)$ in the scattering region, the notation being as in Sec. II. In the N-θ c.m. system (we neglect the kinetic energy of the N and V-quantum),

$$V_E(y) = -i\, \frac{e^{-i\frac{1}{2}Ey^0}}{M_V^0 - E - i\epsilon} , \quad E = M_N + \omega(\underset{\sim}{k}) , \qquad \text{(E4)}$$

$$G_E(x) = -\frac{1}{(2\pi)^4}\int \frac{d^4q}{2\omega(\underset{\sim}{q})} \frac{e^{iq\cdot x}}{[M_N - (\frac{E}{2} + q^0) - i\epsilon][\omega(\underset{\sim}{q}) - (\frac{E}{2} - q^0) - i\epsilon]} . \tag{E4}$$

Here M_N is the bare (and physical) mass of the N-quantum, M_V^0 is the mass of the bare V-quantum and $\omega(\underset{\sim}{q}) = (\underset{\sim}{q}^2 + \mu^2)^{\frac{1}{2}}$, μ being the bare (and physical) mass of the Θ-quantum.

It follows from the form (E2) of the source function, that Eq. (E3) is really time independent. Inspite of this, even this trivial sector of the Lee model still serves to illustrate our point, and we will continue to use a covariant notation. In momentum space Eq. (E3) takes the form

$$\phi_E(p) = (2\pi)^4 \delta^4(p - k) + i\lambda_0^2 \frac{\tilde{G}_E(p)\,\vartheta(p)}{M_V^0 - E - i\epsilon}\int \frac{d^4p'}{(2\pi)^4}\,\vartheta(p')\,\phi_E(p') \tag{E5}$$

where $\phi_E(p)$, $\tilde{G}_E(p)$ and $\vartheta(p)$ are the Fourier transforms of $\psi_E(x)$, $G_E(x)$ and $\rho(x)$ respectively. On account of (E2), $\vartheta(p)$ is of course only a function of $\underset{\sim}{p}^2$ so that we may write $\vartheta(p) = f(\omega_p)$, where $\omega_p \equiv \omega(\underset{\sim}{p})$. If we continue Eq. (E5) from the scattering region into the bound state region, then $\phi_E(p)$ will exhibit a pole at the mass M_V of the physical V particle (by definition of the physical V particle). Hence writing

$$\phi_E(p) = (2\pi)^4 \delta^4(p - k) + \frac{R_E(p)}{E - M_V} \tag{E6}$$

and defining the bound state wave function $\varphi(p)$ as $R_E(p)$ evaluated at the bound state pole $E = M_V$, we arrive at the following homogeneous eigenvalue equation for $\varphi(p)$:

$$\varphi(p) = i\,\lambda_0^2 \frac{\tilde{G}_{M_V}(p)\, f(\omega_p)}{M_V^0 - M_V} \int \frac{d^4q}{(2\pi)^4}\, f(\omega_q)\, \varphi(q) \,. \tag{E7}$$

This eigenvalue problem can be solved, the desired eigenvalue M_V being given as the solution to the equation (notice the dependence of the "Greens function" on M_V itself)

$$M_V = M_V^0 - i\,\lambda_0^2 \int \frac{d^4p}{(2\pi)^4}\, f(\omega_p)^2\, \tilde{G}_{M_V}(p) \tag{E8}$$

which reduces of course to the familiar eigenvalue equation in the Lee model if we perform explicitly the p^0 integration. The function $R_E(p)$ may also be obtained in closed form (in terms of M_V)

$$R_E(p) = i\,\lambda_0^2 \frac{f(\omega_p)\, \tilde{G}_E(p)\, f(\omega_k)}{1 + \tilde{F}(E)} \tag{E9}$$

where, after performing the p^0 integration

$$\tilde{F}(E) = \frac{1}{(2\pi)^3} \int \frac{d^3\underset{\sim}{p}}{2\omega(\underset{\sim}{p})} \frac{f(\omega_p)^2}{[M_N + \omega(\underset{\sim}{p}) - E - i\epsilon]\,[M_N + \omega(\underset{\sim}{p}) - M_V - i\epsilon]} \,. \tag{E10}$$

We have thus obtained by our slightly different, <u>time dependent</u> approach a few of the familiar results usually obtained by solving the eigenvalue problem directly in the form of Eq. (9.1). In exactly the same spirit we may discuss the Lee model in the $V\theta$ sector. We will not do so since we note that the discussion of this sector (neglecting the self-energy effects for the V-quantum) has essentially been given

already in Part B of Sec. X. We note, however, that a discussion of the $V\theta$ sector in terms of BS amplitudes will lead to a genuine time-dependent integral equation, the role of the relative time variable being here to combine the contribution of the two and three particle states communicating with the $V\theta$ channel into a single, linear integral equation.

TABLE I. Leading behavior at $R \sim 0$ of the radial functions $F_{nk}^{\ell\pm}(R)$ and $G_{n,k\mp1}^{\ell\pm}(R)$ in the expansion (5.22) of the solution $\varphi_n^{\ell\pm}(x)$ to first order in E. We use the notation $F_{n,k}^{\ell\pm}(R) \sim G_{n,k\mp1}^{\ell\pm}(R) \sim R^{\gamma}$ as $R \to 0$; α and $\tilde{\alpha}$ are the solutions to the indicial equations (5.11a) for the particular value of n considered.

k	γ	γ
$n - 1$	$\alpha + 1$	$\tilde{\alpha} + 1$
n	α	$\tilde{\alpha}$
$n + 1$	$\alpha + 1$	$\tilde{\alpha} - 1$
$n + 2$	$\alpha + 2$	$\tilde{\alpha}$

TABLE II. Calculated values of the critical coupling constants λ_1 and λ_2 corresponding to the solution $\varphi^{\ell\pm}(x)$ for several values of angular momentum and parity.

$\ell_\pm$	λ_1	λ_2
0_+ , 1_-	-0.3849	0.3849
1_+ , 2_-	-0.6311	2.1126
2_+ , 3_-	-6.0646	0.8794
3_+ , 4_-	-1.1284	13.1284
4_+ , 5_-	-24.1926	1.3778

TABLE III. Computed values of E in the $J^P = 3/2^+$, $I = 3/2$ channel at several values of the exponential parameter γ and at several matrix sizes, for the choice of potential (7.9a) with $a = b = 1.6$ and $m_1 = m_2 = M = 1.0$. The form of the trial function is specified following the notation (7.12). The computed eigenvalues are seen to converge to the same extrapolated value $E = 1.649$ independent of the value of γ.

Form of trial fn.	Matrix size	(γ=0.5) E	(γ=.75) E	(γ=1.0) E	(γ=1.25) E	(γ=1.5) E
2;10	6	1.31352	1.53813	1.65943	1.69954	1.74158
3;210	12	1.40361	1.60520	1.64251	1.65616	1.67156
4;3200	18	1.51648	1.63484	1.65056	1.65775	1.66522
5;43100	26	1.58347	1.64345	1.64919	1.65171	1.65514
5;54200	32	1.61660	1.64671	1.64928	1.65044	1.65211
5;65300	38	1.63243	1.64804	1.64931	1.64984	1.65060
5;76410	46	1.64045	1.64863	1.64938	1.64975	1.65019
5;87520	54	1.64440	1.64893	1.64942	1.64968	1.64995
Extrapol. value	-	1.649	1.649	1.649	1.649	1.649

TABLE IVa,b. Computed values of E in the $J^P = 3/2^+$, $I = 3/2$ channel at two values of γ and at several matrix sizes, for the choice of potential (7.9a) with $a = b = 1.6$ and $m_1 = m_2 = M = 1.0$. Part (a) shows the effect of improving only on the radial dependence of the trial solution, and (b) shows the effect of improving only on the angular dependence of the trial solution. The correct value of E is 1.649.

