UCRL-17902
cy. 2

University of California

Ernest O. Lawrence Radiation Laboratory

PHYSICAL-REGION DISCONTINUITY EQUATIONS FOR MANY-PARTICLE SCATTERING AMPLITUDES. II

Joseph Coster and Henry P. Stapp

August 15, 1967

TWO-WEEK LOAN COPY

This is a Library Circulating Copy which may be borrowed for two weeks. For a personal retention copy, call Tech. Info. Division, Ext. 5545

UCRL-17902 cy. 2

DISCLAIMER

This document was prepared as an account of work sponsored by the United States Government. While this document is believed to contain correct information, neither the United States Government nor any agency thereof, nor the Regents of the University of California, nor any of their employees, makes any warranty, express or implied, or assumes any legal responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by its trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof, or the Regents of the University of California. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof or the Regents of the University of California.

Submitted to
Journal of Mathematical Physics

UCRL-17902
Preprint

UNIVERSITY OF CALIFORNIA

Lawrence Radiation Laboratory
Berkeley, California

AEC Contract No. W-7405-eng-48

PHYSICAL-REGION DISCONTINUITY EQUATIONS FOR MANY-PARTICLE SCATTERING AMPLITUDES. II

Joseph Coster and Henry P. Stapp

August 15, 1967

PHYSICAL-REGION DISCONTINUITY EQUATIONS FOR MANY-PARTICLE SCATTERING AMPLITUDES. II*

Joseph Coster and Henry P. Stapp

Lawrence Radiation Laboratory
University of California
Berkeley, California

August 15, 1967

ABSTRACT

Discontinuity equations are derived for physical-region normal thresholds in all direct and crossed channels. The discontinuity is given as a unitarity-type integral with an integrand that contains, as factors, the two physical scattering functions corresponding to the two vertices of the Landau diagram associated with the normal threshhold. There is also a third factor, which for the case of a leading normal threshold with any number of particles is the Hermitian conjugate of the elastic scattering matrix associated with the set of internal lines of the Landau diagram. For non-leading normal threshholds below the lowest four-particle threshold the extra factor is defined by an integral equation that resembles unitarity, but has a restricted set of intermediate particles.

I. INTRODUCTION

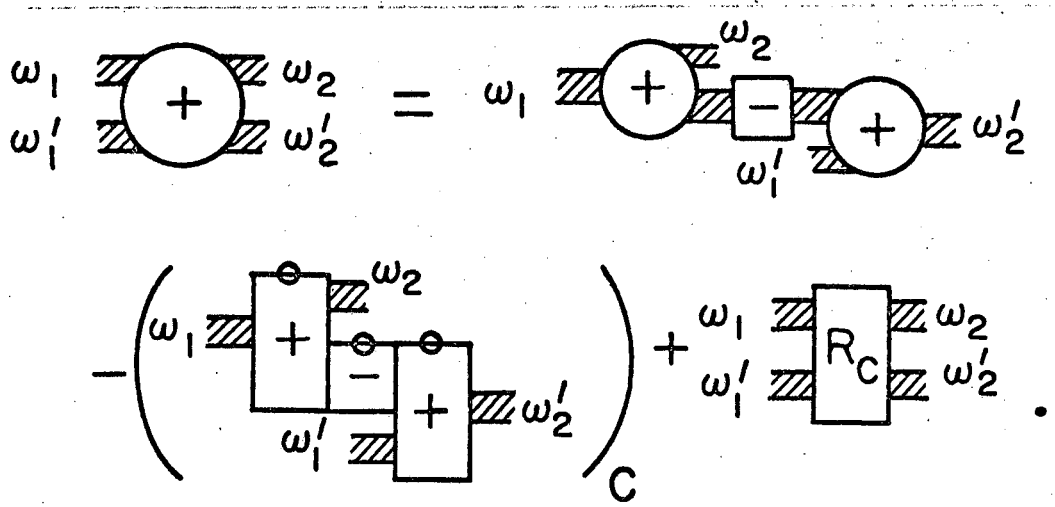
This paper is the second in a series[1] devoted to calculating discontinuities around physical-region singularities of multiparticle scattering functions. The aim here is to obtain discontinuity formulas for normal thresholds. These normal-threshold formulas are important both in their own right and as the basic ingredients of the discontinuity formulas for more complicated singularities.

The main content in this paper consists in the derivation of some physical-region identities. These identities express, typically, any physical-region scattering function as a sum of terms each consisting of a unitarity-type integral over a product of physical scattering functions, or their conjugates. Each term is conveniently represented by a <u>bubble diagram</u>, in which plus and minus bubbles represent the connected parts of scattering amplitudes and their conjugates, respectively, and the lines connecting these bubbles represent physical particles. The identities are derived from unitarity and cluster properties alone, no analyticity property being invoked. Like unitarity, these identities hold at all real values of the external momentum vectors. Their importance lies in the fact that they explicitly display the discontinuity around normal thresholds.

The result that the discontinuity is explicitly displayed by certain terms in these identities follows from certain topological properties of the diagrams that represent the other terms, together with some structure theorems derived earlier[2] that specify the analytic structure of bubble diagram functions. These structure theorems say that the bubble diagram function M^B corresponding to the bubble diagram B has

the following properties: M^B is singular only at points lying on Landau surfaces, and only on those Landau surfaces $L(D)$ that correspond to Landau diagrams $D \supset\subset B$. A Landau diagram $D \supset\subset B$ is a diagram that is a contraction of a diagram D' constructed by replacing the bubbles of B by connected diagrams. Moreover, the signs of the Landau α's of the lines of this D' are restricted by the condition that the (internal) lines of diagrams replacing plus or minus bubbles must be positive or negative, respectively. If only one $D \supset\subset B$ gives a $L(D)$ passing through a point P lying inside the physical region, then the function M^B can be continued around $L(D)$ near P by passing into a well-defined "upper-half plane," which can be defined geometrically in terms of the diagram that generates P. If several surfaces pass through P then the continuation can be made through the intersection of the various upper half planes, provided this intersection is nonempty. The relation between the diagram $D \supset\subset B$ and the corresponding half plane is such that if two diagrams are identical except for a single overall reversal of the signs of all the α's, then the two corresponding half planes are opposite half planes. Thus if a point P lies on the $L(D)$ of two such $D \supset\subset B$, then no continuation is possible, in general. The hypothesis of the structure theorems is the analyticity property of the physical region scattering functions obtained from S-matrix macroscopic causality conditions, as is discussed in Section VIII.

Our key identity reads, in box notation,[1]

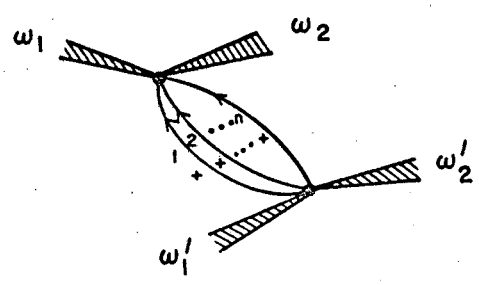


(1.1)

The shaded strips represent sets of any number of lines. A plus box represents the scattering matrix S; a plus circle its connected part. A minus box represents $S^\dagger$; a minus circle its connected part. A plus or minus box with a little circle on it represents S or $S^\dagger$ minus its connected part. The subscript c denotes connected part. Finally the box R_c denotes a well-defined set of bubble diagrams, each of which represents a well-defined integral over a product of physical-region scattering amplitudes or their conjugates.

The importance of (1.1) arises from the following property of R_c: No bubble diagram B in the sum represented by the box R_c supports any Landau diagram D' that contracts to any diagram of the form $D^+_n(\omega' \to \omega)$, where

$D^+_n(\omega' \to \omega) \equiv$



(1.2)

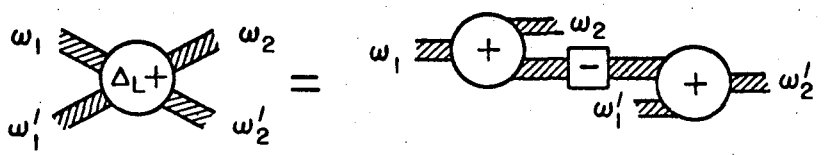
The plus signs on the lines indicate that the corresponding Landau α's are positive. The similar diagram with minus signs on all lines is denoted by $D^-_n(\omega' \to \omega)$. A diagram of the form $D^+_n(\omega' \to \omega)$ or $D^-_n(\omega' \to \omega)$, for any positive integer n, is called a $\omega' \to \omega$ normal threshold diagram, and will be denoted by $D(\omega' \to \omega)$.

Because the box R_c has the property just described, we know from the quoted structure theorems the following fact: The function R_c represented by the box R_c continues into itself via a "minus $i\epsilon$ continuation" past any singularity corresponding to any pure (positive or negative) - α Landau diagram D that contracts to any $D(\omega' \to \omega)$. [The minus $i\epsilon$ continuation is the one opposite to the physical continuation. That R_c must continue in this way follows from the fact that these pure-α diagrams must be pure minus-α diagrams.] Using this fact, one immediately deduces from (1.1) the discontinuity formula for the leading $\omega' \to \omega$ normal threshold, near points lying "inside" the physical region. [The "leading" $\omega' \to \omega$ normal threshold is the multiparticle $(n \geqslant 2)$ $\omega' \to \omega$ normal threshold with the smallest value, M_L, of the $\omega' \to \omega$ exchange c.m. energy $E(\omega' \to \omega)$. A point on this threshold lies "inside" the physical region if any neighborhood of that point contains physical points lying below the thresholds (i.e., at $E(\omega' \to \omega) < M_L$) and physical points lying above the threshold (i.e., at $E(\omega' \to \omega) > M_L$).] The argument goes as follows: Let P' be a point in the physical region lying just below the leading $\omega' \to \omega$ normal threshold. The first two terms on the right of (1.1) vanish at P'. Let P'' be a point such that there is a physical-region path $P'P''$ from P' to P'' with the

following two properties: (a) Every singularity of R_c on $P'P''$ corresponds to a Landau diagram that can be contracted to a $\omega' \to \omega$ normal threshold diagram. (b) No singularity of R_c on $P'P''$ corresponding to a D having both positive and negative α's coincides with a singularity corresponding to a diagram D^*, unless D^* is the same as D up to an overall change of sign of all the α's. [This second condition says that mixed-α singularities of R_c on $P'P''$ do not "accidentally" lie on top of singularities associated with diagrams of essentially different topological structure.] For any such P'' the function $R_c(P'')$ is an explicit expression in terms of strictly physical-region scattering functions of the continuation of the physical scattering function from P' to P'' along a path that has a minus $i\epsilon$ continuation around every pure-α singularity. This is because any pure-α singularity (i.e., one corresponding to a diagram in which all α's have the same sign) must be a pure minus-α singularity, around which the function continues via a minus $i\epsilon$ rule, and each mixed-α singularity has a well-defined rule for continuation, since the possibility that D^* in (b) differs from D by an overall change of signs of the α's is precluded by property (a) plus the fact that R_c has no singularities corresponding to $D^+_n(\omega' \to \omega)$. Since the last term on the right of (1.1) is the continuation of the scattering function from P' to P'', the sum of the first two terms is precisely the discontinuity of the scattering function at P'' corresponding to the path $P'P''$. This argument is given in more detail, and is generalized, in Section VIII.

At points P'' sufficiently near almost any point P on the threshold the second term on the right of (1.1) is zero. Thus the discontinuity

around the threshold singularity alone is given by the first term on the right of (1.1). This result is represented by the equation

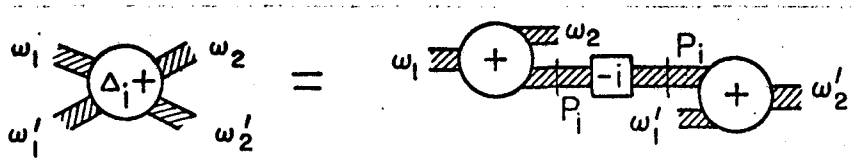


(1.3)

which is valid just above almost all points on the leading threshold T_L. The symbol on the left represents the discontinuity of the scattering amplitude around T_L. Equation (1.3) is a special case of a general rule first conjectured and discussed in Ref. 4.

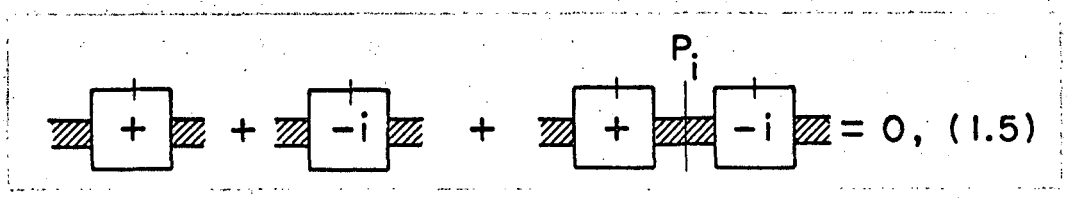
The above arguments apply equally well to the one-particle $\omega' \to \omega$ normal threshold. In that case the minus box in (1.3) becomes unity and the discontinuity formula becomes the well-known pole-factorization property.[5]

Results similar to those for the leading normal threshold described above are obtained for the nonleading normal thresholds lying inside the physical region below the four-particle threshold in the $\omega' \to \omega$ channel. The analog of (1.3) reads



(1.4)

The P_i bar imposes the restriction that the sum of the rest masses of the particles associated with the lines cut by the bar be not less than the mass M_i associated with the threshold T_i in question. The -i box is defined by the equation



where the bar on a box signifies that an I box has been subtracted off. This equation becomes the unitarity equation if the P_i bar is omitted, and the (-i) boxes are replaced by minus boxes. Equation (1.4) is closely connected to a formula obtained by Olive[6] for two-particle thresholds.

Equation (1.1), and the similar equation leading to (1.4), are physical-region identities. Thus they may be substituted into the plus bubble appearing in the first term on the right. In this way discontinuity equations for more complicated singularities can be obtained.

The main task in this paper is to derive the identity (1.1), and ones similar to it, and to substantiate the claims made regarding the properties of R_c. This involves repeated use of only the unitarity and cluster properties; analyticity properties are not involved. The first step is to develop a diagram calculus to deal with the cluster properties and unitarity equations for indeterminate numbers of particles. This is done in Section II. Sections III and IV establish some terminology and place on a firm basis some simple preliminary propositions. These two sections can be skimmed on first reading. The main proofs are in Sections V and VI. Section VII contains some incidental remarks concerning other forms of the results. The analyticity properties that follow from the identities established in Sections V and VI are described in detail in Section VIII.

II. REPRESENTATION OF THE CLUSTER PROPERTY

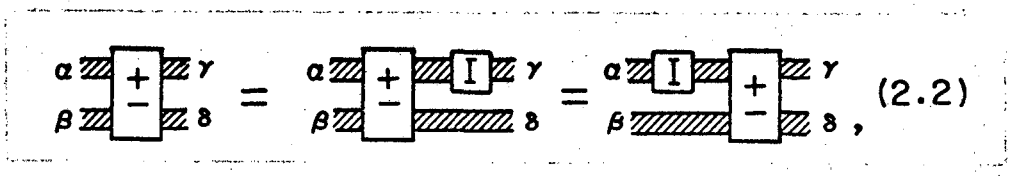
A box labelled by a symbol and connected to a set of lines represents a certain sum of bubble diagrams. The lines on the right and left will be called the incoming and outgoing lines, respectively. A plus (minus) box represents a sum over a column of plus (minus) bubbles, the sum being over all different ways that the given incoming and outgoing lines can be connected to each other by bubbles, subject only to the conditions that each line touch precisely one bubble and each bubble touch at least one incoming line and at least one outgoing line. [Each incoming and outgoing line terminates at or emerges from, respectively, the bubble it touches.]

An "I" box is constructed by the same rules, with the added condition that each bubble touch precisely one incoming line and precisely one outgoing line. No distinction is drawn between a plus, minus, or I bubble that satisfies this condition.

The unitarity equation

(2.1)

is regarded as an equivalence relation connecting different box diagrams. As explained in I, the rule for multiplication of diagrams is that topologically equivalent diagrams of the natural product are counted precisely once. This leads to the second fundamental equivalence relation



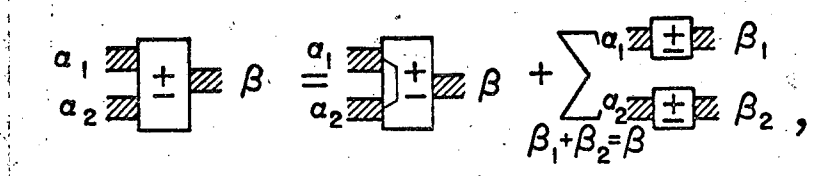
(2.2)

where the same choice of $\pm$ is to be used throughout. These two fundamental equivalences, when combined with cluster properties, will yield our results.

Often a set of intermediate lines connecting two boxes will not be explicitly shown: the two boxes will simply be moved into contact.

In the above equations, and in what follows, sets of lines are labelled by Greek letters. <u>These sets are allowed to be empty, unless otherwise stated</u>. The equation $\alpha = 0$ means the set α is empty, $\alpha \neq 0$ means it is not empty, and $\alpha > 1$ means the set has more than one line. The lines are considered to run from right to left and symbols $X^+(\alpha)$ and $X^-(\alpha)$ denote the sets of leading and trailing end points, respectively, of the lines of the set α.

The cluster properties reside in the definitions of the plus and minus boxes in terms of their respective bubbles. To exploit these properties we do not fully decompose the boxes into their constituent bubbles, but make, rather, partial decompositions into sets of terms with different connectedness properties. The first of these partial decompositions is expressed by the equation

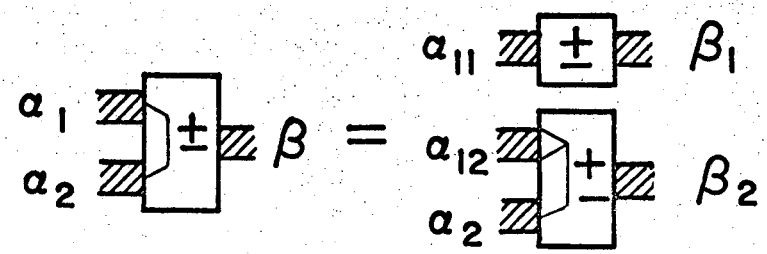


(2.3)

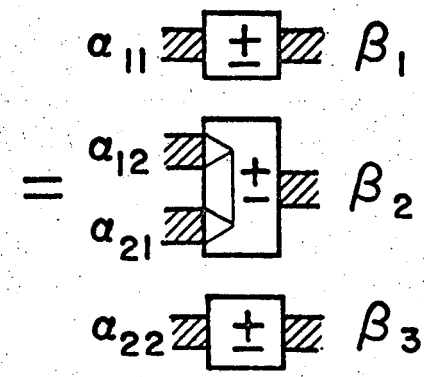
The first term on the right represents the sum of those terms of the left that contain at least one bubble touching lines in both α_1 and α_2. The second term consists of a sum over all decompositions of the set β into two sets β_1 and β_2. Each term of this sum consists of the indicated boxes combined according to a product rule of composition. This rule gives, for each pair consisting of one term from each box, a column consisting of the sum of the bubbles of the two members of this pair. The validity of (2.3) is proved in Appendix A.

Henceforth we adopt a summation convention that serves to eliminate the summation sign in (2.3), and in similar equations that follow. If a set $\emptyset$ appearing on the left is partitioned in all possible ways into certain sets appearing on the right, then these latter will be denoted by $\emptyset_j$: A summation over all partitions of the $\emptyset$ into the various $\emptyset_j$ is always implied. Occasionally the set on the left will already have an index, but the rule still applies. For example, α_1 is partitioned into sets α_{1j}.

