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NUMERICAL SOLUTIONS OF THE BETHE-SALPETER EQUATION

William B. Kaufmann

May 1, 1968

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TABLE OF CONTENTS

Abstract v

I. Introduction 1

II. Theory 4

A. Review of Fundamental Equations 5

B. Angular Separation; Form of the Spectrum. 9

C. Reality Properties of 11

D. Choice of Trial Functions 16

E. Variational Principles. 18

F. Matrix Elements: Rayleigh-Ritz. 25

G. Matrix Elements: Schwinger. 29

H. Example: One Trial Function 35

III. Numerical Results 38

A. Bound States. 38

B. Scattering States 44

IV. Two Channel Calculation 51

V. Summary and Outlook. 63

Acknowledgements. 65

Appendix I. 66

Appendix II 68

Appendix III. 70

Appendix IV 72

Footnotes and References. 80

Figure Captions 85

Figures 89

Tables. 112

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ABSTRACT

After summarizing the application of the Rayleigh-Ritz and Schwinger variational principles to the unequal mass ϕ^3 Bethe-Salpeter equation we present, in graphical and tabular form, the solution of the bound-state problem. The dependence of the coupling-parameter eigenvalue on the exchange mass, external mass ratio, and binding energy is examined in detail for s and p ground states. Mixing of excited levels leading to complex solutions is briefly studied, and some Regge trajectories are also calculated. Scattering phase shifts for unequal-mass scattering have been calculated and representative examples given. The fact that certain levels do not appear to contribute to Levinson's theorem is also examined. Finally the foregoing methods are generalized to two-channel systems, and channel phase shifts and inelasticities are computed.

I. INTRODUCTION

Perturbation theory has been the single most powerful tool in analyzing electromagnetic scattering processes from the point of view of field theory. On the other hand, in strong-interaction processes, the coupling strength of the interaction Hamiltonian may be large enough that bound states occur, in which case the perturbation series diverges and this approach fails. One solution to this problem is to reduce the coupling strength until the perturbation series converges, express the series in closed form, and analytically continue the result as a function of the coupling constant to the desired value. In terms of potential theory this corresponds to expressing the Born series (inside its domain of convergence) in the form of an (Schrödinger) integral equation. The integral equation can then be continued in coupling strength to values for which the Born series diverges. This trick can be used to sum the perturbation series of field theory, and leads to the Bethe-Salpeter equation (BSE),[1] a completely relativistic wave equation describing two interacting particles.

In a classic paper,[2] Wick reduced the ladder approximation of the BSE to Fredholm form and showed that it possessed a discrete coupling-parameter (λ) spectrum at fixed energy below the elastic threshold. Even in the ladder approximation the equation still has two nontrivial independent variables, and hence was considered somewhat intractible. Upon setting the exchange quantum mass (M) equal to zero, Wick[2] and Cutkosky[3] were able to reduce the equation to a

one-dimensional form, from which they obtained a clear picture of the spectrum. Using a form of the Rayleigh-Ritz variational principle, Schwartz[4] numerically solved the more general equation with M nonzero (for the lowest four levels). He restricted the calculation to a system for which the external particles have equal mass for simplicity, but his method is easily generalized to unequal-mass systems also. In a subsequent paper[5] written in collaboration with Charles Zemach, the Schwinger variational method was adapted to the BSE and was solved for scattering phase shifts as well as for bound states.

The methods of solution of the BSE given in Refs. 4 and 5 have the following in common. Both begin with stationary expressions for λ (or the phase shift δ) in terms of a trial function $\psi_T(x)$. The trial function is expanded in a complete set of functions; this reduces the variational problem to a linear matrix equation in the expansion coefficients. The set of equations is truncated, and the eigenvalues are found by standard matrix techniques with the aid of an electronic computer. Successively larger matrices are used to compute λ. The resultant sequence of λ's gives an indication of the convergence of the sequence of approximations.

Our basic aim in this paper is to extend the numerical calculation by Schwartz and Zemach to systems of unequal mass and to provide a "dictionary" of s-wave solutions in graphical and tabular form. The complete range of mass ratios for the external particles was examined; the mass M of the exchanged quantum was varied from 0.125 to 10, where the units are defined by fixing the elastic threshold at 2.

Also p-wave states were examined for $M = 1$. Since the variational solution also gives us excited-state solutions at little extra cost, we have included some examples of these. In response to a suggestion by Daniel Z. Freedman, we found that the Rayleigh-Ritz method could be interpolated in the angular momentum, thus allowing the calculation of Regge trajectories in the region below threshold. Some examples of trajectories are included. Next, we include elastic phase shifts for unequal-mass ϕ^3-type interactions and also for (cutoff) ϕ^4 interactions. In the final section we consider the extension to two (two-body) channels. In a later article we hope to use the variational wave functions to calculate some form factors for the interaction of the Bethe-Salpeter atom with a weakly interacting scalar field.

II. THEORY

The systems discussed in this article may be described in terms of three basic quantum fields. Fields ϕ_1 and ϕ_2 generate particles of rest mass m_1 and m_2 which are to be bound together. Field ϕ_3 describes a quantum of mass M whose exchange provides the binding forces. The interactions of these fields to be considered here are

$$H_A(x) = g_{A1} : \phi_1(x)\phi_1(x)\phi_3(x): + g_{A2} : \phi_2(x)\phi_2(x)\phi_3(x): \tag{2.1}$$

$$H_B(x) = g : \phi_1(x)\phi_2(x)\phi_3(x): \quad . \tag{2.2}$$

These fields are taken to be scalar and without any internal quantum numbers. It is essentially a trivial point to include an internal symmetry superstructure, and so long as the symmetry is perfect, this amounts only to making the coupling parameter in the BS equation a function of the Casimir operators of the symmetry group (Appendix II). Another interaction which has been briefly studied is

$$H(x) = g : \phi(x)^4: \quad . \tag{2.3}$$

In the following subsections we review the basic equations and notation, following closely Refs. 4 and 5. This is followed by a brief discussion of the bound-state spectrum and some comments about complex solutions. Next the variational principles used in the numerical calculations are introduced and the trial functions chosen. Finally the calculation of the matrix elements is briefly discussed along with their Regge continuations. The section closes with an example

computation using one trial function.

A. Review of the Fundamental Equations and Notation

Two-particle systems are most conveniently expressed in terms of center-of-mass coordinates, since in an isolated system translational invariance allows us to express the dynamics in terms of the relative coordinate only. First define the coordinates (x_1, x_2) as the positions of the particles of masses (m_1, m_2). In terms of these coordinates and their corresponding momenta (p_1, p_2) we define a set of total (X, P) and relative (x, p) variables by

$$P = p_1 + p_2 , \qquad X = \mu_1 x_1 + \mu_2 x_2 ,$$
$$p = \mu_2 p_1 - \mu_1 p_2 , \qquad x = x_1 - x_2 . \tag{2.4}$$

The parameters μ_i are restricted by $\mu_1 + \mu_2 = 1$ in order to leave volume elements unchanged. This definition places the "center of mass" X at an arbitrary point on the line connecting points x_1 and x_2. Natural choices from the point of view of nonrelativistic quantum mechanics are

$$\mu_i = m_i/(m_1 + m_2) \quad \text{or} \quad \mu_i = \omega_i/(\omega_1 + \omega_2) , \tag{2.5}$$

where $\omega_i = (\underset{\sim}{k}^2 + m_i)^{\frac{1}{2}}$ and $E = \omega_1 + \omega_2$ equals the total c.m. energy; however, there is no need to commit ourselves to a choice now.

