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ON INTEGRAL REPRESENTATIONS FOR N-PARTICLE FUNCTIONS

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ABSTRACT

An integral representation of the Bergman-Weil type is derived for a function defined on an algebraic variety. This formula is useful for constructing Mandelstam-type integral representations for N-particle functions.

I. INTRODUCTION

The Mandelstam representation expresses a four-particle function on its physical sheet in terms of the multiple discontinuities across the cuts bounding this sheet.[1] The present work is concerned with the analogous representations for N-particle functions. Such representations are evidently needed for a complete dispersion-theoretic dynamics.

The problem of obtaining a Mandelstam-type representation for an N-particle function has three parts. First one must define the physical sheet of this function and identify its boundary cuts. Some work[2,3] has been done on this, and more is in progress. Having identified the boundary cuts, one must obtain formulas for the discontinuities and multiple discontinuities across these cuts. Considerable work[4,5,6] has been done on this, and more is in progress. Having identified the cuts and obtained formulas for their discontinuities, one must finally express the function on the physical sheet in terms of these discontinuities. This last problem is the one discussed here. That is, we assume that the cuts bounding the physical sheet and their discontinuities are known, and seek a representation of the function on this sheet.

The Bergman-Weil representation[7] can be used to represent a function in terms of its multiple discontinuities. Indeed, the representation given by Mandelstam is essentially a special case of the B-W formula. For the general N-particle function one needs,

however, a generalization of the B-W formula to the case in which the function is defined only on an algebraic variety. The reason for this is discussed next.

The N-particle function is originally defined only on the mass shell $\mathcal{M}$, which is the set of points in momentum-energy space that satisfy the mass constraints and momentum-energy conservation laws. Since these conditions are expressed by the simultaneous vanishing of several analytic functions of the momentum-energy vectors, the set $\mathcal{M}$ is an <u>analytic variety</u>. Since these functions are in fact polynomials, $\mathcal{M}$ is moreover an <u>algebraic variety</u>. The B-W formula refers to functions defined over a full space and hence is not immediately globally applicable to a function defined only on the algebraic variety $\mathcal{M}$.

One can try to avoid this difficulty by invoking Lorentz invariance and reducing the problem to a corresponding problem in the space of scalar invariants. This works fine for $N = 4$ and $N = 5$. But for $N > 5$ it doesn't. In particular, for $N > 5$ there is no choice of independent scalar invariants such that all others can be expressed in terms of them as single-valued functions. This means that the mass shell $\mathcal{M}$ maps into a multisheeted surface over any space $\mathcal{S}$ of independent scalars.

One can apply the B-W formula to the function defined on an individual sheet. However, such a sheet corresponds to only part of the mass shell $\mathcal{M}$. Accordingly it is bounded in part by cuts that are not the images of cuts that bound the physical sheet in $\mathcal{M}$.

These extra cuts, which arise solely from the multisheeted nature of the image of $\mathcal{M}$, are "kinematical", in that they have no counterpart in $\mathcal{M}$ itself, and depend on the particular choice of independent scalars. Their discontinuities are not given by unitarity, and hence, in distinction to those for the cuts in $\mathcal{M}$, must not be considered to be given.

In simple cases one can eliminate the unknown discontinuites across these kinematic cuts by considering all the sheets simultaneously. However, the algebra becomes intractable for all but the simplest cases, because one needs solutions of fifth-order algebraic equations.

In order to resolve this problem one can regard the mass shell in invariant space not as a multisheeted surface over some space $\mathcal{S}$ of independent scalar invariants, but rather as an algebraic variety imbedded in a space $\mathcal{S}'$ of a larger set of invariants. Asribekov[8] has shown how the mass shell in invariant space can be regarded as an algebraic variety V imbedded in a space $\mathcal{S}'$ of $(N^2 - 3N)/2$ scalar invariants. Thus for either the invariant-space or momentum-energy-space approach one is led to consider the mass shell as an algebraic variety. Our problem is thus to adapt the B-W formula to functions defined only on algebraic varieties.

By considering the mass shell to be an algebraic variety we achieve an important simplification: The Landau surfaces L_i are given as the zeros of polynomials $Z_i(z)$ in the components of $z = (z_1, \cdots z_n)$, the set of coordinates of the imbedding space.* It is natural to define the cut associated with a singularity at $Z_i(z) = 0$

* See Appendix D.

by a curve through the origin in the Z_i-plane. We shall in fact assume that the cuts bounding the physical sheet are confined to sets of the form $\mathrm{Im}\, Z_i(z) = 0$, where the $Z_i(z)$ are polynomials in the coordinates of the imbedding space. More general cases can be obtained, for example, by approximating a curve in Z_i by a set of straight-line segments, etc.

As in the Mandelstam representation there will be contributions associated with contours at infinity, unless the function falls off "sufficiently fast." These can be taken into account, but in order to concentrate on the essential problem we simply assume that the function falls off "sufficiently fast."

Our procedure is to apply the B-W formula to an appropriate extension $F(z)$ into the imbedding space of the function $f(z)$ defined on the variety V, and then to reduce the formula in a way such that it finally refers only to $f(z)$. This results in a formula

$$f(z) = \frac{1}{(2\pi i)^{n-m}} \sum_{\lambda} \int_{I_\lambda} \Delta_\lambda (Z'_\lambda) \frac{J_\lambda(Z'_\lambda; z)}{J_\lambda(Z'_\lambda)} \prod_{i=1}^{n-m} \frac{dZ'_{\lambda i}}{Z'_{\lambda i} - Z_{\lambda i}(z)} . \tag{1.1}$$

Here λ labels the various sets of $n-m$ of the boundary surfaces, and $\mathrm{Im}\, Z_{\lambda i} = 0$ is the ith surface of the λth set. The region of integration is

$$I_\lambda = \left[\bigcap_{i=1}^{n-m} \{\mathrm{Im}\, Z'_{\lambda i} = 0\} \right] - \{J_\lambda(Z') = 0\},$$

which is generally multisheeted. The integer $n-m$ is the dimension

of the variety V. The function $\Delta_\lambda(Z')$ is the appropriate (n-m)-fold multiple discontinuity and $J_\lambda(Z')$ is a certain Jacobian polynomial. The function $J_\lambda(Z';z)$ is a polynomial with the property $J_\lambda(Z'(z);z) = J_\lambda(Z'(z))$. It is essentially a B-W kernel function. An alternative form of (1.1) is given that does not contain the function $J_\lambda(Z')^{-1}$, and that expresses the integration region as a variety imbedded in V.

It should be emphasized that the variety V is not required to be an analytic manifold. In particular V can have points P such that no neighborhood of P is topologically equivalent to a neighborhood in a Euclidean space of any number of dimensions. The possibility of such "singular points" of V must be allowed because the mass shell has them. It is the presence of such points that causes the main difficulty of this work. Our original motivation was to see whether these points cause any real problem in the construction of generalized Mandelstam-type representations: Are there extra contributions associated with such points? Our answer is that there are none.

Our notation is standard: $\underset{\sim}{R}^n$ and $\underset{\sim}{C}^n$ denote the spaces of n real and n complex variables, respectively. For a function f defined on an algebraic variety V, holomorphy means strong holomorphy. Hepp[9] has shown that weak holomorphy on a domain $\mathcal{D}$ in the mass shell $\mathcal{M}$ implies strong holomorphy on $\mathcal{D}$, and also that strong holomorphy on $\mathcal{D}$ implies the strong holomorphy of appropriate corresponding functions on the image $\mathcal{D}'$ of $\mathcal{D}$ in the space of invariants,

provided $\mathcal{D}$ is a domain of holomorphy bounded by cuts described by scalar invariants.

A set of m functions differentiable on a neighborhood of a set V are said to be functionally independent over V if and only if their matrix of first derivatives has rank m at almost all points of V. In particular, if V is an algebraic variety defined by the m equations $G_i(z) = 0$, $i=1,\cdots m$, where the G_i are polynomials that are functionally independent over V, then the set of points of V where the rank of the matrix $\partial G_i/\partial z_j$ of first derivatives is less than m is the set $\mathcal{S}V$ of singular points of V, and it is confined to a set of dimension less than that of V. The dimension of V is $n-m$, where n is the dimension of the imbedding space. References to proofs of these well-known results are cited in the course of the proof.