A second important decomposition rule is

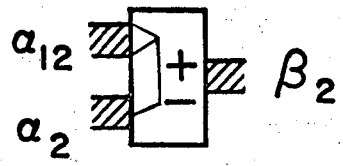


(2.4a)



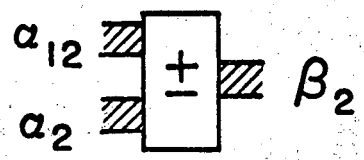
(2.4b)

Here



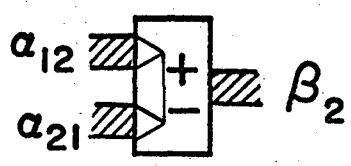
(2.5)

is defined to be empty if $\alpha_2 = 0$ or $\alpha_{12} = 0$. Otherwise it is the sum of all contributions to



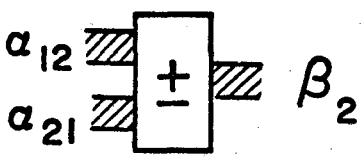
(2.6)

with the property that each line of α_{12} touches a bubble that touches at least one line of α_2. The box diagram



(2.7)

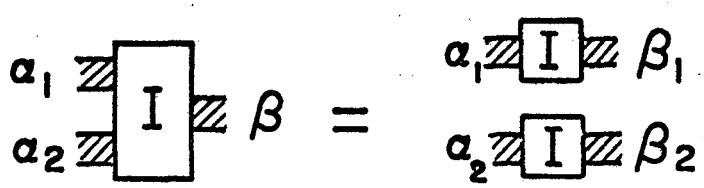
is defined to be empty if $\alpha_{12} = 0$ or $\alpha_{21} = 0$. Otherwise it is the sum of all contributions to



(2.8).

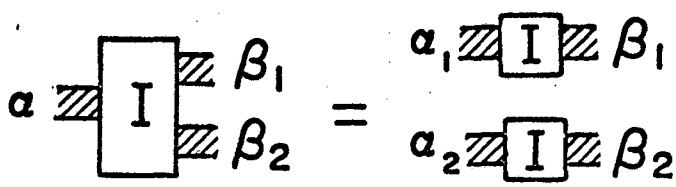
that have the property that each line of α_{12} touches a bubble that touches at least one line of α_{21}, and vice versa. Equations (2.4a) and (2.4b) are proved in Appendix A.

The decomposition (2.3) also applies to the I box, but the first term on the right of (2.3) is then empty. Thus,



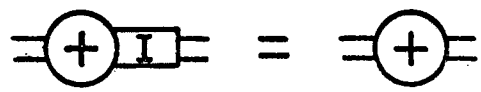
(2.9)

and



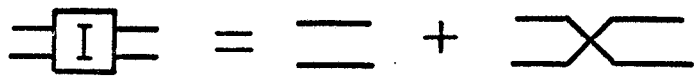
(2.9)′

The I box (2.9) or (2.9)′ is equivalent to the identity when postmultiplying a bubble diagram symmetric in the set of lines α or when premultiplying a bubble diagram symmetric in the set of lines β. This follows from the definition of the I box and from the fact that in products of boxes topologically equivalent contributions are counted only once. Thus, for instance,



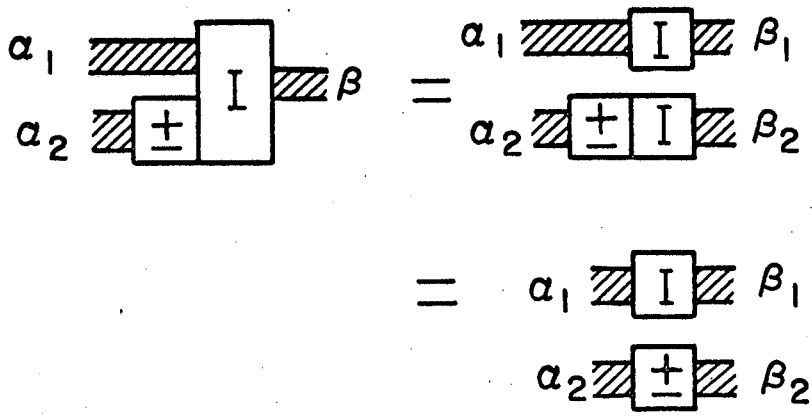
(2.10)

even though



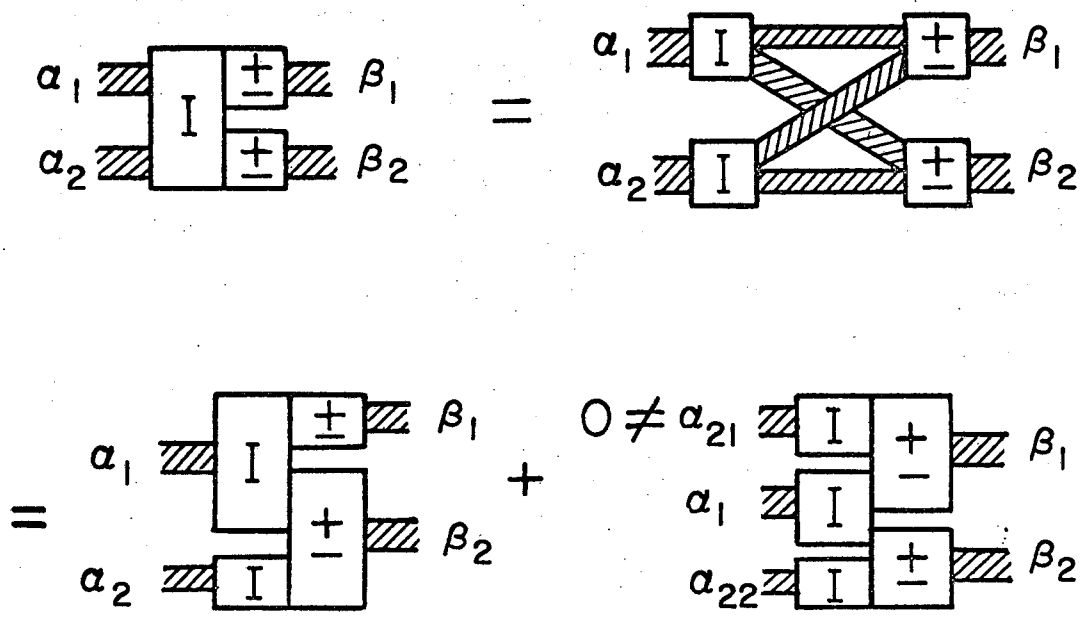
(2.11)

From (2.9), (2.9)′, and (2.2) we obtain



(2.12)

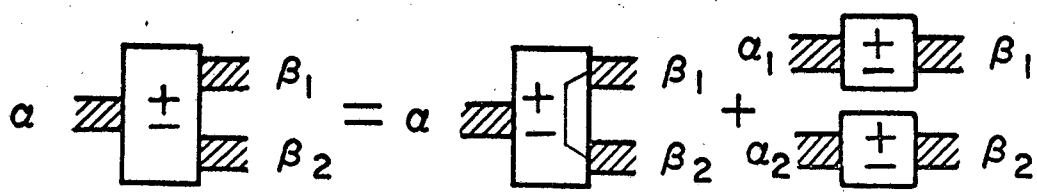
and



(2.13)

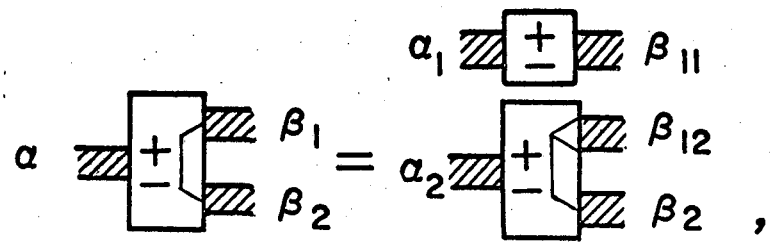
[The number of lines crossing an interface between boxes or represented by a shaded strip is always allowed to be zero, unless otherwise stated.]

<u>Remark 2.1</u>. Equations completely analogous to those discussed in this section but with the role of the incoming and outgoing particles switched are denoted by a primed equation number. We shall use, in particular, the equations



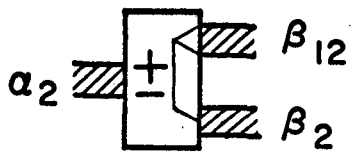
(2.3)′

and



(2.4a)′

where the box diagram



(2.5)′

is defined in analogy with the definition of (2.5). We also record for future use the definition

(2.14)

III. CUT SETS AND DIAGRAMS THAT CANNOT BE CONTRACTED TO POSITIVE-α NORMAL-THRESHOLD DIAGRAMS

Consider a transition from the system of m initial particles to the system of n final particles. Any separation of the $n + m$ particles into two disjoint subsets defines a channel. If the two subsets are precisely the initial and final sets, then the channel is called the direct channel. If each subset contains both initial and final particles, then the channel is called a cross channel. If one subset contains only initial or only final particles and the other subset contains both initial and final particles, then the channel is called a subchannel. The three cases are indicated in Fig. 1 (following page) where we have labelled one of the two subsets of lines ω and the other ω'. By convention the set ω does not consist only of initial lines, and ω' does not consist only of final lines. The sets ω and ω' are further subdivided into the sets $\omega_1 + \omega_2$ and $\omega_1' + \omega_2'$, respectively, where $\omega_1 + \omega_1'$ is the set of final lines and $\omega_2 + \omega_2'$ is the set of initial lines. The sets of external end points of the lines of ω and ω' are denoted by $X(\omega) \equiv X^+(\omega_1) + X^-(\omega_2)$ and $X(\omega') \equiv X^+(\omega_1') + X^-(\omega_2')$, respectively. The "$\omega' \to \omega$ channel" will mean the channel labelled in this way.

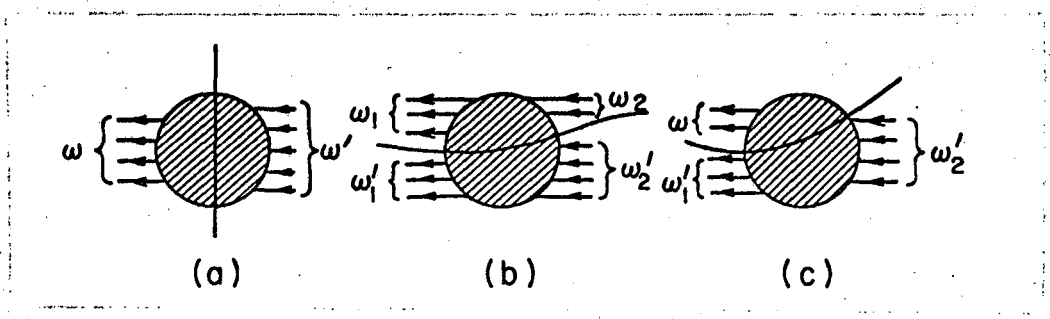


Fig. 1. Three types of channels.

(a) The direct channel.

(b) A cross channel.

(c) A subchannel.

The $\omega' \to \omega$ channel energy $E(\omega' \to \omega)$ is

$$E(\omega,\omega') = \sum_{i \in I(\omega_1)} p_i^{\,o} - \sum_{j \in I(\omega_2)} p_j^{\,o} \tag{3.1}$$

$$= \sum_{i \in I(\omega_2')} p_i^{\,o} - \sum_{j \in I(\omega_1')} p_j^{\,o},$$

evaluated in a frame where the corresponding three-momentum is zero. The set $I(\alpha)$ is the set of indices labelling the lines of α. Thus the reaction energy $E(\omega' \to \omega)$ is the net center-of-mass energy flowing from $X(\omega')$ to $X(\omega)$.

<u>Definition 3.1</u>. An <u>(ω, ω') cut set</u> of a Landau diagram D supported[1] by the bubble diagram B [i.e., $D \supset\subset B$] is a (possibly empty) collection C of lines of D with the following properties:

(1), Every path in D that starts in $X(\omega')$ and ends in $X(\omega)$ passes along the interior of some line of C, and (2), Property (1) is not satisfied by any proper subset of C.

<u>Proposition 3.1</u>. Let C be an (ω, ω') cut set of a $D \supset\subset B$ and let C' be the set of all interior points of all lines of C. Let $\overline{X}(C, \omega)$ and $\overline{X}(C, \omega')$ be the parts of D connected in $D - C'$ to $X(\omega)$ and $X(\omega')$ respectively. Then,

(i) $\overline{X}(C,\omega) \cap \overline{X}(C,\omega') = 0$ (the empty set),

(ii) each line of C has one of its end points in $\overline{X}(C,\omega)$ and its other end point in $\overline{X}(C, \omega')$,

(iii) $D - C' = \overline{X}(C,\omega) \cup \overline{X}(C,\omega')$.

<u>Proof</u>. (i) If $\overline{X}(C,\omega)$ and $\overline{X}(C,\omega')$ had a common point, then $X(\omega)$ could be connected to $X(\omega')$ in $D - C'$. But then C could not be an (ω, ω') cut set, contrary to assumption. (ii) The lines of D can intersect (by definition) only at their end points. Let C_ω be the set of lines of C that have an end point in $\overline{X}(C,\omega)$. Since $\overline{X}(C,\omega)$ and $\overline{X}(C,\omega')$ are disjoint, every path in D from $X(\omega)$ to $X(\omega')$ must leave $\overline{X}(C,\omega)$ via a line of C_ω. Thus, $X(\omega)$ and $X(\omega')$ cannot be connected by any path in $D - C_\omega'$, where C_ω' is the set of all interior points of all lines of C_ω. But then C_ω must be just C, for if C_ω were a proper subset of C then the second requirement on the (ω, ω') cut set C would be violated. Similarly, every line of C must have an end point in $\overline{X}(C,\omega')$. Since $\overline{X}(C,\omega)$ and $\overline{X}(C,\omega')$ are disjoint, one end point of each line of C must lie in $\overline{X}(C,\omega)$ and the other must lie in $\overline{X}(C,\omega')$. (iii) Suppose there were a part of $D - C'$ not connected in $D - C'$ to $X(\omega)$ or to $X(\omega')$. Then, by virtue of (ii) this part cannot be connected in D to $X(\omega)$ or to $X(\omega')$. But all parts of any $D \supset\subset B$ are connected in D to external lines of B, and hence to $X(\omega)$ or to $X(\omega')$, by virtue of the conditions on $D \supset\subset B$. In particular, each bubble of B has both incoming and outgoing lines. The partial ordering condition on the

bubbles of B ensures that each bubble lies on a path that starts at some incoming line of B and ends at some outgoing line of B. In forming $D \supset\subset B$ the bubbles are replaced by connected diagrams. Thus every point of $D \supset\subset B$ is connected to both the incoming and the outgoing lines of B, and hence to $X(\omega) \cup X(\omega')$.

A schematic representation of Proposition 3.1 is shown in Fig. 2, (following page).

<u>Definition 3.2</u>. A <u>simple (ω, ω') cut set</u> C is an (ω, ω') cut set C such that every line of C has its leading end point in $\overline{X}(C,\omega)$ and its trailing end point in $\overline{X}(C,\omega')$

<u>Definition 3.3</u>. A <u>simple positive (ω, ω') cut set</u> is a simple (ω, ω') cut set having no "minus lines". [A <u>minus line</u> is a line of $D \supset\subset B$ that is an internal line of a Landau diagram D_c^b corresponding to a minus bubble b of B. Such lines must carry negative Landau-α's, according to the rules set down in Ref. 1.]

<u>Definition 3.4</u>. $\mathcal{R}(\omega, \omega')$ is the set of all bubble diagrams B with the property that no $D \supset\subset B$ contains a simple positive (ω, ω') cut set.

<u>Definition 3.5</u>. $\mathcal{R}_c(\omega, \omega')$ is the set of all bubble diagrams B for which at least one of the following two conditions holds:

(1) B is not a connected diagram.

(2) No $D \supset\subset B$ contains a simple positive (ω, ω') cut set such that $\overline{X}(C,\omega)$ and $\overline{X}(C,\omega')$ are both connected diagrams.

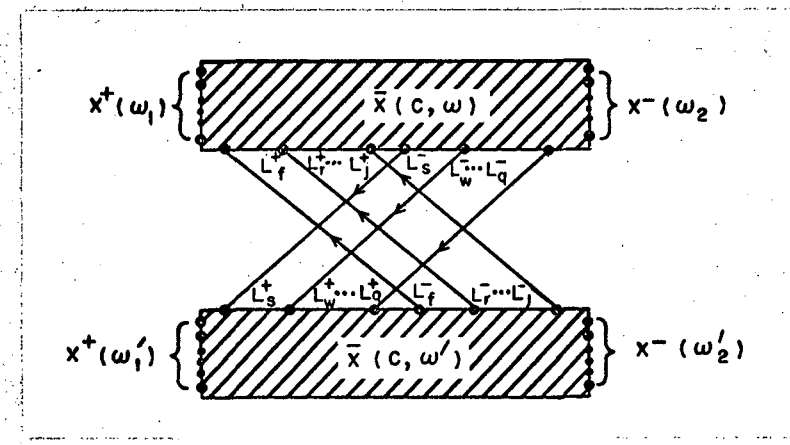


Fig. 2. The topological structure induced by an (ω,ω') cut set C. The directed line segments $L_i \epsilon C$ are shown. The lines are all shown directed to the left in accordance with the discussion at the end of this section. The $\bar{X}(C,\omega)$ and $\bar{X}(C,\omega')$ parts of $D \supset\subset B$ (indicated by the shaded areas of the figure) may be disconnected diagrams. Each point of $\bar{X}(C,\omega)$ $[\bar{X}(C,\omega')]$ is connected in $\bar{X}(C,\omega)$ $[\bar{X}(C,\omega')]$ to one or more of the external end points $X(\omega)$ $[\bar{X}(\omega')]$ exhibited in the figure. Some of the leading end points $L_f^+, L_r^+, \cdots L_j^+$ of the upward-directed line segments of C may lie in $X^+(\omega_1)$, and some of the trailing end points $L_s^-, L_w^-, \cdots L_q^-$ of the downward-directed line segments of C may lie in $X^-(\omega_2)$, and similarly for the lower end points of these lines. The ordering of the end points [i.e., $L_i^+ > L_j^-$] is not significant. The set C is called a <u>simple</u> (ω,ω') cut set (see Def. 3.2) if and only if the set of downward-directed line segments [i.e., those leading from $\bar{X}(C,\omega)$ to $\bar{X}(C,\omega')$] is empty.

It is evident that

$$\mathcal{R}(\omega, \omega') \subset \mathcal{R}_c(\omega, \omega'). \tag{3.2}$$

These definitions isolate the class of bubble diagrams that cannot support any $D_n^+(\omega' \to \omega)$: It is clear from the definition of "support" (see Ref. 1 Sec. II) that no bubble diagram belonging to $\mathcal{R}_c(\omega, \omega')$ can contain any $D_n^+(\omega' \to \omega)$, or any D that contracts to any $D_n^+(\omega' \to \omega)$.

The importance to us of this classification arises from the Third Structure Theorem, which ensures that the bubble-diagram function M^B represented by any $B \in \mathcal{R}_c(\omega, \omega')$ has a minus $i\epsilon$ continuation into itself around the singularity associated with any $D \subset B$ that can be contracted to any $\omega' \to \omega$ normal threshold diagram. Our principal task will be to prove that certain bubble diagrams are equivalent to diagrams belonging to $\mathcal{R}_c(\omega, \omega')$.

It should be made clear that the diagrams $D \supset\subset B$ we are discussing are simply topological structures. They should not be confused with the geometric structures $\bar{D}$ obtained from them by mapping each line L_j of D into a four-vector $\Delta_j = \alpha_j p_j$. The diagrams $\bar{D}$ are a geometric representation of all the Landau equations, including the loop equations, whereas the diagrams $D \supset\subset B$ represent only the conservation laws. In particular, the significance of the arrows on the lines of D arises from the conservation laws

$\sum_j p_j \epsilon_{jn} = 0$: Since each p_j^o is by definition positive and the arrow on each line L_j points to the end at which ϵ_{jn} is plus one, these arrows simply indicate the direction of the flow of positive energy. This interpretation of the arrows in terms of energy flow is independent of the sign of α_j.