(a)

Form of trial func.	(γ=0.5) E	(γ=1.0) E
2;22	1.44970	1.66377
2;33	1.55127	1.67021
2;44	1.61259	1.67060
2;55	1.64267	1.67119
2;66	1.65673	1.67138
2;77	1.66357	1.67138

(b)

Form of trial func.	(γ=0.5) E	(γ=1.0) E
3;111	1.349512	1.644824
4;1111	1.350293	1.645020
5;11111	1.350098	1.644629
6;111111	1.350098	1.644629
7;1111111	1.350098	1.644629
8;11111111	1.350098	1.644629

TABLE V. Values of the coupling constant required to give a bound state with $E = 1.0$ in the $J^P = \frac{1}{2}^+$, $I = \frac{1}{2}$ channel, as computed for a sequence of increasing matrix sizes and for two values of the parameter μ in (7.13). The potential was taken to be of the form (7.9b) with $R_c = 0.7$, and $m_1 = M = 1.0$, $m_2 = 0.144$ (the pion mass). The "convergence" of the numerical results is seen to be extremely poor for the choice $\mu = 0$, but is considerably better for the choice $\mu = 0.856$.

Form of trial func.	$\mu = 0.0$ λ	$\mu = 0.856$ λ
2;10	-0.9937	-0.9546
4;3200	-0.9612	-0.8636
5;54200	-0.7266	-0.8436
5;76410	-1.0129	-0.8361

TABLE VI. Computed values of E in the $J^P = 3/2^+$, $I = 3/2$ channel for several values of the parameters a and b in (7.9a), with $m_1 = m_2 = M = 1.0$.

(b = 1.6)		(b = 1.4)		(b = 1.2)	
a	E	a	E	a	E
1.6	1.649	1.9	1.757	1.9	1.946
1.9	1.401	2.5	1.375	2.5	1.667
2.5	0.856	3.0	1.012	3.0	1.394
3.0	0.359	4.0	0.18	4.0	0.75
5.0	1.890	6.0	1.82	6.0	1.94
6.0	1.642	7.0	1.61	7.0	1.78
7.0	1.341	8.0	1.32	8.0	1.55
8.0	0.987	9.0	1.01	10.0	0.96

TABLE VII. Computed values of E in the $J^P = 3/2^+$, $I = 3/2$ channel for a range of values of the parameter a in (7.9a), with b = 1.6 and $m_1 = M = 1.0$, $m_2 = 0.144$ (the pion mass).

a	E	a	E
1.6	1.04	2.2	0.478
1.9	0.77	2.3	0.373
2.0	0.67	2.4	0.266
2.1	0.579	2.5	0.156

TABLE VIII. Computed values of the coupling constant λ required to obtain a bound state at $E = 1.0$ in the $J^P = 3/2^+$, $I = 3/2$ channel, at several values of the cutoff parameter R_c in (7.9b), with $m_1 = M = 1.0$, and $m_2 = 0.144$ (the pion mass). (We present only the smallest positive value of λ in the spectrum.)

R_c	λ
0.3	2.11
0.5	2.25
0.7	2.38
1.0	2.53

TABLE IX. Computed bound state energies in the $J^P = 3/2^+$, $I = 3/2$ channel at various values of the exchanged mass M, for the choice of potential (7.9a) with $a = b = 1.6$ and $m_1 = m_2 = 1.0$.

M	E
1.0	1.649
0.8	1.702
0.6	1.783
0.5	1.832
0.4	1.890

TABLE X. Computed bound state energies in the $J^P = 3/2^+$, $I = 3/2$ channel at several values of m_2, for the choice of potential (7.9a) with $a = 1.9$, $b = 1.6$ and $m_1 = M = 1.0$.

m_2	E
1.0	1.401
0.9	1.309
0.8	1.218
0.6	1.047
0.4	0.903
0.2	0.83
0.144	0.77

TABLE XI. Computed bound state energies in the $J^P = 1/2^+$, $I = 3/2$ channel at several values of the parameters a and b in (7.9a), and $m_1 = m_2 = M = 1.0$.

(b = 0.35)		(b = 0.25)		(b = 0.1)	
a	E	a	E	a	E
3.5	1.75	3.0	1.85	2.5	1.896
4.0	1.52	3.5	1.625	3.0	1.652
4.5	1.23	4.0	1.32	3.5	1.341
5.0	0.90	4.5	1.01	4.0	0.963
8.0	1.72	5.0	0.58	4.5	0.504
10.0	1.25	7.0	1.80		

TABLE XII. Positive and negative energy spectrum in the $J^P = \frac{1}{2}^+$, $I = \frac{1}{2}$ channel as computed at several values of the coupling constant λ for the choice of potential (7.9b) with $R_c = 1.0$ and $m_1 = m_2 = M = 1.0$.

λ	E	λ	E
-0.5	-1.848	-1.0	-0.573
-0.6	-1.678	-1.1	-0.173
-0.7	-1.468	-1.2	0.302
-0.8	-1.215	-1.3	0.890
-0.9	-0.918	-1.4	1.705

TABLE XIII. Values of the coupling constant λ required to obtain the "nucleon" $(E = 1.0)$ as a bound state in the $J^P = \frac{1}{2}^+$, $I = \frac{1}{2}$ channel, as computed for several values of the cutoff parameter R_c in (7.9b), with $m_1 = M = 1.0$, and $m_2 = 0.144$ (the pion mass). (We present only the smallest value of $|\lambda|$ in the negative coupling constant spectrum.)

R_c	λ
0.3	-0.69
0.5	-0.77
0.7	-0.83
1.0	-0.90

TABLE XIV. Computed values of λ for $J^P = 3/2^+$, $I = 3/2$, $E = 1.8$, $m_1 = m_2 = M = 1.0$, at various values of Λ for the choice of cutoff (9.23). Data (A), (B) and (C) refer to Eqs. (3.1), (9.14) and (9.20) respectively.

Λ	(A) λ	(B) λ	(C) λ
0.5	8.37	6.77	2.92
1.5	3.16	2.15	0.86
3.0	2.23	1.27	0.47
5.0	1.91	0.95	0.32
10.0	1.69	0.68	0.20
20.0	1.63	0.53	0.15

FOOTNOTES AND REFERENCES

1. E. E. Salpeter and H. Bethe, Phys. Rev. 84, 1232 (1951).

2. C. Schwartz, Phys. Rev. 137, B717 (1965).

3. C. Schwartz and C. Zemach, Phys. Rev. 141, 1454 (1966).

4. M. Levine, J. Tjon, and J. Wright, Phys. Rev. Letters 16, 962 (1966) and Phys. Rev. 154, 1433 (1967).

5. Throughout this paper we refer to the pseudoscalar meson as the "pion" and to the spin-$\frac{1}{2}$ particle as the "nucleon," although we do not restrict the masses to have the experimentally measured values.