II. BERGMAN-WEIL FORMULA ON AN ALGEBRAIC VARIETY

<u>Theorem 1.</u> Let $z = \{z_1, z_2, \cdots z_n\}$ be a set of n complex variables. Let V be the algebraic variety

$V \equiv \{z \mid G_i(z) = 0,\ i = 1,2,\cdots m < n\}$,

where the $G_i(z)$ are a set of polynomials that are functionally independent over V. Let W be the set

$$W \equiv \bigcup_{i=1}^{N} W_i,$$

with

$$W_i \equiv \{z \mid \operatorname{Im} Z_i(z) = 0\},$$

where the $Z_i(z)$ are polynomials. Suppose $f(z)$ is defined and holomorphic on V-W. If $f(z)$ goes to zero sufficiently fast as $|z|^2 \equiv \Sigma|z_i|^2 \to \infty$ on V-W, then for any z in V-W

$$f(z) = \frac{1}{(2\pi i)^{n-m}} \sum_{\lambda} \int_{I_\lambda} \frac{dz'\ \Delta_\lambda(z')\left[\prod_{i=1} \delta(G_i(z'))\right] J_\lambda(z';z)}{\prod_{j=1}^{n-m}\left(Z_{i(\lambda,j)}(z') - Z_{i(\lambda,j)}(z)\right)} . \tag{2.1}$$

The index λ labels the various subsets consisting of n-m elements from the set of N indices(i, and $i(\lambda,j)$ is the <u>j</u>th element of the <u>λ</u>th set. The region of integration I_λ is

$$I_\lambda \equiv \bigcap_{j=1}^{n-m} W_{i(\lambda,j)} - \partial V \tag{2.2}$$

and $\Delta_\lambda(z')$ is a corresponding multiple discontinuity, which will be further specified in the course of the proof. The Dirac delta functions

$\delta(G_i)$ effectively restrict the integration regions to $V \cap I_\lambda$. The functions $J_\lambda(z';z)$ are Bergman-Weil kernels, which will be specified in the course of the proof. $\mathcal{S}V$ is the set of singular points of V.

Proof: A fundamental theorem of complex variable theory (See Appendix A) ensures the existence of a function $F(z)$ holomorphic in $\underset{\sim}{C}^n - W$ and satisfying $F(z) = f(z)$ for z in V-W. The B-W formula will be applied to $F(z)$, or, more precisely, to the restriction of $F(z)$ to the region

$$\mathcal{R}(\delta,\epsilon,\rho) \equiv \{z \mid |\mathrm{Im}\, Z_i(z)|^2 \geq \delta_i,\ |G_i(z)|^2 \leq \epsilon_i,\ |z_i|^2 \leq \rho_i\}. \tag{2.3}$$

This region, which has various disconnected parts, is contained in the domain of holomorphy of $F(z)$. Its boundary surfaces are

$$\sigma_i'(\delta, \epsilon, \rho) \equiv \{z \in \mathcal{R}(\delta, \epsilon, \rho) \mid |\mathrm{Im}\, Z_i(z)|^2 = \delta_i\}, \quad i=1,\cdots N, \tag{2.4}$$

$$\sigma_i''(\delta, \epsilon, \rho) \equiv \{z \in \mathcal{R}(\delta, \epsilon, \rho) \mid |G_i(z)|^2 = \epsilon_i\}, \quad i=1,\cdots m \tag{2.5}$$

and

$$\sigma_i'''(\delta, \epsilon, \rho) \equiv \{z \in \mathcal{R}(\delta, \epsilon, \rho) \mid |z_i|^2 = \rho_i\}, \quad i=1,\cdots n. \tag{2.6}$$

The B-W formula[7] gives, formally, for $z \in V$ in the interior of $\mathcal{R}(\delta, \epsilon, \rho)$,

$$f(z) = \frac{(-1)^{\frac{n(n-1)}{2}}}{(2\pi i)^n} \sum_\gamma \int_{I_\gamma(\delta,\epsilon,\rho)} dz' \quad F(z') \quad J_\gamma(z';z) \times \left[\prod_{j=1}^{N'_\gamma} \left(Z_{i(\gamma,j)}(z') - Z_{i(\gamma,j)}(z) \right) \prod_{j=1+N'_\gamma}^{N'_\gamma+N''_\gamma} G_{i(\gamma,j)}(z') \prod_{j=1+N'_\gamma+N''_\gamma}^{N'_\gamma+N''_\gamma+N'''_\gamma} \left(z'_{i(\gamma,j)} - z_{i(\gamma,j)} \right) \right]^{-1}. \tag{2.7}$$

The symbol γ labels the various subsets consisting of n indices from among the set of $N+m+n$ indices $(i_1,\cdots i_N;\ i_1,\cdots i_m;\ i_1,\cdots i_n)$. Thus $N'_\gamma + N''_\gamma + N'''_\gamma = n$ for all γ. The polynomial $J_\gamma(z';z)$ is the determinant of the matrix of polynomials $P_{ji}(z';z)$ defined in reference 7.

The region of integration $I_\gamma(\delta, \epsilon, \rho)$ is the set

$$I_\gamma(\delta,\epsilon,\rho) = \bigcap_{j=1}^{N'_\gamma} \sigma'_{i(\gamma,j)}(\delta,\epsilon,\rho) \bigcap_{j=1+N'_\gamma}^{N''_\gamma+N'_\gamma} \sigma''_{i(\gamma,j)}(\delta,\epsilon,\rho) \bigcap_{j=1+N'_\gamma+N''_\gamma}^{n} \sigma''_{i(\gamma,j)}(\delta,\epsilon,\rho). \tag{2.8}$$

The proof of the theorem consists of four parts. First it is shown that the parameters δ, ϵ, and ρ can be constrained in such a way that the right side of (2.7) is well defined, and equals the left side. The limit $\epsilon \to 0$ is then studied and it is shown that the factors $G_i(z')^{-1}$ in (2.7) can be replaced by factors $2\pi i\delta\left(G_i(z')\right)$, and that only the terms with $N''_\gamma = m$ contribute. This restricts the formula to the variety V. Next the limit $\rho \to \infty$, $\delta \to 0$ is taken. The terms with $N'''_\gamma > 0$ drop out by virtue of the condition $f(z) \to 0$, and the various remaining contributions combine to give the multiple discontinuities Δ_λ appearing in (2.1). The index λ labels the sets γ for which $N'_\gamma = n - m$.

The first problem is to establish conditions under which the right side of (2.7) is well defined. If the parameters ρ_i, ϵ_i, and δ_i are all finite and strictly positive then the functions of the integrand are all bounded and continuous on $I_\gamma(\rho, \epsilon, \delta)$. The question is whether for fixed ρ, ϵ and δ the quantity

$$\int_{I_\gamma(\rho,\epsilon,\delta)} dz' \; \phi(z') \equiv \int_{I_\gamma(\rho,\epsilon,\delta)} dz'_1 \wedge dz'_2 \cdots \wedge dz'_n \phi(z') \tag{2.9}$$

is well defined when $\phi(z')$ is well defined.

The integral (2.9) is locally well defined in a neighborhood $N(z'_0) \subset I_\gamma(\rho, \epsilon, \delta)$ of a point z'_0 of $I_\gamma(\rho, \epsilon, \delta)$ provided there is a set of functions $x_i(t)$ and $y_i(t)$ $[i=1,\cdots n;\ t \equiv (t_1,\cdots t_n)\epsilon \underset{\sim}{R}^n]$ that are continuously differentiable on the unit ball $B \equiv \{t|\Sigma t^2_i \leqslant 1\}$, and that define via $z_i(t) = x_i(t) + iy_i(t)$ a one-to-one mapping of B onto $\bar{N}(z'_0)$. In this case one has, by definition,

$$\int_{\bar{N}(z'_0)} dz' \quad \phi(z') \equiv \pm \int_B dt_1 \cdots dt_n |\partial z/\partial t| \ \phi\Big(z(t)\Big), \tag{2.10}$$

where the sign is determined by the orientation of $N(z'_0)$.

To establish conditions under which the requisite functions $x_i(t)$ and $y_i(t)$ exist, consider the real polynomial mapping $\Gamma_\gamma : \underset{\sim}{R}^{2n} \to \underset{\sim}{R}^n$ defined by

$$r_j = \delta_{i(\gamma,j)} \equiv |\mathrm{Im}\ Z_{i(\gamma,j)}(w)|^2 \qquad j=1,\cdots N'_\gamma$$

$$r_j = \epsilon_{i(\gamma,j)} = |G_{i(\gamma,j)}(w)|^2 \qquad j=1+N'_\gamma,\cdots N''_\gamma+N'_\gamma$$

$$r_j = \rho_{i(\gamma,j)} = |z_{i(\gamma,j)}(w)|^2 \qquad j=1+N'_\gamma+N''_\gamma,\cdots n \tag{2.11}$$

where $w = (w_1,\cdots w_{2n}) = (x_1,\cdots x_n; y_1,\cdots y_n)$. Let Ω_γ be the set of points $w \in \underset{\sim}{R}^{2n}$ such that the rank of the n by $2n$ matrix $||\partial r/\partial w||$ is less than n. According to Sard's theorem[10] the image $\Omega'_\gamma \equiv \Gamma_\gamma(\Omega_\gamma)$ in $\underset{\sim}{R}^n$ of Ω_γ has measure zero. This means Ω'_γ contains no open set in $\underset{\sim}{R}^n$, but it does not preclude the possibility that Ω'_γ is dense in $\underset{\sim}{R}^N$.

