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APPLICATION OF CONFORMAL MAPPING TO EVALUATION AND DESIGN OF MAGNETS CONTAINING IRON WITH NONLINEAR B(H) CHARACTERISTICS

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ABSTRACT

It is shown that for evaluation of two dimensional magnets with nonlinear iron in a conformally transformed geometry, only minor changes of the magnetostatic equations are required. The advantages resulting from evaluation or design of a magnet in a suitably transformed geometry are discussed in detail.

* Work performed under the auspices of the U.S. Atomic Energy Commission.

1. Introduction

Conformal mapping is a powerful technique for finding solutions, or for simplifying the process of finding solutions, to Laplace's differential equation in two dimensions, and a large number of applications to many fields can be found in any textbook dealing with this subject. This method has also been applied successfully to the design of long magnets with infinite permeability of the iron that, far away from the ends, can be described with sufficient accuracy by the two dimensional Laplace equation[1,2]. However it does not seem to be generally known that application of a conformal transformation to a two dimensional multipole can greatly simplify the evaluation or design of that type of magnet even when the iron has nonlinear B(H) characteristics. It is the purpose of this paper to point out some of the advantages that result if such a magnet is conformally transformed and then, in the new geometry, evaluated with a digital computer that solves Poisson's equation numerically. To this end, we write first the magnetostatic equations in the original coordinate system in such a way that it will be easy to transform them to the new coordinate system. From the transformed equations we then deduce the modifications that have to be incorporated in the computer code that numerically integrates the normal magnetostatic equations with nonlinear B(H) characteristics. In the discussion of the application of conformal transformations to two dimensional magnets with nonlinear iron, emphasis is on the description of the advantages that result when the magnetostatic equations are solved numerically. However, to give a complete picture, we will also point out some of the generally known benefits associated with conformal transformations when applied to this kind of problem.

2. Magnetostatic Differential Equations in the Original and Conformally Transformed Coordinates

2.1. NOTATION

MKS units are used throughout. Complex numbers and operators are identified by underlining; their complex conjugate by an asterisk. The absolute value of a complex number is indicated by two vertical bars, and its real part by Re. The Cartesian coordinates of the original problem are x and y; the Cartesian coordinates of the transformed problem are u and v, and they are related to x and y through a suitably chosen conformal transformation

$$u + iv \equiv \underline{w} = \underline{w}\,(x+iy) \equiv \underline{w}\,(\underline{z}) \;. \tag{1}$$

Quantities that depend directly on x and y, or are of special significance in the x, y coordinate system, carry the subscript z, and similarly carry the subscript w when they depend directly on u and v, or are of special significance in the u, v coordinate system.

The reason to consider only conformal transformations is the well known fact that the structure of the magnetostatic equations is destroyed under any other than conformal transformations.

2.2. MAGNETOSTATIC DIFFERENTIAL EQUATION IN ORIGINAL COORDINATE SYSTEM

Without loss of generality, we can derive the two components B_x and B_y of the magnetic flux density from a vector potential which has only a component (A_z) perpendicular to the x-y plane. Introducing the complex field quantity

$$\underline{B}_z = B_x + i\,B_y = \frac{d}{dy}A_z - i\,\frac{d}{dx}A_z = -i\left(\frac{d}{dx}A_z + i\,\frac{d}{dy}A_z\right) \tag{2}$$

and the complex operator

$$\underline{D}_z = -i\left(\frac{d}{dx} + i\,\frac{d}{dy}\right), \tag{3a}$$

we obtain

$$\underline{B}_z = \underline{D}_z\,A_z\ . \tag{4}$$

It should be noted that $\underline{D}_z$ acting on any analytical function $\underline{K}(\underline{z})$ is zero:

$$\underline{D}_z\underline{K}(\underline{z}) = 0\ . \tag{5}$$

Assuming an isotropic medium, and introducing γ, the reciprocal of the relative permeability μ (which may depend both on location and flux density),

$$\gamma_z(x,y,\ |\underline{B}_z|) = 1/\mu_z(x,y,\ |\underline{B}_z|)\ ,$$

we obtain for the field components H_x, H_y, and the complex field quantity

$$\begin{aligned} \underline{H}_z &= H_x + i\,H_y: \\ \underline{H}_z &= \frac{1}{\mu_o}\,\gamma_z\,\underline{B}_z = \frac{1}{\mu_o}\,\gamma_z\,\underline{D}_z\,A_z\ . \end{aligned} \tag{6}$$