The lines of a bubble diagram are defined to be directed from right to left. The lines in the interior of each plus bubble will also be directed from right to left, provided the vector diagram $\bar{D}_c^b$ is mapped into the topological space by a monotonic mapping that places points of greater energy further left. The interior lines of each minus bubble will be leftward-directed if the opposite rule is used. Thus, if these rules are adopted, then all the vectors of any $D' \subset B$ will point from right to left, and one has a diagrammatic representation of the conservation laws in which positive energy flows from right to left.

The convention just established plays no essential role in our arguments. But it makes geometrically obvious the fact that each point of any $D \supset\subset B$ lies on a continuous directed path P that runs from the trailing end point of some incoming line of B to the leading end point of some outgoing line of B, and is such that each line segment L_j lying on P has the same direction as P itself. That such a P exists follows analytically from energy conservation and the partial ordering requirement imposed on bubble diagrams (see Section II of I).

IV. A CHARACTERIZATION OF BUBBLE DIAGRAMS THAT SUPPORT NO POSITIVE-α DIAGRAMS THAT CONTRACT TO A $\omega' \to \omega$ NORMAL-THRESHOLD DIAGRAM

The characterization we seek is expressed in terms of "paths." A path in D is a continuous directed curve composed of an ordered sequence of line segments L_j of D. Neighboring segments meet at coincident end points. Each segment L_j of a path P can be independently directed either along the path or against it. The direction of the path is specified by specifying its origin and its destination. The direction of the line segment L_j of D is from L_j^- to L_j^+, as already mentioned.

<u>Definition 4.1</u>. A $P \in \mathcal{P}_{om}(D \supset\subset B)$ is a path in $D \supset\subset B$ having the property that each L_j lying on P is either directed against P or is a minus line [as defined in Def. (3.3)].

<u>Proposition 4.1</u>. If every $D \supset\subset B$ contains a $P \in \mathcal{P}_{om}(D \supset\subset B)$ that runs from $X^+(\omega_1')$ to $X^-(\omega_2)$, then B belongs to $\mathcal{R}(\omega, \omega')$, and conversely.

<u>Proof</u>. If B does not belong to $\mathcal{R}(\omega, \omega')$, then some $D \supset\subset B$ must have a simple positive (ω, ω') cut set C. Any path P from $X^+(\omega_1')$ to $X^-(\omega_2)$ must contain lines of C. The first of these lines encountered as P is traced out is a nonminus line directed along P. Thus this $D \supset\subset B$ can contain no $P \in \mathcal{P}_{om}(D \supset\subset B)$.

To prove the converse, suppose some $D \supset\subset B$ contains no $P \in \mathcal{P}_{om}(D \supset\subset B)$ that runs from $X^+(\omega_1')$ to $X^-(\omega_2)$. Then any path

p in this $D \supset\subset B$ from $X^+(\omega_1')$ to $X^-(\omega_2)$ must have a first segment L_p that is both nonminus and directed along the path. Let the union of the L_p over all paths p from $X^+(\omega_1')$ to $X^-(\omega_2)$ be denoted by $\bigcup L_p$, and define $\overline{C} = (\bigcup L_p) \bigcup \omega_1 \bigcup \omega_2'$. The sets $X(\omega)$ and $X(\omega')$ certainly cannot be connected in $D - \overline{C}'$, where $\overline{C}'$ is the set of interior points of $\overline{C}$. Some subset C of $\overline{C}$ is therefore an (ω, ω') cut set. Each line of C is a nonminus line with either its trailing end point in $\overline{X}(C,\omega')$ or its leading end point in $\overline{X}(C,\omega)$. Thus, by virtue of Proposition 3.1, each leading end point of C is in $\overline{X}(C,\omega)$ and each trailing end point is in $\overline{X}(C,\omega')$. Thus C is a simple positive (ω, ω') cut set for this $D \supset\subset B$. But then B cannot belong to $\mathcal{R}(\omega, \omega')$.

A few examples illustrating Proposition 4.1 and the various definitions of Section III are given in Fig. 3. In these examples one should insert various connected Landau diagrmas for the bubbles. However, the results are independent of the form of these diagrams D_c^b. We need only the general result that there is a path from any incoming line of any D_c^b to some outgoing line such that the direction of the path agrees with the directions of all its segments. This follows from energy conservation and the fact that energy flows in along incoming lines and out along outgoing lines. (See the discussion at the end of Section III.)

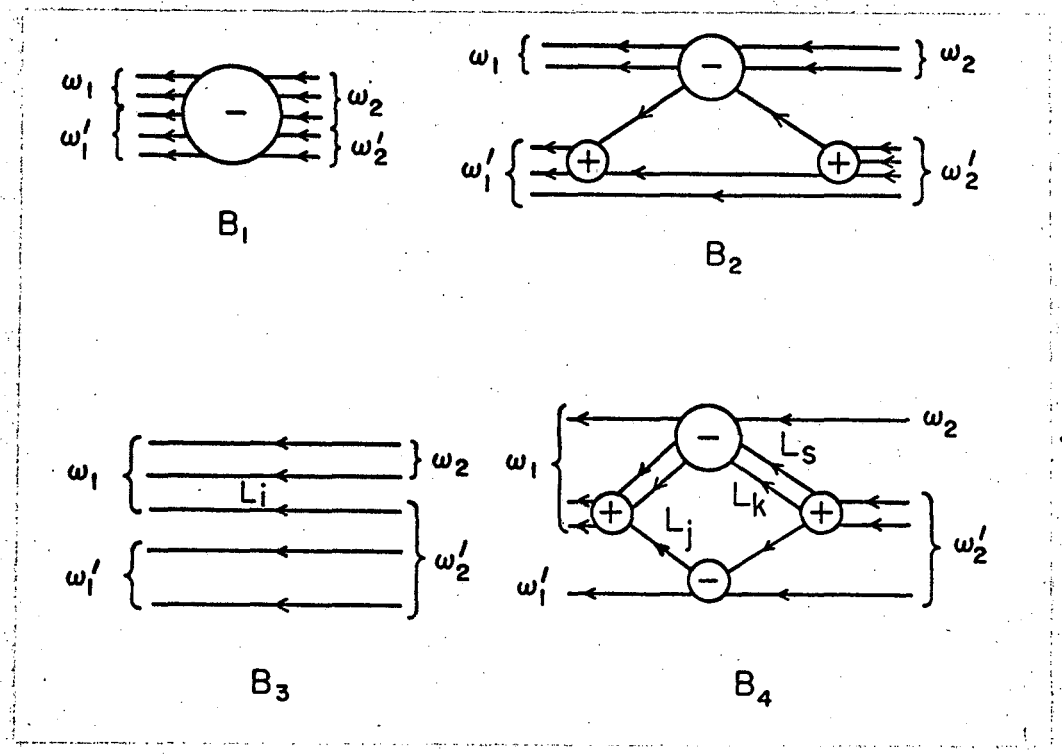


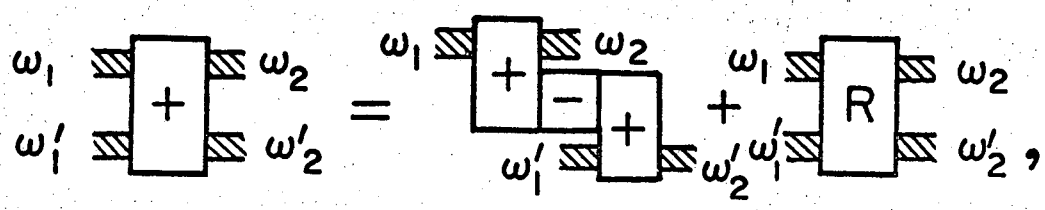
Fig. 3. The bubble diagrams B_1 and B_2 belong to $\mathcal{R}(\omega,\omega')$, whereas B_3 and B_4 do not. This follows from Proposition 4.1: For every $D \supset\subset B_1$ or $D \supset\subset B_2$ there is a path P running from $X^+(\omega_1')$ to $X^-(\omega_2)$ such that P is in $\mathcal{P}_{om}(D \supset\subset B_1)$ or $\mathcal{P}_{om}(D \supset\subset B_2)$, respectively. But not every $D \supset\subset B_3$ or $D \supset\subset B_4$ has a path P running from $X^+(\omega_1')$ to $X^-(\omega_2)$ such that P is in $\mathcal{P}_{om}(D \supset\subset B_3)$ or $\mathcal{P}_{om}(D \supset\subset B_4)$, respectively. That B_3 and B_4 do not belong to $\mathcal{R}(\omega, \omega')$ may also be seen directly from Definition 3.4, since the lines L_i of B_3 and the lines L_j, L_k, and L_s of B_4 are a simple positive (ω,ω') cut set for some $D \supset\subset B_3$ or $D \supset\subset B_4$, respectively.

V. NORMAL-THRESHOLD EXPANSIONS OF SCATTERING FUNCTIONS

A. Two Basic Identities

Unitarity can be regarded as an equivalence relationship between different box diagrams. In this section certain box diagrams are converted by repeated use of unitarity and the cluster properties to certain equivalent box diagrams.

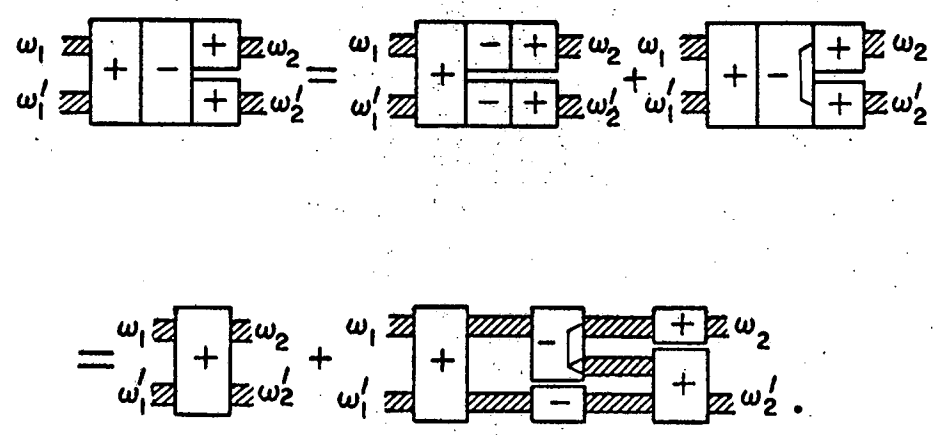
Proposition 5.1. The cluster properties and unitarity imply that



(5.1)

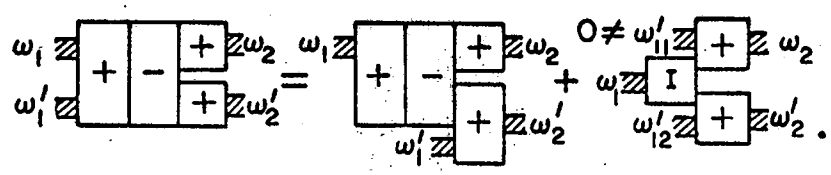
where the R box represents a sum of bubble diagrams all of which belong to $\mathcal{R}(\omega, \omega')$.

Proof. The proposition follows immediately from (2.1) and (2.2) if $\omega_2 = 0$ or if $\omega_1' = 0$: then $R = 0$. Thus we assume that ω_2 and ω_1' are nonempty. The use first of $(2.3)'$, and then of (2.1), (2.2), and $(2.4a)'$ gives



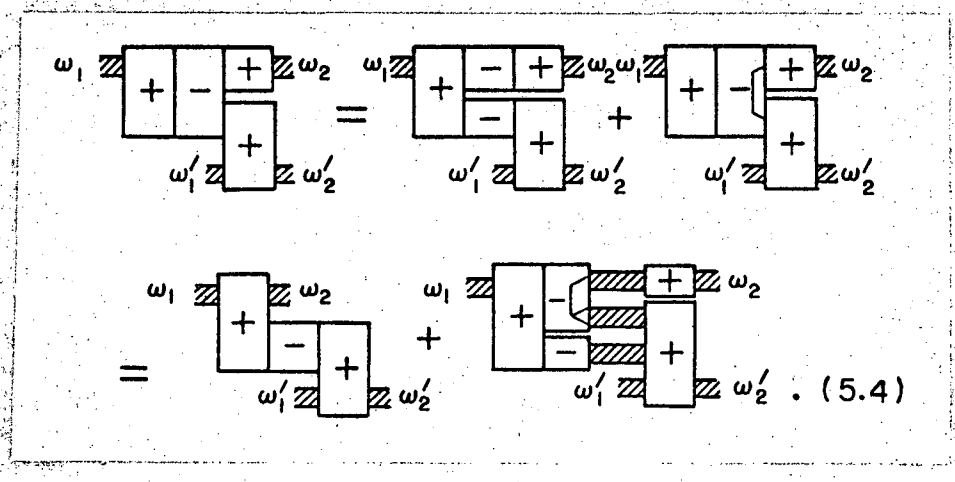
(5.2)

[The various summations required by the formulas of Section II are implied by the summation convention for internal lines introduced in Section II of I.] Alternatively, the use first of unitarity, then of (2.13), (2.2), and again unitarity gives



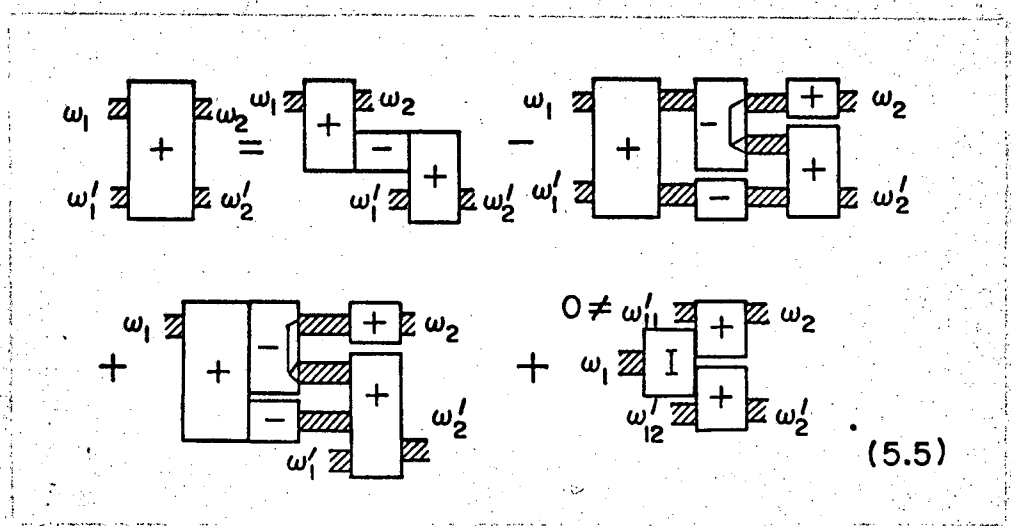
(5.3)

The application first of (2.3)′, and then of (2.1), (2.2), and (2.4a)′ to the first term on the right side of (5.3) gives



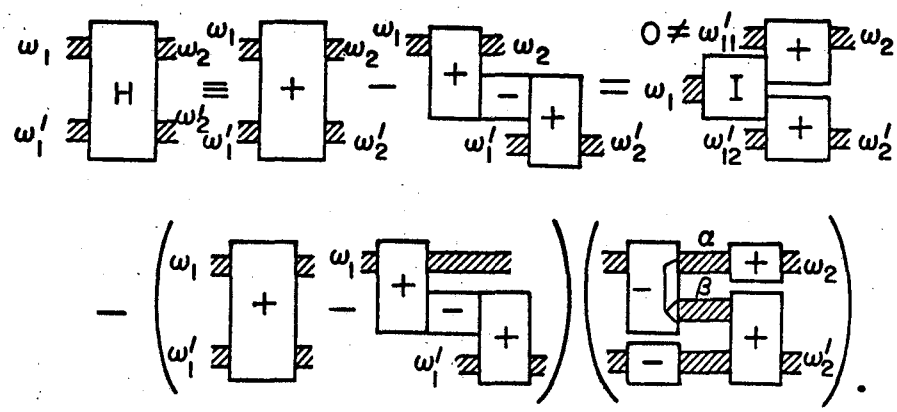
. (5.4)

Combining (5.2), (5.3), and (5.4), one obtains



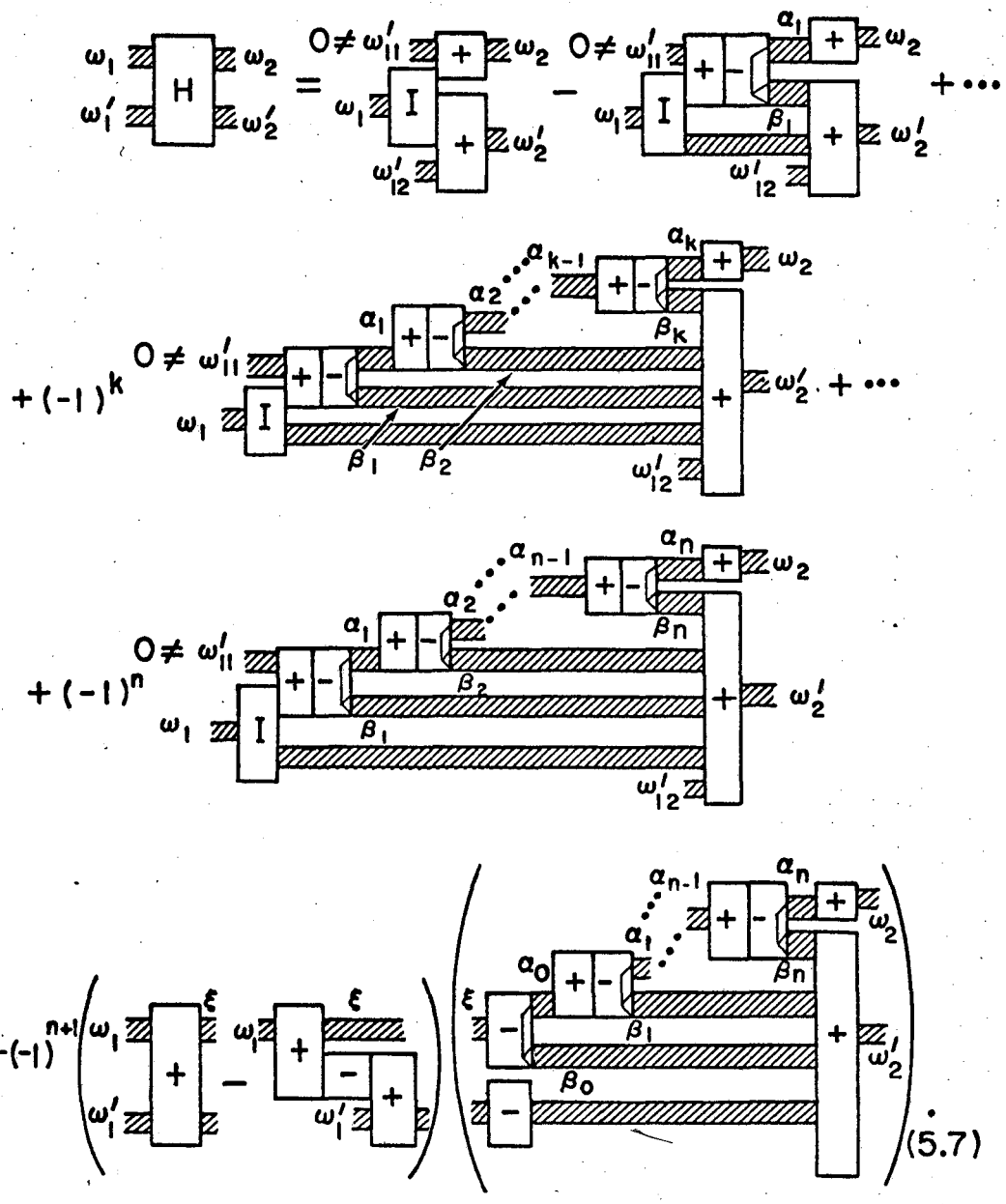
. (5.5)

A rearrangement of terms, and an application of (2.2) and unitarity converts this to



(5.6)

Equation (5.6) can be iterated. Iterating n times, each time simplifying by means of unitarity and (2.2), one obtains


H
I
+
−
ω_1
ω_1'
ω_2
ω_2'
ω_{12}'
$0 \neq \omega_{11}'$
α_1
α_2
α_{k-1}
α_k
α_{n-1}
α_n
α_0
β_1
β_2
β_k
β_n
β_0
ξ
$+(-1)^k$
$+(-1)^n$
$+(-1)^{n+1}$

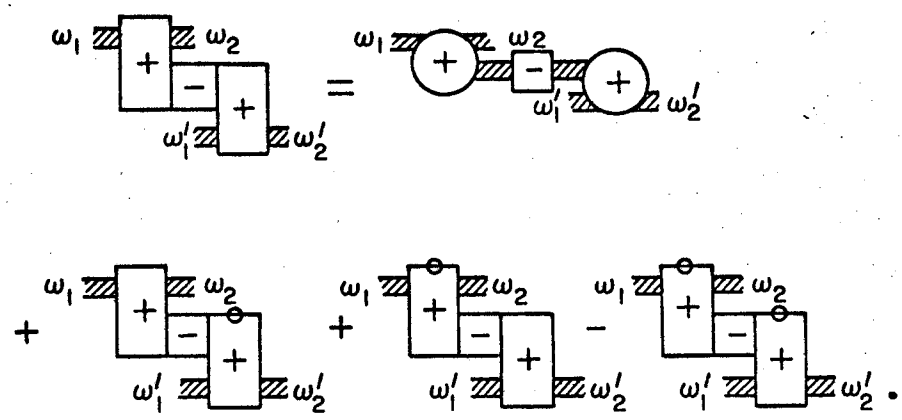
(5.7)

The number of iterations n is fixed so that $E(\omega, \omega') < (n + 1)E_0$, where E_0 is the rest energy of the lightest particle. The definition of the box diagram $(2.5)'$ requires that the sets α_i and β_i be nonempty: Otherwise the set of diagrams containing these sets is empty.