We work in a four-dimensional space where components $x = (\underset{\sim}{x}, x_4)$ are taken as real, hence it is useful to define spherical coordinates R, θ through[4]

$$\underset{\sim}{x} = R \sin\Theta \; \hat{\underset{\sim}{e}}(\vartheta,\varphi), \qquad \tau(=x_4) = R \cos\Theta, \tag{2.6}$$

where $\hat{\underset{\sim}{e}}(\vartheta,\varphi)$ is a unit three-vector in direction (ϑ,φ). In this coordinate system the volume element d^4x is

$$R^3 dR \sin^2\Theta \, d\Theta d\Omega_2, \quad \text{with } d\Omega_2 = \sin\vartheta \, d\vartheta \, d\varphi. \tag{2.7}$$

We will need also the "unit four-vector" $\hat{\underset{\sim}{x}} = (\underset{\sim}{x}/R, x_4/R)$. The analogous formulas in momentum space will be distinguished by a subscript "p." As a differential operator we define

$$p_{op} = -i\underset{\sim}{\nabla} \quad \text{and} \quad p_{4\,op} = -i(\partial/\partial x_4) \; . \tag{2.8}$$

The Bethe-Salpeter wave function in ladder approximation satisfies the integral equation

$$\psi_{kK}(x_1,x_2) = \psi_{kK}^{\,o}(x_1,x_2) + \lambda \int G_A(x_1-x'_1)G_B(x_2-x'_2)V(x'_1-x'_2) \times \psi_{kK}(x'_1-x'_2)d^4x'_1 d^4x'_2 \; , \tag{2.9}$$

Where (A,B) equals $(1,2)$ for direct and $(2,1)$ for exchange ladders (Fig.1). The essential feature of the exchange ladder is the switching of particles 1 and 2 at each successive rung, and occurs whenever particles 1 and 3 carry a conserved "charge" which particle 2 does not possess. The rungs are V, a sum of irreducible Feynman graphs, and the sides are free-particle propagators G_i defined by

$$(p_i^{\,2} + m_i^{\,2})G_i(x) = \delta^4(x) \quad \text{(causal boundary conditions).} \tag{2.10}$$

We will take

$$V(x) = \frac{1}{\pi^2}\int d^4q \, \frac{\exp(-iq\cdot x)}{q^2+M^2} \underset{R^2>0}{=} \frac{4M}{R} K_1(MR) \, , \tag{2.11}$$

which corresponds to the one-particle exchange potential. The inhomogeneous term

$$\psi_{kK}{}^{o}(x_1,x_2) = \exp(ik_1\cdot x_1 + ik_2\cdot x_2) = \exp(ik\cdot x + iK\cdot X) = \psi_k{}^{o}(x)\exp(iK\cdot X) \tag{2.12}$$

represents the incoming plane wave. If we are dealing with bound states, the inhomogeneous term is absent, hence the wave function depends only on K, the total c.m. energy.

Equation (2.9) may be reduced to a differential equation by use of (2.10):

$$(p_1{}^2 + m_1{}^2)(p_2{}^2 + m_2{}^2)\psi_{kK}(x_1,x_2) = \lambda V(x_1-x_2)\, \psi_{kK}(x_A,x_B) \, . \tag{2.13}$$

Translational invariance implies

$$\psi_{kK}(x_1,x_2) = \psi_{kK}(x) \exp(i\, K\cdot X) \, , \tag{2.14}$$

and using the identity

$$\mu_1 x_2 + \mu_2 x_1 = X + (\mu_1 - \mu_2)x \quad , \tag{2.15}$$

we find

$$L(p)\psi_{kK}(x) = D_1^+(p)D_2^-(p)\psi_{kK}(x) = \lambda V(x)\psi_{kK}(x) \tag{2.16}$$

for nonexchange and

$$L(p)\psi_{kK}(x) = \lambda\hat{V}(x)\hat{\psi}_{kK}(x) \tag{2.17}$$

for exchange ladders.[6] In the foregoing expressions we have

$$D_i^{\pm}(p) = (p \pm \mu_i K)^2 + m_i^{\,2} - i\epsilon \,, \qquad \hat{\psi}(x) = \psi(-x) \,, \tag{2.18}$$

$$\hat{V}(x) = \exp\left[i(\mu_1-\mu_2)x\cdot K\right] \; V(x), \tag{2.19}$$

and the p's are interpreted as differential operators.

Equations (2.16) and (2.17) [with boundary conditions implicit in (2.9)] are equivalent to the integral equations

$$\psi_{kK}(x) = \overset{\circ}{\psi}_{kK}(x) + \int G_K(x-x') \; \hat{V}(x') \; \hat{\psi}_{kK}(x') \; dx' \tag{2.20}$$

for exchange (and, with "hats" omitted, for nonexchange) processes, where we have defined

$$G_K(x) = \int \frac{d^4p}{(2\pi)^4} \; \frac{\exp(i\; p\cdot x)}{D_1^+(p)\; D_2^-(p)} \equiv \int \frac{d^4p}{(2\pi)^4} \exp(i\; p\cdot x) \; \tilde{G}_K(p). \tag{2.21}$$

The two-body relative Green's function G_K is reducible to the single integral

$$G_K(x-x') = \frac{i}{8\pi^2} \int_{-\omega_2}^{\omega_1} \frac{d\beta}{E} \; \exp\left[i\beta(t-t')\right] \; K_o(QR) \quad , \tag{2.22}$$

where

$$\omega_i = (\underset{\sim}{k}^2 + m_i^{\,2})^{\frac{1}{2}} \quad \text{and} \quad Q = (\beta^2 - k^2)^{\frac{1}{2}} \quad ,$$

as is derived in Ref. 5. This function is continuous except for a logarithmic singularity at $R = 0$ (i.e., on the light cone).

In the remainder of this article we deal with the continuation of the preceding integral equations to the region with $p_4 (=ip)$ and $x_4 (=ix_o)$

real. With the choice $\mu_i = \omega_i/E$ it is shown in Section IIC of Ref. 5 that this continuation presents little difficulty when $E < m_1 + m_2 + M$ for nonexchange and (using the same reasoning but adding the extra exponential) $E < 2m_1 + M$ for exchange processes, where $m_L = \min(m_1, m_2)$. We look briefly at a different choice of μ_i in Section G.

