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Submitted to Physical Review

UCRL-17857
Preprint

UNIVERSITY OF CALIFORNIA

Lawrence Radiation Laboratory
Berkeley, California

AEC Contract No. W-7405-eng-48

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LOW ENERGY THEOREMS FOR PHOTON PROCESSES*

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September 1967

ABSTRACT

We review and discuss various methods for obtaining low energy theorems for photon processes; (1) Low's method (2) the tensor method and (3) the S-matrix approach. The purely kinematical nature of these theorems is emphasized for they are found to follow primarily from the identification of the correct kinematical singularity and zero free amplitudes. Gauge invariance serves to inform us of the presence of additional kinematical zeros in certain physical amplitudes so that the unknown continuum contribution is surpressed relative to the known singular Born terms arising from single particle exchange at the physical threshold. Besides the well known low energy theorems specifying Compton amplitudes to first order in the photon

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frequency one can show that some pieces of the amplitude satisfy higher order theorems; in fact all 2J+1 multipoles of a spin J target have an associated low energy theorem. We explicitly establish a low energy theorem for the quadrupole moment of a J=1 target to supplement the known theorems for the total change and magnetic moment. An additional theorem for photopion production is obtained along with the well known Kroll-Ruderman theorem and serves to specify the E_2-multipole at threshold.

I. INTRODUCTION

Low energy theorems specifying the exact behavior of scattering amplitudes in the low energy region have been of interest since the original work of Thirring,[1] Gell-Mann, Goldberger and Low[2] on the Compton scattering of low frequency photons from spin 0 and spin $\frac{1}{2}$ systems. This early work, proceeding from the standpoint of field theory, demonstrated that the Compton amplitudes to first order in the frequency of the incident photon was completely specified in terms of the renormalized change and possible magnetic moment of the target particle. The immediate application of this exact low energy information is that it specifies the threshold behavior of the electric and magnetic multipole excitations of the target which admit of direct experimental comparison. Secondly if one assumes the scattering amplitude has sufficiently mild behavior at high energy the threshold theorem can be converted into a sum rule establishing a constraint on the inelastic spectrum. Subsequently low energy theorems have been obtained for a variety of processes involving photons[3] and general methods of obtaining them discussed[4].

The purpose of this present investigation is to reexamine and review the derivation of low energy theorems for photon processes and the specific

assumptions which enter their proof. In the next section we discuss three methods generally used to establish the low energy behavior of amplitudes: 1) Low's method 2) the tensor method 3) S-matrix method. We emphasize the almost purely kinematical nature of these theorems, not to suppose they are devoid of dynamical content but rather to indicate the crucial importance of establishing the kinematical singularity free amplitudes, the kinematical constraints imposed by gauge invariance and analyticity properties in establishing these theorems. It is our conviction that once the purely kinematical factors have been separated from the amplitude by including the content of gauge invariance then simple spectral assumptions such as pole dominance suffice to establish the low energy theorems. The primary task is to separate out the kinematical factors from the amplitudes correctly by knowing which amplitudes are kinematical singularity free (KSF) and kinematical zero free (KZF).

We apply the tensor method to Compton scattering and photopion production. What emerges from this investigation is the recognition that for Compton scattering processes, besides the usual low energy theorems valid to first order in the photon frequency, there are additional threshold theorems imposed by the general requirements of analyticity and gauge invariance. Examples of such theorems are provided by the spin non flip piece of the Compton amplitude $F_1(s,t)$ (to be defined in Section III) which satisfies

$$F_1(m^2,t) = e^2 \tag{1.1}$$

where s and t are the usual Mandelstam variables and e is the total change of the target. At t=0 (forward direction) this is just the Thomson limit; however, letting $s \to m^2$ with

$$t = -\frac{(s-m^2)^2}{2s}(1-\cos\theta)$$

and with fixed $\cos\theta$ yields additional information. Quite generally, there are threshold conditions on the invariant amplitudes of the same form as Eq. (1.1) and hence impose an infinite set of constraints on the physical multipoles. This conclusion has been anticipated and explicitly demonstrated by V. Singh[5] whose work has motivated the present investigation.

Although it is well known[6] that to second order in the photon frequency the unknown polarizability of the target particle influences the scattering, it can be shown that to this order and higher orders certain specific pieces of the amplitude do not depend on the polarizability and do admit of low energy theorems. Furthermore one may conclude that in Compton scattering from a spin = J system all 2J+1 static multipoles of the target enter a low energy theorem, a conclusion independently established by A. Pais.[7] For example for a spin = 1 system characterized by charge e, magnetic moment μ in units e/2m and quadrupole moment Q in units $e/2m^2$ we find the following threshold theorem

$$\begin{aligned}
\underset{\nu\to 0}{4\pi F} = {} & (\underset{\sim}{\epsilon}'\cdot\underset{\sim}{\epsilon})\left(-\frac{e^2}{m}\underset{\sim}{\lambda}'\cdot\underset{\sim}{\lambda}+\frac{eQ}{2m^3}\nu^2(\underset{\sim}{\lambda}'\cdot\hat{k}\underset{\sim}{\lambda}\cdot\hat{k}+\underset{\sim}{\lambda}'\cdot\hat{k}'\underset{\sim}{\lambda}\cdot\hat{k}')\right) \\
& - 2(\mu-e/2m)(e/m)\nu(\underset{\sim}{\lambda}'\times\underset{\sim}{\lambda})\cdot(\underset{\sim}{\epsilon}'\times\underset{\sim}{\epsilon}) \\
& - \mu^2\nu(\underset{\sim}{\lambda}'\times\underset{\sim}{\lambda})\cdot((\underset{\sim}{\epsilon}\times\hat{k})\times(\underset{\sim}{\epsilon}'\times\hat{k}')) \\
& - \frac{e\mu}{m}\nu[(\underset{\sim}{\epsilon}\cdot\hat{k}')(\underset{\sim}{\lambda}'\times\underset{\sim}{\lambda})\cdot(\hat{k}'\times\underset{\sim}{\epsilon}')-(\underset{\sim}{\epsilon}'\cdot\hat{k})(\underset{\sim}{\lambda}'\times\underset{\sim}{\lambda})\cdot(\hat{k}\times\underset{\sim}{\epsilon})] \\
& + O(\nu^2)\times(\text{tensors not included in above})\,. \qquad (1.2)
\end{aligned}$$

Here ν = frequency of photon in barycentric system and $(\underset{\sim}{\epsilon},\underset{\sim}{\lambda},\hat{k})$ and $(\underset{\sim}{\epsilon}',\underset{\sim}{\lambda}',\hat{k}')$ are unit vectors denoting the initial and final polarization of the photon

and target and direction of photon.