Let the box diagrams on the right of (5.7) be denoted by H_0, $H_1, \cdots, H_k, \cdots, H_n, H_{n+1}$, where H_0 is the first term on the right, H_1 the second, etc. Any Landau diagram supported by any H_i contains a path $P \in \mathcal{P}_{om}(D \supset\subset H_i)$ from $X^+(\omega_1')$ to $X^-(\omega_2)$, as will now be shown. For any H_i but H_{n+1} each end point of $X^+(\omega_{11}')$ is connected to the trailing end point of some incoming line of the leftmost upper plus box by a path whose sense is opposed to that of each line of that plus box. This follows from energy conservation. (See the discussion at the end of Section III.) This concludes the argument for H_0. For the remaining H_i, $0 < i \leqslant n$, the path just constructed can, by virtue of the definition of $(2.5)'$, be continued to some point of $X^+(\alpha_1)$ by a path composed only of minus lines and hence lying in $\mathcal{P}_{om}(D \supset\subset H_i)$. The same argument allows the path to be continued to α_2, then to α_3, and so on to $X^-(\omega_2)$. Thus, according to Proposition 4.1, all H_i, but H_{n+1} belong to $\mathcal{R}(\omega, \omega')$. Consider finally H_{n+1}. Since the sets β_i are nonempty, the energy $E(\xi)$ of the set ξ satisfies $E(\xi) \geqslant E(\omega_2) + (n + 1)E_0$. On the other hand, the condition on n is that

$E(\omega, \omega') \equiv E(\omega_1) - E(\xi) + E(\xi) - E(\omega_2) < (n+1)E_0$. These inequalities combine to give $E(\xi) > E(\omega_1)$. But then energy conservation implies that the second term in the first parenthesis vanishes.[7] Furthermore, in the first term in the first parenthesis some energy must flow from $X^-(\xi)$ to $X^+(\omega_1')$. This ensures that there is a path $P \in \mathcal{P}_{om}(D \supset\subset H_{n+1})$ from $X^+(\omega_1')$ to $X^-(\xi)$. This path can then be extended to $X^-(\omega_2)$ by means of the same arguments as before. Thus, all terms on the right of (5.7) belong to $\mathcal{R}(\omega, \omega')$, and Proposition 5.1 is proved.

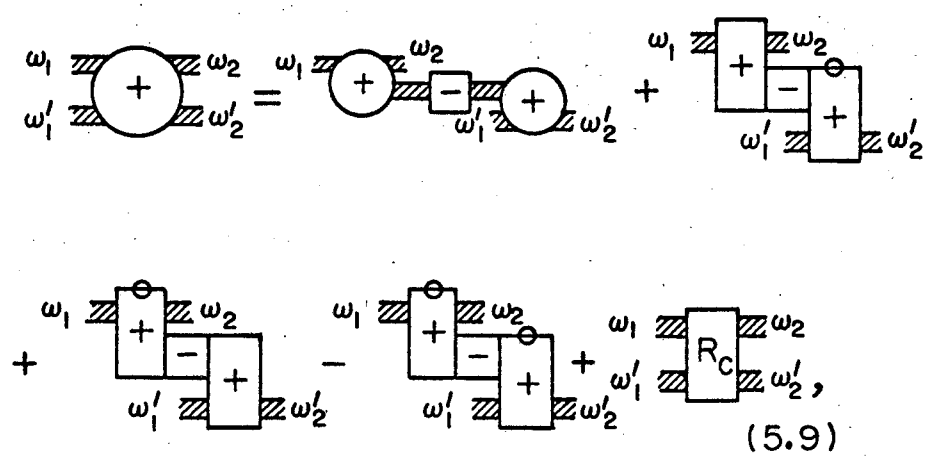
By (2.14) the first term on the right of (5.1) can be written in the form



(5.8)

Then Def. 3.5 [which implies that B belongs to $\mathcal{R}_c(\omega, \omega')$ if it

belongs to $\mathcal{R}(\omega, \omega')$] together with (2.14) and (5.8) allows (5.1) to be written in the form

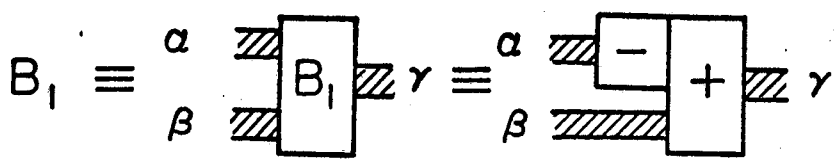


(5.9)

where the R_c box consists of a sum of terms of $\mathcal{R}_c(\omega, \omega')$.

Our next objective is to show that the second and third terms on the right of (5.9) can be placed in the last term. To this end we first prove

<u>Proposition 5.2</u> The box diagram



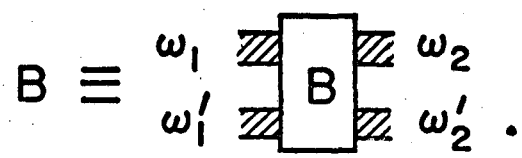
(5.10)

is equivalent to a diagram B_1' such that the only simple positive $(\alpha, \beta + \gamma)$ cut set C of any $D' \supset\subset B_1'$ is the set of lines α.

<u>Proof</u>. Let B_1' be the right side of (B.4). In each term let α_I be the subset of α connected to the I box. The set C evidently contains α_I. For the first term on the right of (B.4) the set α_I is α and the required result clearly holds. For any D' supported by any other term the definition of (2.5) guarantees that any point of $X^-(\alpha - \alpha_I)$ lies at the end of some path $P \in \mathcal{P}_{om}(D' \supset\subset B_1')$ that starts in $X^+(\beta)$. [One uses the properties of (2.5), to trace a path P' consisting only of minus lines from each point of $X^-(\alpha_i)$ to some point of $X^-(\delta_i)$. This path can then be extended to a path P'' ending at some point of $X(\beta)$: One uses the properties of (2.5) to get through each minus box of the form (2.5) encountered on the path from $X^-(\delta_i)$ to $X(\beta)$. The desired P is the negative of P''.] No point of such a path P can belong to any simple positive cut set. Thus, all points of $X^-(\alpha)$ must belong to $\overline{X}(C, \beta + \gamma)$, by the definition of $\overline{X}(C, \beta + \gamma)$. Thus, the only simple positive $(\alpha, \beta + \gamma)$ cut set is α.

This proof also shows that for any $D' \supset\subset B_1'$, all points of $D' - \alpha$ lie in $\overline{X}(C, \beta + \gamma)$. This gives the following

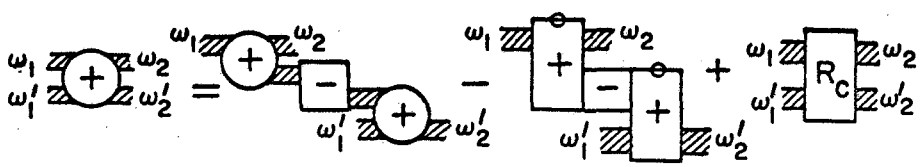
<u>Corollary to 5.2</u>. Suppose the B_1 of (5.10) is part of some box diagram



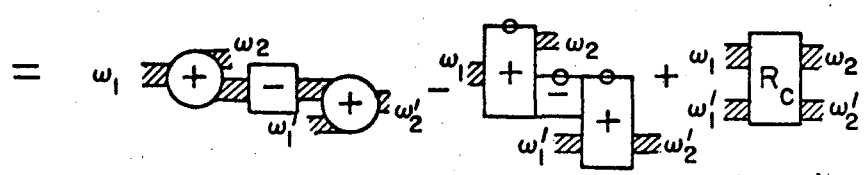
(5.11)

Then the replacement of B_1 in B by the equivalent B_1' of Proposition 5.2 gives a B' with the property that, for any simple positive (ω, ω') cut set C of any $D' \supset\subset B'$, all the points of $X^-(\alpha)$ will belong to $\bar{X}(C, \omega')$ if all points of $X^+(\beta)$ and $X^-(\gamma)$ do. In fact, all points of $D_1' - \alpha$, where D_1' is that part of D' which is supported by B_1', lie in $\bar{X}(C, \omega')$ if all points of $X^+(\beta)$ and $X^-(\gamma)$ do.

<u>Proposition 5.3</u>. The cluster properties and unitarity imply that



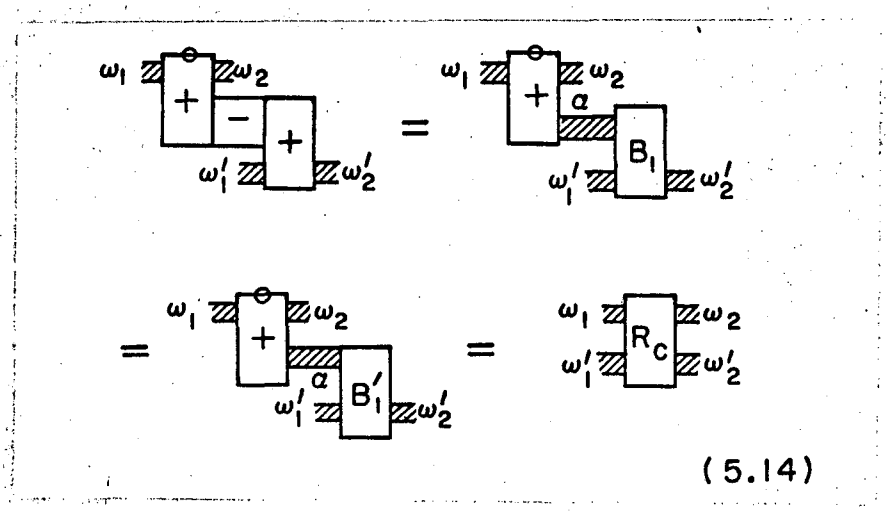
(5.12)



(5.13)

The R_c box is now, and hereafter, a generic symbol used to denote any sum of bubble diagrams each term of which belongs to $\mathcal{R}_c(\omega, \omega')$.

Proof. The corollary to 5.2 applied to the third term on the right of (5.9) gives

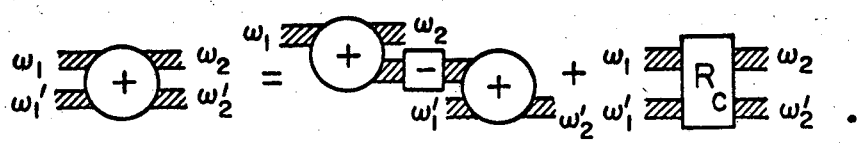


(5.14)

The last step follows from the fact that any simple positive (ω, ω') cut set C that leaves $X^-(\alpha)$ in $\overline{X}(C, \omega')$ must make $\overline{X}(C, \omega)$ a disconnected diagram. A similar argument applies to the second term on the right of (5.9). This proves (5.12). The contribution to the fourth term on the right of (5.9) coming from the connected part of the minus box clearly has no positive cut set C that leaves $\overline{X}(C, \omega)$ and $\overline{X}(C, \omega')$ both connected diagrams. Thus it belongs to $\mathcal{R}_c(\omega, \omega')$, and the proof is complete.

The form of the first term on the right of (5.13) is invariant under the crossing (incoming $\longleftrightarrow$ outgoing) of lines of ω or of ω'. However, no analyticity or crossing properties have been used to derive (5.13) the result is obtained strictly from unitarity and cluster properties in the direct channel.

For $E(\omega, \omega')$ less than the lowest four-particle $\omega' \to \omega$ threshold, E_4, the second term on the right side of (5.13) belongs to $\mathcal{R}_c(\omega, \omega)$. This is proved in Section VI. Thus, for this energy range Eq. (5.13) reduces to

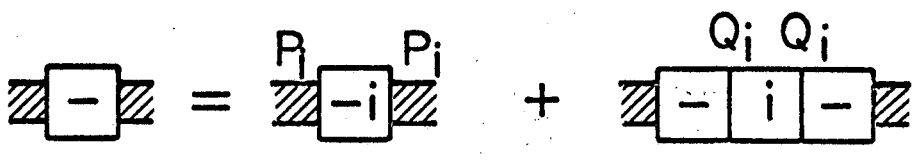


$$\left[E(\omega, \omega') < E_4\right] \quad (5.15)$$

B. Expansion Exhibiting the Discontinuity of the Scattering Function for Nonleading Normal Thresholds

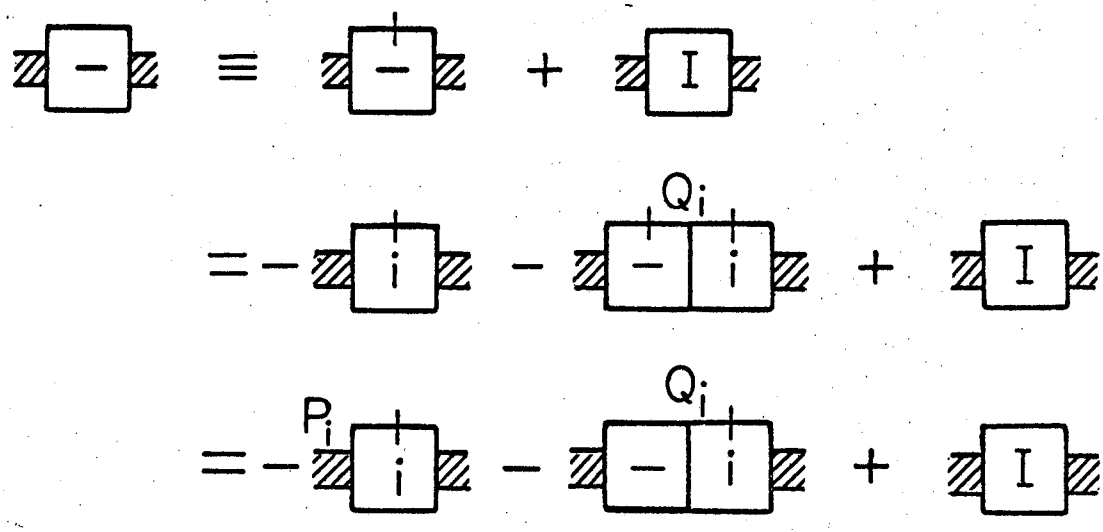
The expression for the scattering function given by Proposition 5.3 exhibits the discontinuity function for the leading normal threshold in the (ω, ω') channel. To exhibit the discontinuity for nonleading normal threshold at channel energy $E(\omega, \omega') = M_i$ we use the i-box formalism of paper I. This formalism applies only if $E(\omega, \omega')$ is below the four-particle threshold of the $\omega' \to \omega$ channel. Thus the following results are similarly restricted.

The basic identity we need is



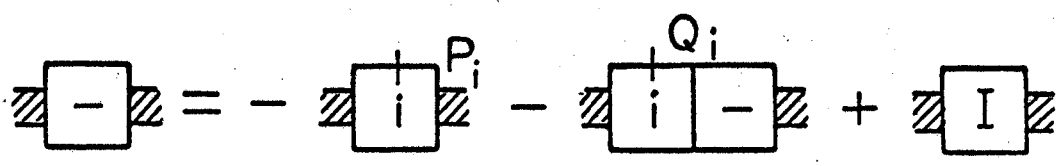
, (5.16)

where P_i and Q_i are the projection operators associated with the mass M_i [see (5.5) of I]. To prove (5.16) we first use (2.9), (5.8), (5.5), and (5.19) of I to obtain



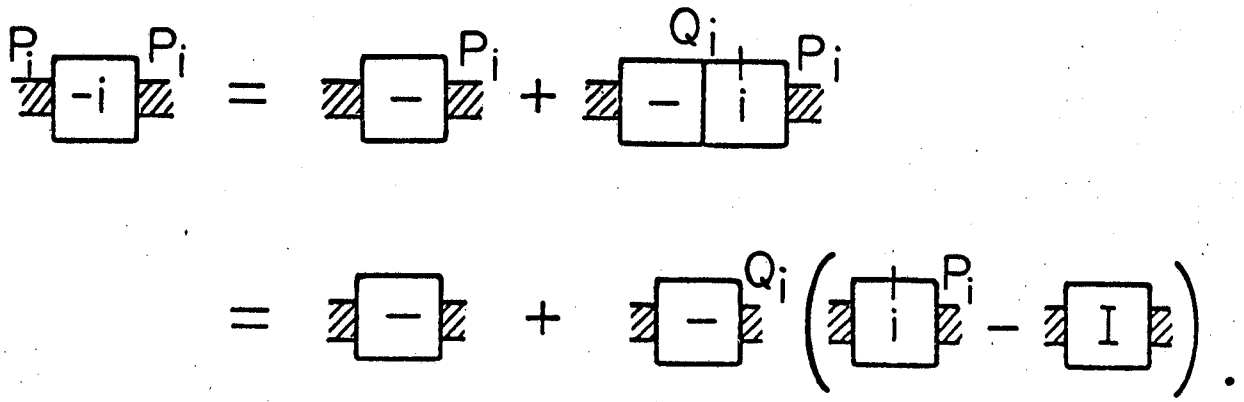
(5.17)

and



(5.18)

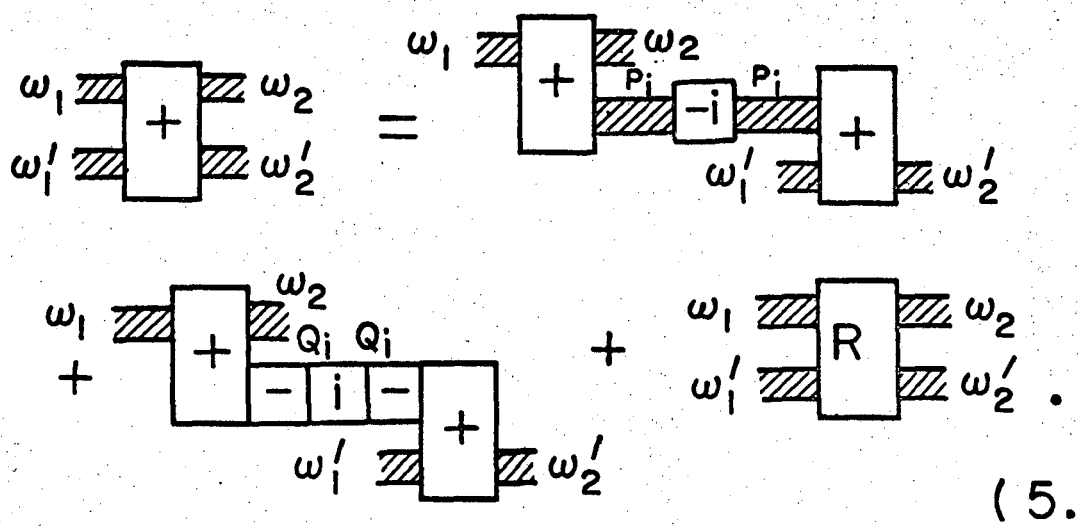
Definition (5.44) of I converts (5.17) to



(5.19)

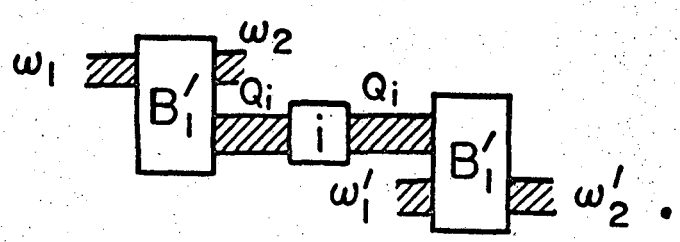
Application of (5.18) to the term in parentheses yields (5.16).