Because the mapping Γ_γ is a real polynomial mapping we can apply Theorem B of Appendix B. This says that Ω'_γ is confined to a finite

union of proper algebraic varieties. In particular, we have

$$\Omega'_\gamma \subset \bigcup_{i=1}^{q} A_i, \tag{2.12}$$

where q is finite,

$$A_i \equiv \{r = (r_i, \cdots r_n) \in \mathbb{R}^n | P_i(r) = 0\}, \tag{2.13}$$

and the P_i are polynomials that are not identically zero:

$$P_i \not\equiv 0 \tag{2.14}$$

We are interested in letting $\delta_i \to 0$, $\epsilon_i \to 0$, and $\rho_i \to \infty$. In place of the variable $r = (\delta_i; \epsilon_i; \rho_i)$ we introduce $r' = (\delta_i; \epsilon_i; \rho_i^{-1})$. Then the limit of interest is $r' \to 0$. The set Ω'_γ, considered as a set in r' space, is readily seen to satisfy

$$\Omega'_\gamma \subset \bigcup_{j=1}^{q'} A'_i \tag{2.15}$$

where q' is finite,

$$A' \equiv \{r' \in R^n | P'_i(r') = 0\}, \tag{2.16}$$

and the P'_i are polynomials $P'_i \not\equiv 0$.

Let two real variables $\bar{\epsilon}$ and $\bar{\delta}$ be introduced and let

$$\epsilon_i(\bar{\epsilon}) = \bar{\epsilon}\,\eta_i, \quad \delta_i(\bar{\delta}) = \bar{\delta}\,\zeta_i, \quad \rho_i^{-1}(\bar{\delta}) = \bar{\delta}\,\xi_i. \tag{2.17}$$

The $(\bar{\epsilon}, \bar{\delta})$ plane is then a linear subspace of the space of variables $r' \in \mathbb{R}^n$. For almost all choices of the set of parameters η_i, ζ_i, and ξ_i the sets Ω'_γ will all intersect the $(\bar{\epsilon}, \bar{\delta})$ plane in a finite set of one dimensional curves. Choose such a set and let Ω'' be the union of these curves.

Let Q be the open first quadrant in the $(\bar{\epsilon}, \bar{\delta})$ plane. Let $\bar{\epsilon}_m(\bar{\delta})$ be the least value of $\bar{\epsilon}$ such that $(\bar{\epsilon}_m(\bar{\delta}), \bar{\delta})$ lies in Q and

on Ω''. Let $\tilde{Q}$ be the set

$$\tilde{Q} = \{(\bar{\epsilon}, \bar{\delta}) \in Q \mid \bar{\epsilon} < \bar{\epsilon}_m(\bar{\delta})\}. \tag{2.18}$$

That is, $\tilde{Q}$ is the part of Q lying below the lowest curve of $Q \cap \Omega''$. For some sufficiently small ball N centered at the origin the set $\tilde{Q} \cap N$ will be a connected open set, as shown in Fig. 1.

Define

$$I_\gamma(\bar{\epsilon}, \bar{\delta}) = I_\gamma(\delta(\bar{\delta}), \epsilon(\bar{\epsilon}), \rho(\bar{\delta}))$$

and

$$I(\bar{\epsilon}, \bar{\delta}) = \bigcup I_\gamma(\bar{\epsilon}, \bar{\delta}).$$

For any $(\bar{\epsilon}, \bar{\delta})$ in $\tilde{Q} \cap N$ the rank of $||\partial r/\partial w||$ is n at every point of $I(\bar{\epsilon}, \bar{\delta})$. This is what has just been shown. But if the rank of this matrix is n, then the set of n real variables r_i can be augmented locally by a set of n real variables t_i to give a set of $2n$ real variables such that the $2n$-by-$2n$ matrix $||\partial r/\partial w; \partial t/\partial w||$ has rank $2n$. These t_i can in fact be taken to be linear functions of the w_j. Because the rank of this matrix is maximal the equations $r_i(w)$ and $t_i(w)$ have a unique holomorphic inverse[11] $w = w(t,r)$. Fixing the $(\bar{\epsilon}, \bar{\delta})$ in $\tilde{Q} \cap N$, we obtain for any z on $I(\bar{\epsilon}, \bar{\delta})$ the requisite set of functions $w_i(t) = (x_i(t), y_i(t))$. Thus the integrations appearing on the right side of (2.7) are locally well defined at all points of $I(\bar{\epsilon}, \bar{\delta})$, for any $(\bar{\epsilon}, \bar{\delta}) \in \tilde{Q} \cap N$.

Because $I(\bar{\epsilon}, \bar{\delta})$ consists of a finite number of real algebraic manifolds, restricted to a bounded region, its measure is finite. Thus the right side of (2.7) is well defined for $(\bar{\epsilon}, \bar{\delta})$ in $\tilde{Q} \cap N$. If the right side of (2.7) is well defined in this sense, then Weil's theorem[7] gives (2.7). [Exceptional cases where several I_γ coincide can be excluded by very slight adjustment of the η_i, ζ_i and ξ_i in (2.17)].

We wish now to take the limit $\bar{\epsilon} \to 0$ in (2.7). To evaluate the result, (2.7) is first converted to an alternative form. For a given term γ the denominator in (2.7) can be written as

$$D_\gamma(z';z) = \prod_{i=1}^{n} \bar{Z}_{\gamma i}(z') - \bar{Z}_{\gamma i}(z), \tag{2.19}$$

where $\bar{Z}_{\gamma i}$ the polynomial $Z_{i(\gamma,j)}$, $G_{i(\gamma,j)}$, or $Z_{i(\gamma,j)}$. Define

$$J_\gamma(z') \equiv \det \| \partial\bar{Z}_{\gamma i}(z')/\partial z'_j \|. \tag{2.20}$$

At points z' such that $J_\gamma(z') \neq 0$ the n functions $\bar{Z}_{\gamma i}$ can serve as local coordinates[12] for $I_\gamma(\bar{\epsilon}, \bar{\delta})$. Thus, apart from contributions from the points $J_\gamma(z') = 0$, we can write

$$f(z) = \frac{1}{(2\pi i)^n} \sum_\gamma \int_{I_\gamma^{-1}(\bar{\epsilon},\bar{\delta})} \epsilon\, \bar{F}(\bar{Z}'_\gamma) \frac{\bar{J}_\gamma(\bar{Z}'_\gamma;z)}{\bar{J}_\gamma(\bar{Z}'_\gamma)} \prod_{i=1}^{n} \frac{d\bar{Z}'_{\gamma i}}{\bar{Z}'_{\gamma i} - \bar{Z}_{\gamma i}(z)}. \tag{2.21}$$

Here $\bar{F}(\bar{Z}'_\gamma) \equiv F\big(z(\bar{Z}'_\gamma)\big)$, $\bar{J}_\gamma(\bar{Z}'_\gamma;z) \equiv J_\gamma\big(z(\bar{Z}'_\gamma);z\big)$, $\bar{J}_\gamma(\bar{Z}'_\gamma) \equiv J_\gamma\big(z(\bar{Z}'_\gamma)\big)$, and ϵ is an orientation-dependent sign. The integration region $I_\gamma^{-1}(\bar{\epsilon},\bar{\delta})$ is the image in $\bar{Z}_\gamma$ of $I_\gamma(\bar{\epsilon},\bar{\delta})$. It is a generally multisheeted surface lying over the set*

$$\{|\mathrm{Im}Z'_{i(\gamma,j)}|^2 = \delta_{i(\gamma,j)}(\bar{\delta}),\ |G'_{i(\gamma,j)}|^2 = \epsilon_{i(\gamma,j)}(\bar{\epsilon}),\ |z'_{i(\gamma,j)}|^2 = \rho_{i(\gamma,j)}(\bar{\delta})\}. \tag{2.22}$$

The function $z(\bar{Z}')$ is defined on this multisheeted surface.

The contributions to (2.21) from the points where $\bar{J}_\gamma(\bar{Z}'_\lambda) = 0$ are not well defined. However, the contribution from these points can be calculated before going to the form (2.21). According to Theorem C of Appendix C

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* For each j, $1 \leq j \leq n$, the appropriate one of these equations holds.

$$\int_{I_\gamma(\bar{\epsilon},\bar{\delta}) \cap \{H(z') = 0\}} dz' \quad \phi(z') = 0, \tag{2.23}$$

for any integrable function $\phi(z')$, and any analytic function $H(z') \not\equiv 0$. Thus the points where $J_\gamma(\bar{Z}') = 0$ can be excluded from $I_\gamma^{-1}(\bar{\epsilon},\bar{\delta})$ in (2.21), unless $J_\gamma(z')$ is identically zero. This last proviso can be dropped. For if $J_\gamma(z')$ were identically zero then one of the $\bar{Z}_i$'s could be expressed locally as an analytic function of the others.[13] This entails that the integral again vanish. (See the Corollary to Theorem C.)

The equivalence of the contributions to the right sides of (2.21) and (2.7) from the region $J_\gamma(z') \geqslant \epsilon'$ for any $\epsilon' > 0$, and the absolute convergence of the right side of (2.7), ensure that the right side of (2.21) exists as a Lebesgue integral.