We have examined photopion production from nucleons in the same way and found in addition to the Kroll-Ruderman theorem[3] a new theorem. In the notation of CGLN[8] we write the production amplitude as

$$4\pi\mathcal{J} = i\underset{\sim}{\sigma}\cdot\underset{\sim}{\epsilon}\,\mathcal{J}_1 + \frac{\underset{\sim}{\sigma}\cdot\underset{\sim}{q}\,\underset{\sim}{\sigma}\cdot(\underset{\sim}{k}\times\underset{\sim}{\epsilon})}{qk}\mathcal{J}_2$$

$$+ \frac{i\underset{\sim}{\sigma}\cdot\underset{\sim}{k}\,\underset{\sim}{q}\cdot\underset{\sim}{\epsilon}}{qk}\mathcal{J}_3 + \frac{i\underset{\sim}{\sigma}\cdot\underset{\sim}{q}\,\underset{\sim}{q}\cdot\underset{\sim}{\epsilon}}{q^2}\mathcal{J}_4$$

where $\underset{\sim}{\epsilon}$, $\underset{\sim}{k}$ and $\underset{\sim}{q}$ are the polarization, momentum of the photon and the momentum of the pion. As $(q) \to 0$ we find neglecting term of $O(m)$

$$4\pi\mathcal{J}_1^{(-)} = eg/2M$$

$$4\pi\mathcal{J}_1^{(+,0)} = 0$$

$$\frac{4\pi\mathcal{J}_4^{(-)}}{q^2} = -\frac{g}{2Mm^2}\left(e + \frac{m^2}{8M^2}e - \frac{m^2}{2M}(\mu_p' - \mu_n')\right)$$

$$\frac{4\pi\mathcal{J}_4^{(+,0)}}{q^2} = \frac{g}{4M^2m}\left(e - \frac{me}{2M} - m(\mu_p' \mp \mu_n')\right)$$

where m = pion mass, M = nucleon mass, g = pion-nucleon coupling constant ($g^2/4\pi \cong 14$) and $\mu'_{p,n}$ are the anomalous nucleon moments. The first two conditions are the old result of Kroll and Ruderman[3] and the second two are new. Taken together these results stipulate the threshold behavior of the E_{0+} and E_{2-} multipoles. We will discuss the kinematical features of photopion production which give rise to these results.

In the next section we will discuss the various methods for establishing low energy theorems followed by a section applying these techniques.

II. METHODS

Here we will describe three methods for establishing the low energy behavior of photon processes. The first, Low's method, has been the traditional method of proving threshold theorems and contains all of the requisite physical assumptions. This method is not manifestly Lorentz invariant and this feature can make it difficult to abstract the full content of Lorentz plus gauge invariance. The subsequently developed tensor method, much utilized in current algebra calculations, has the advantage of manifest covariance without loss of generality. Moreover it is easy to see how the additional low energy theorems arise. Different in spirit if not detail from both Low's method and the tensor method is the S-matrix approach to proving threshold theorems. In this approach first applied to nucleon Compton scattering by Goldberger and Abarbanel[9] one utilizes the helicity decomposition of the physical amplitude from the start. By using the standard criterion for removing kinematic zeros from the helicity amplitudes and the crossing properties of the amplitudes it is possible to establish the KSF and KZF amplitudes and then the low energy theorem follows directly from the dispersion integral representation of the amplitude.

A. Low's Method

To be explicit we will consider the processes of Compton scattering from a target of mass M to illustrate the method (see Fig. 1) although it can be generally applied to any photon process.[4] We introduce the usual Mandelstam variables $s = (p+k)^2 = (p'+k')^2$, $u = (p-k')^2 = (p'-k)^2$, $t = (k'-k)^2$ satisfying $s+u+t = 2M^2$. We denote the scattering amplitude by

$$F = \epsilon_\mu \epsilon_\nu' T_{\mu\nu} \tag{2.1}$$

where the Lorentz tensor $T_{\mu\nu}$ admits of a general decomposition in terms of KSF invariant amplitudes consistent with the requirements of Lorentz invariance, parity and time reversal invariance. Differential current conservation and the observation that the change densities commute at equal times then imposes the additional requirement of gauge invariance $k'^{\nu}T_{\mu\nu} = k^{\mu}T_{\mu\nu} = 0$ or

$$k_0 k_0' T_{00} = k_i k_j' T_{ij} \tag{2.2}$$

To proceed further it is essential to make assumptions regarding the spectrum of intermediate states. If it is assumed that there is one state degenerate in mass with the target[10] and all excited states have higher mass that the target then one can make the explicit separation

$$T_{\mu\nu} = B_{\mu\nu} + E_{\mu\nu} , \tag{2.3}$$

where $B_{\mu\nu}$ is the singular Born term which can be precisely computed in Schrodinger perturbation theory and depends on the change and current distributions in the target and $E_{\mu\nu}$ represents the contribution of the excited states. The assumed analyticity requirements of the full amplitude plus the spectral assumption then permit the expansion of $E_{\mu\nu}$ about threshold in powers of the frequency of the photon. Evidently to compute T_{ij} from Eq. (2.2) to same order ν^n of the frequency of the photon what is required is knowledge of T_{00} or equivalently E_{00} to $O(\nu^n)$. Once this is established one may use Eq. (2.2) to identify the contributions to the various invariant amplitudes in T_{ij} invoking the known crossing properties of the amplitudes and thus constructing the low energy theorem.

It has long been known that E_{00} vanishes to first order in the frequency of the photon and hence there is a low energy theorem for the <u>full</u> amplitude to this order. Most of the "classical" low energy theorems are of

this type. Moreover it has also been known that to second order the unknown polarizability structure of the target must enter the expansion of the amplitude in frequency.[6] Recently it has been pointed out by Singh[5] that this polarizability contribution to the total amplitude can be isolated and second and higher order theorems be established for just those pieces of the amplitude to which it does not contribute. Singh's formal observation is that the excited state contribution has a definite structure imposed by Lorentz invariance and current conservation in the zero frequency limit e.g.

$$\underset{k_0, k_0' \to 0}{E_{00}} = k_i k_j' [\alpha \delta_{ij} + \beta (s_i s_j + s_j s_i)] + O(k_0^3) \tag{2.4}$$

where α and β are unknown constants characterizing the low energy polarizability of the target and $s_i^{(ab)}$ is the spin vector of the particle. That the requirement Eq. (2.4) lends to "higher" order low energy theorems has been demonstrated.[5,7]

B. Tensor Method

The evident content of Low's method even when supplemented with Singh's lemma on the spin structure of the excited state contribution is abstracted from the general requirements on the scattering amplitude of (1) Lorentz invariance and discrete conservation laws (2) analyticity, crossing and spectral assumptions (3) gauge invariance. With the development of current algebra the tensor method has emerged as a manifestly covariant formulation of Low's method which implements the above assumptions and is perfectly general.