The magnetostatic equation relating H_x, H_y to the current density j_z in the direction perpendicular to the x - y plane is:

$$j_z(x,y) = \frac{d}{dx}\,H_y - \frac{d}{dy}\,H_x = -\,\mathrm{Re}\left(i\left(\frac{d}{dx} - i\,\frac{d}{dy}\right)\right)\ (H_x + i\,H_y)\ .$$

With eq. (3a) and (6), we obtain therefore for the magnetostatic differential equation in the x - y coordinate system:

$$\mathrm{Re}\ \underline{D}_z^* \gamma_z\ (x,y,\ |\underline{B}_z(x,y)|)\ \underline{D}_z\ A_z\ (x,y) = -\ \mu_o\ j_z\ (x,y) \tag{7}$$

It should be noted that when $\underline{D}_z^*$ acts on γ_z, it acts not only on the explicit dependence of γ_z on x, y, but also on the x, y dependence that results from the dependence of γ_z on $|\underline{B}_z(x,y)|$. This is, of course, the reason why total and not partial derivatives are used in defining $\underline{D}_z$.

2.3. MAGNETOSTATIC DIFFERENTIAL EQUATION IN TRANSFORMED COORDINATION SYSTEM

Introducing new coordinates u, v through the conformal transformation, eq. (1), we can express x and y in A_z (x,y) through u and v, obtaining a new function A_w (u,v). The implicit dependence of A_w on x, y is of course the same as the direct dependence of A_z on x, y:

$$A_z(x,y) = A_w\Big(u(x,y),\ v(x,y)\Big)\ . \tag{8}$$

To obtain the magnetostatic equation in u, v, we express $\underline{D}_z$ through derivatives with respect to u, v. From eq. (3a) we get:

$$\underline{D}_z = -i \left(\frac{du}{dx}\frac{d}{du} + \frac{dv}{dx}\frac{d}{dv} + i \left(\frac{du}{dy}\frac{d}{du} + \frac{dv}{dy}\frac{d}{dv} \right) \right) .$$

Using the Cauchy-Riemann relations:

$$\frac{du}{dy} = -\ \frac{dv}{dx}\ ;\ \frac{dv}{dy} = \frac{du}{dx}\ ,$$

we obtain, with

$$\underline{D}_w = -\ i\ \left(\frac{d}{du} + i\ \frac{d}{dv}\right) \tag{3b}$$

and

$$\underline{w}' = \frac{d\underline{w}}{d\underline{z}} = 1/\underline{z}' :$$

$$\underline{D}_z = \left(\frac{du}{dx} - i\,\frac{dv}{dx}\right) \underline{D}_w = \underline{w}'^* \,\underline{D}_w \,. \tag{9}$$

For the relation between $\underline{B}_w = \underline{D}_w A_w$ and $\underline{B}_z$ we obtain from eq.'s (4), (8), and (9):

$$\underline{B}_z = \underline{w}'^* \,\underline{B}_w = \underline{B}_w/\underline{z}'^* \,. \tag{10}$$

It should be noted that eq. (10) is identical to the transformation formulas that one obtains if one assumes that the fields can be derived from a complex potential function, although that has obviously not been assumed in the derivation of eq. (10).

Using, similarly to (8), and with eq. (10)

$$\gamma_z\,(x,y,\,|\underline{B}_z|) = \gamma_w\,(u(x,y),\,v(x,y),\,|\underline{B}_w|/|\underline{z}'|)\;, \tag{11}$$

we obtain from eq. (7), (8), and (9):

$$\mathrm{Re}\;\underline{D}_z^*\,\underline{w}'^*\gamma_w\,\underline{D}_w\,A_w = -\,\mu_o j_z$$

Because of eq. (5), we can write $\underline{w}'^*$ to the left of $\underline{D}_z^*$. We thus obtain, using eq. (9) again:

$$|\underline{w}'|^2 \cdot \mathrm{Re}\;\underline{D}_w^*\,\gamma_w\,\underline{D}_w\,A_w = -\,\mu_o j_z \,.$$

Introducing

$$j_z(x(u,v),\,y(u,v)) \cdot |\underline{z}'|^2 = j_w(u,v) \tag{12a}$$

we get for the magnetostatic differential equation in the u, v coordinate system:

$$\mathrm{Re}\ \underline{D}_w^* \gamma_w(u,v,\ |\underline{B}_w|/|\underline{z}'|)\ \underline{D}_w\ A_w = -\ \mu_o j_w(u,v)\ . \tag{12b}$$

Of the many quantities that are of interest in the design or evaluation of magnets, we want to discuss only the transformation properties of two more quantities. Although both transformation properties are trivial, they are of such practical importance that it is worthwhile to state them.