Introduce the notation

$$Z_\gamma \equiv \{Z_{i(\gamma,j)}, z_{i(\gamma,j)}\} \quad \text{and} \quad G_\gamma \equiv \{G_{i(\gamma,j)}\}.$$

For any fixed $(\bar{\epsilon},\bar{\delta}) \in \mathcal{Q} \cap N$ the region $I_\gamma^{-1}(\bar{\epsilon},\bar{\delta})$ can be considered the product of a region

$$I_\gamma(\bar{\delta}) \equiv \{Z'_\gamma \,|\, |\mathrm{Im}\, Z'_{i(\gamma,j)}|^2 = \delta_{i(\gamma,j)}(\bar{\delta}), |z'_{i(\gamma,j)}|^2 = \rho_{i(\gamma,j)}(\bar{\delta})\} \tag{2.24}$$

times a multisheeted Z'_γ-dependent region

$$I_\gamma^{-1}(\bar{\epsilon}, \bar{\delta}; Z'_\gamma) \equiv \{G'_\gamma \,|\, \text{for some } z \text{ in } \mathcal{R}(\bar{\epsilon},\bar{\delta}):$$

$$Z_\gamma(z) = Z'_\gamma, |G_{i(\gamma,j)}(z)|^2 = |G_{i(\gamma,j)}|^2 = \epsilon_{i(\gamma,j)}(\bar{\epsilon})\}$$

$$\equiv G_\gamma[\mathcal{R}(\bar{\epsilon},\bar{\delta}) \cap \{z \,|\, Z_\gamma(z) = Z'_\gamma\}] \,|\, [|G'_{i(\gamma,j)}| = \epsilon_{i(\gamma,j)}(\epsilon)] \tag{2.25}$$

where $G_\gamma[A]$ is the image of A under $G_\gamma(z)$ and $G_\gamma[A]|B$ is the restriction of $G_\gamma[A]$ to B. Performing the G'_γ integration first, one obtains from (2.21)

$$f(z) = \frac{1}{(2\pi i)^n} \sum_\gamma \int_{I_\gamma(\bar\delta)} A_{\bar\delta\bar\epsilon}(Z'_\gamma; z)dZ'_\gamma, \tag{2.26}$$

where $A_{\bar\delta\bar\epsilon}$ is Lebesgue integrable, by virtue of Fubini's theorem.

Let $\bar Z'_\gamma = (G'_\gamma, Z'_\gamma)$ be a point on the multisheeted surface $I^{-1}(\bar\epsilon', \bar\delta; Z'_\gamma) \otimes I(\bar\delta)$ and let $I^{-1}(\bar\epsilon', \bar\delta; \bar Z'_\gamma)$ be the part of $\left[\bigcup_{0 \leqslant \bar\epsilon'' \leqslant \bar\epsilon'} I^{-1}_\gamma(\bar\epsilon'', \bar\delta; Z'_\gamma)\right] \equiv A$ connected in A to $\bar Z'_\gamma$. For $\bar\epsilon > 0$ define

$$\begin{aligned} E_\gamma(\bar\epsilon,\bar\delta) = \Big\{ & \bar Z'_\gamma \,|\, I^{-1}(\bar\epsilon, \bar\delta; \bar Z') \text{ is nonempty and either } \bar J(G''_\gamma, Z'_\gamma) = 0 \\ & \text{for some } G''_\gamma \in I^{-1}_\gamma(\bar\epsilon, \bar\delta; \bar Z'_\gamma) \text{ or } \bar J(G''_\gamma, Z'_\gamma) \neq 0 \\ & \text{for all } G''_\gamma \in I^{-1}_\gamma(\bar\epsilon, \bar\delta; \bar Z'_\gamma) \text{ but} \\ & I^{-1}(\bar\epsilon, \bar\delta; \bar Z') \neq [|G_{i(\gamma,j)}|^2 \leqslant \epsilon_{i(\gamma,j)}(\bar\epsilon)] \Big\} \end{aligned} \tag{2.27}$$

and

$$\begin{aligned} E'_\gamma(\bar\epsilon,\bar\delta) \equiv \Big\{ & \bar Z'_\gamma | \bar J(G''_\gamma; Z'_\gamma) \neq 0 \text{ for all } G''_\gamma \in I^{-1}_\gamma(\bar\epsilon, \bar\delta; \bar Z'_\gamma) \\ & \text{and } I^{-1}(\bar\epsilon, \bar\delta; \bar Z'_\gamma) = [|G_{i(\gamma,j)}|^2 \leqslant \epsilon_{i(\gamma,j)}(\bar\epsilon)] \Big\} \\ & \bigcup \left\{ \bar Z'_\gamma | I^{-1}_\gamma(\bar\epsilon, \bar\delta; \bar Z') \text{ is empty} \right\}. \end{aligned} \tag{2.28}$$

The separation of the space of integration in (2.21) into the two sets $I^{-1}_{\gamma}(\bar{\epsilon},\bar{\delta}) \cap E_{\gamma}(\bar{\epsilon},\bar{\delta})$ and $I^{-1}_{\gamma}(\bar{\epsilon},\bar{\delta}) \cap E'_{\gamma}(\bar{\epsilon},\bar{\delta})$ gives for any $(\bar{\epsilon},\bar{\delta})$ in $\tilde{Q} \cap N$

$$f(z) = f(z;\, \bar{\epsilon},\bar{\delta}) + f'(z;\, \bar{\epsilon},\bar{\delta}). \tag{2.29}$$

The second part of the region $E'_{\gamma}(\bar{\epsilon},\bar{\delta})$ does not contribute to (2.29). In the first part of $E'(\bar{\epsilon},\bar{\delta})$ one can use the fundamental theorem of Cauchy-Poincare[14] to shrink the contours in G'_{γ} to infinitesimal contours about $G_{i(\gamma,j)} = 0$. This yields

$$f'(z;\, \bar{\epsilon},\bar{\delta}) = \frac{1}{(2\pi i)^{n-m}} \sum_{\gamma} \int_{\hat{I}^{-1}_{\gamma}(\bar{\epsilon},\bar{\delta})} \epsilon\, F^{0}(Z'_{\gamma}) \frac{J^{0}_{\gamma}(Z'_{\gamma};z)}{J^{0}_{\gamma}(Z'_{\gamma})} \prod_{i=1}^{n-m} \frac{dZ'_{\gamma i}}{Z'_{\gamma i} - Z_{\gamma i}(z)}. \tag{2.30}$$

The region of integration $\hat{I}^{-1}_{\gamma}(\bar{\epsilon},\bar{\delta})$ consists generally of several mutually disjoint connected parts. Each connected part consists of a multisheeted covering of $I_{\gamma}(\bar{\delta}) \cap \hat{E}'_{\gamma}(\bar{\epsilon},\bar{\delta})$, where $\hat{E}'_{\gamma}(\bar{\epsilon},\bar{\delta})$ is the projection of $E'_{\gamma}(\bar{\epsilon},\bar{\delta})$ onto Z'_{γ} space. The functions $F^{0}(Z'_{\gamma}) \equiv F\big(z(0;\, Z'_{\gamma})\big)$, etc., are evaluated by evaluating the $\bar{Z}'_{\gamma} = (G'_{\gamma};\, Z'_{\gamma})$ occurring in $z(\bar{Z}'_{\gamma})$ at $G'_{\gamma} = 0$ on the appropriate sheet of $I^{-1}_{\gamma}(0,\, \bar{\delta};\, Z'_{\gamma})$. The condition $I^{-1}(\bar{\epsilon},\, \bar{\delta};\, \bar{Z}'_{\gamma}) = [\,|G_{i(\gamma,j)}|^{2} \leqslant \epsilon_{i(\gamma,j)}(\bar{\epsilon})\,]$ in (2.28) ensures that the integration region in (2.30) is restricted to the image of V. The condition $J_{\gamma} \neq 0$ then ensures that $N''_{\gamma} = m$.

The problem is now to justify passing to the limit $\bar{\epsilon} = 0$ in (2.29), and to show that $f(z;\, 0,\bar{\delta}) = 0$. As a first step we show that $I_{\gamma}(\bar{\delta}) \cap \hat{E}_{\gamma}(0,\bar{\delta})$ is of measure zero, or, more specifically, that

$$\int_{I_\gamma(\bar\delta)\cap \hat E_\gamma(0,\bar\delta)} \Phi(Z'_\gamma)\, dZ'_\gamma = 0. \tag{2.31}$$

for any locally integrable function $\Phi(Z'_\gamma)$. Here $\hat E_\gamma(0,\bar\delta) = \bigcap_{\bar\epsilon>0} \hat E_\gamma(\bar\epsilon,\bar\delta)$.