To begin one utilizes Lorentz invariance and the assumed discrete symmetries of the system to decompose the amplitude into its general form in

terms of a tensor basis and invariant amplitudes. The tensor method essentially requires the construction of a KSF and KZF gauge invariant tensor basis. Since there appears to be no general solution to this problem in the literature in the case of mass zero particles[11] we indicate a general method in what follows and apply it in the next section to special cases.

The fundamental lemma,[12] suggested by perturbation theory, says that if one takes all possible tensors constructed of external momenta and spin matrices allowed by relativistic invariance and other discrete conservation laws, their scalar coefficients can be chosen to be KSF. In establishing a KSF basis it is important to construct the tensors for the amplitudes with the external wave functions removed -- that is one considers the general tensor form <u>before</u> sandwiching between Dirac spinors or polarization vectors. Since a basis so chosen is not in general linearly independent, one has to reduce it to a linearly independent basis, taking care that no kinematical singularities are introduced in this processes. A rule of thumb is to always reduce tensors that contain higher powers of momenta in terms of the ones that contain lower powers of momenta, since this appears not to introduce kinematical singularities while the inversion does.

There is one simplification worth noting. The only singularities relevant to the low energy theorems are the ones that occur at $k \to 0$, $k' \to 0$ (or $s \to M^2$, $t \to 0$). Singularities that occur elsewhere are harmless. Therefore, one should reduce terms containing higher powers of the four vectors k and k' into terms containing lower powers of k and k'.

Now expand the amplitude for the scattering of photon from an arbitrary target into a KSF and linearly independent tensor basis as follows:

$$T^{\mu\nu}_{\alpha\beta\gamma\ldots} = A^{\mu\nu}_{\alpha\beta\gamma\ldots} + k_i B^{\mu\nu,i}_{\alpha\beta\gamma\ldots} + k_i' B'^{\mu\nu,i}_{\alpha\beta\gamma\ldots}$$

$$+ k_i k_j C^{\mu\nu,ij}_{\alpha\beta\gamma\ldots} + \ldots + (\text{terms of higher power in } k \text{ and } k') \qquad (2.5)$$

The indices μ and ν go with photons of momenta k and k' respectively, and $\alpha\beta\gamma\ldots$ etc. are the indices of the initial and final targets. A,B,C etc. are linearly independent Lorentz tensors <u>which do not depend on k or k'</u>; they are constructed from momenta independent of k and k'. In accordance with the previous discussion, we assume that such a representation is always possible.

Using gauge invariance in Eq. (2.5) gives

$$k_\mu A^{\mu\nu}_{\alpha\beta\gamma\ldots} + k_\mu k_i B^{\mu\nu,i}_{\alpha\beta\gamma\ldots} + k_\mu k_i' B^{\mu\nu,i}_{\alpha\beta\gamma\ldots} + \ldots$$

$$= k_\nu' A^{\mu\nu}_{\alpha\beta\gamma\ldots} + k_\nu' k_i B^{\mu\nu,i}_{\alpha\beta\gamma\ldots} + k_\nu' k_i' B^{\mu\nu,i}_{\alpha\beta\gamma\ldots} + \ldots = 0 \qquad (2.6)$$

Differentiating this equation with respect to k (or k') and setting k=0, we get the result that A must vanish. This is the tensor analogue of Low's classical result. In general, however, there are Born terms singular in the limit $k \to 0$, $k' \to 0$, which invalidate the representation given by (2.5). In this case, we can write,

$$T^{\mu\nu}_{\alpha\beta\gamma\ldots} = T^{\mu\nu}_{\alpha\beta\gamma\ldots}(\text{Born}) + A^{\mu\nu}_{\alpha\beta\gamma\ldots} + k_i B^{\mu\nu,i}_{\alpha\beta\gamma\ldots} + \ldots \qquad (2.7)$$

The previous conclusion is now changed to

$$A^{\mu\nu}_{\alpha\beta\gamma\ldots} = A^{\mu\nu}_{\alpha\beta\gamma\ldots}(\text{Born}) \qquad (2.8)$$

where

$$A^{\mu\nu}_{\alpha\beta\gamma\ldots}(\text{Born}) = - \lim_{\substack{k \to 0 \\ k' \to 0}} \frac{\partial}{\partial k_\mu} \{k_{\mu'} T^{\mu'\nu}_{\alpha\beta\gamma\ldots}(\text{Born})\} .$$

There are, however, higher order low energy theorems contained in (2.7). For example, differentiating twice with respect to k_a and k_b, we obtain

$$B^{a\nu,b}_{\alpha\beta\gamma\ldots}+B^{b\nu,a}_{\alpha\beta\gamma\ldots} = \text{Born term.} \tag{2.9}$$

and this process can be continued to higher powers of k and k'. Equations of the form of (2.9) clearly impose restrictions on the threshold behavior of multipoles of arbitrarily high order.

One disadvantage of the above procedure is that it is fairly laborious to relate the tensors defined by Eq. (2.6) to the physical scattering parameters. Some of them may not even contribute to observable amplitudes. In practice, a convenient procedure is to express these tensors in terms of the scalars s and t and the natural tensors constructed from the momenta. Using the gauge conditions, one then goes to a gauge invariant tensor basis, without introducing any kinematical singularities or zeros at k=0, k'=0 in the scalar coefficients. The amplitude, written in this fashion, satisfies the constraints given by (2.6) automatically. It is then fairly easy to relate this new tensor expansion to physical amplitudes like helicity amplitudes.