6. E. Abers and C. Zemach, Phys. Rev. 131, 2305 (1963).

7. Inspite of this we were still forced in several instances to introduce a cutoff.

8. For a discussion of this question see J. S. Goldstein, Phys. Rev. 91, 1516 (1953); the nucleon-nucleon problem has also been discussed by M. Gourdin, Nuovo Cimento 7, 364 (1958), and by A. R. Swift and B. W. Lee, Phys. Rev. 131, 1857 (1963).

9. A. Bastai, L. Bertocchi, S. Fubini, G. Furlan and M. Tonin, Nuovo Cimento 30, 1512 (1963); A. Bastai, L. Bertocchi, G. Furlan and M. Tonin, ibid. 30, 1532.

10. M. Gell-Mann and F. Low, Phys. Rev. 84, 350 (1951).

11. See for instance Silvan S. Schweber, An Introduction to Relativistic Quantum Field Theory (Row, Peterson and Company, Evanston Illinois, and Elmsford, New York, 1961), p. 690.

12. Throughout this paper we label the variables refering to the nucleon and pion by the indices 1 and 2 respectively.

13. F. J. Dyson, Phys. Rev. 75, 486 (1949); F. J. Dyson, ibid. 75, 1736 (1949).

14. See G. N. Watson, A Treatise on the Theory of Bessel Functions (University Press, Cambridge 1962), 2nd ed.

15. This choice turns out to be undesirable in the case where the masses of the "pion" and "nucleon" are very different. This point will be discussed in Sec. VII-B.

16. We use the following normalization of the Fourier transforms:

$$F(x) = \int \frac{d^n p}{(2\pi)^n} \tilde{F}(p) \; e^{ip\cdot x}$$

17. G. C. Wick, Phys. Rev. 96, 1124 (1954).

18. Rigorously speaking, the continuation should be performed by analytically continuing first in the t-variable and distorting subsequently the t' integration contour as desired, picking up the contribution from the singularities which are encountered in this process. Since $K_0(z)$ is regular in the z-plane cut along the negative real axis, it follows from the definition (2.16) of the Green's function that we need to specify the sheet of $f(t) = \sqrt{x(t)} = [(\underset{\sim}{r} - \underset{\sim}{r}')^2 - (t - t')^2]^{\frac{1}{2}}$ ($\underset{\sim}{r}$, $\underset{\sim}{r}'$ and t' being treated as parameters) when performing the continuation in t. If we cut the x-plane along the negative real axis, the argument of the Bessel function in (2.16) will remain in the first and fourth quadrants so that the desired continuation in t and subsequent distortion in the t'-integration contour may be completed.

19. See, for example, P. M. Morse and H. Feshbach, Method of Theoretical Physics, Part I (McGraw-Hill Book Company, Inc., New York, 1953), 437.

20. S. W. MacDowell, Phys. Rev. 116, 774 (1960).

21. J. D. Stack, A Study of Regge Dynamics in pion-nucleon scattering (Ph. D. thesis), UCRL-16115, Apr. 27, 1965.

22. We omit in general the label M for the magnetic quantum number unless its presence is strictly required, in order to simplify the notation.

23. We take the definitions from W. Magnus and F. Oberhettinger, Functions of Mathematical Physics (Chelsea Publishing Company, New York, 1949).

24. To prove these transformation laws we consider an infinitesimal rotation $R = 1 - \frac{1}{2} i\omega_{\alpha\beta} J_{\alpha\beta} = 1 - i \underset{\sim}{M}\cdot\underset{\sim}{\theta} - i\underset{\sim}{N}\cdot\underset{\sim}{\phi}$, where the Hermitian matrices M_i and N_i $(i = 1, 2, 3)$ are evidently related to the generators $J_{\alpha\beta}$ as follows: $M_i = \frac{1}{2} \epsilon_{ijk} J_{jk}$, $N_i = J_{i4}$; and where $\theta_i = \frac{1}{2} \epsilon_{ijk} \omega_{jk}$ and $\phi_i = \omega_{i4}$. In particular we have in the $(\frac{1}{2},0)$ representation $\underset{\sim}{M} = \underset{\sim}{N} = \frac{1}{2} \underset{\sim}{\sigma}$; and in the $(0,\frac{1}{2})$ representation $\underset{\sim}{M} = - \underset{\sim}{N} = \frac{1}{2} \underset{\sim}{\sigma}$. Notice that in some other conventions $M_i = J_{4i}$. In these cases our representation (s,s') would correspond to the representation (s',s) in the other convention.

25. We do not label the radial functions by the quantum numbers J and M, since they are independent of them.

26. I should like to thank Dr. L. Schlessinger for suggesting this approach to the problem.

27. (Reference 27 has been skipped).

28. This statement is not quite so empty as it may appear. To give an example, the Schrodinger equation is left unchanged by the substitution $\ell \to -(\ell + 1)$, so that we conclude that if $u^{\ell}(r)$ is a solution, then so is $u^{-(\ell+1)}(r)$. However, if $u^{\ell}(r)$ is "regular" at $r = 0$ then $u^{-(\ell+1)}(r)$ is not; hence only $u^{\ell}(r)$ solves the eigenvalue problem.

29. K. M. Case, Phys. Rev. 80, 797 (1950).

30. It follows from (5.12) and the property $\alpha_m = -\tilde{\alpha}_{-m}$ that it is sufficient to calculate $\lambda_{1,2}$ for the first equation in (5.11a) only.

31. This does not represent an improvement, since our basis functions (7.7) place the time coordinate on equal footing with the space coordinates. This is of course the basic difficulty of dealing with weakly bound states in our approach.

32. For a review of the Gaussian Integration techniques see, for instance, R. E. Kreps, "Comments on Gaussian Integration and Integral Equations," Dept. of Physics, University of Pittsburgh.

33. By "act" attractively (repulsively) we mean that the binding energy of the bound state increases (decreases) as we increase the magnitude of the coupling constant. We will call a force attractive whenever it is capable of giving a bound state.

34. Note that the terms "ground state" and "excited state" refer here to a given quantum number n, and in particular to $n = \ell$ for the ℓ_+ solution, and to $n = \ell - 1$ for the ℓ_- solution.

35. G. F. Chew and F. E. Low, Phys. Rev. 107, 1570 (1956).

36. R. Blankenbecler and R. Sugar, Phys. Rev. 142, 1051 (1956).

37. T. D. Lee, Phys. Rev. 95, 1329 (1954); see also R. D. Amado, Phys. Rev. 122, 696 (1961); Taizo Muta, Phys. Rev. 33, 666 (1965).

38. We refer here specifically to the pion-nucleon problem. We will denote the nucleon and pion mass by M and μ respectively. The notation $E(\underset{\sim}{p}) = (\underset{\sim}{p}^2 + M^2)^{\frac{1}{2}}$ and $\omega(\underset{\sim}{p}) = (\underset{\sim}{p}^2 + \mu^2)^{\frac{1}{2}}$ will replace here our earlier notation $\omega_1(\underset{\sim}{p})$ and $\omega_2(\underset{\sim}{p})$.