Each Z'_γ in $\hat E_\gamma(0,\bar\delta)$ is the image of a z in V. The points of V can be separated into two parts; those lying at the "regular" points $R \subset V$, which are the points of V at which rank $||\partial G/\partial z|| = m$; and those lying at the "singular" points $S \subset V$, which are the points of V at which rank $||\partial G/\partial z|| < m$. At any point of R one can introduce a set of local coordinates $(G'_\gamma; \tilde G'_\gamma)$ with G'_γ the m functions of $G_i(z)$. Then the contribution to (2.31) from points Z'_γ that are the images of no points $z \in S$ can be converted (locally) to

$$\int_{[\tilde I_\gamma(\bar\delta)\cap \tilde E_\gamma(0,\bar\delta)] - \tilde S_\gamma} \tilde\Phi(\tilde G'_\gamma)\, |\partial Z'_\gamma/\partial \tilde G'_\gamma|\, d\tilde G'_\gamma \tag{2.31'}$$

where the tildes indicate the transformation to the new variables, $S_\gamma = Z_\gamma[S]$ is the image of S, and the Jacobian $|\partial Z'_\gamma/\partial \tilde G'_\gamma|$ is analytic. Because of the continuity properties of the functions defining $E_\gamma(\bar\epsilon,\bar\delta)$, a point $\tilde G'_\gamma \in \tilde I_\gamma(\bar\delta)$ belongs to $[\tilde I_\gamma(\bar\delta)\cap \tilde E_\gamma(0,\bar\delta)] - \tilde S_\gamma$ only if there is a z in R such that $\tilde G_\gamma(z) = \tilde G'_\gamma$ and either, (a) $J_\gamma(z) = 0$ or, (b) for all sufficiently small $\bar\epsilon' > 0$ the set $I^{-1}(\bar\epsilon',\bar\delta;\, \bar Z'(z))$ intersects a part of the boundary of $\mathcal{Q}(\bar\epsilon',\, \bar\delta)$ not associated with γ. Points $\tilde G'_\gamma$ for which (b) holds are [with perhaps slight adjustments of the parameters in (2.17)] confined to sets of

lower dimension not contributing to (2.31′). Condition (a) gives $J_\gamma(z(G'_\gamma;\tilde{G}'_\gamma)) \equiv \tilde{J}_\gamma(G'_\gamma;\tilde{G}'_\gamma) = 0$. But at points of R the vanishing of $\tilde{J}_\gamma(G'_\gamma;\tilde{G}'_\gamma) \equiv |\partial(G'_\gamma;Z'_\gamma)/\partial(G'_\gamma;\tilde{G}'_\gamma)|\cdot|\partial(G'_\gamma;\tilde{G}'_\gamma)/\partial z|$ implies the vanishing of $|\partial Z'_\gamma/\partial\tilde{G}'_\gamma|$, since $||\partial G'_\gamma/\partial G'_\gamma||$ is a unit matrix, and $\partial G/\partial\tilde{G} = 0$. Thus (2.31′) is zero. This means that for any $(\bar{\epsilon},\bar{\delta}) \in \tilde{Q} \cap N$ one can omit from (2.26) the contributions from $[I_\gamma(\bar{\delta}) \cap E_\gamma(0,\bar{\delta})] - S_\gamma$.

We must show also that

$$\int_{I_\gamma(\bar{\delta})\cap E_\gamma(0,\bar{\delta})\cap S_\gamma} \Phi(Z'_\gamma)\, dZ'_\gamma = 0, \tag{2.31''}$$

where

$$S_\gamma = Z_\gamma[S] \tag{2.32}$$

is the image of S under $Z_\gamma(z)$. The mapping $Z_\gamma(z)$ is a polynomial mapping. Thus, by virtue of the Chevalley Theorem (see Appendix B) either (a); S_γ is confined to the zeros of a finite set of polynomials $P_i(Z'_\gamma) \not\equiv 0$, or (b); S_γ includes all points not lying on the zeros of a finite set of polynomials $P_i(Z'_\gamma) \not\equiv 0$. In the first case (2.31″) is assured by Theorem C of Appendix C. In the second case S_γ must contain open sets. To prove this is impossible, note first that by virtue of the Chevalley Theorem and Lemma B of Appendix B the set of variables $Z_\gamma(z)$ must be locally linearly independent when $Z'_\gamma = Z_\gamma(z)$ lies in an open set

$$S' = \{Z'_\gamma | P'_i(Z'_\gamma) \neq 0\ ,\ i=1,\cdots s < \infty\}, \tag{2.33}$$

where the $P'_i(Z'_\gamma)$ are polynomials that are not identically zero. Thus at points of $S' \cap S_\gamma$ one can find a set of local coordinates that includes the Z_γ. Since the number of coordinates in the set Z_γ is at least

n-m, the set S_γ cannot include an open neighborhood of any point of S' unless the dimension of S is at least n-m. However, the dimension of the set of singular points S of V is less than the dimension n-m of V.[15] This proves (2.31"), hence also (2.31).

Equation (2.26) is true for any $(\bar{\epsilon},\bar{\delta})$ in $\tilde{Q}\cap N$. By virtue of (2.31) we can for any such $(\bar{\epsilon},\bar{\delta})$ restrict the range of integration to $I_\gamma(\bar{\delta}) - \hat{E}_\gamma(0;\bar{\delta})$. For any point Z'_γ in $I_\gamma(\bar{\delta}) - \hat{E}_\gamma(0;\bar{\delta})$ there is some $\bar{\epsilon}(Z'_\gamma) < \bar{\epsilon}$ such that Z'_γ lies in $I_\gamma(\bar{\delta})\cap \hat{E}'_\gamma(\bar{\epsilon}',\bar{\delta})$ for all $\bar{\epsilon}' \leqslant \bar{\epsilon}(Z'_\gamma)$. This follows from the definition of $\hat{E}_\gamma(0,\bar{\delta})$ and the fact that for all $\bar{\epsilon}' \leqslant \bar{\epsilon}''$, $\hat{E}_\gamma(\bar{\epsilon}'',\bar{\delta};\ z'_\gamma)\subset \hat{E}_\gamma(\bar{\epsilon}',\bar{\delta};Z'_\gamma)$. Thus as we let $\bar{\epsilon}\to 0$, each point Z'_γ in the region of integration $I_\gamma(\bar{\delta}) - \hat{E}_\gamma(0;\bar{\delta})$ is eventually included in the set of points that contribute to the $f'(z;\bar{\epsilon},\delta)$ of (2.30).

This result is not sufficient to give the desired result

$$f(z) = f'(z;\ 0,\ \bar{\delta}). \tag{2.34}$$

For it is conceivable that the function $A_{\bar{\delta}\bar{\epsilon}}$ could depend on $\bar{\epsilon}$. If this were the case then the contribution from $I_\delta(\bar{\delta})\cap\left[\hat{E}_\gamma(\bar{\epsilon},\bar{\delta}) - \hat{E}_\gamma(0,\bar{\delta})\right]$ could remain significant, even though this region itself shrinks to zero as $\bar{\epsilon}\to 0$. However, $A_{\bar{\delta}\bar{\epsilon}}$ is in fact independent of $\bar{\epsilon}$, for $(\bar{\epsilon},\bar{\delta}) \in \tilde{Q}\cap N$.

To prove this, let $\bar{\epsilon}'$ and $\bar{\epsilon}''$ be two different values of $\bar{\epsilon}$. Then (2.26) and (2.31) give

$$0 = \sum_\gamma \int_{I_\gamma(\bar{\delta}) - \hat{E}_\gamma(0,\bar{\delta})} [A_{\bar{\delta}\bar{\epsilon}'}(Z'_\gamma;z) - A_{\bar{\delta}\bar{\epsilon}''}(Z'_\gamma;\ z)]\, dZ'_\gamma \tag{2.35}$$

for all z not lying on some one of the surfaces $|\mathrm{Im}\, Z_i(z)| = \delta_i(\bar{\delta})$ or $|z_i(z)| = \rho_i(\bar{\delta})$. The dependence of the right side on z comes through the analytic functions $J_\gamma(\bar{Z}; z)$ of (2.21), and through the denominator functions that appear there. The pole singularities of these denominators entail that the numerator at $I_\gamma(\bar{\delta})$ be the multiple discontinuity of the function on the left. This is zero. Thus the integrand on the right must be zero. (See Appendix E)

The validity of (2.34) now follows immediately. One first fixes $(\bar{\epsilon},\bar{\delta}) \in \tilde{Q} \cap N$. The contributions to (2.26) from points $I_\gamma(\bar{\delta}) \cap \hat{E}(0,\bar{\delta})$ can then be discarded, by using (2.31). The contribution from near any point Z'_γ of $I_\gamma(\bar{\delta}) \cap \hat{E}'(0,\bar{\delta})$ can be evaluated by reducing $\bar{\epsilon}$ to a value $\bar{\epsilon}' < \bar{\epsilon}'(Z'_\gamma)$. This gives (2.34).