Substitution of (5.16) into (5.1) yields



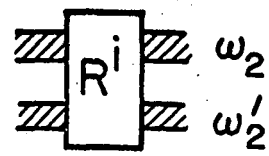
(5.20)

By Proposition 5.2 the second term on the right of (5.20) is equivalent to



(5.21)

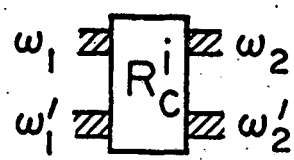
Some new definitions are now needed. Let $C^i(\omega, \omega') \equiv C^i$ denote any simple positive (ω, ω') cut set composed of lines such that the sum of their masses is greater than or equal to M_i. Let $\mathcal{R}^i(\omega, \omega')$ represent the set of all bubble diagrams B with the property that no $D \supset\subset B$ has a cut set C^i. Let

$R^i(\omega,\omega') \equiv$ 

(5.22)

represent any sum of terms of $\mathcal{R}^i(\omega, \omega')$.

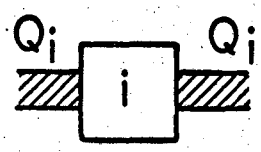
Similarly, let

$R^i_c(\omega,\omega') \equiv R^i_c \equiv$ 

(5.23)

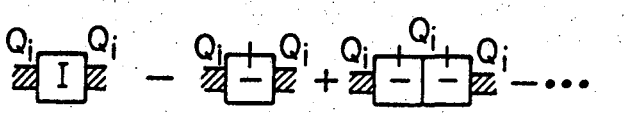
represent any sum of bubble diagrams each term of which either is disconnected, or does not support any Landau diagram containing a cut set C^i such that $\bar{X}(C^i, \omega)$ and $\bar{X}(C^i, \omega')$ are both connected diagrams. Evidently, the sets $\mathcal{R}(\omega, \omega')$, $\mathcal{R}_c(\omega, \omega')$, and $\mathcal{R}^i(\omega, \omega')$ are subsets of $\mathcal{R}_c{}^i(\omega, \omega')$. The function corresponding to any $B \in \mathcal{R}_c{}^i(\omega, \omega')$ has, by virtue of the Third Structure Theorem, a minus $i\epsilon$ continuation into itself past any normal threshold singularities at $E(\omega, \omega') = M_i$ that is associated with a $\omega' \to \omega$ normal threshold diagram.

By the Corollary to (5.2), any simple positive (ω, ω') cut set, and in particular any $C^i(\omega, \omega')$, contained in a Landau diagram D supported by (5.21) must be composed only of lines of that part of D that is supported by

$J_i \equiv$ 

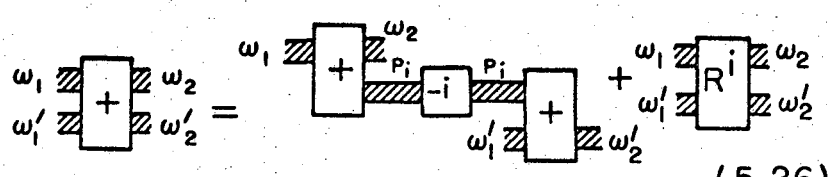
. (5.24)

The function J_i is defined by Fredholm theory (see Appendix B of I). Its analytic structure is, however, exhibited by the terms of the formal expansion:

$J_i =$ 

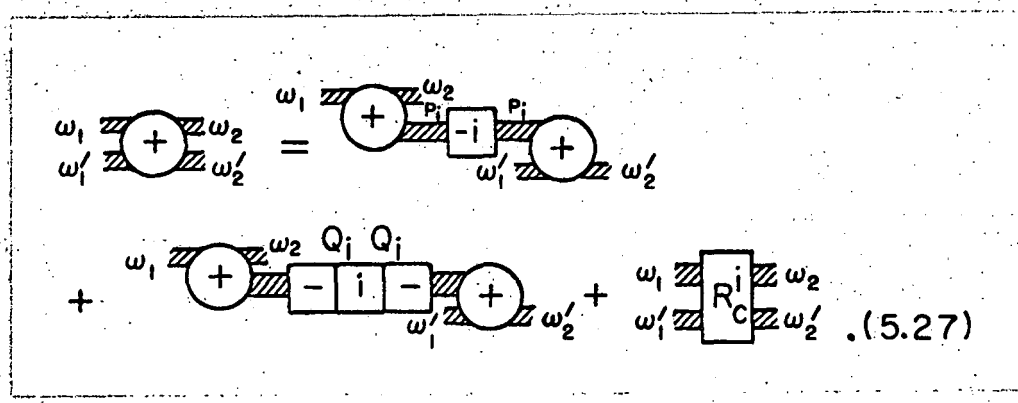
. (5.25)

Each term in this expansion consists of minus bubbles and sets of lines restricted by the Q_i bar conditions. Consequently, no Landau diagram supported by any term on the right of (5.25) can contain a cut set $C^i(\omega, \omega')$. Therefore (5.21) belongs to $\mathcal{R}^i(\omega, \omega')$, and (5.20) can be written



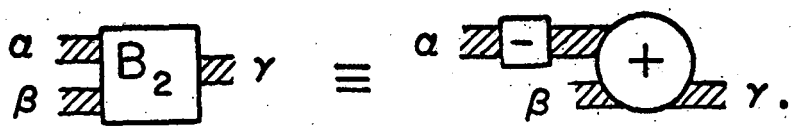
. (5.26)

The formula just obtained is the i-box version of Proposition 5.1. It can be simplified by further manipulations. Alternatively, one can apply (5.16) directly to (5.15). Then one obtains



The second term on the right can be incorporated into the last, by virtue of the following proposition.

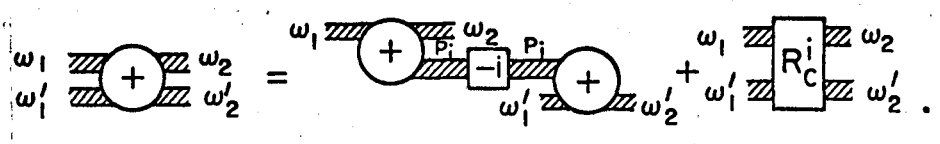
<u>Proposition 5.2C</u>. Consider the diagram



(5.28)

If the center-of-mass energy of the set α is below the four-particle threshold in the $\beta + \gamma \to \alpha$ channel, then B_2 is equivalent to a bubble diagram B_2' having the property that for any $D' \supset C B_2'$ the only simple positive $(\alpha, \beta + \gamma)$ cut set C with connected $\bar{X}(C, \beta + \gamma)$ is α itself.

This proposition, which is similar to Proposition 5.2, upon which it is based, is proved in Appendix C. Combined with the properties of J_i of (5.25) it shows that the second term on the right of (5.27) belongs to $\mathcal{R}_c^{\,i}(\omega, \omega')$. Thus we obtain



(5.29)

The second term on the right has a minus $i\epsilon$ continuation into itself past the normal threshold at $E(\omega, \omega') = M_i$. The first term is therefore the discontinuity of M^+ around this threshold.

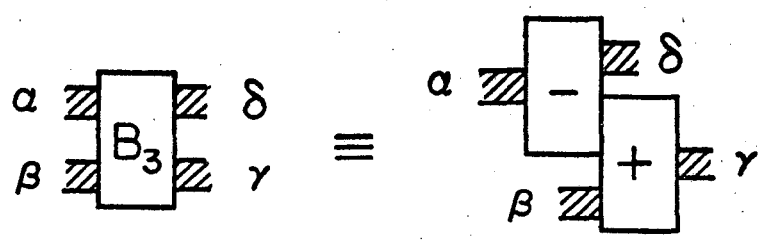
VI. ANALYSIS OF THE SECOND TERM

A. General Result

In this section the second term on the right of (5.12) is further analyzed. That is, various bubble diagrams included in this term, but belonging to $\mathcal{R}_c(\omega, \omega')$, are identified and separated out.

A generalization of Proposition 5.2 is needed:

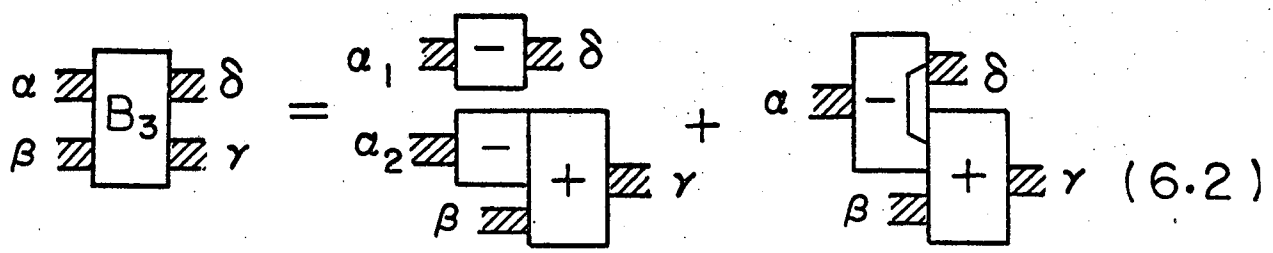
Proposition 6.1. The box diagram

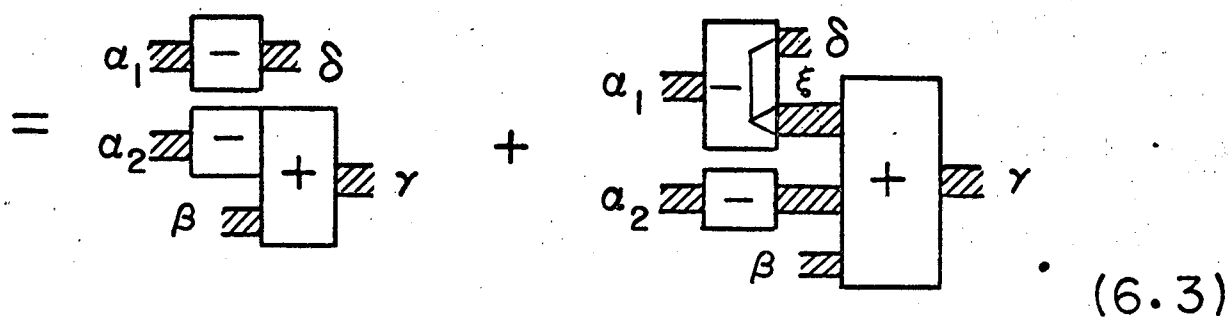


(6.1)

is equivalent to a box diagram B_3' such that, for any $D' \supset\subset B_3'$ the only simple positive $(\alpha, \beta + \gamma + \delta)$ cut set C containing no line of δ is the set α.

Proof. Equations $(2.3)'$ and $(2.4a)'$ give



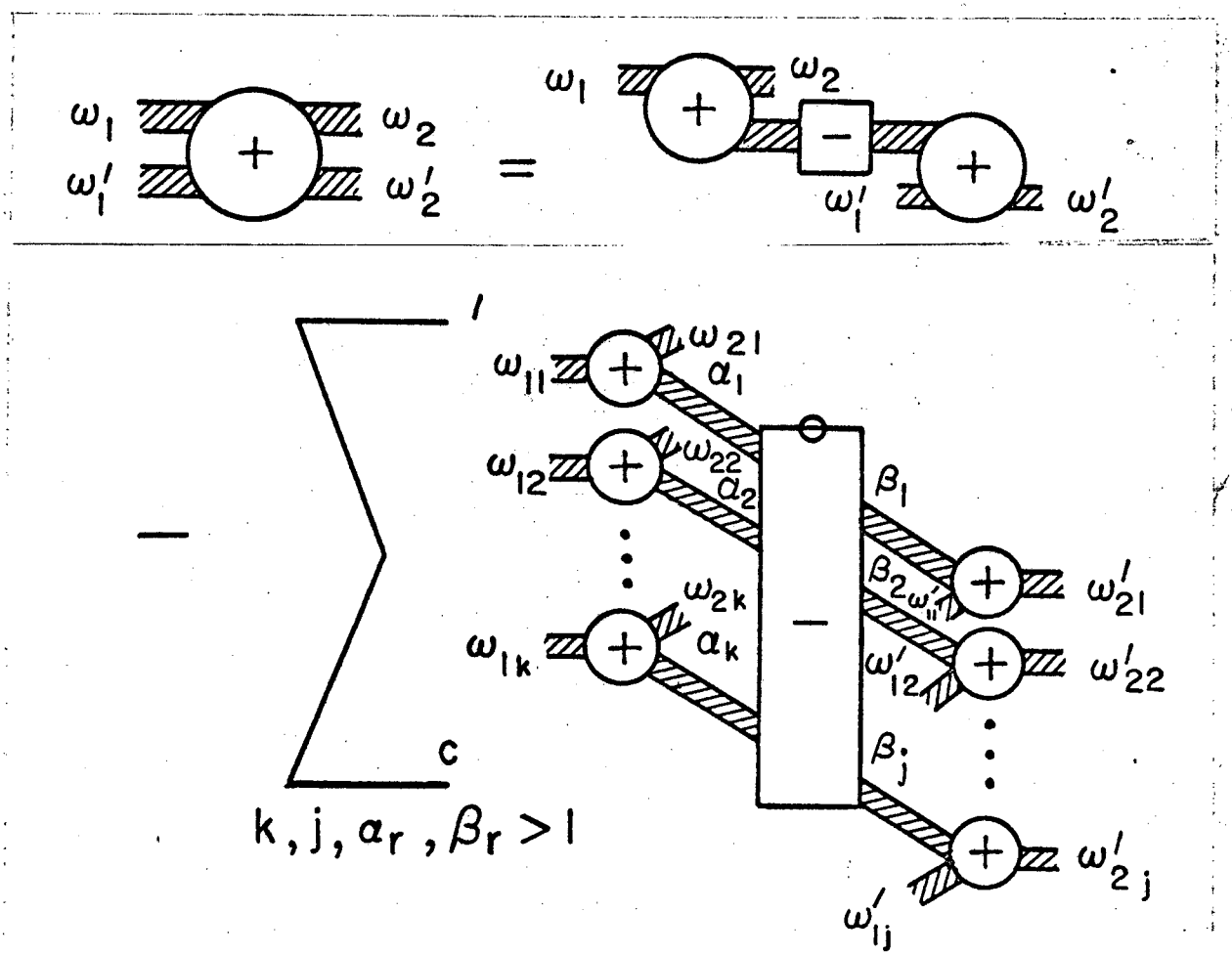


Since no line of δ can be in the cut set C, every line of α_1 in the first term of (6.3) evidently must be. [By definition, no line from any minus bubble can be included in this cut set.] Then the Corollary to 5.2 completes the proof for this first term. For the second term the definition of the box diagram (2.5)′, together with the requirement that each bubble has both incoming and outgoing lines, ensures that each line of α_1 is connected to a minus bubble (including trivial bubbles) that is connected to a line of δ. Thus each line of α_1 must belong to C. Each point of $X^+(\xi)$ lies on a minus bubble that is connected to a line of δ and it must therefore belong to $\overline{X}(C, \beta + \gamma + \delta)$. Then the Corollary to 5.2 completes the proof.

<u>Corollary 6.1</u> Suppose the B_3 of (6.1) is part of the box diagram B of (5.11). The replacement of the two B_1 parts of (6.3) by the B_1'

of Proposition 5.2 converts B_3 to B_3' and converts B to an equivalent B' having the property that, for any simple positive (ω, ω') cut C of any $D' \supset\subset B'$, all the points of $X^-(\alpha)$ belong to $\overline{X}(C, \omega')$ if all points of $X^+(\beta)$, $X^-(\gamma)$, and $X(\delta)$ do. Let the part of D' supported by B_3' be called D_3'. Then all points of $D_3' - \alpha$ belong to $\overline{X}(C, \omega')$ if all points of $X^+(\beta)$, $X^-(\gamma)$ and $X(\delta)$ do.

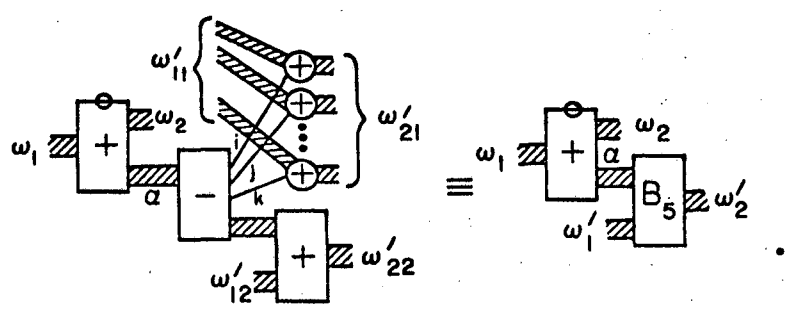
<u>Proposition 6.2</u>. Unitarity and the Cluster Properties imply that



(6.4)

The subscript c on Σ_c indicates a sum over connected terms only. The condition $\alpha > 1$ means the set α must have more than one line. The prime on the summation symbol indicates that it includes only one of the $k!\ j!$ topologically equivalent terms coming from the relabeling of the sets of initial and final lines. [And, as throughout this paper, once the sets of external lines are fixed, the sums are only over topologically distinct diagrams.]

Proof. Let the right-hand plus box in the second term on the right of (5.12) be expanded according to (D.11) of Appendix D, with the lines $i, j, \cdots n$ of σ identified with the lines that go to the minus box. The terms $G^{+}_{ij\cdots k}$ of (D.11) give structures of the form



(6.5)

One observes that no D supported by any term of this structure can have a simple positive (ω, ω') cut set C such that $\overline{X}(C, \omega')$ and $\overline{X}(C, \omega)$ are both connected diagrams. For if $\overline{X}(C, \omega')$ is connected, then the lines $i, j, \cdots k$ must evidently belong to $\overline{X}(C, \omega')$. But then by Corollary 6.1 all the points of $X^{-}(\alpha)$ must belong to $\overline{X}(C, \omega')$.