Since x, y and u, v are related through a conformal transformation, infinitesimal areas da_w and da_z are related through

$$da_z = |\underline{z}'|^2\ da_w\ . \tag{13}$$

With eq. (12a) we obtain therefore for the total current I_w passing through any given area in the u, v coordinate system:

$$I_w = \int j_w\ da_w = \int j_z |\underline{z}'|^2\ da_w = \int j_z\ da_z = I_z\ ,$$

i.e. total currents passing through conformally mapped areas are identical.

For the field energy E stored in any given area per unit length of magnet, we obtain from $E_w = \int\ |\underline{B}_w|^2\ \gamma_w\ da_w/2\mu_o$ and eq.'s (10), (11), and (13)

$$2\mu_o\ E_w = \int\ |\underline{B}_z|^2\ \gamma_z\ |\underline{z}'|^2\ da_w = \int\ |\underline{B}_z|^2\ \gamma_z\ da_z = 2\mu_o\ E_z\ ,$$

i.e. conformally mapped areas store equal field energy per unit magnet length.

2.4. MAGNETOSTATIC COMPUTER CODE MODIFICATIONS FOR INTEGRATION OF EQ. 12

Comparing eq.'s (7) and (12), one notices two differences:

a. The current density j_w appearing in (12b) is related to the current density j_z through eq. (12a). The proper current density j_w can obviously be obtained by modifying the input data according to eq. (12a). Since in most practical magnets, the current density j_z is constant within the boundary of each conductor, it is convenient to add a small subroutine that allows to input $\underline{w}(\underline{z})$ and $|\underline{z}'|^2$, and the boundary and current density j_z for each conductor, and that then prepares the input data to give the correct j_w. That same routine can of course also be used to transform all other boundaries from the x, y coordinates to the u, v coordinates. Since this routine does not interact with the integration routine, it is clearly a simple task to add this routine to the program.

b. The major discrepancy between eq.'s (7) and (12b) is that γ_z depends on $|\underline{B}_z|$, whereas γ_w depends on $|\underline{B}_w|/|\underline{z}'|$. It would be an easy task to store $1/|\underline{z}'|$ for each iron mesh zone and then to multiply each flux density value by $1/|\underline{z}'|$ before finding the value of γ_w when integrating eq. (12b). However, since in most integration routines, the value for the flux density in a mesh zone is derived by appropriate numerical procedures from potentials at mesh points surrounding that zone, it will in general be easy to modify the algorithm so that it gives $|\underline{B}_w|/|\underline{z}'|$ instead of $|\underline{B}_w|$. Both this and the above mentioned modifications were incorporated into POISSON[3]) with very little effort and without increasing the storage requirements or the execution time for evaluation of magnets.

Although we used vector potentials for the derivation of eq. (12), the resulting conclusions concerning the necessary modifications of j and γ are, of course, independent of the method that was used to derive them, and are valid no matter what algorithm is used to actually integrate the magnetostatic equations.

3. Consequences of Application of Conformal Transformation to Evaluation of Iron Magnets

3.1. INTRODUCTORY REMARKS

When discussing the advantages resulting from using conformal transformations in conjunction with a magnet analysis program, we will talk especially about the analysis program POISSON[3]. Although some comments apply only to POISSON, or a program similar to it, most remarks are valid no matter what analysis program is used. Also, we will talk mostly about evaluation or design of quadrupoles, since their discussion is representative for all higher multipoles.

3.2. CONFORMAL TRANSFORMATION OF A QUADRUPOLE

To provide a good quadrupole field in a circular aperture, it is desirable to build a magnet with the highest degree of symmetry possible. Figure 1 shows the schematic outline of 1/8 of such a magnet, with the 0° and 45° lines being lines of constant scalar and vector potentials respectively. Within the aperture, the field $\underline{B}_z$ can be derived from a complex potential

$$\underline{F}(\underline{z}) = A_z + i\,V_z$$

and because of the symmetry of the magnet shown in Fig. 1, the power series for $\underline{F}(\underline{z})$ has to have the form

$$\underline{F}(\underline{z}) = \sum_{n=0}^{\infty} a_{2(2n+1)} \; \underline{z}^{2(2n+1)} \; . \tag{14a}$$