The rest is simple. Equation (2.34) refers only to functions defined on V. Now let $\bar{\delta} \to 0$. The terms with $N''_\gamma > 0$ drop out by virtue of the fall-off condition. The integration regions collapse to the intersections of regions $\mathrm{Im} Z_i = \pm \epsilon_i$. The orientations of these regions reverse with the reversal of the signs in front of $\pm i\epsilon_i$ and one obtains discontinuities. Specifically one obtains* from (2.34) and (2.30) the result (1.1) with

$$\Delta_\gamma(Z'_\gamma) = \sum \left(\prod_{j=1}^{n-m} \sigma_j \right) F\left(Z_{i(\gamma,j)} + i(\sigma_j \epsilon)\right), \tag{2.36}$$

where the sum is over the 2^{n-m} combinations of signs of the $\sigma_j = \pm 1$, and ϵ is infinitesimal. $\Delta_\gamma(Z'_\gamma)$ need not exist as a function. We consider it to be defined in the mean as limit $\epsilon \to 0$ of integrals.

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* For the normal case where no two regions I_λ coincide. Abnormal cases can be regarded as limiting cases.

The function $J_\lambda(Z'_\lambda; z)$ appearing in (1.1) is the determinant of the polynomials $P_{\lambda ji}(z(Z'_\lambda); z)$. These are defined by[7]

$$Z_{\lambda j}(z') - Z_{\lambda j}(z) = \sum_{i=1}^{n} P_{\lambda ji}(z'; z)\,(z'_i - z_i). \tag{2.37}$$

In the limit $z' \to z$ we have

$$P_{\lambda ji}(z; z) = \partial Z_{\lambda j}/\partial z_i. \tag{2.38}$$

Thus $J_\lambda\big(Z_\lambda(z); z\big) = J_\lambda(z)$. On the other hand if $Z_\lambda(z') = Z_\lambda(z)$ but $z' \neq z$ then $J_\lambda\big(Z_\lambda(z'); z\big) = 0$. The factor $J_\lambda(Z'_\lambda; z)/J_\lambda(Z'_\lambda)$ has therefore the effect of separating the contributions from different points z' that correspond to the same point Z'_λ: Because of this factor the multiple discontinuity for a point z' satisfying $Z_\lambda(z') \in \big[I_\lambda - E_\lambda(0,0)\big]$ will be $\Delta_\lambda(z')$, as is required. We thus understand the role of the factor $J_\lambda(Z'_\lambda; z)/J_\lambda(Z'_\lambda)$ in (1.1).

The representation (2.1) is just an alternative form of (1.1). A delta function occurring under an integral sign signifies that we should introduce a set of integration variables that includes the argument of the delta function, and then hold this variable fixed at zero. Applying this prescription to (2.1), with $(G_\gamma; Z_\gamma)$ as the new variables, one gets (1.1). By choosing other variables one obtains from (2.1) other equivalent representations.

In (2.1) the integration region I_λ occurs as an algebraic variety of real dimension n-m imbedded in a space of complex dimension n, whereas in (1.1) it is a multisheeted surface lying over a λ-dependent space of real dimension n-m.

III. COMMENTS

The denominators appearing in the integral representations are the direct generalizations of the s, t, and u variables of Mandelstam: they are polynomials in the components of the energy-momentum vectors that are also polynomials in certain invariants. These polynomials are the ones corresponding to the Landau singularities whose cuts bound the physical region. The invariant denominator functions formed from them are the same in the momentum-space and invariant-space forms of the representation. One anticipates that the momentum-space form will be more convenient for large values of N, since the mass shell constraints are much simpler. Moreover, one can deal directly with the scattering functions themselves, rather than with invariant amplitudes. The problem of constructing global invariant amplitudes has been explicitly solved in the general spin case only for $N = 4$.[16]

If one uses the momentum-space form then a problem arises: Lorentz orbits are noncompact. This means that the scattering function is essentially constant over noncompact regions. In order to obtain fall-off at large distances, and compact regions of integration, one must effectively factor out the integrations over Lorentz orbits. One can simply set certain components to zero, or alternatively, use Toller variables.[17]

The question of how fast is "sufficiently fast" is, of course, important. Evidently the general formula will have terms with fewer denominators than appear in (2.1), just as in the case of the Mandelstam representation. However, this question is not embarked upon here, where the concern has been with contributions from the singular points of the variety, rather than points at infinity.

APPENDIX A

<u>Theorem A</u>: Let D be a domain of holomorphy in $\mathbb{C}^n$. Let V be an analytic variety in $\mathbb{C}^n$. Let f be a function holomorphic on $V \cap D$. Then there exists a function F holomorphic on D such that $f(z) = F(z)$ for all z in $V \cap D$.

This theorem is a consequence of Cartan's Theorem B.[18]

Theorem B (Wright): Let $\Gamma : \underset{\sim}{R}^p \to \underset{\sim}{R}^n$ can be any real polynomial mapping. That is, Γ is defined by a set of n real polynomials $r(w) = \left(r_1(w), \cdots r_n(w)\right)$ in the components of $w = (w_1, \cdots w_p)$, where the w_j are real variables. Let Ω be the set of points $w \in \underset{\sim}{R}^p$ at which the rank of $||\partial r/\partial w||$ is less than n. Then the image $\Omega' = \Gamma(\Omega)$ in $\underset{\sim}{R}^n$ of Ω is contained in the union of a finite set of proper algebraic varieties. In particular,

$$\Omega' \subset \bigcup_{i=1}^{q} A_i, \tag{B.1}$$

where q is finite,

$$A_i = \{r \in \underset{\sim}{R}^n \mid P_i(r) = 0\}, \tag{B.2}$$

and the $P_i(r)$ are polynomials in the components of r that are not identically zero: $P_i(r) \not\equiv 0$.

Proof: This theorem is based on the

Chevalley Theorem:[19] Let $\tilde{\Gamma} : \underset{\sim}{C}^p \to \underset{\sim}{C}^n$ be any polynomial mapping of $\underset{\sim}{C}^p$ to $\underset{\sim}{C}^n$. Then the image of $\tilde{\Gamma}$ is a constructible set in $\underset{\sim}{C}^n$. More generally, $\tilde{\Gamma}$ maps constructible sets in $\underset{\sim}{C}^p$ to constructible sets in $\underset{\sim}{C}^n$.

A constructible set of $\underset{\sim}{C}^n$ (or $\underset{\sim}{C}^p$) is a set that is a finite union of locally closed subsets of $\underset{\sim}{C}^n$ (or $\underset{\sim}{C}^p$). A locally closed subset is a set that can be expressed as the intersection of an open set with a closed set. The closed sets referred to here are the closed set of the Zariski topology. In this topology the closed sets are the algebraic varieties. That is, the closed sets are sets of the form

$$\{z \mid P_1(z) = P_2(z) = \cdots P_s(z) = 0\}, \tag{B.3}$$

where s is finite, and the $P_j(z)$ are polynomials in the components of $z \in \underset{\sim}{C}^n$ (or $\underset{\sim}{C}^p$). The open sets are complements of closed sets. [These open and closed sets will, outside this paragraph, be called Z-open and Z-closed, to distinguish them from the ordinary open and closed sets induced by the Euclidean metric in $\underset{\sim}{R}^{2n}$ (or $\underset{\sim}{R}^{2p}$).]

To apply the Chevalley theorem to the present problem, the mapping $\tilde{\Gamma}$ is taken to be the natural extension of Γ to the complex domain: $r \to \tilde{r}$, $w \to \tilde{w}$. The set

$$\tilde{\Omega} \equiv \{\tilde{w} \in \underset{\sim}{C}^p \mid \text{rank } \|\partial\tilde{r}/\gamma\tilde{w}\| < n\} \tag{B.4}$$

is a Z-closed and hence constructible set in $\underset{\sim}{C}^p$. Thus Chevalley tells us that $\tilde{\Omega}' \equiv \tilde{\Gamma}(\tilde{\Omega})$ is constructible in $\underset{\sim}{C}^n$. Lemma B below ensures that $\tilde{\Omega}'$ is of $2n$ Lebesgue measure zero. This, together with constructibility, means that

$$\tilde{\Omega}' \subset \bigcup_{j=1}^{q} \tilde{A}_i \tag{B.5}$$

where q is finite,

$$\tilde{A}_i \equiv \{\tilde{r} \in \underset{\sim}{C}^n \mid \tilde{P}_i(\tilde{r}) = 0\}, \tag{B.6}$$

and the $\tilde{P}_i$ are polynomials that are not identically zero:

$$\tilde{P}_i \not\equiv 0. \tag{B.7}$$

The set Ω is the restriction of $\tilde{\Omega}$ to $\underset{\sim}{R}^p$. And Γ is the restriction of $\tilde{\Gamma}$ to $\underset{\sim}{R}^p$. Thus

$$\Omega' \equiv \Gamma(\Omega) = \tilde{\Gamma}(\Omega) \subset \tilde{\Gamma}(\tilde{\Omega}) \equiv \tilde{\Omega}' . \tag{B.8}$$

The set Ω' is restricted to $\underset{\sim}{R}^n$:

$$\Omega' = \Omega' \mid \underset{\sim}{R}^n . \tag{B.9}$$

Thus

$$\Omega' \subset \tilde{\Omega}' \mid \underset{\sim}{R}^n \subset \bigcup_{j=1}^{q} \tilde{A}_i \mid \underset{\sim}{R}^n$$

$$\equiv \bigcup_{j=1}^{q} A_i , \tag{B.10}$$

where

$$A_i \equiv \{r \in \underset{\sim}{R}^n \mid \left(\tilde{P}_i(r) \mid \underset{\sim}{R}^n\right) \equiv P_i(r) = 0\}. \tag{B.11}$$

The condition $\tilde{P}_i \not\equiv 0$ of (B.7) ensures that $P_i \not\equiv 0$, since an analytic function that vanishes over a real environment vanishes identically.