A few comments about the calculation of the singular Born amplitudes are in order. A typical term in the expansion of the Compton amplitude $T^{\mu\nu}_{\alpha\beta\gamma\ldots}$ appears as follows:

$$T^{\mu\nu}_{\alpha\beta\gamma\ldots} = f(s,t)a^{\mu\nu}_{\alpha\beta\gamma\ldots} + \ldots \tag{2.10}$$

where $a^{\mu\nu}_{\alpha\beta\gamma\ldots}$ is a guage invariant tensor constructed out of various momenta. The singular part of f is given by

$$f_{\pm}^{B} \sim \frac{1}{s-M^2} \pm \frac{1}{u-M^2} \tag{2.11}$$

The plus sign goes with a tensor $a^{\mu\nu}$ even under the photon crossing, and the

minus sign with an odd tensor. For fixed $\cos\theta = 1+2st/(2-M^2)^2$, f_-^B becomes singular as $s \to M^2$. This singularity cannot, of course, appear in the physical amplitude. The tensor $\epsilon^{\mu\nu}$, dotted into the photon polarization vectors, must develop a compensating zero at the same point. This kinematical zero suppresses the non-singular continuum contribution at the threshold, and hence the amplitude is exactly given by the uniquely determined residue of the Born pole. The kinematical zero that destroys the pole singularity corresponds to the well-known absence of $0 \to 0$ transitions.

The even Born term f_+^B is not singular at the threshold, if $(\cos\theta)$ is kept fixed. Hence, the Born term in general cannot dominate the continuum contribution. However, for the amplitudes for which low energy theorems are valid, gauge invariance implies that f_+ must contain a kinematical factor t. We can, therefore, write

$$f_+ = t\bar{f}_+, \text{ where } \bar{f}_+ \text{ is KSF}$$

and

$$\bar{f}_+^B \sim \frac{1}{(s-M^2)(u-M^2)}$$

The continuum is now suppressed by the factor of t, which vanishes at the physical threshold, and a low energy theorem holds. We stress again that the result depends on the unambiguous residue of the Born pole, and not on the ambiguous finite part. For example, explicit inclusion of the non-singular graphs like the sea gull graph is not really necessary, since gauge invariance takes care of this problem automatically. In pion photoproduction and charged photon Compton scattering,[13] there are usually singular t-channel Born terms. Again, these terms can be obtained by gauge invariance requirements.

C. S-Matrix Method

From the recognition of the central role played by kinematics in the derivation of low energy theorems, the possibility of a pure S-matrix approach suggests itself. As is well known[14] the requirement of gauge invariance is implemented in S-matrix theory by the requirement of strictly zero mass and two helicity states for the photon. These conditions if demanded in every Lorentz frame are equivalent to gauge invariance.

Goldberger and Abarbanel[9] have applied this method to Compton scattering. One assumes that the helicity amplitudes in the s and t channel are KSF and factors the kinematical zeros from the spin flip amplitudes corresponding to their vanishing in the forward and backward direction. One then assumes that the resulting s and t channel amplitudes are KSF and KZF in t and s respectively. Finally using the crossing properties relating s and t channel amplitudes one can construct the KSF and KZF amplitudes in both s and t. We will exhibit this procedure in the simple case of Compton scattering from a spin 0 target where it gives the same results as the tensor method.

In the S-matrix method the amplitude is constructed via a dispersion integral and the unitarity conditions for the absorptive part. Hence in this approach it appears necessary to assume a high energy behavior sufficiently damped to allow the dispersion integral to converge and also such that no unknown polynomial pieces are added to the amplitude. Such asymptotic assumptions are completely foreign to the methods previously considered and it is not clear why they play an important role in proving low energy theorem in S-matrix theory.

One may calculate the correct singular pole terms from unitarity irrespective of the number of subtractions required in the dispersion integral

providing the subtraction terms do not introduce additional unwarranted kinematical zeros or singularities. Then the low energy theorem can be established and is independent of any unknown subtraction terms depending as it does only on the known residue of the pole term piece. This is certainly true of spin zero Compton scattering and it is a matter of conjecture for more complex systems.

III. APPLICATIONS

A. Compton Scattering

We now apply the tensor method described above to the process of physical Compton scattering. The case of a spinless target is considered first since it simply exemplifies all of the essential features of the methods. Later on we also consider the case of a spin 1 target and also characterize some of the features of scattering for arbitrary spin targets.

Let the J=0 Compton amplitude be (see Fig. 1)

$$F = \epsilon_\mu \epsilon_\nu{}' T_{\mu\nu}(P,k,k') \tag{3.1}$$

and

$$P = (p'+p),\ s = (p+k)^2,\ t = (k-k')^2$$

$s+t+u=2m^2$. If we set $Q_1=k$, $Q_2=k'$, $Q_3=P$ the tensors $g_{\mu\nu}, Q_{i\mu}Q_{j\nu}$ form a linearly independent basis. However we can introduce further tensors like $Q_4{}^\mu = \epsilon^{\mu\epsilon\beta\gamma}k_\epsilon k_\beta{}' P_\gamma$ and obtain a redundant system. Observe that $Q_4{}^\mu Q_4{}^\nu$ can be expressed in terms of $Q_i{}^\mu Q_j{}^\nu$ with $i,j = 1,2,3$ without introducing any denominators involving the scalars s and t. Therefore $Q_4{}^\mu Q_4{}^\nu$ can be eliminated in favor of the other tensors without introducing possible singularities and the resulting linear independent basis of tensors is KSF. We therefore set

$$T_{\mu\nu} = f_1 P_\mu P_\nu + f_2 k_\mu k_\nu + f_3 k_\mu' k_\nu'$$
$$+ f_4 k_\mu P_\nu + f_5 k_\nu P_\mu + f_6 k_\mu' P_\nu + f_7 k_\nu' P_\mu$$
$$+ f_8 k_\mu k_\nu' + f_9 k_\mu' k_\nu + f_{10} g_{n\nu} \tag{3.2}$$

and find the $f_i(s,t,u)$ are KSF.