From this follows for the fields:

$$\underline{B}_z^* = i \; \underline{F}'(\underline{z}) = 2 \; i \sum_{n=0}^{\infty} (2n+1) \; a_{2(2n+1)} \; \underline{z}^{4n+1} \; . \tag{14b}$$

The transformation

$$\underline{w} = \underline{k} \; \underline{z}^2 \tag{15a}$$

leads to

$$\underline{F}_w(\underline{w}) = \sum_{n=0}^{\infty} a_{2(2n+1)} \cdot (\underline{w}/\underline{k})^{2n+1} \tag{16a}$$

and

$$\underline{B}_w^* = \frac{i}{\underline{k}} \sum_{n=0}^{\infty} (2n+1) \cdot a_{2(2n+1)} \; (\underline{w}/\underline{k})^{2n} \; . \tag{16b}$$

When the magnet is a good quadrupole magnet, in the aperture the term proportional to a_2 dominates in eq. (14), and therefore dominates also in eq. (16). But eq. (16) then describes an essentially homogeneous field in the aperture, and this is of course highly desirable for evaluation as well as poleface design of a magnet. Before discussing the resulting advantages in detail, it is convenient to introduce a particular value for the scale factor $\underline{k}$ in eq. (15a).

To get simple relations for saturation considerations, we introduce r_o as the distance from the center of the original magnet to the iron nearest the center and require that for $|\underline{z}| = r_o$, $|\underline{w}'| = 1$, giving $|\underline{B}_z| = |\underline{B}_w|$ there.

Using for simplicity a real k, we thus get from eq. (15a) for $|\underline{z}| = r_o$: $|\underline{w}'| = 2k\,r_o = 1$.

Using this in eq. (15a) we obtain

$$\underline{w} = \rho_o\,(\underline{z}/r_o)^2\,;\quad \rho_o = r_o/2 \tag{15b}$$

and

$$\underline{w}' = (\underline{z}/r_o) = (\underline{w}/\rho_o)^{1/2}\,. \tag{15c}$$

For a 2n-pole, one would use similarly:

$$\underline{w} = \rho_o\,(\underline{z}/r_o)^n\,;\quad \rho_o = r_o/n$$

$$\underline{w}' = (\underline{z}/r_o)^{n-1} = (\underline{w}/\rho_o)^{1-\frac{1}{n}} \tag{15d}$$

Applying the transformation described in eq. (15b) to the magnet shown in Fig. 1 leads to the configuration shown in Fig. 2, which is drawn to the same scale as Fig. 1.

When evaluating a quadrupole magnet, the quantity of interest is usually the gradient of the field. From eq.'s (10) and (15c) we obtain

$$\underline{B}_z^* = \underline{B}_w^*\,\underline{z}/r_o = \underline{B}_w^*\,(\underline{w}/\rho_o)^{1/2} \tag{17a}$$

$$\underline{B}_w^* = r_o\,\underline{B}_z^*/\underline{z}\,. \tag{17b}$$

From eq. (17b) follows the $\underline{B}_w^*$ is directly a measure for the gradient in the real magnet even if the quadrupole is not perfect. To obtain the local gradient in the aperture region, we can differentiate eq. (17a) with respect to $\underline{z}$:[†]

$$\frac{d\,\underline{B}_z^*}{d\underline{z}} = \left(\underline{B}_w^* + 2\,\underline{w}\cdot\frac{d\,\underline{B}_w^*}{d\underline{w}}\right)/r_o \qquad (17c)$$

It is clear that if the magnet is a good quadrupole magnet, the derivative on the right side of eq. (17c) contributes very little to the local field gradient inside the good field aperture.

3.3. ADVANTAGES OF TRANSFORMED MAGNETS

The most obvious reason for gaining advantages through conformally transforming a magnet is, of course, the same reason why conformal mapping has been used advantageously for a long time in many fields: The simplifications of the geometry make many aspects of a problem so transparent that they become outright trivial, whereas they are often quite obscure in the original geometry. For instance, it is quantitatively much easier to see what kind of an effect a modification of the magnet near the useful field aperture has on the field of an essentially homogeneous-field magnet than it is to see what the effect of the equivalent modification is on the gradient of a quadrupole magnet. Or, to take a specific drastic case: if a sextupole magnet has a circular useful field aperture r_o, and a

[†] This is, of course, correct only where $\underline{B}_z^*$ can be derived from a complex potential, i.e. where γ = const. and j = o.

significant modification of the magnet is made at the distance $3r_o$ from the center, it is not entirely obvious what its effect is on the second derivatives of the field. However, if the circular aperture of the transformed magnet is ρ_o, the modification in this geometry is then at the distance $27 \cdot \rho_o$ and it is obvious that this will have very little effect on the homogenity of the field inside ρ_o, allowing the conclusion that the modification will also have very little effect on the second derivatives of the field in the original magnet. From these considerations follows that the task of designing a multipole magnet with a reasonably pure multipole field in the useful field aperture becomes greatly simplified through a conformal transformation, particularly since one can apply many of the fairly simple and well understood rules that one has for the design of homogeneous field magnets.