<u>Lemma B:</u> Let $\tilde{\Gamma} : N \subset \underset{\sim}{C}^p \to \underset{\sim}{C}^n$ be a holomorphic mapping of N into $\underset{\sim}{C}^n$. Let $Z(z) \equiv \left(Z_1(z), \cdots Z_n(z)\right)$ be the n functions that define $\tilde{\Gamma}$, and let

$$\tilde{\Omega} \equiv \{z \in N \mid \text{rank} \parallel \partial Z/\partial z \parallel < n \} . \tag{B.12}$$

Then the set

$$\tilde{\Omega}' \equiv \tilde{\Gamma}(\tilde{\Omega}) \tag{B.13}$$

has 2n-dimensional Lebesgue measure zero.

Proof: This lemma is akin to Sard's theorem,[10] which is used to prove it. Let the $2n$ real variables that comprise $Z = (Z_1, \cdots Z_n)$ be $W = (W_1, \cdots W_{2n}) = (X_1, \cdots X_n; Y_1, \cdots Y_n) = (X;Y)$. Let the $2p$ real variables that comprise $z = (z_1, \cdots z_p)$ be $w = (w_1, \cdots w_{2p}) = (x_1, \cdots x_n; y_1, \cdots y_n) = (x;y)$. Let N_r be the image of N in R^{2p}, and let

$$\tilde{\tau} = \{w \in N_r \mid \text{rank} \|\partial W/\partial w\| < 2n\} . \tag{B.14}$$

If we regard sets in $C^n(C^p)$ as equivalent to their images in $R^{2n}(R^{2p})$, then $\tilde{\Gamma}$ can be regarded as a map from R^{2p} to R^{2n}. Then by Sard's theorem the set $\tilde{\tau}' \equiv \tilde{\Gamma}(\tilde{\tau})$ is of $2n$ dimensional Lebesgue measure zero. The lemma then follows if we can prove the equivalence $\tilde{\tau} \approx \tilde{\Omega}$.

The equivalence $\tilde{\tau} \approx \tilde{\Omega}$ is an immediate consequence of the identity

$$\det \| \partial W/\partial \bar{w} \| = \left| \det \| \partial Z/\partial \bar{z} \| \right|^2 , \tag{B.15}$$

which follows from the Cauchy-Riemann equations. Here $\bar{z}$ is any subset of n of the variables $(z_1, \cdots z_p)$ and $\bar{w}$ is the corresponding subset of w. To prove (B.15) let the $2n \times 2n$ matrix $\| \partial W/\partial \bar{w} \| \equiv M$ be considered a set of four $n \times n$ matrices:

$$M \equiv \| \partial X/\partial \bar{x};\ \partial X/\partial \bar{y};\ \partial Y/\partial \bar{x};\ \partial Y/\partial \bar{y} \| . \tag{B.16}$$

Let E be the unit $n \times n$ matrix and define the $2n \times 2n$ matrices

$$L \equiv \|E;\ iE;\ E;\ -iE\| \tag{B.17}$$

and

$$N \equiv \|E/2;\ \ E/2;\ \ -iE/2;\ \ iE/2\| \ . \tag{B.18}$$

Then

$$LMN = \|\partial Z/\partial\bar{z};\ \ 0;\ \ 0;\ \ \partial Z^*/\partial\bar{z}^*\| \ , \tag{B.19}$$

where use has been made of the formulas

$$\frac{\partial}{\partial z_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} - \frac{i\partial}{\partial y_i}\right) \tag{B.20}$$

and

$$\frac{\partial}{\partial z_i^*} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} + \frac{i\partial}{\partial y_i}\right) , \tag{B.21}$$

and the Cauchy-Riemann equations. Since $\det L \cdot \det N = 1$, one obtains from (B.19) the desired (B.15).

APPENDIX C

Theorem C: Let $z = (z_1, \cdots z_n)$ be a set of n complex variables. Let $w = (w_1, \cdots w_{2n}) = (x_1 \cdots x_n; y_1 \cdots y_n)$ be the corresponding set of $2n$ real variables. Let M be a compact real analytic manifold of real dimension n imbedded in the space of the $2n$ real variables w. Let $H(z) \not\equiv 0$ be a function holomorphic at all points of M. Let $M' = M \cap \{H(z) = 0\}$. Let $\phi(z)$ be any integrable function on M'. Then

$$\int_{M'} \phi(z)dz \equiv \int_{M'} \phi(z)dz_1 \wedge dz_2 \cdots \wedge dz_n = 0 \tag{C.1}$$

Proof: Let M_i' be the subset of M' such that $\partial H(z)/\partial z_i \neq 0$. Then at any point of M_i' a coordinate system $(Z_1^i, Z_2^i \cdots Z_n^i)$ can be constructed with $Z_1^i = H(z)$. For some neighborhood $N \subset M_i'$ of this point the set of functions $z(Z^i)$ are well defined and holomorphic and one can write

$$\int_N \phi(z)dz = \int_{N'} \phi(z(Z^i)) \left| \frac{\partial z}{\partial Z^i} \right| dZ^i = 0 . \tag{C.2}$$

The integral (C.2) vanishes due to the constraint $Z_1^i = 0$ imposed by the definition of M_i'. Let M_{ij}' be the subset of $M' - \bigcup M_i'$ such that $\partial H(z)/\partial z_i \, \partial z_j \neq 0$. At any point of M_{ij}' one can construct a coordinate system $(Z_1^{ij}, \cdots Z_n^{ij})$ with $Z_1^{ij} = \partial H/\partial z_i$. In some neighborhood $N \subset M_{ij}'$ of this point the set of functions $z(Z^{ij})$ are well defined and holomorphic and one can write

$$\int_N \phi(z)dz = \int_{N'} \phi\left(z(Z^{ij})\right) \left|\frac{\partial z}{\partial Z^{ij}}\right| dZ^{ij} = 0. \tag{C.3}$$

The integral (C.3) vanishes due to the constraint $Z_1^{ij} = 0$ imposed by the definition of M'_{ij}.

Proceeding in the same way one shows that

$$\int_{\mathcal{U}} \phi(z)dz = 0, \tag{C.4}$$

where $\mathcal{U}$ contains any point of M' where some finite-order partial derivative of $H(z)$ is nonzero. Since $H(z)$ is holomorphic at each point of M', and is not identically zero, we have $\mathcal{U} = M'$. This proves the theorem.

<u>Corollary</u>: Let z, w and M be as in Theorem C. Suppose M is the intersection of the sets Γ_i defined by rectifiable curves $C_1, C_2, \cdots C_n$ in the planes $Z_1, Z_2, \cdots$ and Z_n, respectively, where Z_i is an analytic function of z on C_i. Let $J(z) = |\partial Z/\partial z|$ be identically zero. Then for any $\phi(z)$ integrable on M,

$$\int_M \phi(z)dz \equiv \int_M \phi(z)dz_1 \wedge dz_2 \cdots dz_n = 0 \tag{C.5}$$

<u>Proof</u>: By a rectifiable curve we mean a curve can be considered to be a limit of a set of straight-line segments. Consider the contribution from a set of n such segments. By an adjustment of the definition of the Z_i's, these segments can be made to be segments of the real axis.

Then the constraint equations are $\mathrm{Im}\, Z_i = 0$, $i=1,\cdots n$. Because the Jacobian $J(z)$ is identically zero one can divide the set $Z = (Z_1,\cdots Z_n)$ into subsets $\hat{Z}$ and $\tilde{Z}$ such that $\hat{Z} = \hat{Z}(\tilde{Z})$, where $\hat{Z}$ is nonempty and the matrix $||\partial\tilde{Z}/\partial Z||$ has maximal rank almost everywhere.[13] The contributions to (C.5) from points where the rank is not maximal vanish by virture of Theorem C. At any point where the rank is maximal one can find a set $\bar{Z}$ such that $|\partial(\bar{Z},\tilde{Z})/\partial z| \neq 0$. Near this point the coordinates $(\bar{Z},\tilde{Z})$ are equivalent to the coordinates z. By virtue of the relation $\hat{Z} = \hat{Z}(\tilde{Z})$ the constraint equations are of the form $F_i(\tilde{W}) = 0$. They do not depend on $\bar{W}$. Thus the manifold is the product of a manifold in $\tilde{Z}$ space times the whole $\bar{Z}$ space. The real dimension of the manifold in the $\tilde{Z}$ space must therefore be less than the complex dimension of this space. Thus the integral over $d\tilde{Z} \doteq d\tilde{Z}_1 \wedge d\tilde{Z}_2 \cdots d\tilde{Z}_p$ vanishes.