Next we make the separation

$$f_i(s,t,u) = b_i(s,t,u) + c_i(s,t,u) \tag{3.3}$$

where b_i contains the single particle piece and c_i, the continuum contribution is assumed nonsingular near the elastic threshold $s=m^2$. The singular contribution to the b_i is calculated to be

$$b_1 = b_9 = -e^2\left(\frac{1}{s-m^2} + \frac{1}{u-m^2}\right)$$
$$b_6 = b_5 = -e^2\left(\frac{1}{s-m^2} - \frac{1}{u-m^2}\right) \tag{3.4}$$

The gauge conditions $k_\mu T_{\mu\nu} = k_\nu' T_{\mu\nu} = 0$ imply

$$(s-u)f_1 = tf_6$$
$$(s-u)f_7 = tf_3$$
$$(s-u)f_5 + 2f_{10} = tf_9$$
$$(s-u)f_1 = tf_5$$
$$(s-u)f_4 = tf_2 \tag{3.5}$$

and we see that f_1 contains a kinematical zero proportional to t so that $A(s,t,u) \equiv f_1(s,t,u)/t$ is KSF. We also remark that the b_i given by (3.4) will satisfy (3.5) automatically providing we define $b_{10} = 2e^2$ corresponding to a contact interaction (seagull). Eliminating all scalar functions in favor of f_1 and f_9 and dropping terms proportional to k_μ or k_ν' we obtain

$$T_{\mu\nu} = Aa_{\mu\nu} + Bb_{\mu\nu} \tag{3.6}$$

where

$$a_{\mu\nu} = tP_\mu P_\nu + tk_\mu{}'k_\nu + (s-u)(P_\nu k_\mu{}' + P_\mu k_\nu)$$
$$+ 2(s-m^2)(u-m^2)g_{\mu\nu}$$
$$b_{\mu\nu} = tg_{\mu\nu} + 2k_\mu{}'k_\nu$$

and

$$A = f_1/t$$
$$B = \frac{1}{2}(f_9-f_1) \tag{3.7}$$

are KSF amplitudes. The singular part of these amplitudes is unambiguous and given by

$$A^B = e^2/(s-m^2)(u-m^2)$$
$$B^B = 0 \tag{3.8}$$

and defining

$$F_1(s,t) = (s-m^2)(u-m^2)A$$
$$F_2(s,t) = (s-m^2)(u-m^2)B$$

the low energy theorem is expressed as

$$F_1(m^2,t) = e^2$$
$$F_2(m^2,t) = 0 \tag{3.9}$$

This first expression if differentiated with respect to t and then setting t=0 is just the result given by Singh.[5] The low energy theorem is now merely the statement that the KSF amplitudes are dominated by the singular Born terms near the physical threshold $s \to m^2$,

$$t = -\frac{(s-m^2)^2}{2s}(1-\cos\theta_s) .$$

At this point it is instructive to make contact with the S-matrix approach.[9] Here we introduce the s and t channel helicity amplitudes which enjoy the simple crossing relation

$$F^s_{+,+} = - F^t_{+,-} \qquad F^s_{+,-} = F^t_{+,+} \; . \tag{3.10}$$

The first step in the procedure is to remove the known kinematical zeros in the amplitudes corresponding to angular momentum conservation in the forward and backward direction by dividing by half angle factors[15]

$$\bar{F}^s_{+,+} = (\cos\theta s/2)^{-2} F^s_{+,+} \qquad \bar{F}^t_{+-} = (\sin\frac{\theta t}{2}\cos\frac{\theta t}{2})^{-2} F^t_{+-}$$

$$\bar{F}^s_{+,-} = (\sin\frac{\theta s}{2})^{-2} F^s_{+,-} \qquad \bar{F}^t_{++} = F^t_{++} \tag{3.11}$$

Then $\bar{F}^s$ and $\bar{F}^t$ are assumed to be KSF and KZF in t and s respectively. Using (3.10) and the expressions for the half angle factors one has

$$\bar{F}^s_{+,+} = \frac{(s-m^2)^2}{t(t-4m^2)} \bar{F}^t_{+,-}$$

$$\bar{F}^s_{+,-} = - \frac{st}{(s-m^2)^2} \bar{F}^t_{++} \tag{3.12}$$

so it follows that

$$a = - \frac{\bar{F}^s_{+,+}}{(s-m^2)^2} = - \frac{\bar{F}^t_{+-}}{t(t-4m^2)}$$

$$b = - \frac{s\bar{F}^s_{+,-}}{(s-m^2)^2} = \frac{1}{t} \bar{F}^t_{++} \tag{3.13}$$

are the amplitudes that are KSF and KZF in <u>both</u> s and t. From the pole term contribute to a and b the low energy theorem follows immediately. The connection of these amplitudes to those of Eq. (3.6)

$$a = -2A$$

$$b = B+2m^2A \tag{3.14}$$

introduces no kinematical factors so the results of the S-matrix method and tensor method agree as to the correct KSF amplitudes.

As a final comment, one can prove the absence of a kinematical 1/t singularity in A using the Hall-Wightman theorem.[16] If $T_{\mu\nu}$ is assumed to be an analytic function in the Cartesian components of the vectors k,k',P then $T_\mu{}^\mu$ must be an analytic function of the scalars in a corresponding domain. From the identity

$$T_\mu{}^\mu = \{t(4m^2 - \tfrac{1}{2}t) - (s-u)^2\}A + 3tB$$

we have since B is KSF that a 1/t factor in A would clearly introduce the same factor in $T_\mu{}^\mu$ contradicting the Hall-Wightman theorem.

One can carry out a similar procedure for Compton scattering from higher spin targets and we here consider explicitly the case J=1. Denoting ϵ,λ and ϵ',λ' as the initial and final polarizations of the photon and target and the kinematics as shown in Fig. 1 the relativistic amplitude $\lambda_\alpha T^{\alpha\beta}{}_{\mu\nu}\lambda_\beta{}'$ can be expressed in terms of a gauge invariant tensor basis according to

$$\begin{aligned}
\lambda_\alpha T^{\alpha\beta}{}_{\mu\nu}\lambda_\beta{}' = {} & a_{\mu\nu}\{f_1(\lambda\cdot\lambda') + f_2[(\lambda'\cdot k)(\lambda\cdot k) + (\lambda'\cdot k')(\lambda\cdot k)] \\
& + f_3(\lambda\cdot k)(\lambda'\cdot k') + f_4(\lambda\cdot k')(\lambda'\cdot k)\} + b_{\mu\nu}\{f_5(\lambda\cdot\lambda') \\
& + f_6[(\lambda'\cdot k)(\lambda\cdot k) + (\lambda'\cdot k')(\lambda\cdot k')] + f_7(\lambda\cdot k)(\lambda'\cdot k') + f_8(\lambda'\cdot k)(\lambda\cdot k')\} \\
& + f_9\{\lambda_\mu\lambda_\nu{}'(k\cdot k') - (\lambda\cdot k)k_\mu{}'\lambda_\nu{}' - (\lambda'\cdot k')k_\nu\lambda_\mu + (\lambda\cdot k)(\lambda'\cdot k')g_{\mu\nu}\} \\
& + f_{10}\{\lambda_\mu{}'\lambda_\nu(k\cdot k') - (\lambda'\cdot k)k_\mu{}'\lambda_\nu - (\lambda\cdot k')k_\nu\lambda_\mu{}' + (\lambda'\cdot k)(\lambda\cdot k')g_{\mu\nu}\} \\
& + f_{11}\{\lambda_\mu\lambda_\nu{}'(k\cdot P)^2 - (k\cdot\lambda)P_\mu\lambda_\nu{}'(k\cdot P) - (k'\cdot\lambda')P_\nu\lambda_\mu(k\cdot P) + (k\cdot\lambda)(k'\cdot\lambda')P_\mu P_\nu\} \\
& + f_{12}\{\lambda_\mu{}'\lambda_\nu(k\cdot P)^2 - (k\cdot\lambda')P_\mu\lambda_\nu(k\cdot P) - (k'\cdot\lambda)P_\nu\lambda_\mu{}'(k\cdot P) + (k\cdot\lambda')(k'\cdot\lambda)P_\mu P_\nu\}
\end{aligned} \tag{3.15}$$