39. In order to establish this connection we remind the reader of the "rules" of "old fashioned perturbation theory" as applied specifically to the pion nucleon problem: 1) with each vertex in a given time-ordered diagram associated a factor

$$\frac{iGM}{(2\pi)^{3/2}} \frac{\overline{W}(\underset{\sim}{p}')\,\gamma_5\, W(\underset{\sim}{p})}{\sqrt{2\omega(\underset{\sim}{k})E(\underset{\sim}{p}')E(\underset{\sim}{p})}} ,$$

where $W(\underset{\sim}{p})_{\alpha\sigma} = U_\alpha(\underset{\sim}{p},\sigma)$ $\left(= V_\alpha(-\underset{\sim}{p},\sigma)\right)$ is the Dirac spinor corresponding to an incoming (outgoing) nucleon (antinucleon) line, and $\underset{\sim}{k}$ is the momentum carried by the meson and $\overline{W}(\underset{\sim}{p}) = \beta\, W(\underset{\sim}{p})\beta$. These factors are to be arranged from right to left as we follow continuously a particular nucleon line, reading the diagram from right to left. 2) For each intermediate state obtained by cutting the diagram vertically in as many distinct places as possible, we introduce a factor $[E - \sum_i E(\underset{\sim}{p}_i) - \sum_i \omega(\underset{\sim}{k}_i)]^{-1}$, where $E(\underset{\sim}{p}_i)$ and $\omega(\underset{\sim}{k}_i)$ are the energies of the fermions and nucleons in the intermediate state, respectively. This gives the T matrix. (Note that the sum over the intermediate spins is implicit in the matrix multiplication.)

40. The fact that $e^{\frac{i\omega_{12}}{2}t}\ \psi(x)$ rather than $\psi(x)$ itself is expected to be a slowly varying function of time in the "non-relativistic" limit is suggested by the inhomogeneous term in Eq. (2.11). The "physical" reason is of course that for our choice $\mu_i = \frac{1}{2}$ in (2.9), the c.m. coordinate X as defined in (2.9) will not have the usual interpretation of c.m. coordinate in the non-relativistic limit. On the other hand, with the definition $X = (\omega_1 x_1 + \omega_2 x_2)/E$ corresponding to the choice $\mu_i = \omega_i/E$ [and to the transformation $\psi(x) \to e^{\frac{i\omega_{12}}{2}t}\ \psi(x)$], we would recover the usual definition of the c.m. coordinate in the nonrelativistic limit.

41. This asymptotic form is to be compared with $\tilde{G}_0(\underset{\sim}{r},0)$ as given by $(E = \omega_1 + \omega_2)$

$$\tilde{G}_0(\underset{\sim}{r},0) = \frac{i}{4E}\int \frac{d^3\underset{\sim}{p}}{(2\pi)^3}\left[\frac{\omega_1}{\sqrt{\underset{\sim}{p}^2 + m_1^2}} + \frac{\omega_2}{\sqrt{\underset{\sim}{p}^2 + m_2^2}}\right]\frac{e^{i\underset{\sim}{p}\cdot\underset{\sim}{r}}}{(\underset{\sim}{p}^2 - \underset{\sim}{k}^2 - i\epsilon)}\,.$$

For the equal mass case this reduces just the Blankenbecler-Sugar Green's function (see Ref. 36).

42. Thus for a potential $V(r)$ with the property $V(r) \sim (1/r)^3$ as $r \to 0$, the solutions $u^\ell(r)$ to the Schrodinger equation behave as follows as $r \to 0$:

$$u^\ell(r) \sim \exp[\pm\sqrt{\lambda/r}\ ] \qquad \text{(repulsive potential)}$$

$$u^\ell(r) \sim \exp[\pm i\ \sqrt{\lambda/r}\ ] \qquad \text{(attractive potential)}\ .$$

43. For the special case where $\Lambda = M$, we obtain upon taking carefully the limit: $\Lambda \to M$:

$$I'(x) = 2M^3[K_0(MR) + i\,\gamma\cdot\hat{x}\,K_1(MR)].$$

44. M. E. Rose, Elementary Theory of Angular Momentum (John Wiley and Sons, Inc., New York, 1957).

45. Paul Roman, Theory of Elementary Particles (North-Holland Publishing Company, Amsterdam; Interscience Publishers Inc., New York, 1960).

FIGURE CAPTIONS

Fig. 1. Integral equation (in the ladder approximation) satisfied by the BS wave function above the elastic threshold.

Fig. 2. Diagram representing the nucleon exchange force.

Fig. 3. Leading asymptotic behavior of the solution to the (Wick rotated) BS equation for the c.m. energy E below the first inelastic threshold and for the choice $\mu_i = \frac{1}{2}$ in (2.9). (Scale corresponds to $m_1 = 1.0$, $m_2 = 0.6$ and $M = 1.6$).

Fig. 4a,b. A plot of the solution $\alpha^{1\pm}$ (solid curve) and $\tilde{\alpha}^{1\pm}$ (dashed curve) to the indicial equations (5.11) as a function of the coupling constant λ.

Fig. 5. Computed values of E as a function of a, for $b = 1.6$ (———), $b = 1.4$ (-----) and $b = 1.2$ (—— - ——), corresponding to the data presented in Table VI.

Fig. 6. A plot of the data presented in Table XI.

Fig. 7. A plot of the data presented in Table XII.

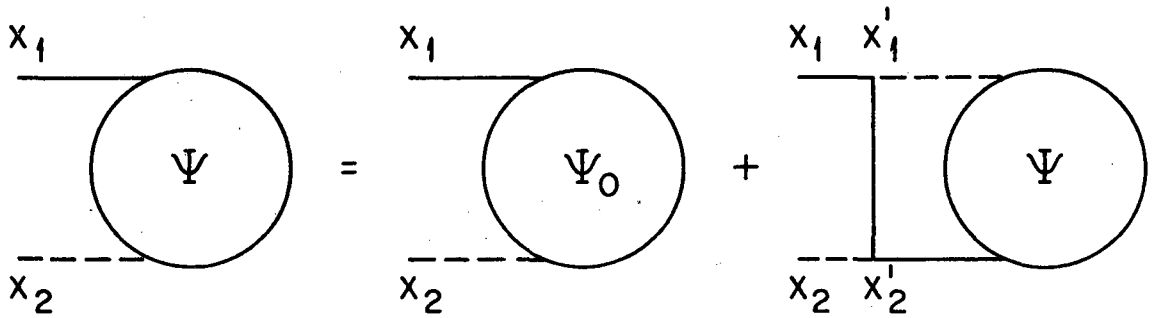
Fig. 8. Ladder graphs corresponding to the V-Θ sector of the Lee model. The solid and shaded line refer to the fully dressed N and V particles, respectively.

Fig. 9. Covariant "box" diagram. Solid and dashed lines refer to nucleon and meson, respectively.

Fig. 10. Separation of the nucleon and antinucleon contributions in the Born diagram of Fig. 2.