B. The Angular Separation and General Form of the Spectrum

If $P = (Q, iE)$, the differential operator $L(p)$ becomes

$$\left[p^2 + 2\mu_1 E(ip_4) + \beta_1\right]\left[p^2 - 2\mu_2 E(ip_4) + \beta_2\right] \tag{2.23}$$

and equals

$$p^4 - (\beta_1 + \beta_2)(ip)^2 - 4\mu_1\mu_2 E^2(ip_4)^2$$

$$- 2E(\mu_1 - \mu_2)(ip)^2 (ip_4) + 2E(\mu_1\beta_2 - \mu_2\beta_1)(ip_4) + \beta_1\beta_2 , \tag{2.24}$$

where $\beta_i = m_i^2 - (\mu_i E)^2$. Both the potential functions V and L are rotational invariants, so we can choose

$$\psi(x) = \chi_\ell(R, \cos\theta)\ Y_{\ell m}(\hat{\underset{\sim}{e}}) . \tag{2.25}$$

Due to the presence of terms proportional to p_4 and p_4^2 in L and the exponential $\exp[(\mu_2 - \mu_1)Ex_4]$ multiplying the exchange potential, the R and $\cos\theta$ dependences are generally not separable. An exception occurs at $E = 0$ (i.e., when the binding is so strong that the bound state is massless).[7] At this point both V and L are invariant under all rotations in Euclidean four-space $[O(4)]$, so have the separation[2,4]

$$\psi(x) = f_n(R)\ Y_{n\ell m}(\hat{\underset{\sim}{x}}) , \tag{2.26}$$

where $Y_{n\ell m}(\hat{\underset{\sim}{x}})$ are the basis functions of the $(n/2, n/2)$ representation of $O(4)$ (See Appendix I) and satisfy

$$-p^2 f_n Y_{n\ell m} = \Box f_n Y_{n\ell m} = \left[f''_n + 3f'_n/R - n(n+2)f_n/R^2\right] Y_{n\ell m}. \qquad (2.27)$$

At zero energy, then, the spectrum of λ eigenvalues is determined by an infinite set of <u>uncoupled</u> ordinary differential equations of fourth order, differing only in their value of index n. When appropriate boundary conditions are imposed, each equation has a discrete spectrum with $\lambda(n,1) \leq \lambda(n,2) \cdots$ as in the usual Sturm-Liouville problem, with the wave function corresponding to $\lambda(n,k)$ having $k-1$ nodes in R. Since these differential equations are independent of ℓ (this is not the case if $E \neq 0$), we have for each radial solution g_n the degenerate set of solutions

$$\psi_{n\ell m}(x) = g_n(R)\, Y_{n\ell m}(\hat{\underset{\sim}{x}}) \quad \text{for} \quad \ell = n,\ n-1,\ \cdots,\ 0. \qquad (2.28)$$

These general features of the spectrum are shown schematically in Fig. 2.

As E is increased from zero, the $O(4)$ symmetry is broken, the set of fourth-order equations becomes coupled, and the degenerate levels split. Although (2.28) is no longer a solution in this case, it has been found that a finite sequence of the form

$$\sum_{n=\ell}^{N+\ell} f_n(R)\, Y_{n\ell m}(\hat{\underset{\sim}{x}}), \quad \text{with} \quad N = 0,1,2,\cdots, \qquad (2.29)$$

can yield a fast converging approximation to the true solution.[4,5]

If $m_1 = m_2$ and $\mu_1 = \mu_2$, both $L(p)$ and $V(x)$ are invariant under "time reflection" (i.e., $\tau \to -\tau$), thus allowing eigenstates to be classified as even or odd under this operation. For this case the exchange and nonexchange solutions are trivially related. Equation (2.17) becomes

$$L(p)\psi(x) = (-)^{\ell+z}\lambda\, V(x)\psi(x) \,, \tag{2.30}$$

where $z = +$ if ψ is even, or $-$ if ψ is odd, under time reflection, a result differing from Schrödinger theory only by the extra factor $(-)^z$. Equation (2.30) is also valid if $m_1 \neq m_2$ at the point $E = 0$.

C. Reality Properties of λ

To discuss reality of λ it is useful to introduce the following scalar products:

$$(\phi_1,\phi_2)_1 = \int \phi^*_1(x)\phi_2(x)\, d^4x, \qquad (\phi_1,\phi_2)_2 = \int \phi^*_1(-x)\phi_2(x)\, d^4x \,. \tag{2.31}$$

For arbitrary total momentum K it follows from (2.18) that

$$(\phi_1, D^{\pm}_i\phi_2)_1 = \pm\, (D^{\pm}_i\, \phi_1,\phi_2)_1, \qquad (\phi_1,\ V\,\phi_2)_1 = (V\,\phi_1,\ \phi_2)_1 \,, \tag{2.32}$$

$$(\phi_1, D^{\pm}_i\phi_2)_2 = \quad (D^{\pm}_i\, \phi_1,\phi_2)_2, \qquad (\phi_1,\ V\,\phi_2)_2 = (V\,\phi_1,\ \phi_2)_2 \,. \tag{2.33}$$

If $m_1 = m_2$, then $L = D^+_1 D^-_2$ is Hermitian with either scalar product. This is another way of saying only <u>even</u> derivatives occur in L. On the other hand, if $m_1 \neq m_2$ only scalar product 2 renders L Hermitian.[8] Further, if $\phi_A(-x) = -\phi_A(-x)$ and $\phi_B(x) = \phi_B(-x)$, then it is clear that $(\phi_B,\phi_B)_2 > 0$, whereas $(\phi_A,\phi_A)_2 < 0$, so the second scalar product is indefinite. It is further evident that although V is a positive

definite operator for the first scalar product, it need not be for the second. For the moment restricting ourselves to nonexchange solutions, we now show that certain λ eigenvalues may be complex even when E and ℓ are physical. Such solutions are generally unfamiliar, as they do not occur with equal-mass kinematics. To see this, recall that the condition $m_1 = m_2$ implies that the operators L and V are Hermitian, the latter being also positive definite (using scalar product 1). In this case, let ψ_1 and ψ_2 be solutions of the BS equation with eigenvalues λ_1 and λ_2 respectively. Combining this with hermiticity, we obtain

$$(\lambda_1 - \lambda^*_2)_1 \, (\psi_1, V\psi_2)_1 = 0. \tag{2.34}$$

If $\lambda_1 = \lambda_2$, then it follows that $\lambda_1 = \lambda^*_2 = \lambda^*_1$, hence λ_1 is real when $(\psi_1, V\psi_1)$ is nonzero. This is true, since V is positive definite. If $m_1 \neq m_2$, however, we must use the second scalar product to make L and V Hermitian, and V is no longer a positive definite operator. Thus the reality proof fails, since now $(\psi_1, V\psi_1)_2$ may vanish. The only term in L which is non-Hermitian contains $E(m_2 - m_1)$, hence a purely real spectrum exists when either of these factors vanishes (i.e., $m_1 = m_2$ or $E = 0$ assures λ of reality). We will see that the violation of these conditions frequently does lead to complex solutions.