This means that $\overline{X}(C, \omega)$ cannot be a connected diagram. A similar argument applies to the left-hand circled plus box of (6.5). Thus all the contributions associated with the $G^{+}_{ij\cdots k}$ terms of (D.11) give terms that belong to $\mathcal{R}_c(\omega, \omega')$. The remaining term becomes the second term on the right of (6.4), after the contribution coming from the connected part of the minus box is shifted to R_c. This completes the proof of Proposition 6.2.

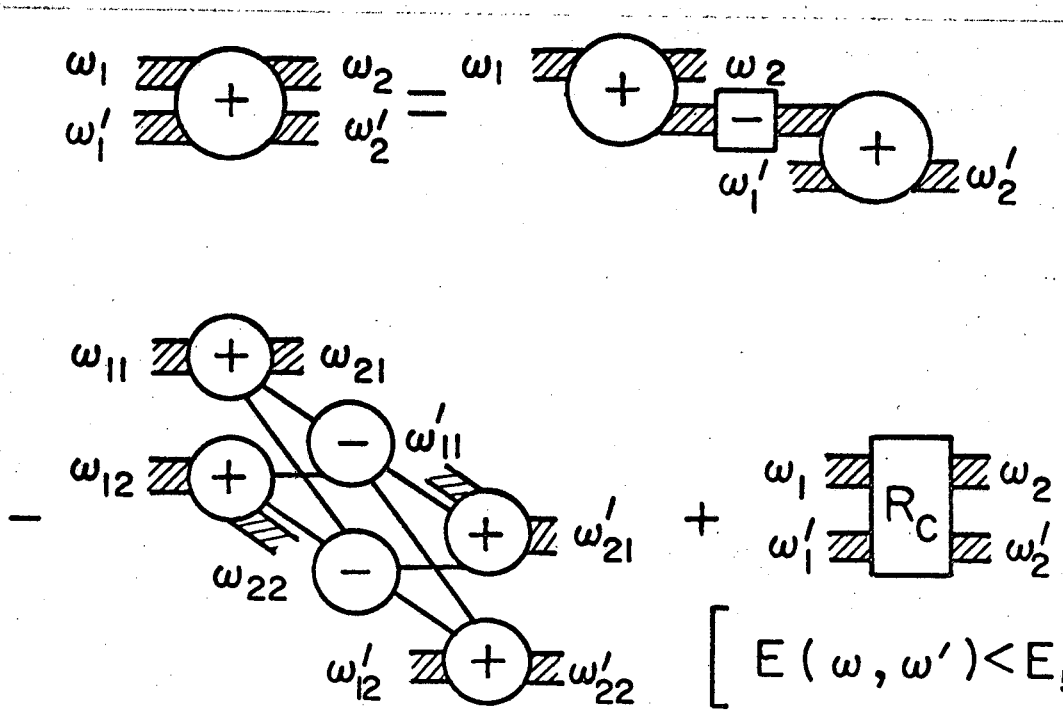
B. Special Cases

Case 1: $E(\omega, \omega')$ below the four-particle threshold

In this case the conditions on the second term on the right of (6.4) cannot be met. This proves (5.15)

Case 2: $E(\omega, \omega')$ below the five-particle threshold E_5

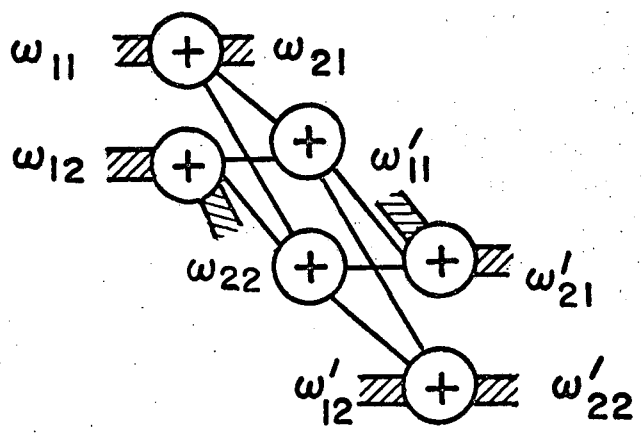
It is shown in Appendix E that for this case (6.4) gives



$$\left[\, E(\omega, \omega') < E_5 \,\right]. \qquad (6.6)$$

The first two terms on the right contain all the singularities associated with positive-α Landau diagrams that can be contracted to any $D_n^{+}(\omega' \to \omega)$. The positive-$\alpha$ double-cross diagram obtained by shrinking the bubbles of the second term on the right to points, and assigning plus signs to the lines, is such a diagram. It is readily confirmed that the

positive-α contributions to the double-cross diagram from the sum of the first two terms in (6.6) are equal to the positive-α contributions from

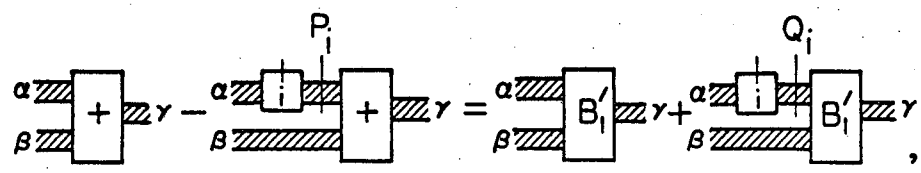


(6.7)

The significance of the results obtained in this section is this: The function represented by R_c in (6.4) or (6.6) must, by virtue of the third structure theorem, continue into itself in a well defined way around any Landau surface $L(D)$, near $P \in L(D)$, for any D that contracts to a $\omega' \to \omega$ normal threshold diagram, provided the only singularity surfaces of R_c through P are $L(D)$ and $L(D^*)$, where D^* is obtained from D by reversing the signs of all the α's. And if D is a pure-α diagram, then this well-defined continuation is the minus $i\epsilon$ (i.e., nonphysical) continuation. Thus the discontinuity corresponding to any path $P'P''$ that encounters only singularities of these types in R_c is precisely the first two terms on the right of (6.4) or (6.6).

VII. SHEET CONVERTERS

Below the lowest four-particle threshold E_4^{α} in the channel $\beta + \gamma \to \alpha$ one obtains from (5.18), (5.10), and Proposition 5.2 the result



(7.1)

where B_1' is the right side of (B.4), and satisfies the following proposition.

<u>Proposition 7.1</u>

Let $\hat{\alpha}$ and $\hat{\beta}$ be sets that satisfy $\hat{\alpha} + \hat{\beta} = \alpha + \beta$, $\hat{\alpha} \subset \alpha$ and $\hat{\alpha} \cap \hat{\beta} = 0$. The only simple positive $(\hat{\alpha}, \hat{\beta} + \gamma)$ cut set C of any $D' \supset\subset B_1'$ is $\hat{\alpha}$.

This proposition is a trivial extension of Proposition 5.2.

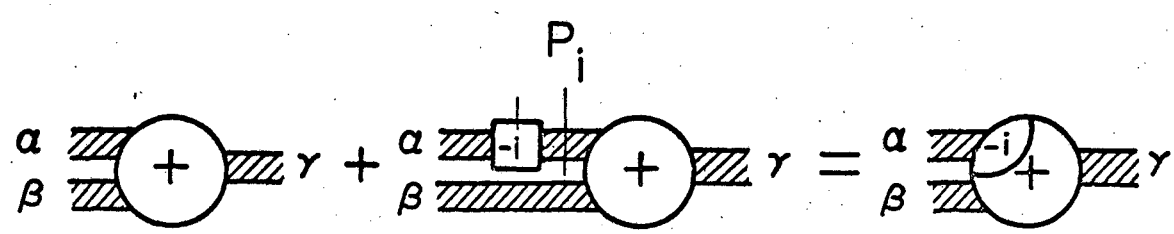
By virtue of Proposition 7.1 and the Third Structure Theorem, we know that the function represented by the connected part of B_1' has a minus $i\epsilon$ continuation into itself around any singularity associated with any diagram $D' \supset\subset B_1'$ that can be contracted to any $\hat{\beta} + \gamma \to \hat{\alpha}$ normal threshold diagram. Using the properties of (5.25) and B_1', one finds a similar result for the second term on the right of (7.1): The connected part of the second term on the right of (7.1) has a minus $i\epsilon$ continuation into itself around any singularity in the interval

$$M_i \leqslant E(\alpha,\beta + \gamma) < E_4^{\alpha}$$

associated with any diagram $D' \supset\subset B_1'$ that can be contracted to any $\hat{\beta} + \gamma \to \hat{\alpha}$ normal threshold diagram. Let the set of singularities just described be called $S_i(\hat{\alpha},\hat{\beta} + \gamma)$.

Since the connected parts of both terms on the right of (7.1) continue into themselves via a minus $i\epsilon$ rule around all singularities of $S_i(\hat{\alpha},\hat{\beta} + \gamma)$, and since (7.1) holds identically throughout the physical region, the connected part of the left side must continue into itself via the same rule. If M_i is chosen greater than the physical threshold in the $\beta + \gamma \to \alpha$ channel, so that the physical region includes points where $E(\alpha, \beta + \gamma) < M_i$, then the connected part of either side of (7.1) can be identified as the continuation of the scattering function from physical points $E(\alpha,\beta + \gamma) < M_i$ to points in $E(\alpha,\beta + \gamma) > M_i$ lying underneath the cuts associated with the singularities of $S_i(\hat{\alpha},\hat{\beta} + \gamma)$. Continuability past other singularities is not guaranteed, however.

The result just obtained is represented by the equation



(7.2)

where the right side represents the continuation of the scattering function to the underside of the cuts associated with the singularities of $S_i(\hat{\alpha},\hat{\beta} + \gamma)$. [Some terms on the left of (7.1) are transferred to the right by the methods of Appendix C (see C.3), in order to get (7.2). The symbol $-i$ in the right-hand term is simply i in References 1, 2, and 6.]

The restriction that M_i be above the lowest physical threshhold in the channel $\beta + \gamma \to \alpha$ means that the lines α of (7.2) can be considered cut by a Q_i bar. One would like to have this equation also with a P_i bar on these lines, since the formula could then be inserted into the discontinuity formula (5.29).

To obtain this result one must use the pole-factorization theorem[5] in (5.26). We do not pursue the matter here, but only remark that the formulas in terms of the i boxes are the more useful ones anyway. For the expression in terms of functions on other sheets introduces, in effect, new unknown functions. And there is a different new unknown function for each choice of the "other" external lines [for example, those in the sets β and γ of (7.2)]. The i box formula gives the discontinuity directly in terms of physical functions alone, and the i box is independent of the "other" lines.

VIII. ANALYTICITY PROPERTIES

The phase-space factors in unitarity equations have singularities. Corresponding singularities must appear in some scattering function. These latter singularities can combine with phase-space singularities to yield still other singularities of scattering functions, and so on. It has been shown[8] that all singularities generated in this way by unitarity must lie on Landau surfaces. The first part of our (maximal) analyticity assumption is, accordingly, that all physical-region singularities of scattering functions lie on Landau surfaces.

Physical-region singularities lying on Landau surfaces are physically interpretable in terms of the notion that momentum-energy is transferred over macroscopic distances only by physical particles. In particular, if the singularities of scattering functions are confined to Landau surfaces, then transition amplitudes can be shown to fall off faster than any power of a scaling parameter τ unless the trajectory regions defined by the wave packets of the initial particles can be connected to those of the final particles by a network of physical particles.[9] In the limit of large τ the distances involved become infinite. If one invokes the macroscopic causality requirement that the transition amplitudes fall off fast unless all the particles of the network move forward in time, then the singularities are confined to the positive-α branches of the Landau surfaces.[9] Moreover, the scattering functions on the two sides of these surfaces are analytically connected by a path that moves into a certain well-defined upper half plane.[9] This rule of continuation is called continuation via a plus $i\epsilon$

rule. Continuation through the opposite half plane is called continuation via a minus $i\epsilon$ rule.

The work of Reference 9 establishes also that if several Landau surfaces intersect, then the rules of continuation are compatible in the following sense: If the various intersecting Landau surfaces are all associated with diagrams that are contractions of any one single parent diagram, then the intersection of the various upper half planes is nonempty, and hence a region of continuation exists; if the diagrams corresponding to one subset of the surfaces intersecting at point $\overline{K}$ have a common parent, but no two that include one of the remaining diagrams have a common parent, then in a neighborhood of $\overline{K}$ the scattering function can be decomposed into a sum of functions each having only certain of the singularities, and having a well-defined rule of continuation past these singularities.

The above results provide a rule for continuation of scattering functions around all known combinations of intersecting singularity surfaces. To include any possible others, we assume that singularities that are "unrelated at $\overline{K}$" are "independent at $\overline{K}$." Singularities "unrelated at $\overline{K}$" are singularities corresponding to diagrams that have no common parent whose surface contains $\overline{K}$. Singularities "independent at $\overline{K}$" are singularities at $\overline{K}$ that can be separated into different terms of an expansion of the scattering function. The assumption that unrelated singularities are independent ensures that there is a well-defined rule of continuation past all combinations of intersecting

surfaces of singularities of the scattering function. This rule is called the "general $i\epsilon$ rule." It ensures that an integral over a scattering function can be defined locally, as a sum of contour integrals that detour around singularities in a manner determined by the plus $i\epsilon$ rules for the individual singularity surfaces.

The general $i\epsilon$ rule is the hypothesis of three "structure theorems" proved in Ref. 2. These theorems provide the analytic basis of the present work. The First Structure Theorem says that the function corresponding to a connected bubble diagram B can have singularities only on the Landau surfaces corresponding to Landau diagrams $D \supset\subset B$. A diagram $D \supset\subset B$ is a diagram that can be obtained by inserting connected Landau diagrams for the bubbles of B, and then contracting some (or no) lines. The Second Structure Theorem says that these diagrams $D \supset\subset B$ can be further restricted by demanding that the α's on the lines of $D \supset\subset B$ that are interior lines of plus or minus bubbles of B have plus or minus signs, respectively. The Third Structure Theorem gives the rule for continuation around the singularity associated with a given $D \supset\subset B$. It says that if $\mathcal{L}[D]$ is the only Landau surface passing through a point $\bar{K}$, and if the momenta p_j and Feynman α_j of the internal lines of the corresponding Landau diagram are given uniquely by continuous functions $\bar{\alpha}_j(K)$ and $\bar{p}_j(K)$ for points on $\mathcal{L}[D]$ near $\bar{K}$, then the function represented by B continues into itself when continued around $\mathcal{L}[D \supset\subset B]$ near $\bar{K}$ by a path in the upper half plane of the variable

$$\sigma_{\overline{K}}(K) \equiv \sigma(K;\overline{K}) \equiv \sum_j \overline{\alpha}_j(\overline{K})\,\overline{p}_j(\overline{K})\cdot p_j(K). \tag{8.1}$$

The sum is over all internal lines, and $p_j(K)$ is any set of p_j consistent with the conservation laws at K. [A number of useful equivalent expressions for $\sigma(K;\overline{K})$ are given in Ref. 2.] The signs of the $\alpha_i(\overline{K})$ in (8.1) are fixed according to the rule given above: α's from lines lying inside plus (minus) bubbles are positive (negative). By virtue of the Third Structure Theorem, the function represented by any bubble diagram B always has a well-defined continuation into itself past the singularity surfaces near a point P unless there are several $D \supset\subset B$ with surfaces $\mathcal{L}[D]$ that intersect at P and have incompatible rules for continuation.

The position of a Landau surface corresponding to a given D is not changed if the signs of all its α's are reversed. But the rule for continuation past the corresponding singularity is reversed. Thus if both these D are supported by some B, then the Third Structure Theorem fails to provide any (single) path of continuation for this function. This is precisely what happens at a threshold: The singularity surface corresponds to two $D \supset\subset B$ that specify opposite rules for continuation. Accordingly, the function represented by B on the side of the threshold is not analytically connected to the function represented by B on the other side.

It should be emphasized that the contours of integration in the definitions of bubble diagram functions are always fixed to be real, except for the infinitesimal distortions away from the singularities of the scattering functions of the integrand: One never distorts (by finite amounts) any contours, but uses always the originally defined (almost) real contours. Thus when we say that a function corresponding to a bubble diagram B continues into itself, we mean the function defined by B with real undistorted contours of integration continues into the function defined by B with real undistorted contours, always excepting of course the infinitesimal distortions required by the definitions of scattering functions.

Our procedure has been to derive universal physical-region identities of the form $M^+ = T + R$, where M^+ is the physical scattering function, T is a threshold term that vanishes below the threshold t corresponding to a certain positive-α normal threshold diagram D^+, and R is a function that has no singularity corresponding to any Landau diagram that can be contracted to D^+. Let D^- be the diagram obtained from D^+ by reversing the signs of all the α's. The singularity surface corresponding to D^- also lies at the threshold t, but the continuation past it is via the minus $i\epsilon$ rule. Thus if the only singularities of R near some point P on t are ones corresponding to D^+ or D^-, then R must have a minus $i\epsilon$ continuation into itself past t near P, since the construction of R rules out singularities corresponding to D^+.

Because T vanishes below threshold, R is equal to M^+ there. Thus R is an explicit expression in terms of physical amplitudes (i.e., scattering functions at physical points) of a function that equals M^+ below t near P but has a minus $i\epsilon$ continuation into itself past t near P. Since M^+ has a plus $i\epsilon$ continuation into itself past t, the discontinuity around t near P is just T.

In the argument just given it was assumed that each singularity of R near P corresponds to one of the threshold diagrams D^+ or D^-. The argument can be extended easily, however, to the case in which each singularity of R near P corresponds, merely, to some pure (positive or negative)-α diagram that contracts to D^+ or D^-. Those contracting to D^+ are ruled out by the construction of R, while those that contract to D^- must be pure negative-α diagrams. Therefore R again continues into itself around t near P via the minus $i\epsilon$ rule. Since only a finite number of pure-α singularity surfaces enter any bounded region,[10] the point P can be taken to lie on no pure-α surface other than t. Then the plus and minus $i\epsilon$ continuations are simply into the upper and lower half $E(\omega' \to \omega)$ plane respectively.

The above arguments cover only singularities corresponding to pure-α diagrams that contract to D^+ or D^-. However, for almost every point P on t the contraction condition can be ignored. For, for almost every P on t there is a neighborhood $N(P)$ of P such that each pure-α surface that intersects $N(P)$ corresponds to a diagram that contracts to D^+ or D^-. This follows from the general

theory of pure-α surfaces developed in Ref. 9. That theory tells us that, apart from a set of points $\mathcal{M}_0$ (of zero measure on t) where certain external lines are parallel, the set of points lying on positive-α surfaces are the union of a set of (codimension 1) analytic manifolds only a finite number of which pass through any bounded region. Thus for almost every point P on t there is a neighborhood $N(P)$ of P that intersects no Landau surface except those that coincide with t. The general theory also tells us that the normal to the surface at any point P lying on just one surface uniquely determines the positions of the external lines of the geometric Landau diagram $\overline{D}(P)$ that generates this point P. This means that any pure-α diagram whose singularity surface contains a P lying on no Landau surface except t must be such that all the lines ω intersect at one point and all the lines ω' intersect at another point. But any pure-α diagram of this kind can be contracted to D^+ or D^-. Thus for almost all points P on t, R has minus $i\epsilon$ continuation into itself around t near P, provided R has no mixed-α singularities passing near P.

A theory of mixed-α Landau surfaces analogous to the theory of pure-α Landau surfaces developed in Ref. 9 is not available. It seems likely, however, that almost every point P on t will have a neighborhood that intersects no Landau surfaces except ones corresponding to diagrams that contract to a normal or pseudonormal $(\omega' \to \omega)$ threshold diagram.[11] Let us assume this is true.

It also seems likely that in equations derived from unitarity and cluster properties alone there will be no systematic cancellations between pure-α and mixed-α singularities. That is, if one side of such an equation has only pure-α singularities, in some neighborhood $N(P)$, then so should the other side. In particular, we do not expect singularities associated with normal threshold diagrams to be cancelled by singularities associated with pseudonormal threshold diagrams, since these will be moved relative to each other by small variations of the masses. The absence of such cancellations will also be assumed. [These assumptions are either implicit or explicit in all derivations of discontinuities from unitarity.]