where $a_{\mu\nu}$ and $b_{\mu\nu}$ are defined by Eq. (3.6). Our observation is that f_1 and f_2 are KSF so that near threshold they approach their Born terms; in particular we have the low energy theorems on $H_{1,2}(s,t) = (s-m^2)(u-m^2)f_{1,2}(s,t)$ which are

$$H_1(m^2,t) = -e^2$$

$$H_2(m^2,t) = -e^2Q/2M^2 \tag{3.16}$$

where Q is the quadrupole moment in units of $1/2M^2$. The theorem on H_1 is just the Thomson limit and the result on H_2 involving the quadrupole moment is an additional "higher" order theorem also independently obtained by A. Pais.[7] There are also theorems involving the magnetic moment which are well known for targets of arbitrary spin[3] and including these we obtain with (3.16) the result Eq. (1.1) quoted in the introduction.

Let us sketch the proof that $f_{1,2}(s,t)$ are KSF. We will consider only those pieces of the general decomposition of the amplitude $T^{\alpha\beta}{}_{\mu\nu}$ (before the application of gauge invariance) needed for the discussion. In the case of f_1 and f_2 it is clear that we need only show that the scalars multiplying the tensors $P_\mu P_\nu g_{\alpha\beta}$, $P_\mu P_\nu k_\alpha k_\beta$, $P_\mu P_\nu k_\alpha' k_\beta'$ are all proportional to $k\cdot k' = -t/2$. If we set

$$T_{\mu\nu}{}^{\alpha\beta} = a_1 P_\mu P_\nu k_\alpha k_\beta + b_1 P_\mu k_\nu k_\alpha k_\beta$$

$$+ a_2 P_\mu P_\nu k_\alpha' k_\beta' + b_2 P_\mu k_\nu k_\alpha k_\beta + \cdots$$

crossing implies $a_1 = a_2$ and $k'^\nu T_{\mu\nu}{}^{\alpha\beta} = 0$ implies $k\cdot P a_1 + k\cdot k' b_1 = 0$ hence $a_1 \approx t$ so that $f_2 \equiv a_1/t$ is KSF. A similar argument works for f_1 (but not for $f_{3,4}$). The low energy theorem then follows by calculating the singular part of the Born term.[17]

The low energy theorem on the quadrupole moment is of interest since it implies that the quadrupole moment, like the change and magnetic moment, is uniquely defined in terms of the threshold behavior of the Compton amplitude. If $H_2(s,t) \to 0$ as $s \to \infty$ then one can write a sum rule for the quadrupole-moment in terms of total photoabsorption amplitudes,

$$\frac{-e^2Q}{2m^2} = \frac{1}{\pi}\int ds\ \mathrm{Im}H_2(s,t)\left[\frac{1}{s-m^2} + \frac{1}{s-m^2+t}\right] . \tag{3.17}$$

We emphasize that unlike the change and magnetic moment the quadrupole moment does not admit of a low energy theorem for strictly forward scattering ($k'=k$) since in this limit the tensors corresponding to $f_{2,3,4}$ are identical and there are no low energy theorems for $f_{3,4}$ (they are related to the polarizability of the particle). Nonetheless, we may still extrapolate $f_2(s,t)$ defined for $t\neq 0$ to $t=0$ with a unique result. It is of practical interest to apply this sum rule Eq. (3.17) to nuclear systems with J=1, in particular the Deuteron.

These theorems can be generalized to higher spin targets in an obvious way. Consider an integer spin target represented by an J index tensor $\lambda_{\alpha_1\alpha_2\cdots\alpha_J}$ so the relativistic Compton amplitude is

$$\lambda_{\alpha_1\alpha_2\cdots\alpha_J} T_{\mu\nu}^{\alpha_1\alpha_2\cdots\alpha_J,\beta_1\beta_2\cdots\beta_J} \lambda'_{\beta_1\beta_2\cdots\beta_J} .$$

Then the J+1 scalar amplitudes multiplying the tensors $a_{\mu\nu}\delta_{\alpha_1\beta_1}\delta_{\alpha_2\beta_2}\cdots\delta_{\alpha_J\beta_J}$, $a_{\mu\nu}\delta_{\alpha_1\beta_1}\delta_{\alpha_2\beta_2}\cdots k_{\alpha_J\beta_J}$, ... $a_{\mu\nu}k_{\alpha_1\beta_1}k_{\alpha_2\beta_2}\cdots k_{\alpha_J\beta_J}$ where $k_{\alpha\beta} = k_\alpha k_\beta + k_\alpha' k_\beta'$ are all KSF and the Born terms receive contributions from the J+1 even multipoles so there are low energy theorems for these multipoles. The J odd multipoles can also be shown to have associated low energy theorems so that all 2J+1 multipoles are defined in terms of the Compton amplitude. These theorems have been extensively studied by Pais.[7]

B. Photopion Production

As a final illustration we derive a new low energy theorem for photopion production. In the barycentric system this process is described in terms of the four CGLN[8] amplitude

$$4\pi\mathcal{F}^{(\pm,0)} = i\underset{\sim}{\sigma}\cdot\underset{\sim}{\epsilon}\,\mathcal{F}_1^{(\pm,0)} + \frac{\underset{\sim}{\sigma}\cdot\underset{\sim}{q}\,\underset{\sim}{\sigma}\cdot(\underset{\sim}{k}\times\underset{\sim}{\epsilon})}{qk}\mathcal{F}_2^{(\pm,0)}$$

$$+ \frac{i\underset{\sim}{\sigma}\cdot\underset{\sim}{k}\,\underset{\sim}{q}\cdot\underset{\sim}{\epsilon}}{qk}\mathcal{F}_3^{(\pm,0)} + \frac{i\underset{\sim}{\sigma}\cdot\underset{\sim}{q}\,\underset{\sim}{q}\cdot\underset{\sim}{\epsilon}}{q^2}\mathcal{F}_4^{(\pm,0)} \qquad (3.18)$$