APPENDIX D LANDAU SURFACES

Theorem D: (C. Chandler) Let K_N be the space of N momentum-energy vectors $(k_1, \cdots k_N)$. Let I_N be the space of the N^2 Lorentz invariant inner products $(k_i \cdot k_j)$. Let D_N be a Landau diagram with N external lines and $n > 0$ internal lines. Let $\mathcal{L}_K(D_N)$ be the union of the Landau surfaces in K_N corresponding to D_N and all of its nontrivial $(n' > 0)$ contractions. Let $\mathcal{L}_I(D_N)$ be the image of $\mathcal{L}_K(D_N)$ in invariant space. Let $\mathcal{M}_K(D_N)$ and $\mathcal{M}_I(D_N)$ be the mass shell for D_N in K_N and I_N, respectively. Then

$$\mathcal{L}_K(D_N) = \mathcal{M}_K(D_N) \cap \left\{ \mathcal{R}_K(k_j; D_N) = 0 \right\} \tag{D.1}$$

and

$$\mathcal{L}_I(D_N) = \mathcal{M}_I(D_N) \cap \left\{ \mathcal{R}_I(k_i \cdot k_j; D_N) = 0 \right\} \tag{D.2}$$

where $\mathcal{R}_K(k_j; D_N) \not\equiv 0$ is a real polynomial in the k_j, and $\mathcal{R}_I(k_i \cdot k_j; D_N) \not\equiv 0$ is a real polynomial in the $(k_i \cdot k_j)$.

Proof: The α-form of the Landau equations is[20]

$$\alpha_i \, \partial D(\alpha_i; D_N) \Big/ \partial \alpha_i = 0 \qquad i=1, \cdots n, \tag{D.3}$$

where $D(\alpha_i; D_N)$ is a homogeneous polynomial in the α_i. The coefficients are real linear functions of the $(k_i \cdot k_j)$. The n equations (D.3) have

a nontrivial solution if and only if the resultant $\mathcal{R}(k_i \cdot k_j; D_N)$ is zero.[21] The resultant of a system of n homogeneous polynomials in n variables is a not-identically-zero integral polynomial in the coefficients.[21] Since the coefficients are real linear combinations of the $(k_i \cdot k_j)$, and since integers are real, the resultant is a real polynomial in the $(k_i \cdot k_j)$. Thus (D.2) is proved, and (D.1) follows from it.

APPENDIX E THE NONDEPENDENCE OF $A_{\bar{\delta}\bar{\epsilon}}$ on $\bar{\epsilon}$

Because of the (polynomial) dependence of $J_\gamma(Z'_\gamma, z)$ on z, the proof of the nondependence of $A_{\bar{\delta}\bar{\epsilon}}$ on $\bar{\epsilon}$ given in the text is incomplete: What proved there is only that the leading term in the polynomial expansion

$$J_\gamma\Big(Z_\gamma(z'), z\Big) = J_\gamma\Big(Z_\gamma(z'), z'\Big) + (z' - z)_i \; J_\gamma^{\,i}(z', z) + \cdots \quad \text{(E.1)}$$

gives a contribution to $A_{\bar{\delta}\bar{\epsilon}}$ that is independent of $\bar{\epsilon}$. This suggests that the other terms should also have this property. A proof of this is outlined here.

The first step is to convert the basic formula (2.21) to a more convenient form. First replace the variables of integration $\bar{Z}'_{\gamma i}$ by the original variables z', thereby eliminating the $\bar{J}_\gamma(\bar{Z}'_\gamma)$ in the denominator. Now let γ be the label of a set of variables $\bar{Z}_\gamma$ that does not include G_1. Let γ_i be the labels of the n different sets corresponding to replacing, in turn, each variable in $\bar{Z}_\gamma$ by G_1. The region of integration in G_1, for fixed values of the other variables in the set specified by γ_i, will generally be bounded by the curve that is the image in G_1 of I_γ. This bound comes from the condition on $\mathcal{R}(\bar{\epsilon},\bar{\delta})$ imposed by the condition on that variable of the set $\bar{Z}_\gamma$ that is replaced in $\bar{Z}_{\gamma i}$ by G_1. The image of I_γ in G_1 is a continuation of the original contour in G_1. The contribution from I_γ can be eliminated from (2.21) if one adds, instead, the contribution from this extra piece of contour in G_1, to the term corresponding to each of the γ_i. This follows from the defining property of the $P_{ji}(z'; z)$,

$$\sum_i \frac{P_{ji}(z';\, z)}{\bar{Z}_j(z') - \bar{Z}_j(z)} \, (z'_i - z_i) = 1 . \tag{E.2}$$

For (E.2) implies the vanishing of the determinant of the $(n + 1)$ by $(n + 1)$ matrix constructed by adding a column of ones to the matrix $P_{ji}(z';z)/\bar{Z}_j(z') - \bar{Z}_j(z)$, where j ranges over the index set corresponding to the n variables of $\bar{Z}_\gamma$, together with G_1. Expansion of this determinant on the column of ones gives the required identity.

By means of the above transformation, applied to every set $\bar{Z}_\gamma$ not containing G_1, one eliminates from (2.21) all contributions corresponding to $\bar{Z}_\gamma$ not containing G_1. Simultaneously the contour in G_1 becomes a cycle (has no boundaries) for each value of the other variables. Applying the same transformation for each of the m variables G_i one obtains a modified form of (2.21) in which γ is restricted to sets having all m of the variables G_i, and the contours in the G_i are cycles for all values of the other variables. Note that the actual regions of integration are not altered by this transformation: One simply considers the contributions from certain regions I_γ to be expressed as sums of contributions from the same regions I_γ inserted into various other terms of (2.21).

In the modified form of (2.21), the contour in the G_i part of space is a cycle. Thus by the theorem of Cauchy-Poincaré[14] the integral is independent of $\bar{\epsilon}$, so long as $(\bar{\epsilon},\bar{\delta})$ is in $\tilde{Q} \cap N$. The modified form of (2.21) can evidently also be used to give a variation of the proof of the theorem of the text.

FOOTNOTES AND REFERENCES

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† This work would have been part of the Doctoral Dissertation of the second-named author, had he lived.

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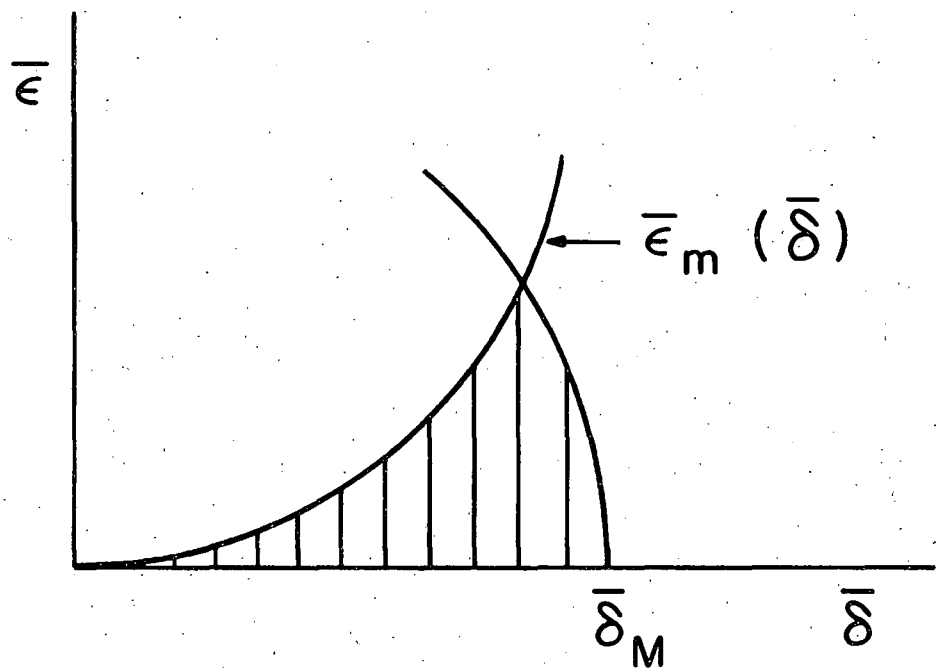

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Fig. 1. The region $\tilde{Q} \cap N$ is a connected open set containing all points in Q where $\bar{\epsilon} < \bar{\epsilon}_m(\bar{\delta})$. The set of points $\bar{\epsilon} = 0$, $0 < \bar{\delta} < \bar{\delta}_M > 0$ is contained in the boundary.

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