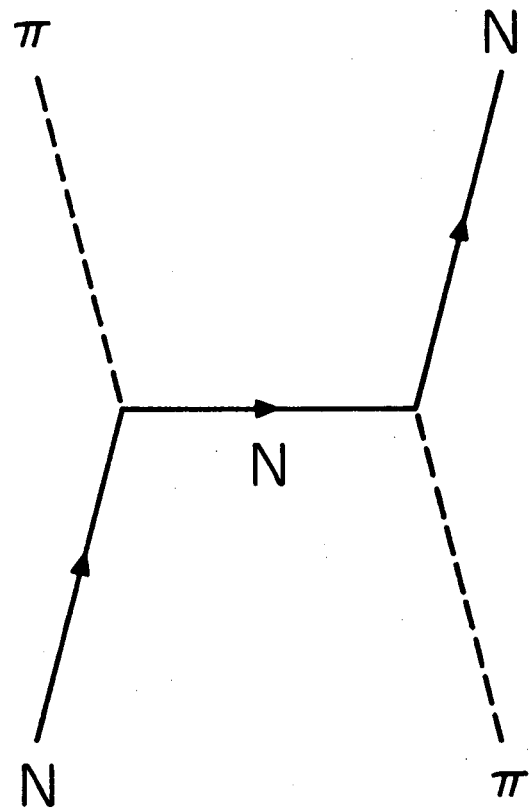
Fig. 11a,b. Typical time-ordered graphs contributing to the fourth order box diagram of Fig. 9.

Fig. 12. A plot of the data presented in Table XIV.