It is well known[3] that as q = pion momentum → 0 the production amplitude is specified by the Kroll-Ruderman theorem

$$4\pi\,\mathcal{F}_1^{(-)} = eg/2M$$

$$4\pi\,\mathcal{F}_1^{(+,0)} = 0 \qquad (3.19)$$

neglecting terms of the order m. To the same order in m one has at threshold[18]

$$\frac{4\pi\,\mathcal{F}_4^{(-)}}{q^2} = \frac{-g}{2Mm^2}\left(e + \frac{m^2}{8M^2}e - \frac{m^2}{2M}(\mu_p'-\mu_n)\right)$$

$$\frac{4\pi\,\mathcal{F}_4^{(+,0)}}{q^2} = \frac{g}{4M^2m}\left(e - \frac{me}{2M} - m(\mu_p'\mp\mu_n)\right) \qquad (3.20)$$

where μ_p',μ_n are the anomalous moments of the nucleons in units of e/2M. These results can be used to derive conditions on the standard multipole moments.[8] Using the definition of the multipole coefficients in terms of the $\mathcal{F}_i$[19] it is straightforward to show that Eq. (3.19) implies as q → 0

$$4\pi E_{0+}^{(-)} = eg/2M$$

$$4\pi E_{0+}^{(+,0)} = 0 \qquad (3.21)$$

corresponding to the observed isotropic production of charged pions at threshold.

Using Eq. (3.20) one also obtains the result valid to O(m) as $q \to 0$

$$120\pi \frac{E_{2-}^{(-)}}{q^2} = \frac{eg}{M^3}\left\{\left(\frac{7}{2}\right)\frac{M^2}{m^2} + \frac{3M}{2m} - \frac{1}{16}\right\}$$

$$- \frac{5g}{4M^2}(\mu_p' - \mu_n)$$

$$24\pi \frac{E_{2-}^{(+,0)}}{q^2} = -\frac{eg}{2M^2 m} + \frac{1}{4M^2}\left(\frac{eg}{2} + g(\mu_p' \mp \mu_n)\right) \tag{3.22}$$

At present there is insufficient experimental information on the E_2-multipole at threshold to test this prediction.

To prove the statements Eqs. (3.19)-(3.22) we expand the covariant amplitude into the following tensor basis without taking into account gauge invariance (we omit isotopic indices)

$$T_\mu = \bar{u}(p')i\gamma_5\{B_1\gamma_\mu k\cdot\gamma + B_2 P_\mu + 2B_3 q_u$$

$$+ B_4 2k_\mu + B_5\gamma_\mu + \frac{1}{2}B_6 P_\mu k\cdot\gamma + B_7 k_\mu k\cdot\gamma - B_8 q_\mu k\cdot\gamma\}u(p)$$

where p,p',k,q are the initial and final nucleon momenta, photon and pion momenta respectively and $P = \frac{1}{2}(p'+p)$. In this form the $B_i(s,t)$ are known to be KSF as shown by Ball.[19] The gauge condition $k^\mu T_\mu = 0$ imposes the following conditions

$$(s-u)B_2 - 2(t-m^2)B_3 = 0 \tag{3.24a}$$

$$B_s + \frac{1}{4}(s-u)B_6 + \frac{t-m^2}{2}B_8 = 0 \tag{3.24b}$$

In this case it is convenient to separate out the singular Born terms explicitly by writing

$$B_i = B_i^B + \bar{B}_i \tag{3.25}$$

where $\bar{B}_i$ has no one particle singularity in s,t or u. If one calculates the B_i^B by including only the s and u channel nucleon poles then the gauge conditions Eq. (3.24) force the presence of the t channel pion pole in some of the $B_i^{(-)B}$ amplitudes. We will include this pole in the B_i^B so that the Born terms are assumed to be calculated in a gauge invariant way. We stress, however, that the low energy theorems are independent of the details of the separation made in Eq. (3.25) and they only depend on the unambiguous singular part of the Born terms.

The gauge conditions given by (3.24) must be satisfied by $\bar{B}_i$ since the Born terms already satisfy them. We can eliminate $\bar{B}_5$ from the second equation and $\bar{B}_3$ from the first. However since both $\bar{B}_3$ and $\bar{B}_2$ are pole free in s,t and u $\bar{B}_2$ must contain the factor $t-m^2$ so that $\bar{B}_2(s,t) = \tilde{B}_2(s,t)(t-m^2)$ where $\tilde{B}_2(s,t)$ is KSF and KZF.

Once the conditions (3.24) have been imposed the B_i are related to the physical CGLN amplitudes A_i[8] according to

$$A_1 = B_1 - MB_6 \qquad A_3 = -B_8$$

$$A_2 = 2B_2/(t-m^2) \qquad A_4 = -\tfrac{1}{2}B_6 \tag{3.26}$$

and the A_i are then directly related to the $\mathcal{F}_i$ appearing in Eq. (3.18). If we make the separation $A_i = A_i^B + \bar{A}_i$ where A_i^B are obtained from B_i^B using (3.26) then our observation is that all the $\bar{A}_i$ in particular $\bar{A}_2 = 2\tilde{B}_2$ are KSF. The contribution of the continuum pieces $\bar{A}_i$ to $\mathcal{F}_1$ and $\mathcal{F}_4$ (and to E_{0+} and E_{2-}) are easily calculated to be higher order in the pion mass at the physical threshold than the singular Born terms and this establishes the low energy theorems Eqs. (3.19)-(3.21).

From the connection between the A_i and $\mathcal{F}_i$[8,19] one sees that the Kroll-Ruderman theorem Eq. (3.19) follows from the fact that $A_{1,3,4}$ are KSF and from the known residue of the dynamical poles. The additional theorem on $\mathcal{F}_4$ Eq. (3.20) incorporates in an essential way the fact that $\bar{A}_2$ is KSF. The fact that the pion mass appears in the denominator of $\mathcal{F}_4$ arises from the $1/(t-m^2)$ pole in A_2^B and the divergence in $\mathcal{F}_4/q^2$ as $m \to 0$ is characteristic of the long range nature of the force associated with the exchange of a mass = 0 particle.