These two assumptions allow us to conclude that R has no mixed-α singularities in a neighborhood $N(P)$ of almost any point P on t. Energy conservation precludes the possibility that T has any pseudonormal threshold singularities near t. Then, since $M^{+}-T$ has only pure-α singularities in a neighborhood $N(P)$ of almost every P of t, so must R.

A more extensive study of the mixed-α singularities is needed. But that is a subject in itself.

APPENDICES

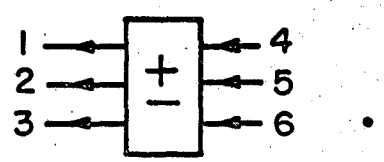
Appendix A. Proof of Decomposition Rules

Every contribution to the left side of (2.3) that is not contained in the first term on the right is a column of bubbles none of which is connected to both α_1 and α_2. Any such term occurs as a contribution to the second term on the right. No two different contributions to the left can occur as the same contribution to the second term on the right. Thus the left side is contained in the right. Every contribution to the second term on the right occurs as a contribution to the left. No two different contributions to the second term on the right can occur as the same contribution to the left. No contribution to the second term on the right is contained in the subset of those contributions to the left that constitute the first term on the right. Thus the right side is contained in the left side. Therefore the two sides of (2.3) are identical sets of diagrams.

Each contribution to the left side of (2.4a) is contained in one and only one term on the right. This term is one in which α_{12} is the subset of α_1 consisting of lines connected to bubbles that are connected to α_2, and β_2 is the subset of β consisting of lines connected to bubbles connected to $\alpha_{12} \cup \alpha_2$. A given contribution to the left occurs as precisely one contribution to this unique term in which it appears. No two different contributions to the left occur as the same contribution on the right. And every contribution on the right

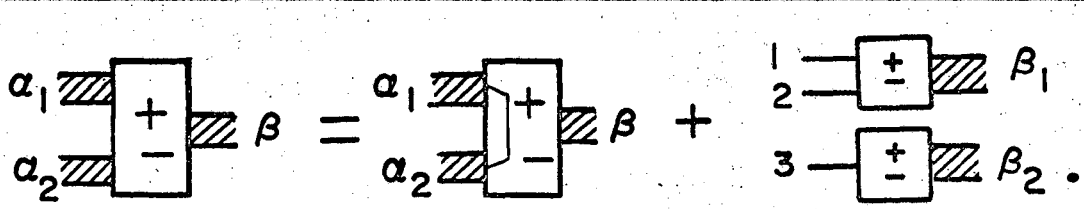
occurs at least once on the left. Thus the two sides of (2.4a) are identical. A completely similar argument proves (2.4b).

An illustration of the decomposition formulas is provided by considering the $3 \to 3$ box



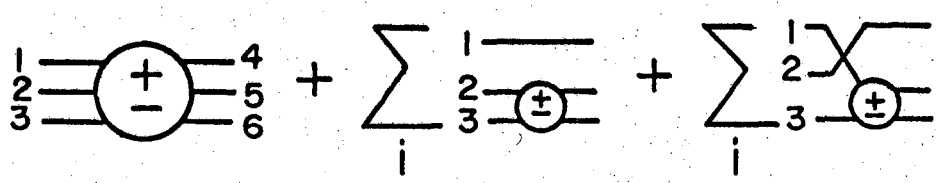
(A.1)

For the case $\alpha_1 = \{1, 2\}$, $\alpha_2 = \{3\}$, and $\beta = \{4, 5, 6\}$, Eq. (2.3) reads



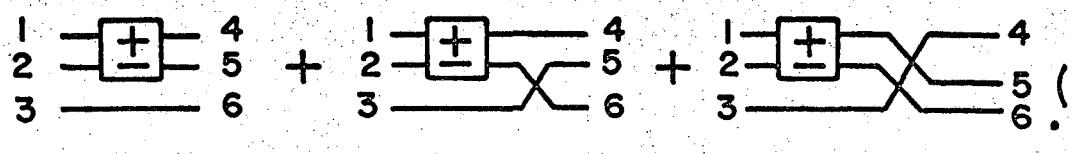
(A.2)

The first term on the right of (A.2) is then by definition the sum of bubble diagrams



(A.3)

where the sum sign labeled "i" has the same significance as in I (see the first paragraph of Section IV of I). The second term on the right of (A.2) is



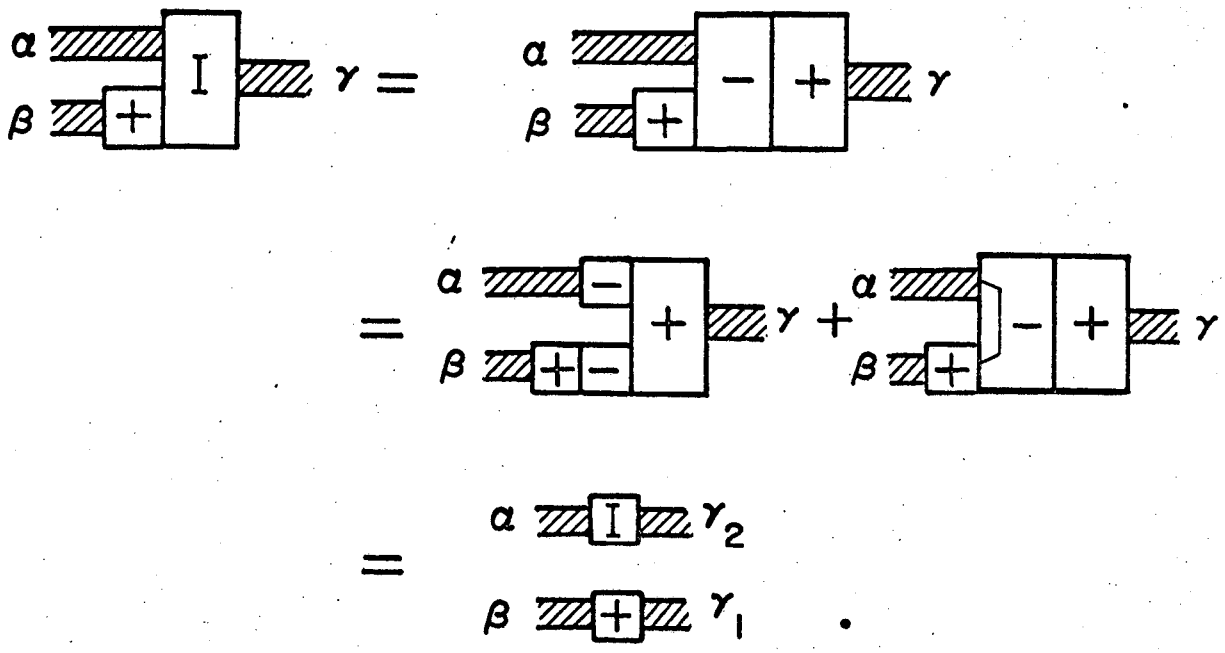
(A.4)

The expressions (A.3) and (A.4) add up to the usual cluster expansion of the $3 \to 3$ box [Eq. (4.4) of I], as we see by substituting the expansion of the $2 \to 2$ box [Eq. (A.8) of I] into (A.4).

Appendix B: Iteration of Certain Box Diagrams

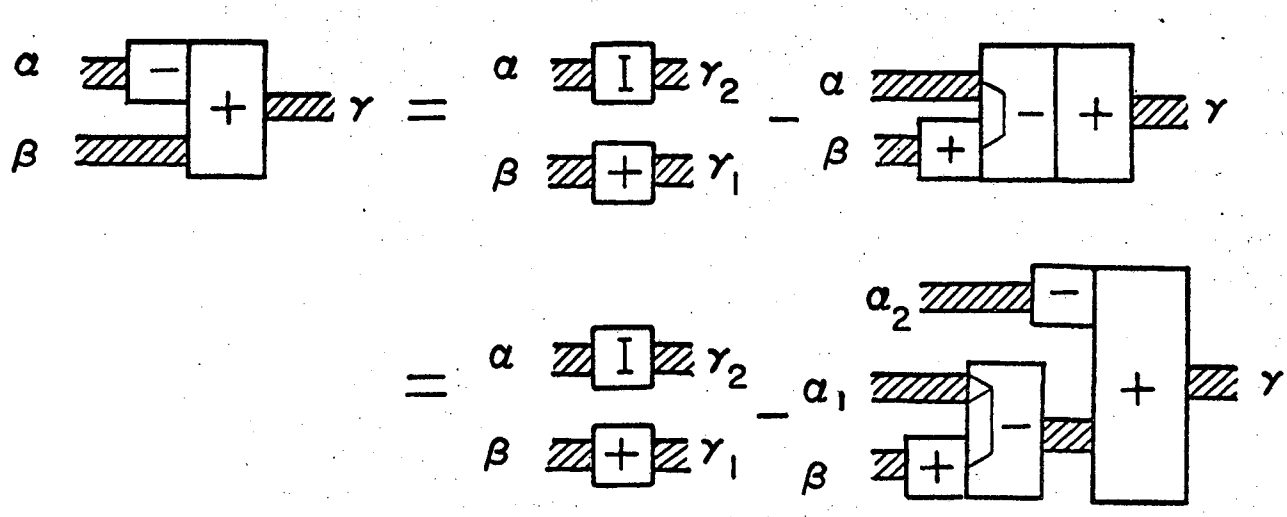
In this appendix the box diagram B_1 defined by (5.10) is converted to an equivalent box diagram B_1' used in the proof of Proposition 5.2.

Equations (2.3) and (2.12) give



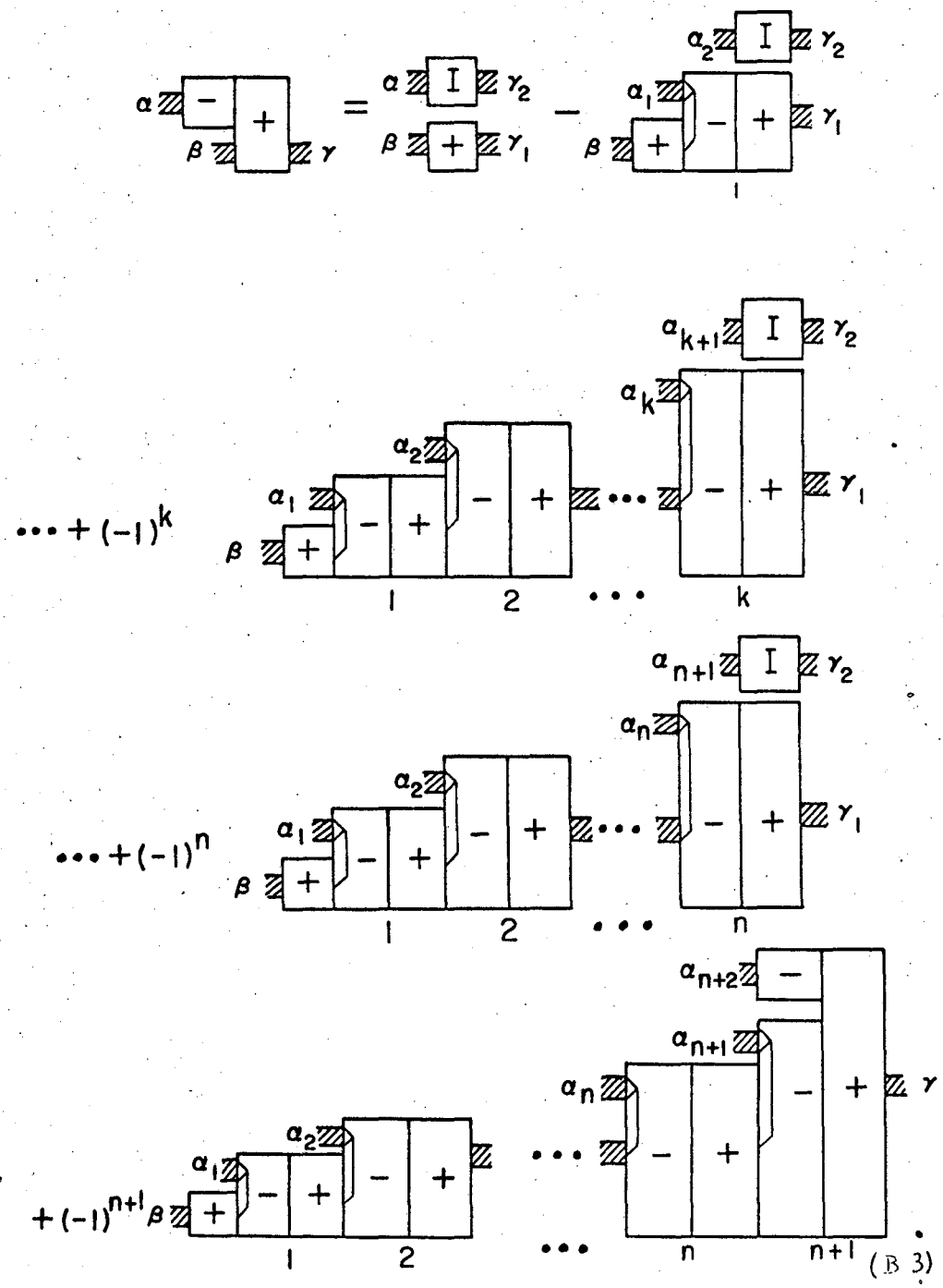
(B.1)

Use of unitarity, and then (2.4a), converts this to



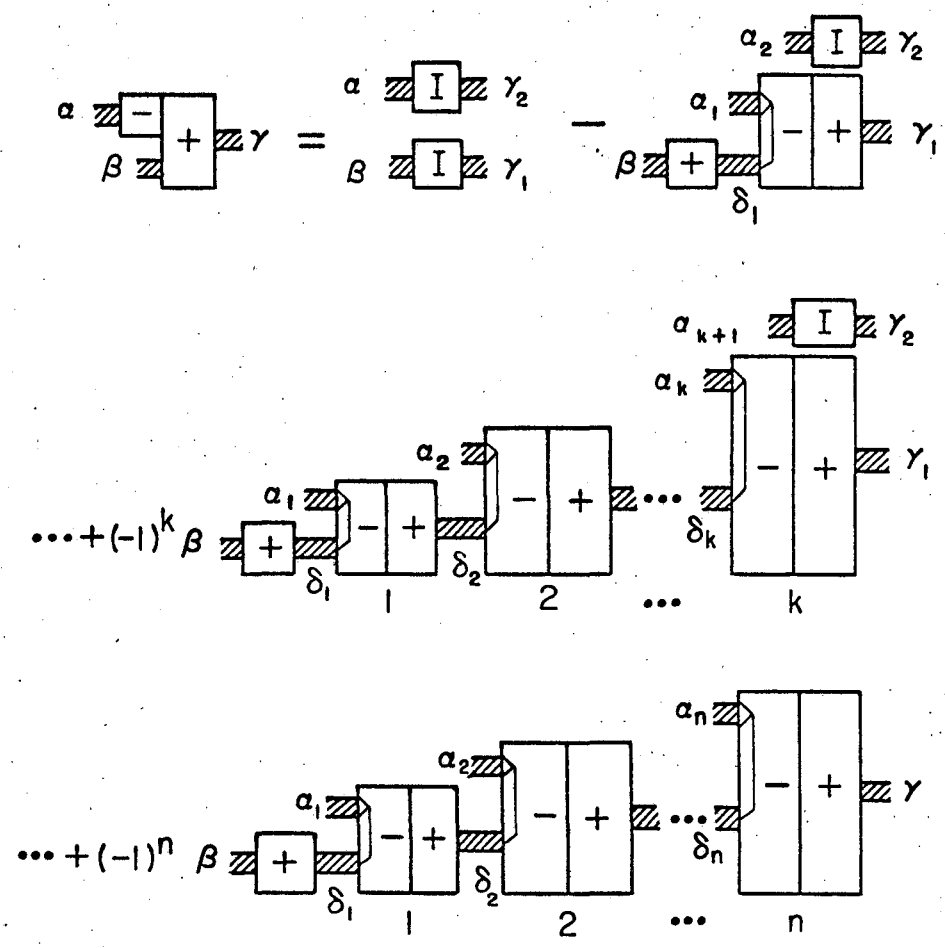
(B.2)

This equation can be iterated by replacing the last factor of the last term by the entire right side. An n-fold iteration gives



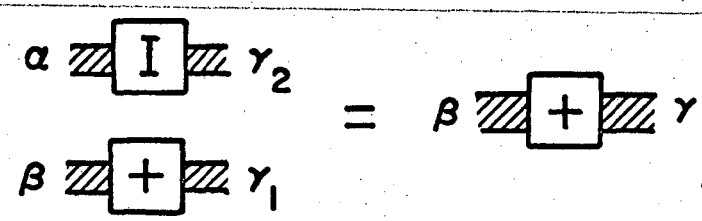
(B 3)

Let the number of iterations n be at least equal to the number of lines $n(\alpha)$ of the set α. Then the last term on the right side of (B.3) is an empty set, since the definition of (2.5) implies that if any of the sets $\alpha_1, \alpha_2, \cdots, \alpha_{n+1}$ is empty, then no diagram satisfying the required conditions exists. Thus for any $n \geqslant n(\alpha)$ we have



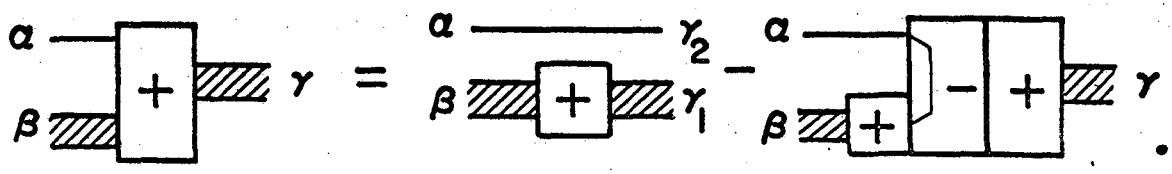
(B.4)

If the set α is empty, then all terms of the right side of (B.4) except the first are empty. Thus, the right side of (B.4) is in this case



(B.5)

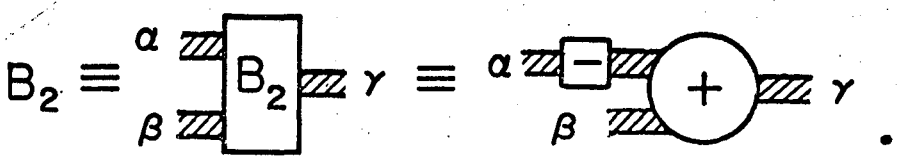
Equation (B.4) is therefore a trivial identity for $n(\alpha) = 0$. If $n(\alpha) = 1$, Eq. (B.4) is



(B.6)

Appendix C. Proposition 5.2C

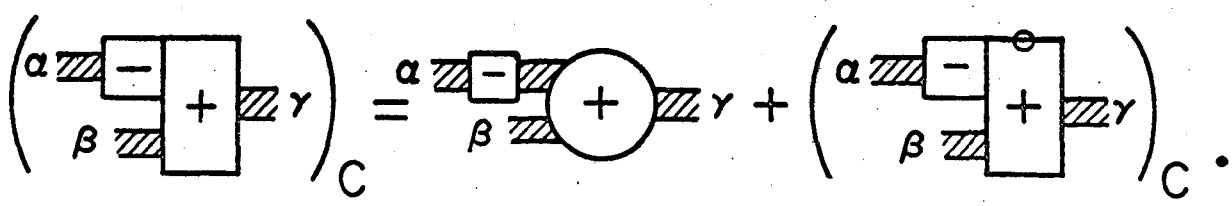
Consider the diagram



(C.1)

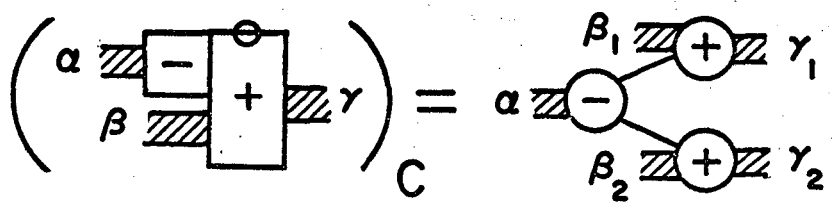
If the center-of-mass energy of the set α is below the four-particle threshold in the channel $\beta + \gamma \to \alpha$, then B_2 is equivalent to a bubble diagram B_2' such that for any $D' \supset\subset B_2'$ the only simple positive $(\alpha, \beta + \gamma)$ cut set C with connected $\overline{X}(C, \beta + \gamma)$ is α itself, Moreover $D' - \alpha$ lies in $\overline{X}(C, \beta + \gamma)$.