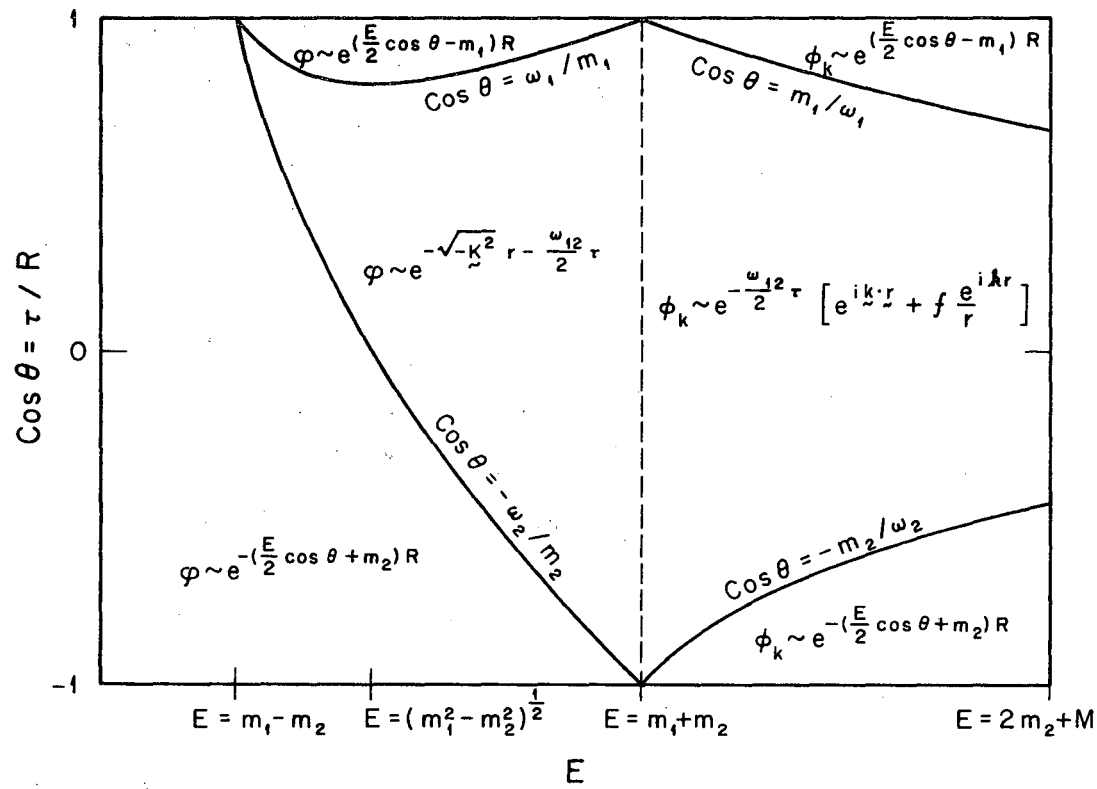
XBL677-3445

Fig. 1



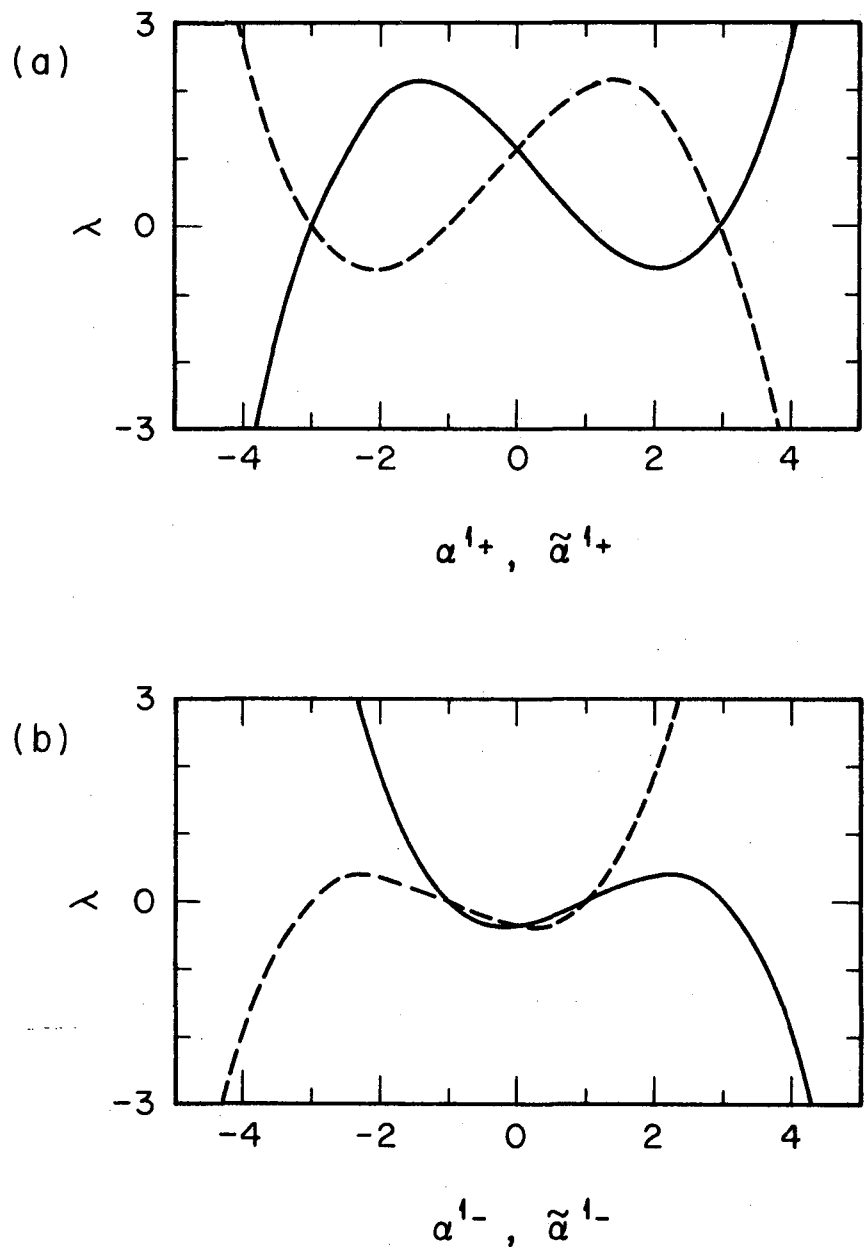
XBL677-3446

Fig. 2



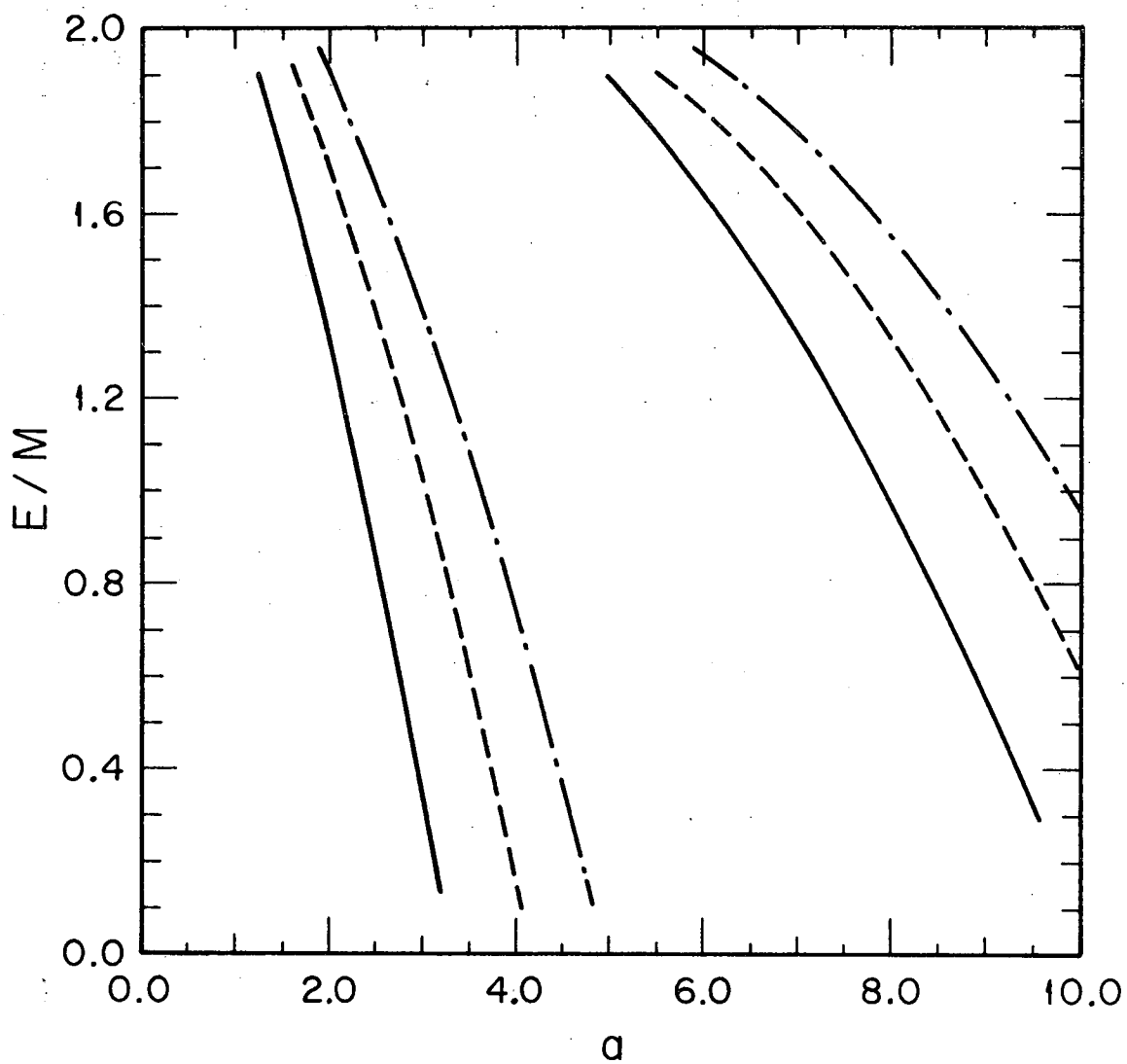
XBL677-3447

Fig. 3



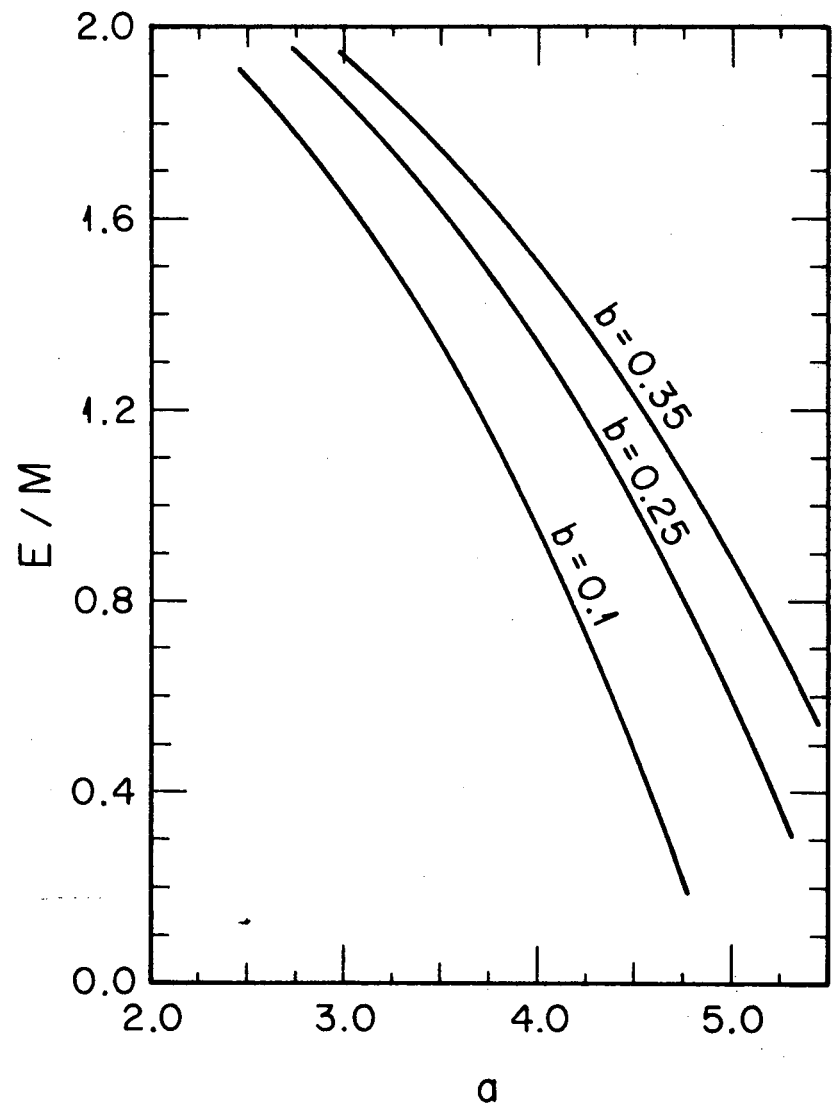
XBL677-3448