To demonstrate this in more detail we consider the design of a quadrupole magnet that is required to have a good quadrupole field within an elliptical aperture. Figure 3 shows one quarter of an aperture ellipse with a ratio of major to minor axis of 2.5, with a rough outline of one quarter of the magnet poleface also indicated. To see how far the poleface has to be carefully designed, we apply the transformation $\underline{w} = k\, \underline{z}^2$, mapping the 1/4 - ellipse of Fig. 3 into the 1/2 - ellipse in Fig. 4. Since we know that in order to obtain a homogeneous dipole field, the poleface should extend at least one quarter of the magnet gap beyond the ends of the aperture region, we also indicate the required poleface width. It is also shown how far the poleface would have to go on the left side in order to get a quadrupole that is symmetrical with respect to the 45° - line (in the original geometry), with the resulting better field quality because of the higher degree of symmetry. Since conformally mapped areas store the same energy per unit magnet length, it is directly evident how much one pays in terms of stored energy for the magnet with the higher symmetry. After fixing the ends of the polefaces in the described manner, one would then transform these endpoints into the original geometry and design the rest of the magnet structure (coils, yokes, etc.) in the original geometry. Then, as the last step, after transforming the whole magnet and generating a mesh in the new geometry, one would evaluate the magnet in the transformed geometry and optimize the poleface to give a highly homogeneous field in the transformed geometry, leading to a high quality quadrupole field in the original geometry.

There are, of course, some basic differences between true homogeneous field magnets and homogeneous field magnets that are obtained through conformal transformation of a multipole magnet. In most magnets the coils have a uniform current density and the air-coil and most air-iron interfaces are straight lines (the magnet shown in Fig. 1 is an exception in this respect). This is of course no longer true after a multipole magnet has been transformed. Although this has generally very little effect on the design of the aperture region, it means for instance that one can not obtain a practical multipole magnet by transformation of a window frame magnet.

A more significant difference arises when one considers saturation effects. When one designs a homogeneous field magnet and other considerations, such as stored field energy, do not preclude such a conservative design, one can get very good field homogeneity over a wide field range by extending the flat poleface significantly beyond the aperture limits. Doing the same in the case of a transformed multipole would lead to a badly saturating magnet because, according to eq.'s (12b) and (15d), the quantity determining the saturation in iron is $|\underline{B}_w| \cdot |\underline{w}/\rho_o|^{1-\frac{1}{n}}$. With $|\underline{B}_w|$ in the poleface region essentially constant, the factor $|\underline{w}/\rho_o|^{1-\frac{1}{n}}$ will lead to stronger saturation effects the more the poleface is extended beyond the aperture limit. Numerically, this effect can be quite significant: if the total width of the symmetrical poleface of a transformed quadrupole is $3\rho_o$ in one case, and $4\rho_o$ in another, the values for $|\underline{w}/\rho_o|^{1/2}$ at the ends of the poleface are 1.344 and 1.5 = 1.344 · 1.11. This points out that in order to design a multipole magnet with a good field distribution at low as well as at high fields, it is exceedingly important to be able to achieve good field distributions with as little iron beyond the aperture limits as possible. For this reason, a performance optimization procedure that allows one to optimize the field distribution simultaneously at low and high fields[4]) is even more important for the design of multipole magnets than it is for the design of dipole magnets.

While the presence of the above mentioned saturation effects is obvious without application of a conformal transformation, their qualitative and quantitative discussion and evaluation is considerably easier in the transformed geometry.

Again with respect to this subject, one gains a better understanding through considering the transformed magnet. The resulting simplifications of the design process will of course in many cases lead to improved design and performance of magnets.