Some insight into the appearance of complex solutions can be obtained from the following simple model. Let states 1 and 2 of the same angular momentum be described by

$$L_o\psi_1 = \lambda_1 V \psi_1 \qquad \text{and} \qquad L_o\psi_2 = \lambda_2 V\psi_2 , \tag{2.35}$$

where L_o is an Hermitian differential operator. The state vectors ψ_1 and ψ_2 correspond to the same c.m. energy and are orthogonal with respect to V. Assume that λ_1 and λ_2 are nearly equal. Add a perturbation P to L_o, and assume (to first order) that $\psi \approx a_1\psi_1 + a_2\psi_2$, where ψ satisfies

$$(L_o + P)\psi = \lambda_{new} V \psi . \tag{2.36}$$

This leads to a secular determinant whose solution is

$$\lambda_{new} = (P_{11} + P_{22} + \lambda_1 + \lambda_2)/2 \pm \left\{ \left[(P_{11} - P_{22} + \lambda_1 - \lambda_2)/2\right]^2 + P_{12}P_{21} \right\}^{\frac{1}{2}} . \tag{2.37}$$

If P is Hermitian, λ_{new} is a pair of real levels. On the other hand, if P is skew Hermitian and real, P_{11} and P_{22} vanish and P_{12} equals P^*_{21}. The new levels than would be

$$\lambda_{new} = (\lambda_1 + \lambda_2)/2 \pm \left\{ \left[(\lambda_1 - \lambda_2)/2\right]^2 - |P_{12}|^2 \right\}^{\frac{1}{2}} . \tag{2.38}$$

Thus λ_{new} can be real or imaginary depending on the closeness of the levels and the size of the perturbation.

To apply this to the BS equation, assume that L_o describes the equal-mass configuration. Since $\tau \to -\tau$ is a symmetry of L_o and V, we can classify solutions by their reflection symmetry under τ. To generate the perturbation, let the masses change by $m_1 \to m_1 + \delta$ and $m_2 \to m_2 - \delta$ in (2.24) while holding the energy fixed. The operation

(to first order in δ) results in

$$L_o + P , \tag{2.39}$$

where

$$P = -2i\delta\epsilon p_4\left[p^2 + m^2(1 - \epsilon^2)\right] , \epsilon = E/(m_1+m_2) \tag{2.40}$$

Note that P is real and skew Hermitian, since p_4 is pure imaginary. Since the operator P is odd in τ, it can connect only states of opposite τ symmetry (and the same ℓ).

The following picture emerges from Eq. (2.38). When $\delta = 0$ assume that there are two real levels with equal E and nearly degenerate λ's. As δ grows $|P_{12}|^2$ increases and the levels approach each other. When $|P_{12}|^2 = (\lambda_1 - \lambda_2)^2/4$ the levels coincide ("collide"). Further increase in δ leads to a pair of complex conjugate levels whose real part is the average of the original levels and whose imaginary part is proportional to the mass difference. Figure 3 shows a typical pair of "colliding" solutions as described by Eq. (2.38).[9]

Numerical computations show that in fact there exist sets of parameters which give rise to complex solutions for a range of values of E. Figure 4 shows the collision of two $\ell = 0$ solutions as the external masses are allowed to vary subject to the constraint $m_1 + m_2 = 2$. When m_1 and m_2 equal 1, hermiticity forces λ to be real. At this point τ symmetry classifies the states. The state labeled $(0^+)^*$ is even under τ-reflection, the asterisk meaning that there is a lower lying state with this symmetry. The state labeled by 0^- is the lowest s

state which is odd under τ-inversion. As m_1 decreases, the eigenvalues approach as in the model, and at $m_1 \simeq 0.6$ the levels intersect. Any further reduction of m_1 causes the solutions to become complex conjugates.

When we apply the preceding reasoning to the exchange potential problem an interesting result is found. Equation (2.30) shows that the equal-mass $(0^+)^*$ and 0^- solutions have λ eigenvalues of different sign, hence they cannot cross. If the external masses are now made unequal these two states are not expected to mix. Since (for a given ℓ) all solutions of a given sign have the same value of z, no "collisions" of λ curves leading to complex solutions are expected to occur. Thus we expect a purely real spectrum even for unequal masses.

To prove this assertion, we merely note that the reality proof is valid for unequal masses now, for

$$(\psi, V\psi)_2 = \int \psi(-x)\, V(x)\, \psi(-x)\, d^4x \tag{2.41}$$

is positive definite. The reality of this spectrum in this simple case unfortunately does not imply that this is a general feature of exchange potentials; in fact with the more general potential

$$V = V_1(x;M_1) + \sigma V_2(x;M_2), \tag{2.42}$$

if $\sigma < -2$ and $M_2 > M_1$ the potential is no longer positive definite, and again complex solutions are possible.

D. Choice of Trial Functions

It remains for us to choose a form for the radial basis functions $f_n(R)$ introduced in (2.29). It is important to have trial functions that match the true solutions closely near $R = 0$, where the potential is the strongest. Since the coupling terms between the set of ordinary differential equations vanishes in this limit, the indicial equations may be solved independently yielding[2] $f_n \sim R^{\gamma_n}$ where $\gamma_n = -(n+2), -n, n, (n+2)$ if $n \neq 0$ and $f_o \sim R^{-2}, \ell nR, 1, R^2$. For both bound and scattering states we reject the solutions that are not finite at the origin (two for each value of n).[10] This pair of conditions along with the condition $\psi \to 0$ as $R \to \infty$ for bound states assures us of a discrete spectrum of eigenvalues.

Since the roots of the indicial equation differ by an integer we might expect a solution of the form

$$f_n(R) = \sum_{k=n}^{\infty} c_k R^k + \ell nR \sum_{k=n+2}^{\infty} d_k R^k . \tag{2.43}$$

Indeed, by direct substitution into (2.13) we find that the $R^{n+2} \ell nR$ term is needed on the left to cancel the <u>leading</u> term in the potential. Further complication, however, occurs due to the presence of the ℓnR term in V. For example, substitution of (2.43) into the differential equation leads to a term in $V\psi$ proportional to $(R^2 \ell nR)(\ell nR)$ in the case $n = 0$. The expansion of $L\psi$ will contain such a term only if f_n has a term such as $R^6(\ell nR)^2$, since its fouth derivative (from L) has a term proportional to $R^2(\ell nR)^2$. Continuing this process, we see that all powers of ℓnR will be needed, although the higher powers of ℓnR are

"shielded" by high powers of R.

For calculational purposes, a simple set of trial functions (which ignores the log dependence) is

$$\phi_n(R) = R^{n+k-1} \exp(-\alpha R) \quad \text{for} \quad k = 1,2,\cdots ; \quad n = \ell, \ \ell+1, \cdots . \tag{2.44}$$

These were used in Ref. 4 to find the bound-state spectrum. The parameter α is a free variable to be adjusted to improve convergence. A serious drawback of the above set is that only the leading power of ψ is matched by the trial function. Since the differential equation involves fourth derivatives, we should try to also match the leading behavior of the <u>derivatives</u>. (For example, although $R^2 \ \ell nR$ vanishes at the origin, $(d^2/dR^2)(R^2 \ \ell nR)$ behaves as $2\ell nR$, while the second derivative of the elements of $\phi_n(R)$ given above are <u>finite</u> at $R = 0$. The higher derivatives of the true solution grow even faster.)