Proof. The arguments establishing Proposition 5.2 apply equally well to the connected part of B_1, which is



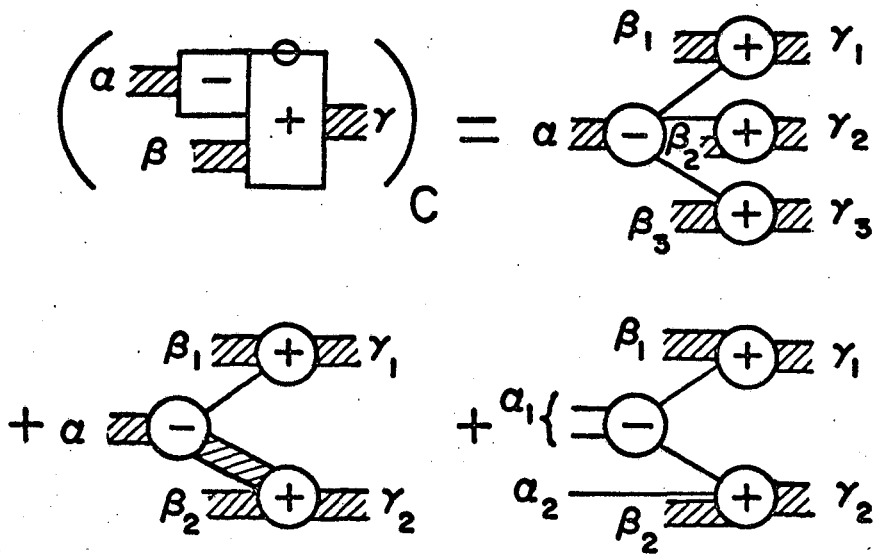
(C.2)

If α is taken to be a single line, the second term on the right of (C.2) drops out, and Proposition 5.2 proves the proposition. If the center-of-mass energy of the set α is less than the three- or four-particle threshold, respectively, then the second term on the right of (C.2) is



(C.3)

or



(C.4)

respectively. (Topologically equivalent contributions to the right side are to be counted only once.) The only simple positive $(\alpha, \beta + \gamma)$ cut set C of any of the bubble diagrams of (C.3) or (C.4) such that $\overline{X}(C, \beta + \gamma)$ is connected is the set α. This follows at once from Definition 3.1, Proposition 3.1, and from the fact that no line of C can be a minus line. Proposition 5.2, applied to the left side of (C.2), completes the proof.

<u>Corollary</u>. Suppose B_2 of (C.1) is part of B of (5.11). Replacement of B_2 by the B_2' of the proposition converts B to B'. Let C be any simple positive (ω, ω') cut set of any $D' \supset\subset B'$. If $\overline{X}(C, \omega')$ is connected, then all points of $X^-(\alpha)$ belong to $\overline{X}(C, \omega')$ if all points of $X^+(\beta) \cup X^-(\gamma)$ belong to $X(\omega')$.

<u>Proof</u>. The above proof still applies, if the $X^+(\beta) \cup X^-(\gamma)$ belong to $X(\omega')$.

Appendix D. The Principle of Inclusion and Exclusion and an Expansion on Unconnected Lines

Let A be a set such that each member of the set either has or has not the property "i." The subset of A consisting of members that have the property "i" are designated by A_i. The subset of A consisting of members that have not the property "i" are designated by $A^i (i = 1, 2, \cdots, n)$. Then,

$$A = A_i + A^i. \tag{D.1}$$

By a repeated application of (D.1) we obtain

$$A = A_1 + A^1$$

$$= A_{12} + A_1{}^2 + A^1{}_2 + A^{12}. \tag{D.2}$$

It follows from (D.1) also that

$$A_1{}^2 = A_1 - A_{12}. \tag{D.3}$$

By the definition of A_i, and by (D.1),

$$A^1{}_2 = (A - A_1)_2 = A_2 - A_{12}. \tag{D.4}$$

Substituting (D.3) and (D.4) into (D.2), we obtain

$$A = A_1 + A_2 - A_{12} + A^{12}. \tag{D.5}$$

Equation (D.5) can be generalized to the formula

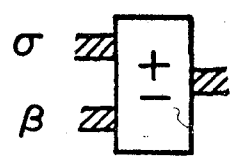
$$A = \sum_i A_i - \sum_{i<j} A_{ij} + \cdots$$

$$- (-1)^s \sum_{i<j<\cdots<k} A_{ij\cdots k} + \cdots$$

$$\cdots - (-1)^n A_{12\cdots n} + A^{12\cdots n}, \tag{D.6}$$

where s is the number of subscripts in $A_{ij\cdots k}$ and where each summation runs from 1 to n. [For a proof of (D.6) see Ref. 12.]

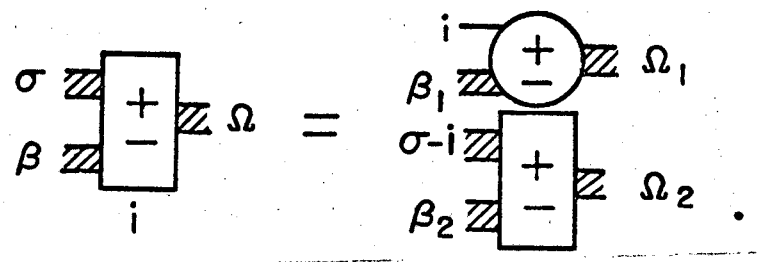
Consider now a plus or minus box of the form

$G^{\pm} =$ 

, (D.7)

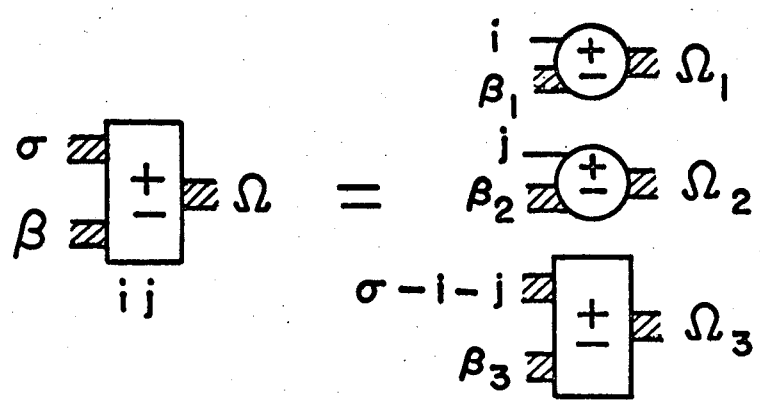
where σ consists of precisely n lines. Let the property "i" be the property that line i of σ is connected in $G^{\pm}$ to no line of

$\sigma - i \equiv \sigma - \{i\}$. Let $G_i^{\pm}$ be the subset of diagrams of $G^{\pm}$ with the property i. Then $G_i^{\pm}$ has the form

$G_i^{\pm} \equiv$ 

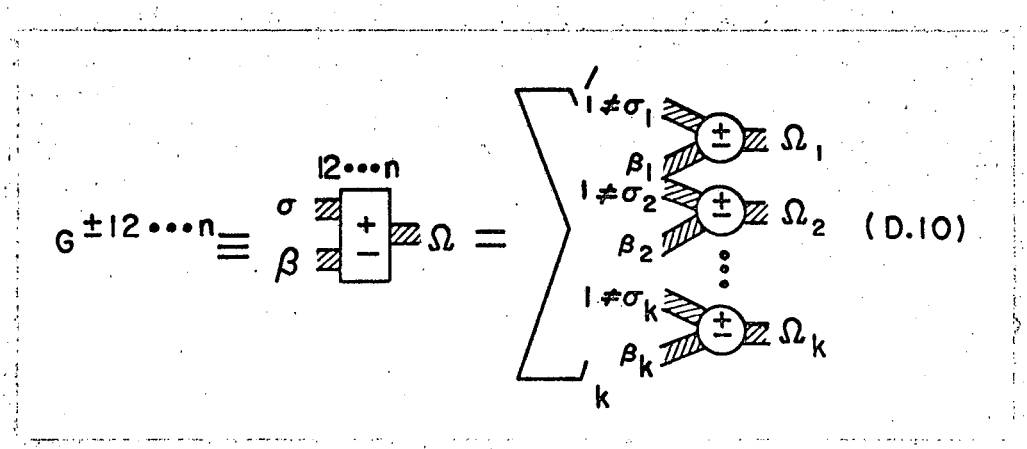
. (D.8)

This is because each term of $G_i^{\pm}$ occurs as one and as only one term on the right; no two different terms of $G_i^{\pm}$ occur as a single term on the right; and each term on the right occurs at least once in $G_i^{\pm}$. Similarly, for $i \neq j$,

$G_{ij}^{\pm} \equiv$ 

(D.9)

etc. The term $G^{\pm 12\cdots n}$ is the term of $G^{\pm}$ such that each line i of σ is connected in G to some line of $\sigma - i$. Thus it has the form



where the prime on the summation symbol indicates that only one of the $k!$ topologically equivalent diagrams obtained by reordering the bubbles is to be counted, and the condition $\sigma_r \neq 1$ means that the set σ_r has two or more lines. That is, the sum is over all ways of partitioning the set σ into sets $\sigma_r \neq 1$, the set β into the sets β_k, and the set Ω into the sets Ω_k, as specified by the summation convention of Section II.

Let $\overset{o}{G}{}^{\pm}$ be $G^{\pm}$ minus its connected part. The expression given above for $G^{\pm}$ also applies to $\overset{o}{G}{}^{\pm}$ except that the sum of columns of bubbles on the right of (D.10) does not include a column consisting of just one bubble. Since $\overset{o}{G}{}^{\pm}_{ij\cdots k} = G^{\pm}_{ij\cdots k}$, the application of (D.6) to the function $\overset{o}{G}{}^{\pm}$ yields

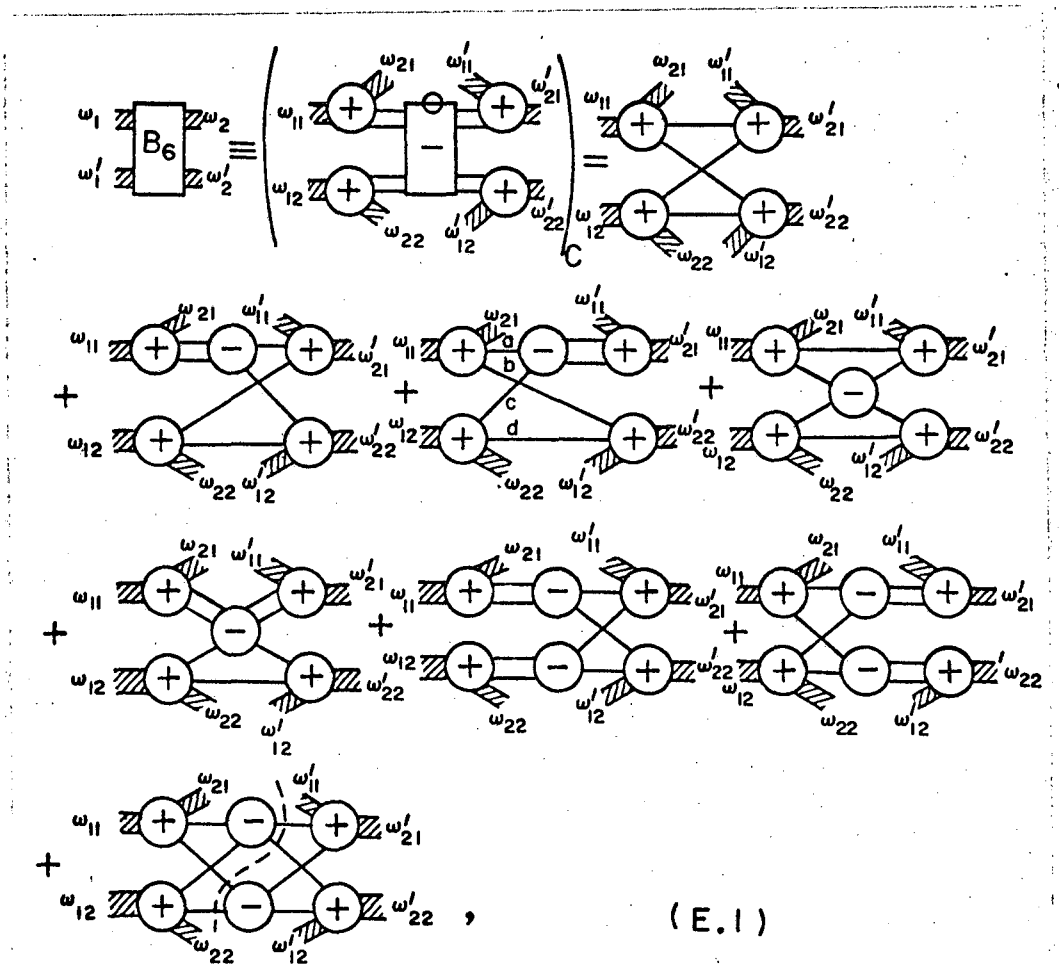
$$\overset{o}{G}{}^{\pm} = \sum_{i} G_{i}^{\pm} - \sum_{i<j} G_{ij}^{\pm} + \cdots + (-1)^{s} \sum_{i<j<\cdots<k} G_{ij\cdots k}^{\pm}$$

$$+ (-1)^{n} G_{12\cdots n}^{\pm} + \overset{o}{G}{}^{\pm 12\cdots n} , \qquad \text{(D.11)}$$

where the $G_{ij\cdots k}^{\pm}$ are of the form indicated in (D.9) and $\overset{o}{G}{}^{\pm 12\cdots n}$ is given by (D.10) with the restriction $k > 1$.

Appendix E. The Double-Cross Term

In this appendix we prove Eq. (6.6). If $E(\omega, \omega')$ is less than the five-particle threshold, then the columns of plus bubbles in the second term on the right of (6.4) each consist of precisely two bubbles, and this term, denoted by B_6, is given by



(E.1)

where topologically equivalent diagrams are to be counted only once.

Only the last bubble diagram on the right of (E.1) can contain a simple positive (ω, ω') cut set C such that $\overline{X}(C, \omega)$ and $\overline{X}(C, \omega')$ are both connected. This follows directly from the properties of cut sets established by Proposition 3.1. In particular, any $D \supset\subset B$ is divided by C into three disjoint sets $\overline{X}(C, \omega)$, $\overline{X}(C, \omega')$ and C', the first two of which must be connected. Since C' contains no minus lines, the part of D contained in any minus bubble must belong to either $\overline{X}(C, \omega)$ or $\overline{X}(C, \omega')$. This precludes the possibility that the other one is connected, in the case of the fourth and fifth terms on the right of (E.1). For the third term the requirement that $\overline{X}(C, \omega)$ be connected implies that either (i) lines a and c belong to $\overline{X}(C, \omega)$, or (ii) lines b and d belong to $\overline{X}(C, \omega)$. The requirement that $\overline{X}(C, \omega')$ be connected implies that either (i') lines a and b belong to $\overline{X}(C, \omega')$, or (i") lines c and d belong to $\overline{X}(C, \omega')$. These conditions are incompatible with the requirement $\overline{X}(C, \omega) \cap \overline{X}(C, \omega') = 0$. Except for the last term, the remaining bubble diagrams of the right side of (E.1) are ruled out in the same way. The last term does not belong to $\mathcal{Q}_c(\omega, \omega')$; the lines intersected by the dotted curve are a simple positive (ω, ω') cut set.

FOOTNOTES AND REFERENCES

* This work was supported by the U. S. Atomic Energy Commission.

1. The first is Joseph Coster and Henry P. Stapp, Physical Region Discontinuity Equations for Many-Particle Scattering Amplitudes. I (scheduled for January publication in J. Math. Phys.). This paper is referred to as I.

2. Henry P. Stapp, Crossing, Hermitian Analyticity, and the Connection Between Spin and Statistics, Lawrence Radiation Laboratory Report UCRL-16816, April 1966 (submitted to J. Math. Physics). Essentially the same results were obtained in Ref. 3, though there without strict observance of the mass shell conditions. A further discussion of the structure theorems and their role in the present work, is given in Section VIII.

3. M. J. Bloxham, D. I. Olive, and J. C. Polkinghorne, S-Matrix Singularity Structure in the Physical Region (Cambridge preprints, June, 1967).

4. Henry P. Stapp, Lectures on S-Matrix Theory (1964), Matscience, Madras, India. See also Lectures on the Analytic Structure of Multiparticle Scattering Amplitudes ICTP 65/17 (1965), Trieste; and D. I. Olive, Nuovo Cimento 37, 1422 (1965).

5. D. I. Olive, Phys. Rev. 135, B745 (1964). J. B. Boyling, Nuovo Cimento 33, 1356 (1964). H. P. Stapp in High Energy Physics and Elementary Particles (IAEA, Vienna, 1965), and Ref. 2.

6. D. I. Olive, Nuovo Cimento 29, 326 (1963). See also R. J. Eden, P. V. Landshoff, D. I. Olive, and J. C. Polkinghorne, The Analytic S-Matrix (Cambridge University Press, 1966). See also J. Gunson, J. Math. Phys. 6, 827, 845, 852 (1965).

7. One can either consider the set of bubble diagrams to be restricted by the conservation laws or, alternatively, allow the diagrams to be restricted only by strictly topological conditions, and note that the corresponding functions will vanish if the conservation laws are not satisfied. It is in the former sense that equations such as (5.1) are strictly true as equivalence relations among diagrams.

8. H. P. Stapp, Phys. Rev. 125, 2139 (1962); J. C. Polkinghorne, Nuovo Cimento 25, 901 (1962).

9. C. Chandler and H. P. Stapp, Macroscopic Causality Conditions and Properties of Scattering Amplitudes, Lawrence Radiation Laboratory report UCRL-17734, June 1967 (submitted to the Physical Review).

10. Henry P. Stapp, Finiteness of the Number of Positive-α Landau Surfaces That Enter Bounded Portions of the Physical Region, J. Math. Physics (in Press).

11. D. Branson, Nuovo Cimento 44, 1081 (1966).

12. Second Ref. 4. The result is well known to Combinatorial Analysts. See John Riordan, An Introduction to Combinatorial Analysis (Wiley, 1958), p. 50.

This report was prepared as an account of Government sponsored work. Neither the United States, nor the Commission, nor any person acting on behalf of the Commission:

A. Makes any warranty or representation, expressed or implied, with respect to the accuracy, completeness, or usefulness of the information contained in this report, or that the use of any information, apparatus, method, or process disclosed in this report may not infringe privately owned rights; or

B. Assumes any liabilities with respect to the use of, or for damages resulting from the use of any information, apparatus, method, or process disclosed in this report.

As used in the above, "person acting on behalf of the Commission" includes any employee or contractor of the Commission, or employee of such contractor, to the extent that such employee or contractor of the Commission, or employee of such contractor prepares, disseminates, or provides access to, any information pursuant to his employment or contract with the Commission, or his employment with such contractor.