While the advantages discussed so far are to a large degree of a qualitative nature, evaluation of multipole magnets in the transformed geometry can also lead to a very significant increase in accuracy. Magnet evaluation codes usually compute potentials at a large number of discrete mesh points that cover the geometry of the magnet. In the algorithm for the calculation of the potentials, the behaviour of the potential in the region around mesh points is generally approximated by polynomials in the coordinates. These polynomials are, to cite two examples, of first order in POISSON, and second order in SYBIL[5]. This basic difference is due to the meshes employed in these two types of programs, SIBYL using a uniform rectangular mesh and POISSON a variable triangular mesh. This gives POISSON the advantage of being more flexible and therefore having virtually no restrictions on the boundaries of the problem and between different materials, at the expense of being less accurate. While these inaccuracies are of very little significance far away from the useful field aperture, they can be of importance in the aperture if the field there is highly inhomogeneous. By evaluating a multipole magnet in the transformed geometry, the aperture field will be very homogeneous for a well designed magnet. Consequently, the potentials are nearly exactly linear functions of the coordinates, thus practically eliminating this source of error. Furthermore, the local field gradients in the original geometry are essentially given by the field in the transformed geometry. This means that in order to obtain the local field gradient in the aperture of a quadrupole, one has to calculate second derivatives of relatively inaccurate potentials if the evaluation is done in the original geometry, whereas one has to take essentially only first derivatives of very accurate potentials if the evaluation is done in the transformed geometry. The cascading of these two main accuracy-improving properties lead to a very significant improvement of overall accuracy: Evaluating a magnet that has an analytical quadrupole field distribution with POISSON gave the gradients in the aperture region with about 1% errors when the evaluation was done in the original geometry, whereas the error was only 0.01% when evaluated in the transformed geometry. While 0.01% accuracy is better than normally needed, 1% errors are more than tolerable in many cases. It is clear that the accuracy improvement is even more urgently needed (and obtainable with this procedure) for higher multipoles, where even an intrinsically more accurate program like SIBYL could not be expected to be quite as accurate as one would need under some circumstances. A minor advantage results when a multipole magnet is to be evaluated with an irregular variable mesh program like POISSON and the evaluation is done in the transformed geometry: Because of the curvature of the poleface in the original geometry, it is very difficult to generate a good mesh point distribution, while it is very easy to generate a practically perfect mesh in the aperture region of the transformed magnet.

Finally, it might be worthwhile to remark that it is possible to check internal consistency and accuracy of a program by computing potentials and fields of a magnet in the original and a conformally transformed geometry, and then comparing the results.

4. Limitations and Drawbacks

Although it is possible to evaluate a transformed magnet geometry that covers more than one leaf of a Riemann surface, this is clearly neither desirable nor practical. Therefore a magnet geometry should be sufficiently symmetrical so that the transformed magnet covers only 360° or less of a plane. While most multipole magnets satisfy this condition, one has to realize that for all practical purposes this makes it impossible to evaluate in the transformed geometry the effects of slight assembly-asymmetries of a basically symmetric magnet.

It is clear that by transforming a 2n-pole with $\underline{w} \sim \underline{z}^n$, the ratio of the aperture area to total magnet area is much smaller in the transformed geometry than it is in the original geometry, leading to a reduced mesh point density in the aperture of the transformed magnet. When using a variable mesh code, this can be partly corrected by adjusting the mesh spacing accordingly; with a fixed mesh code, one gains a small advantage because the fraction of the magnet that has to be evaluated is generally larger in the original geometry than it is in the transformed geometry. (For instance, one has to evaluate 1/4 of a symmetrical sextupole in the original geometry, but only 1/12 in the transformed geometry.) Depending on the details of the magnet under consideration, sometimes neither one of these gains is enough to compensate sufficiently the reduced mesh point density in the aperture region of the transformed magnet. We want to discuss briefly two methods that can be used to improve the mesh point density in the aperture.

Instead of using the transformation that produces exactly the desired mapping, one can use one that gives the desired mapping in the aperture region in very good approximation, and compresses the transformed magnet far away from the aperture.

For instance, instead of using the really desired transformation $\underline{w} = \frac{r_o}{n} (\underline{z}/r_o)^n$ for a 2n-pole, one can use the transformation

$\underline{w} = \frac{r_o}{n\epsilon} \ln (1 + \epsilon(\underline{z}/r_o)^n)$ with $\epsilon \ll 1$. In the aperture region $(|\underline{z}|/r_o \leq 1)$ the latter transformation gives approximately the same mapping as the former, whereas for $\epsilon(|\underline{z}|/r_o)^n \geq 1$ they differ markedly, making the overall size of the second transformed magnet relative to its aperture smaller than the first. The slight deviation from the transformation $\underline{w} = \frac{r_o}{n} (\underline{z}/r_o)^n$ should not decrease the evaluation accuracy in the aperture if ϵ is not too large; also all the gains of qualitative nature described at the beginning of this section are preserved. Since the exact form of the used transformation is known, it is of course very easy to take the difference between it and the transformation $\underline{w} \sim \underline{z}^n$ quantitatively into account. In magnets with extreme dimensions one could even think of using the transformation:

$$\underline{w} = \frac{r_o}{\epsilon n} \ln [1 + \ln (1 + \epsilon(\underline{z}/r_o)^n)] .$$