The most obvious solution to this problem is to simply choose basis functions of the form

$$\phi_n{}^k(R) = R^{n+k-1}(\ell nR)^m \ e^{-\alpha R} \quad \text{for} \quad m = 0, \ 1, \ \cdots . \tag{2.45}$$

The minimum k is a function of m, for example, if $m = 1$, then $k_{min} = 2$. Instead of this basis, however, we consider[11]

$$\phi_n{}^k(R) = R^{n+k} \ K_k(\alpha R) \qquad \text{for} \quad k = 1, \ 2, \ \cdots , \tag{2.46}$$

where $K_k(\alpha R)$ is the modified Bessel function of order k. From the expansion of $K_k(\alpha R)$ about the origin we see that this function has the correct leading <u>logarithmic</u> behavior at the origin as well as the correct

power behavior, so that all derivative terms will also have the correct leading form. Of course these trial functions do not have $R^n(\ell nR)^2$ terms in their expansion, but we use them anyway since they have the further advantage of being simple in momentum space:

$$\tilde{\phi}_n^{\,k}(p)\, Y_{n\ell m}(\Omega_p) = \int \exp(-ip\cdot x)\phi_n^{\,k}(R)\, Y_{n\ell m}(\Omega_x)\, d^4x$$

$$= \frac{p^n}{(p^2+\alpha^2)^{n+k+2}}\, Y_{n\ell m}(\Omega_p) \quad . \tag{2.47}$$

Trial functions that have simple forms in both x and p space are very useful in simplifying calculations with the Schwinger variational principle. These trial functions are also a great value even with the Rayleigh-Ritz method if one wishes to use a free Green's function whose legs have been renormalized by the sum of all bubbles.[12]

E. Variational Principles

For calculation of the bound-state spectrum, we deal with two fundamental variational principles; the first is the Rayleigh-Ritz:[4]

$$\lambda[\psi_T] = \frac{(\psi_T,\ L\psi_T)_2}{(\psi_T,\ V\psi_T)_2} \quad . \tag{2.48}$$

It is clear that if ψ satisfies the BS equation, then $\lambda[\psi] = \lambda$, where λ is the true eigenvalue. If we set $\psi_T = \psi - \delta\psi$, it is easy to see that $\lambda[\psi_T] - \lambda$ is of order $(\delta\psi)^2$.

A particularly useful parameterization of ψ_T is that of the linear expansion

$$\psi_T = \sum_{i=1}^{N} c_i\phi_i \ . \tag{2.49}$$

With this choice, $\delta(\lambda[\psi]) = 0$ is equivalent to the set of N linear equations

$$\sum_{j=1}^{N} (L_{ij} - \lambda_T V_{ij}) c_{ij} = 0; \qquad i = 1,2,3,\cdots, N. \tag{2.50}$$

Solving this set of equations gives the N eigenvalues (if no degeneracy) $\lambda_1, \lambda_2, \cdots, \lambda_N$. This method is ideally suited to machine computations, since it gives a systematic procedure for generating a sequence of approximations. It also has the advantage of providing approximations to higher eigenvalues. If $m_1 = m_2$ one can easily generalize the Hylleraas-Undheim theorem[13] to prove that this sequence of successive approximations monotonically approaches the true eigenvalues (from above). If the masses are unequal, the theorem is not true (essentially due to the presence of complex solutions); however, it appears to be valid for those levels which do not mix.

If we choose to consider the integral equation

$$\psi(x) = \int G_o(x-x') \, V(x') \, \psi(x') \, d^4x' , \tag{2.51}$$

we are led to the Schwinger variational expression

$$\lambda[\psi_T] = \frac{\int \hat{\psi}_T(x) \, V(x) \, \psi_T(x) \, d^4x}{\sqrt{\int \hat{\psi}_T(x) \, V(x) \, G(x-x') \, V(x') \, \psi_T(x') \, d^4x \, d^4x'}} , \tag{2.52}$$

which is stationary about solutions to (2.51), as is easily checked.

Following Ref. 5 we define $U(x) = V(x) \, \psi(x)$,

$$\tilde{U}_T(p) = \int d^4x \, \exp(-ip\cdot x) \, U_T(x) \, , \tag{2.53}$$

and

$$\tilde{G}(p) = \int d^4x \, \exp(-ip\cdot x) \, G(x) \, , \tag{2.54}$$

hence (2.52) becomes

$$\lambda[\psi_T] = \frac{\int \hat{U}_T(x) \, V^{-1}(x) \, U_T(x) \, d^4x}{\int \tilde{\hat{U}}_T(p) \, \tilde{G}(p) \, \tilde{U}_T(p) \, d^4p/(2\pi)^4} \tag{2.55}$$

The advantage of choosing trial functions that have simple Fourier transforms is now clear, since the Green's function matrix element of (2.55) is far simpler to compute than that of (2.52). Actually a modification of the trial functions (2.47) is used, since U is taken to be the variational function. Finally, it is clear that a linear expansion of $U(x)$ analogous to (2.49) will lead to a linear set of matrix equations of the form (2.50).

We next set up expressions to determine the scattering phase shift. It is possible to use the Rayleigh-Ritz method to calculate phase shifts through an anlytic continuation procedure,[14] however, for the problem at hand, the Schwinger method seems more reliable.

To obtain a real variational expression for the scattering phase shift δ_ℓ it is convenient to first break up the Green's function (Wick-rotated form) into real and imaginary parts in the scattering region

$$G(x) = G_R(x) + i \, G_I(x), \tag{2.56}$$

or, in terms of formal Fourier transforms,[15]

$$\tilde{G}(p) = \tilde{G}_R(p) + i\,\tilde{G}_I(p) \ . \tag{2.57}$$

The function $\tilde{G}_R(p)$ need <u>not</u> be real, although it is for equal-mass scattering. From the expression[5]

$$G(x-x') = \frac{\exp(i\,k\,|\underset{\sim}{r}-\underset{\sim}{r}'|)}{8\pi E\,|\underset{\sim}{r}-\underset{\sim}{r}'|} - \left(\frac{1}{8\pi^2 E}\right)\left(\int_{-\infty}^{-\omega_2} + \int_{\omega_1}^{\infty}\right) d\beta\; e^{\beta(\tau-\tau')} K_o(Q|R-R'|) \ , \tag{2.58}$$

where $Q = (\beta^2 - k^2)^{\frac{1}{2}}$ and $\omega_i^{\,2} = (\underset{\sim}{k}^2 + m_i^{\,2})^{\frac{1}{2}}$, it follows that

$$\tilde{G}_I(p) = \frac{\pi^2}{E}\;\delta(p_4)\;\delta(\underset{\sim}{p}^2 - k^2) \ , \tag{2.59}$$

and, for $k > 0$ and $x - x'$ large and spacelike,

$$G_R(x-x') \sim \frac{\cos k\,|\underset{\sim}{r}-\underset{\sim}{r}'|}{8\pi E\,|\underset{\sim}{r}-\underset{\sim}{r}'|} \ . \tag{2.60}$$

The standing-wave solution $\psi_{\underset{\sim}{k}}^{\,P}$ to the BS equation is defined by the integral equation[16]

$$\psi_{\underset{\sim}{k}}^{\,P}(x) = \psi_{\underset{\sim}{k}}^{\,o}(x) + \int G_R(x-x')\;V(x')\;\psi_{\underset{\sim}{k}}^{\,P}(x')\;d^4x' \tag{2.61}$$

and the K matrix by

$$K_{\underset{\sim}{k}',\,\underset{\sim}{k}} = \int \psi_{\underset{\sim}{k}'}^{\,P}(-x)\;V(x)\;\psi_{\underset{\sim}{k}}^{\,o}(x)\;d^4x = \int \psi_{\underset{\sim}{k}'}^{\,o}(-x)\;V(x)\;\psi_{\underset{\sim}{k}}^{\,P}(x)\;d^4x \ . \tag{2.62}$$

Equations (2.61) and (2.62) can be analytically continued below the elastic threshold where $k^2 < 0$.[17]

Here the subscripts refer to the relative momentum. The second integral of Eq. (2.62) follows from

$$G_R(x,x') = G_R(-x',-x) \tag{2.63}$$

which in turn follows trivially from (2.58).