Fig. 4



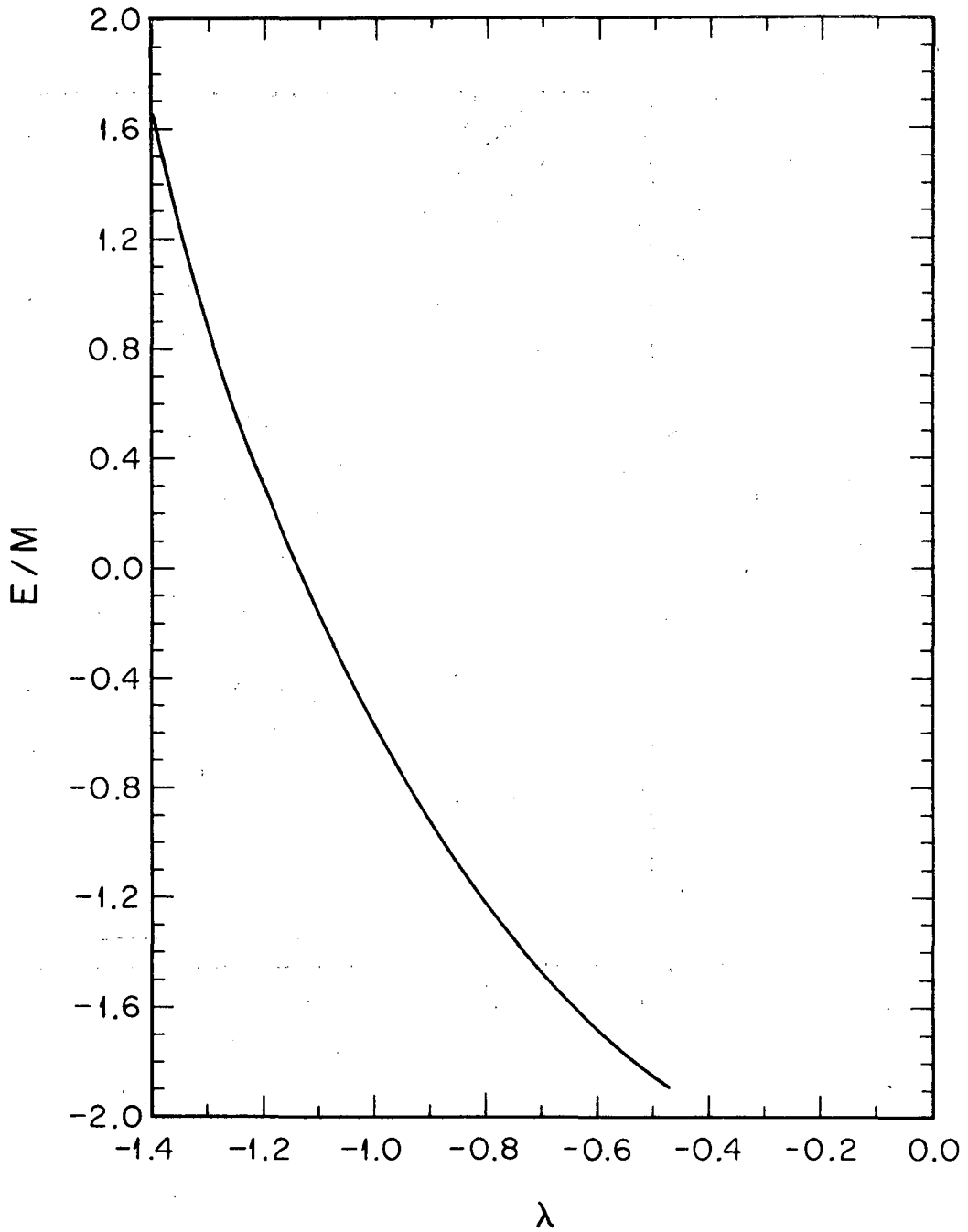
XBL677-3449

Fig. 5



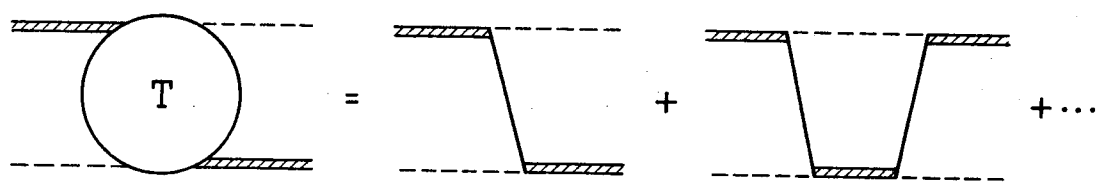
XBL677-3450

Fig. 6



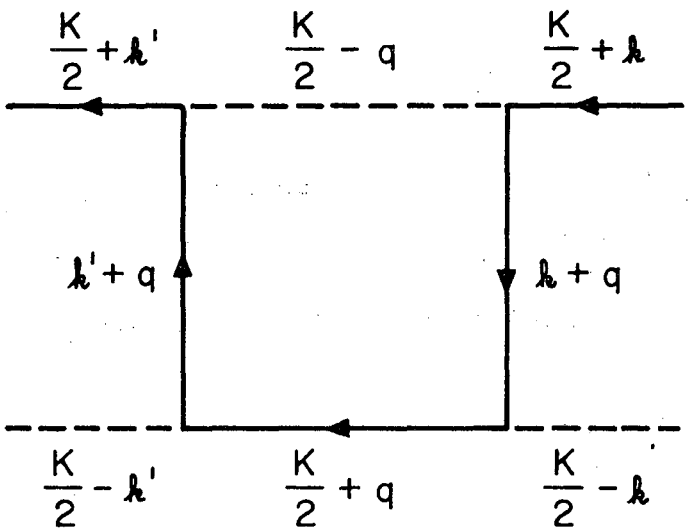
XBL677-3451

Fig. 7



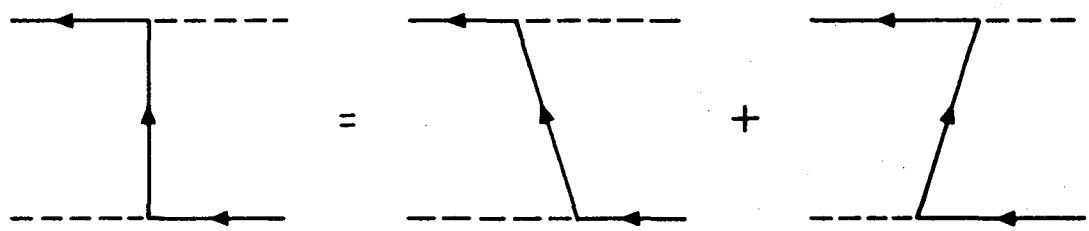