A somewhat simpler procedure would be to evaluate the magnet in two steps: First in the original geometry, giving the overall potential distribution and all the gross-saturation characteristics, but very poor accuracy in the aperture region. In the second step one evaluates, in the "ideally" transformed geometry, only a part of the magnet, extending, in the transformed geometry, from the center to about 5-20 times the aperture dimension. Depending upon whether or not this region contains coils, one can then evaluate this partial magnet with boundaries parallel or perpendicular to field lines calculated in the first step, or with boundary values of the potentials obtained in the first step. Since the boundaries are far removed from the aperture region, the accuracy of the evaluation in the aperture region will be practically independent of the choice of the boundaries or the boundary values. If the saturation behaviour is of no interest, it will in most cases not even be necessary to make the first evaluation since one can guess in general with sufficient accuracy what one has to do at the boundaries.

Refering to eq.'s (10) and (12a), it is evident that $\underline{z}' \neq o$; $\underline{w}' \neq o$ has to hold in all regions containing iron and $\underline{w}' \neq o$ has to hold in coil regions of magnets. In the rare cases where one would like to use a transformation that violates these conditions, it is usually possible to modify the ideally wanted transformation such that it still gives essentially the desired mapping in the area of interest, but avoids the violation of the above mentioned conditions, just as was suggested to compress the outside portions of a mapped magnet.

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FIGURE CAPTIONS

1. 1/8 of quadrupole in original geometry.
2. 1/8 of quadrupole in transformed geometry.
3. 1/4 of aperture ellipse and poleface in original geometry.
4. 1/4 of aperture ellipse and poleface in transformed geometry.