The relation of $K_{\underset{\sim}{k}'\,\underset{\sim}{k}}$ to the physically measurable phase shifts is found by partial-wave expansion of Eq. (2.62). Define the partial wave functions by

$$\psi_{\underset{\sim}{k}}{}^{P}(x) = \sum_{\ell} (2\ell + 1)\, i^{\ell}\, \psi_{\ell}{}^{P}(r, \cos\theta)\, P_{\ell}(\hat{\underset{\sim}{x}}\cdot\hat{\underset{\sim}{z}}) , \tag{2.64}$$

$$\psi_{\underset{\sim}{k}}{}^{o}(x) = \sum_{\ell} (2\ell + 1)\, i^{\ell}\, j_{\ell}(kr)\, P_{\ell}(\hat{\underset{\sim}{x}}\cdot\hat{\underset{\sim}{z}}) , \tag{2.65}$$

and

$$K_{\underset{\sim}{k}'\,\underset{\sim}{k}} = \sum_{\ell} (2\ell + 1)\, i^{\ell}\, K_{\ell}\, P_{\ell}(\hat{\underset{\sim}{k}}\cdot\hat{\underset{\sim}{k}}') , \tag{2.66}$$

where

$$K_{\ell} = \left(\frac{k}{8\pi E}\right) \int j_{\ell}(kr)\, V(x)\, \psi_{\ell}{}^{P}(r, \cos\theta)\, d^4x . \tag{2.67}$$

From the equation of motion, (2.61), the asymptotic expression (2.60) and the expansion

$$\frac{\cos k\,|\underset{\sim}{r}-\underset{\sim}{r}'|}{|\underset{\sim}{r}-\underset{\sim}{r}'|} = k \sum_{\ell=0}^{\infty} (2\ell + 1)\; j_{\ell}(kr_{<})\, n_{\ell}(kr_{>})\, P_{\ell}(\hat{\underset{\sim}{r}}_1\cdot\hat{\underset{\sim}{r}}_2) , \tag{2.68}$$

we find

$$u_\ell^P(r, \cos\theta) \sim j_\ell(kr) - n_\ell(kr)\, K_\ell \,, \tag{2.69}$$

and so we recognize

$$K_\ell = \tan \delta_\ell \tag{2.70}$$

in the scattering region. Furthermore, since u_ℓ^P solves a <u>real</u> inhomogenous equation it is clear from the definition (2.61) that K_ℓ is a real function so long as the Wick rotation is valid (i.e., below the inelastic threshold). This proves that δ_ℓ is real in the elastic region (elastic unitarity).

The following expression for $K_{\underset{\sim}{k}', \underset{\sim}{k}}$ is stationary about solutions to Eq. (2.61):

$$\int \psi_{\underset{\sim}{k}}^P(-x)V(x)\psi_{\underset{\sim}{k}}^o(x)d^4x + \int \psi_{\underset{\sim}{k}'}^o(-x)V(x)\psi_{\underset{\sim}{k}}^P(x)d^4x - \int \psi_{\underset{\sim}{k}'}(-x)V(x)\psi_{\underset{\sim}{k}}^P(x)\, d^4x$$
$$+ \iint \psi_{\underset{\sim}{k}'}^P(-x)V(x)G_R(x-x')V(x')\psi_{\underset{\sim}{k}}^P(x')d^4x'\, d^4x \,. \tag{2.71}$$

Employing the same Fourier transform technique as used with the bound-state calculation, we find, on partial-wave analysis

$$[K_\ell] = 2u_\ell^P(k,0) - \int u_\ell^P(r,\tau)V^{-1}(R)\, u_\ell^P(r,-\tau)\, d^4x$$
$$+ \int \frac{d^4p}{(2\pi)^4} \left[\tilde{u}_\ell^P(p,p_4)\right]^2 \tilde{G}_R(p) \,, \tag{2.72}$$

where we have restricted ourselves to the region $k^2 < 0$ (below threshold) so that the Green's function integral is well defined. The u_ℓ are the partial-wave coefficients of $U(x)$ $[=V(x)\psi(x)]$ and are defined analogously

to ψ_ℓ in Eq. (2.64).

To get into the scattering region, we must express the Green's function integral in a form that is easily analytically continued. We have seen that K_ℓ is real below the inelastic threshold, so the left-hand side of (2.72) is real. Since $\tilde{u}_\ell(k,0)$, $u_\ell{}^P(x)$, and V^{-1} are real, we conclude that the Green's function integral is also real, although $\tilde{u}_\ell(p)$ and $\tilde{G}_R(p)$ are not necessarily real individually. We have, then,

$$\int [\tilde{u}^P(p)]^2 \, \tilde{G}_R(p) \, \frac{d^4p}{(2\pi)^4} = \mathrm{Re}\left\{ \int [\tilde{u}_\ell{}^P(p)]^2 \, \tilde{G}(p) \, \frac{d^4p}{(2\pi)^4} - i \int \tilde{G}_I(p) \, [\tilde{u}_\ell{}^P(p)]^2 \, \frac{d^4p}{(2\pi)^4} \right\} . \tag{2.73}$$

Finally, using (2.58) and the reality of $[\tilde{u}_\ell(k,0)]^2$, we find

$$\int [\tilde{u}_\ell{}^P(p)]^2 \, \tilde{G}_R(p) \, \frac{d^4p}{(2\pi)^4} = \mathrm{Re} \int [\tilde{u}_\ell{}^P(p)]^2 \, \tilde{G}(p) \, \frac{d^4p}{(2\pi)^4} . \tag{2.74}$$

The integral on the right side of (2.74) is analytic and may be continued into the scattering region. Combining this result with Eq. (2.72) then gives our stationary expression for $\tan\delta_\ell$. The resulting expression differs from Ref. 5 only in the sign change $(-\tau)$ in the potential integral, and generalizes that result to unequal-mass scattering.

Finally, it is easy to see that the preceding results are adapted to exchange potentials by only the addition of the exponential factor to the inverse potential term and the sign change of the argument of $U(x)$, provided the latter is defined by

$$U(x) = \exp\,[\tau(\mu_1-\mu_2)E] \; V(x) \; \psi(x) \quad . \tag{2.75}$$

F. Calculation of the Matrix Elements and Continuation in ℓ: Rayleigh-Ritz

In this subsection we compute matrix elements of operators L and V with respect to the bases (2.44) and (2.46).