Recently Jones and Frautchi[20] have given a complete helicity and Regge analysis of the photopion process. The low energy theorem in their notation, is essentially the fact that the following combination of t channel helicity amplitudes has the indicated kinematical factor

$$\bar{f}^t_{01,\frac{1}{2}\frac{1}{2}} - \bar{f}^t_{01,-\frac{1}{2}-\frac{1}{2}} \sim \frac{(t-4M^2)^{\frac{1}{2}}(t-m^2)}{t^{\frac{1}{2}}} \tag{3.27}$$

where the $\bar{f}$'s are related to the helicity amplitudes by half angle factors

$$f^t_{cd,ab} = (\sin\frac{\theta_t}{2})^{|\lambda-\mu|}(\cos\frac{\theta_t}{2})^{|\lambda+\mu|}\bar{f}^t_{cd,ab} .$$

The extra factor of $t-M^2$ in Eq. (3.27), compared to what Jones and Frautchi have, only correct for the continuum part of the combination $\bar{f}^t_{01,\frac{1}{2}\frac{1}{2}}-\bar{f}^t_{01,-\frac{1}{2}-\frac{1}{2}}$. The t channel Born term clearly does not have this kinematical zero because of the Feynman propagator $1/(t-m^2)$. However since the singular part of this graph is well known its contribution can be explicitly calculated. The continuum piece does have the factor $(t-m^2)$, which near the physical threshold, suppresses the continuum contribution by a factor proportional to the pion mass. Hence the low energy theorem can be easily expressed as

$$(\bar{f}^t_{01,\frac{1}{2}\frac{1}{2}} - \bar{f}^t_{01,-\frac{1}{2}-\frac{1}{2}}) - (\text{Born term}) \to O(m)$$

near the physical threshold.

ACKNOWLEDGMENTS

One of us (HP) would like to thank Prof. V. Singh for a helpful discussion.

We also would like to thank Prof. G. Chew for the hospitality of the Lawrence Radiation Laboratory where this work was done.

REFERENCES

1. W. Thirring, Phil. Mag. 41, 1193 (1950).

2. F. Low, Phys. Rev. 96, 1428 (1954); M. Gell-Mann and M. Goldberger, Phys. Rev. 96, 1433 (1954).

3. N. M. Kroll and M. A. Ruderman, Phys. Rev. 93, 233 (1954), (Photopion production); L. I. Lapidus and Chou Kuang-Chao, JETP, 12, 898 (1961), (Compton scattering from arbitrary spin targets); B. Sakita, Phys. Rev. 127, 1800 (1962), (Photodisintegration of the Deuteron).

4. F. E. Low, Phys. Rev. 110, 974 (1958); S. L. Adler and Y. Dothan, Phys. Rev. 151, 1267 (1966).

5. V. Singh, Phys. Rev. Letters (to be published).

6. V. A. Petrunkin, JETP 13, 808 (1961), see also Ref. 2.

7. A. Pais, Phys. Rev. Letters (to be published); also CERN preprint (to be published).

8. G. F. Chew, M. L. Goldberger, F. E. Low and Y. Nambu, Phys. Rev. 106, 1345 (1957). We adopt the CGLN notation for the photopion production amplitude.

9. H. Abarbanel and M. L. Goldberger, Princeton University preprint (to be published); This approach has also been suggested by D. Horn, Caltech preprint: CTSL INT REPORT 34, May 1967 (unpublished).

10. If there is a single particle state mass degenerate with the target its contribution can be included in a trivial way. However a mass zero quanta, like the photon, can produce inelastic cuts which begin at the elastic threshold. Precisely how such cuts influence the low energy theorem is not known. The contribution of intermediate states with photons in them is assumed to contribute higher order terms in $\alpha = 1/137$ (provided

perturbation theory is valid!) so the low energy theorems are presumed to be true to first order in α and all orders in the strong interactions. It is an interesting question whether or not the low energy theorems are valid to higher orders in the electromagnetic coupling.

11. Solutions for finite mass particles also fail to take into account parity and time reversal invariance. See K. Hepp, Helv. Phys. Acta 37, 55 (1964); D. N. Williams, Construction of Invariant Scalar Amplitudes without Kinematic Singularities for Arbitrary Spin Non-zero Mass, Two-Body Scattering Process, Lawrence Radiation Laboratory Report UCRL-11113 (1963).
12. A. C. Hearn, Nuovo Cimento 21, 333 (1961).
13. M. A. B. Beg, Phys. Rev. Letters 17, 333 (1966).
14. S. Weinberg, Phys. Rev. 138, B988 (1965).
15. M. Gell-Mann, M. L. Goldberger, F. E. Low, E. Marx and F. Zachariasen, Phys. Rev. 133, B145 (1964).
16. D. Hall and A. S. Wightman, Mat. Fys. Medd. Dan. Vid. Selsk., 31, No. 5 (1957).
17. The electromagnetic vertex for $J \leq 3/2$ has been defined by V. Glaser and B. Jaksić, Nuovo Cimento 5, 1197 (1957).
18. We remark that this additional theorem has nothing directly to do with the PCAC result of S. Fubini, G. Furlan, and C. Rossetti, Nuovo Cimento 40, 1171 (1965).
19. J. Ball, Phys. Rev. 124, 2014 (1961).
20. L. Jones and S. Frautschi, Caltech preprint, (to be published).

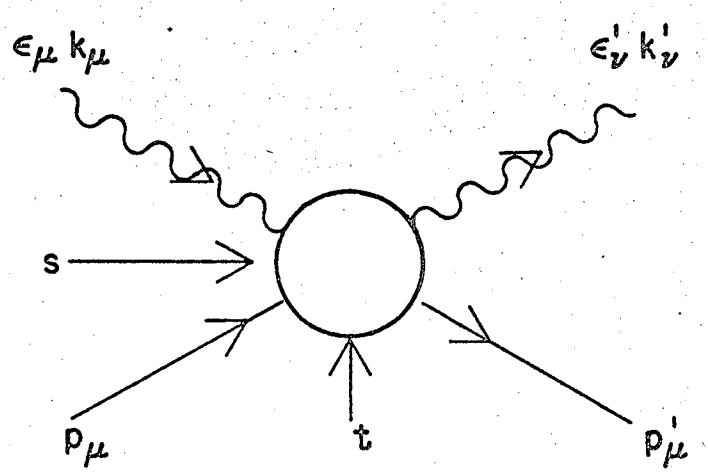

Figure 1

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