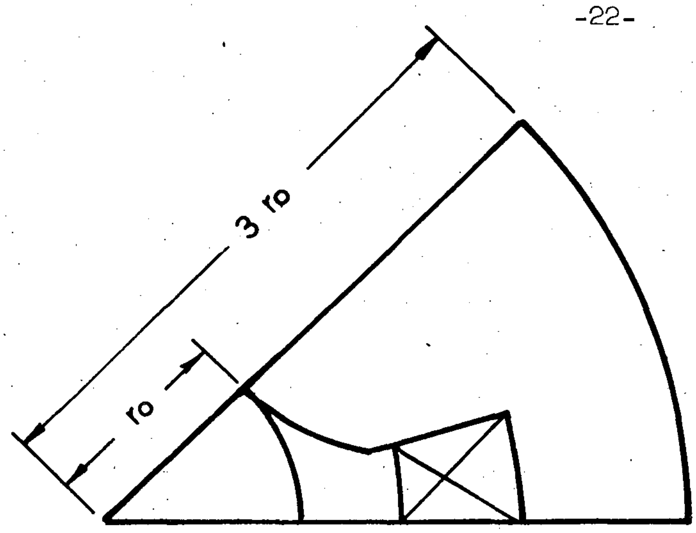

Fig. 1

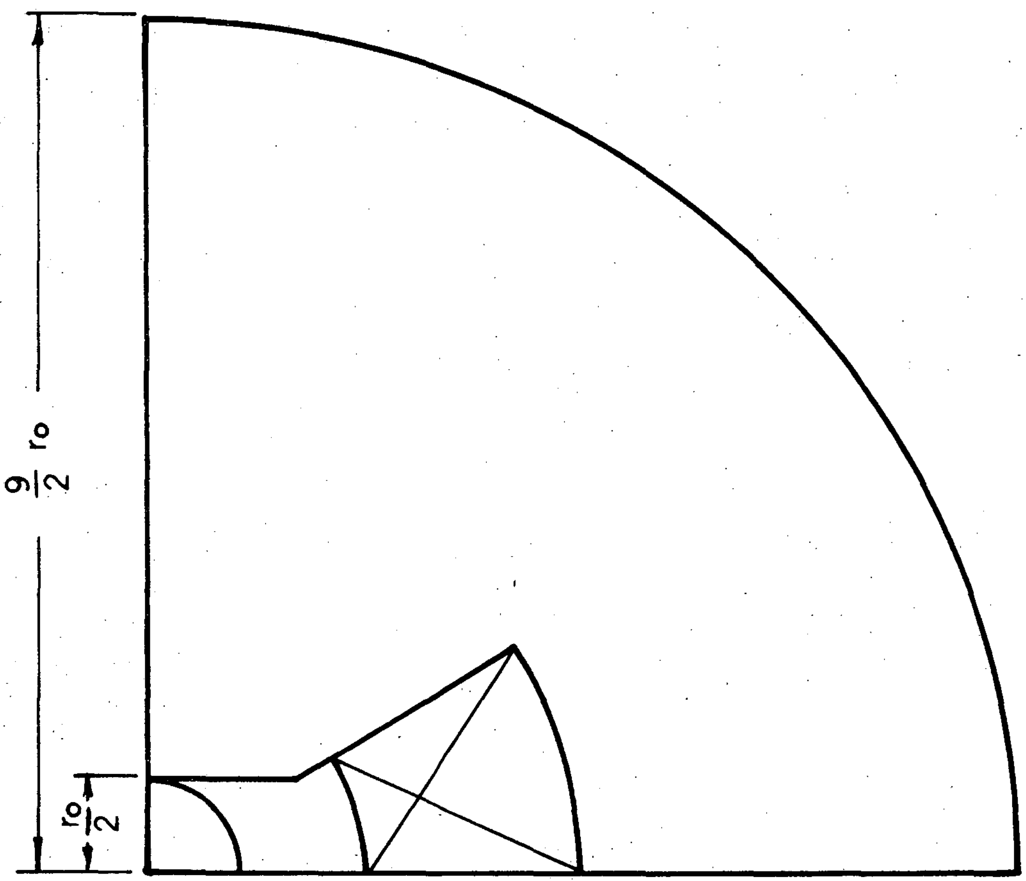

Fig. 2

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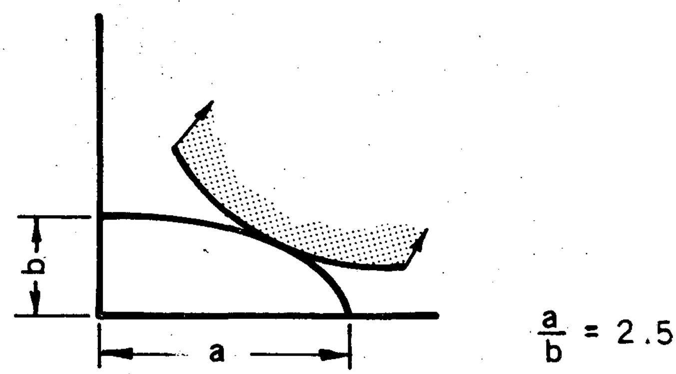

Fig. 3

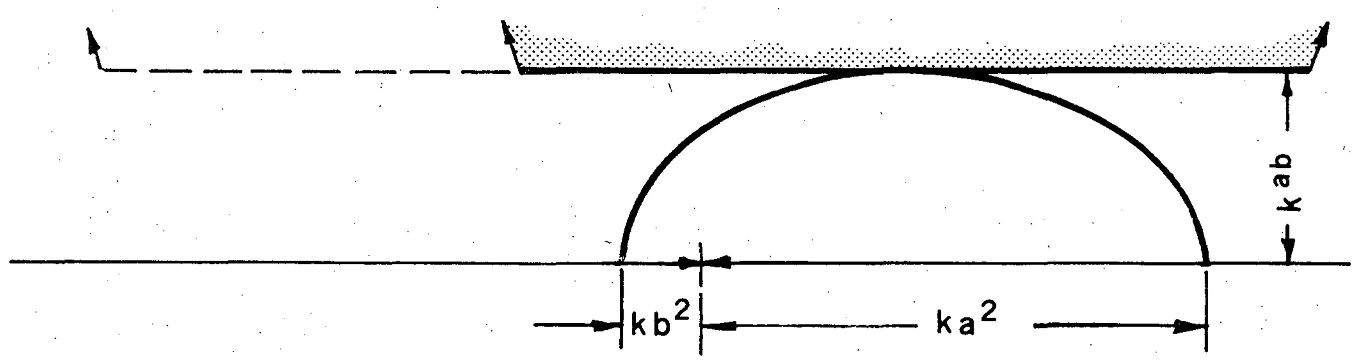

Fig. 4

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