The effect of the differential operator L(p) on a single trial function $f_{nk}(R)\, Y_{n\ell m}(\hat{\underset{\sim}{x}})$ is easily calculated from Eqs. (A5) and (A6) of Appendix I, and is

$$L\,[f_{nk}(R)\, Y_{n\ell m}(\hat{\underset{\sim}{x}})] = \sum_{n',k'} c(n',k')\, f_{n'k'}(R)\, Y_{n'\ell m}(\hat{\underset{\sim}{x}}) \,, \qquad (2.76)$$

$$k-k' = 0,1,2,3,4; \quad n-n' = 0,\pm 1,\pm 2,$$

where $c(n',k')$ are known constants. Using the orthogonality of $Y_{n\ell m}$, the L matrix elements are reduced to one-dimensional integrals of the form

$$\int_0^\infty f_{n'k'}(R)\, f_{nk}(R)\, R^3\, dR \quad . \qquad (2.77)$$

For nonexchange processes V is a function only of R, hence the V matrix elements are simply

$$\delta_{nn'} \int_0^\infty f_{nk'}(R)\, [K_1(MR)/R]^s \; f_{nk}(R)\, R^3\, dR, \qquad (2.78)$$

where $s = 1$ for ϕ^3 and $s = 2$ for ϕ^4 potentials. (For ϕ^4 potentials n is no longer integral, but this does not introduce numerical problems.) For exchange processes we have two simple alternatives. The first is to expand the exponential $\exp[\tau E(\mu_1-\mu_2)]$ by use of

$$e^{xy} = (2/y) \sum_{n=0}^{\infty} (n+1)\, I_{n+1}(y)\, C_n^{\,1}(x) \ . \tag{2.79}$$

The resulting angular integrals are $O(4)$ Clebsch-Gordan coefficients, and are easily calculated by using the explicit form of $C_n^{\,1}(x)$. The radial integrals also present no problems. The second alternative is to simply set $\mu_1 = \mu_2$, thus eliminating the unpleasant factor. Provided that the mass difference of the external particles is not too extreme, this technique proves quite adequate.

With the basis (2.44), integrals (2.77-8) reduce to the form

$$Q(N) = \int_0^{\infty} \exp(-2\alpha R)\, R^{N+1}\, dR \ , \tag{2.80}$$

$$P_a(N) = \int_0^{\infty} \exp(-2\alpha R)\, R^{N+1}\, K_a(MR)\, dR \ , \tag{2.81}$$

where $a = 1$; P_0 will also be needed below. The first integral is simply a gamma function, hence is easily calculated. The second is more difficult, but can be done numerically without any trouble. In both cases only one value of N need be explicitly calculated, because of the existence of the recursion formulas

$$Q(N-1) = 2\ [Q(N)/(N+1)] \ , \tag{2.82}$$

$$P_0(N-1) = [2\ P_0(N) + M\ P_1(N)]/(N+1) \ , \tag{2.83}$$

$$P_1(N-1) = [M\ P_0(N) + 2\ P_1(N)]/N \ . \tag{2.84}$$

For the basis (2.46) the integrals (2.77-8) become

$$Q_{ij}(N) = \int_0^\infty K_i(x)\, K_j(x)\, x^{N+1}\, dx\ , \tag{2.85}$$

$$P_{ijk}(N) = \int_0^\infty K_i(x)\, K_j(x)\, K_k(x)\, x^{N+1}\, dx\ , \tag{2.86}$$

where $i,j = 0,1,2,\cdots$ and $k = 1$. Due to the Bessel recursion formulas we need compute only $i,j = 0,1$ numerically. By analogy to (2.83-4) it is easy to show

$$\begin{aligned} Q_{01}(N-1) &= [Q_{11}(N) + Q_{00}(N)]/N, \\ Q_{00}(N-1) &= 2Q_{01}(N)/(N+1)\ , \\ Q_{11}(N-1) &= 2Q_{01}(N)/(N+1) \end{aligned} \tag{2.87}$$

(which obviously simplify). Six independent recursion formulas for $P_{ijk}(N)$ can be constructed from the relation

$$P_{ijk}(N-1) = [P_{i\pm1jk}(N) + P_{ij\pm1k}(N) + b\, P_{ijk\pm1}(N)]/[N+1\pm(i+j+k)]. \tag{2.88}$$

A similar set of recursion formulas exists if $K_k(bx)$ is replaced by J_k or I_k, a fact which is useful for exchange potential and form factor calculations.

Were we to use a "square well" potential of the form $V(R) = V_o$ if $R < R_o$ and $V(R) = 0$ otherwise, all the necessary integrals are of the form (2.85) except that now the limits are altered. The recursion formulas are still valid provided that we include "surface" terms that arise

from integrations by parts used in their derivation.

In the foregoing we have assumed that ℓ is a nonnegative integer. on the other hand, once the $Y_{\ell m}$ dependence has been factored off there is nothing to prevent us from using the resulting equations to analytically continue in ℓ.[18] Equation (2.29) becomes in this case

$$\psi_\ell = \sum_{n=\ell}^{\ell+S} f_n(R)\, Z_n(\cos\theta) \quad \text{with} \quad S = 0,1,2,\cdots; \tag{2.89}$$

where

$$Z_{n\ell}(\cos\theta) = N_{n\ell}\, C_{n-\ell}^{\ell+1}(\cos\theta)\, \sin^\ell\theta\ . \tag{2.90}$$

The normalization constants $N_{n\ell}$ are given in Appendix I, and $C_n^{\alpha}(x)$ are, as before, Gegenbauer polynomials. The index n $(=\ell, \ell+1, \cdots)$ is no longer integer, but since $n-\ell$ remains integer, $C_{n-\ell}^{\ell+1}$ are still polynomials, and the equation (2.76) remains valid (provided the $Y_{\ell m}$ is first cancelled). The integrals (2.77-8) are trivially continued in n (or N) for both sets of trial functions, and all the recursion formulas are still valid. The recursion formulas have the further function of giving meaning to the integrals in the left half of the n plane, where the integral definitions diverge. In effect, these formulas extract poles in the matrix elements at negative integers and serve as a device to analytically continue the matrix elements. This method of computation of trajectories suffers from the defect of giving only points below the elastic threshold; however, it may be possible to use this information to learn about points in the scattering region.[19]

G. Calculation of the Matrix Elements: Schwinger

In this section we generalize the calculational forms of Ref. 5 to unequal-mass kinematics. At the close of the section we comment about the Regge continuation.

The basis functions (2.46) are most convenient, since their Fourier transforms (2.47) are very simple. Actually the trial functions used were

$$2\,\pi \exp\left[i\pi\,(n-\ell)/2\right] \cos^{n-\ell}\theta_p \,\sin^{\ell}\theta_p \; y^n (y^2+1)^{-n-q}\,, \qquad (2.91)$$

$$n-\ell = 0,1,\cdots\;;\quad q = 1,2,\cdots\,,$$

where

$$y = |p|/\alpha\;\left[\equiv (|p|^2 + p_4{}^2)^{\frac{1}{2}}/\alpha\right] \quad \text{and} \quad \cos\theta_p = p_4/|p|\;.$$