XBL677-3452

Fig. 8



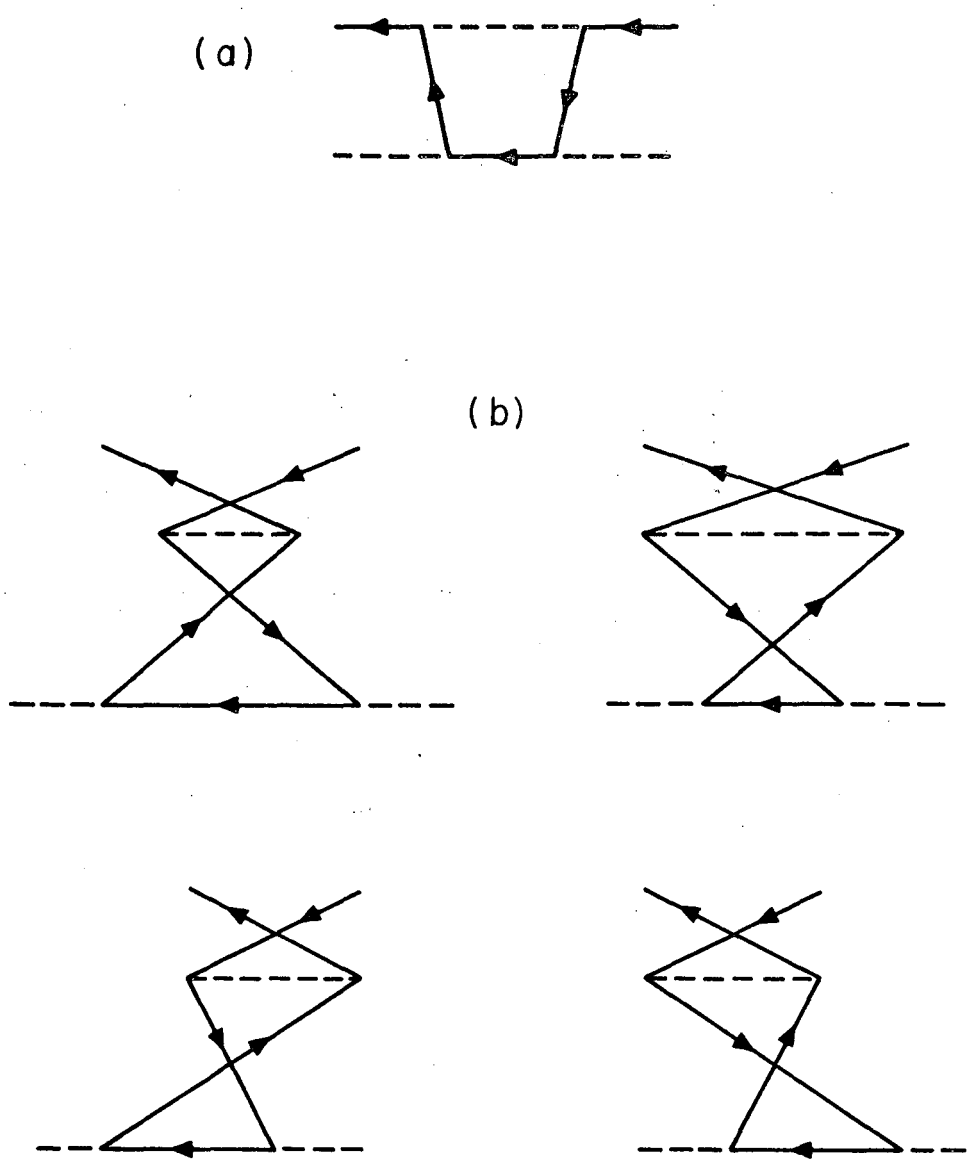
XBL677-3453

Fig. 9



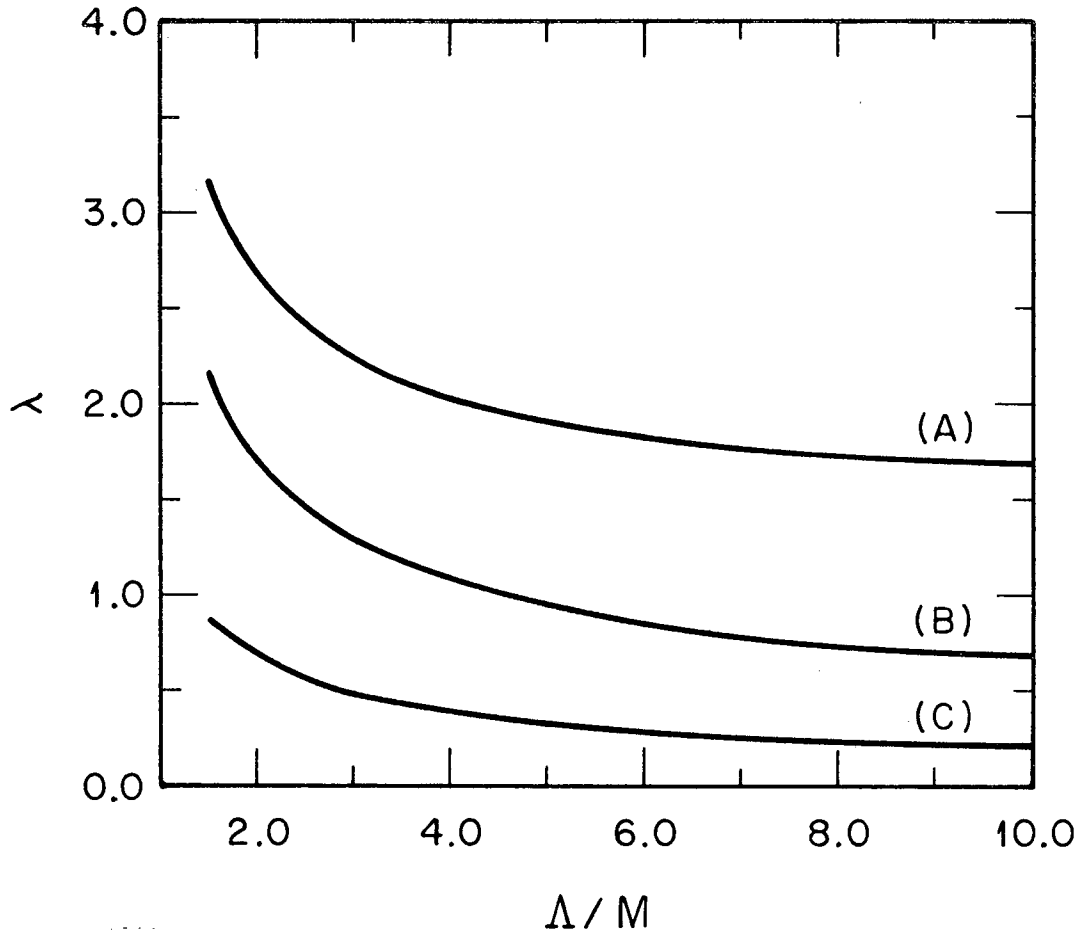
XBL677-3454

Fig. 10



XBL677-3455

Fig. 11



XBL677-3456

Fig. 12

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