The $Y_{\ell m}$ have been factored out, and the phase has been chosen to make the matrix equations real. The integer q has values $1,2,\cdots$ instead of $3,4,\cdots$ as with ψ, because $V \sim 1/R^2$ for small R. The angular functions have been simplified to make the Green's function integrals simpler. Equation (2.91) can be Fourier transformed to yield, after some algebra,

$$u_{n\ell q}(R,\tau) = \sum_k (\alpha^4/4\pi)\exp\left[i\pi(n-\ell+k)/2\right] B(n-\ell,k)\,\sin^{\ell}\theta\; C_k{}^{\ell+1}(\cos\theta)$$

$$\times \sum_{\lambda=0}^{j} \binom{j}{\lambda} (-)^{j-\lambda}\left[\Gamma(\sigma+2)2^{\sigma}\right]^{-1} (\alpha R)^{\sigma}\, K_{k-\sigma}(\alpha R)\;, \qquad (2.92)$$

where j is the maximum integer in $(n-\ell-k)/2$ and $\sigma = n + q - \ell - 2$

and $k = 0,2,\cdots,n-\ell$ if $n-\ell$ is even or $1,3,\cdots,n-\ell$ if $n-\ell$ is odd. The matrix $B(n-\ell,k)$ is defined by

$$\cos^{n-\ell}\theta = \sum_k B(n-\ell,k)\, C_k^{\ell+1}(\cos\theta) \,. \tag{2.93}$$

Taking matrix elements of the Green's function G_o with respect to a single term of the momentum-space trial function (2.91) shows us that we need to evaluate expressions of the form

$$A(K,M,N) = e^{i\pi K/2}\, c \int_0^\infty \frac{dx\, x^N}{(1+x^2)^M} \left[\frac{1}{\pi}\int_0^\pi \frac{d\theta\, \cos^K\theta}{G_o^{-1}(\theta,x)}\right] , \tag{2.94}$$

where $c = \alpha^2/4\mu_1\mu_2E^2$. The inverse two-particle propagator function $\left[G_o^{-1}(\theta,x)\right]$ is

$$x^2\cos^2\theta + i\, c'\, x\cos\theta(\Delta\mu x^2 + \gamma_-) + c\, x^4 + (\beta_1+\beta_2)x^2 + \beta_1\beta_2 \,, \tag{2.95}$$

where

$$\Delta\mu = \mu_1 - \mu_2 \,, \qquad \beta_i = [m_i^2 - (\mu_i E)^2]/\alpha^2,$$

$$\gamma_\pm = \mu_1\beta_2 \pm \mu_2\beta_1 \,, \qquad c' = 2\,E\,c\,/\,\alpha \,.$$

The bracketed expression in (2.94) will be denoted by $J(K)$. The substitution $z = \exp(i\theta)$ reduces (2.91) to a contour integration around the unit circle (using symmetry). The four poles in the denominator occur in pairs which are reciprocals, so (except if the poles are <u>on</u> the unit circle) we pick up exactly two residues. The prescription is to evaluate

the integral for a value of the parameters (μ's and k) such that the branches are known and then analytically continue the result to the desired parameter values. We find

$$J(0) = \frac{1}{\pi} \int_0^{\pi} G_o(\theta,x)d\theta = \frac{1}{c} \frac{1}{x^2+\gamma_+} \left[\frac{\mu_1}{D_1} + \frac{\mu_2}{D_2} \right], \tag{2.96}$$

$$J(1) = \frac{1}{\pi} \int_0^{\pi} G_o(\theta,x)\cos\theta \, d\theta = \frac{i}{xc \; 2E(x^2+\gamma_+)} \left[\frac{x^2+\beta_1}{D_1} - \frac{x^2+\beta_2}{D_2} \right], \tag{2.97}$$

where $D_i = \left[(x^2 + \beta_i)^2 + (2\mu_i Ex/\alpha)^2 \right]^{\frac{1}{2}}$, thus reducing (2.91) to a single integral if $K = 0$ or 1. It is possible to evaluate $J(K)$ for arbitrary K by contour integrations, but these two values of K will be sufficient for calculational purposes due to the existence of the recursion formulas

$$A(K,M+1,N) = -A(K,M+1,N+2) + A(K,M,N) \tag{2.98a}$$

and

$$\begin{aligned} A(K+2,M,N) = {} & -c \exp(i \pi K/2) \, I(K) \, E(M,N-2) \\ & + c' \left[\Delta\mu \, A(K+1,M-1,N-1) + (\gamma_- - \Delta\mu) \, A(K+1,M,N-1) \right] \\ & + \left[A(K,M-1,N) + (\beta_1 + \beta_2 - 1) \, A(K,M,N) + \beta_1\beta_2 \, A(K,M,N-2) \right], \end{aligned} \tag{2.98b}$$

where

$$E(M,N) = \int_0^\infty \frac{x^N dx}{(1+x^2)^M} = \Gamma\left(\frac{N+1}{2}\right)\Gamma\left(M - \frac{N+1}{2}\right) \Big/ 2\Gamma(M)$$

and

$$I(K) = \int_0^\pi \cos^K\theta \, d\theta = \begin{cases} 0 & \text{if } K \text{ odd} \\ K!/2^K\,[(K/2)!]^2 & \text{if } K \text{ even} \end{cases} .$$

The proof of (2.98a) is trivial, and (2.98b) is derived by simply multiplying numerator and denominator of the integrand of $E(M,N)\,I(K)$ by the propagator.

At this point we digress briefly to see for what range of E the Green's function integral can be evaluated when $\mu_1 = \mu_2$ and $m_1 \neq m_2$. We used this for calculating exchange potential bound-state problems. The denominator has its zeros on the imaginary axis at

$$\cos\theta = \pm\, i(p^2 + \beta_i)/(2\mu_i pE),$$

so the only way that this point could cross the integration contour is if $\beta_i < 0$. For $\mu_i = \frac{1}{2}$, then, we have $E < 2m_L$, where m_L is the smaller of m_1 and m_2. As long as $m_L > 0$ we can evaluate the integral in the allowed region and then analytically continue the resulting expression.

Finally we need the precise relationship between $A(K,M,N)$ and the matrix elements of G_o:

$$(u_{n\ell q} G_o u_{n'\ell q'}) = \sum_{\sigma=0}^{\ell+1} \binom{\ell+1}{\sigma} A(n+n'-2\ell+2\sigma,\ n+n'+q+q',\ n+n'+3). \quad (2.99)$$

If m_1 or m_2 is zero, the recursion formula (2.98b) cannot be used, as it becomes indeterminate, but a slight modification of it can be used.

The Green's function integral is an improvement over its Rayleigh-Ritz analog $(\psi, L\psi)$ for two reasons. First, the $V(x)\,V(x')$ appearing under the integral suppresses the tail of ψ, which can be quite large if E is near threshold. Second, the asymptotic behavior is treated more accurately, since the combinations $\int G(x,x')V(x')\psi_T(x')d^4x'$ and $\int \psi_T(x)V(x)G(x,x')d^4x$ have the same form at ∞ as $\psi(x)$ and $\psi(x')$ respectively, even though ψ_T does not. The net result is that the behavior at ∞ is taken care of automatically and all the freedom of the trial function can go into matching ψ and ψ_T near the origin.

There is no essential difference in the inverse potential matrix elements introduced by unequal-mass kinematics, hence we merely summarize the calculational formulas. We define

$$BB(m,m';N) = \int_0^\infty \frac{K_{m-2}(t)\,K_{m'-2}(t)\,t^N}{K_1(Mt/\alpha)} \quad . \tag{2.100}$$

These integrals are evaluated for $m,m' = 1$ and 2 for all N needed, and the Bessel function recursion formula is used to evaluate (2.100) for all other m. Using the orthogonality relations for the Gegenbauer polynomials, we find

$$(u_{n\ell q}V^{-1}u_{n'\ell q'}) = \alpha^3(-1)^{(n-\ell+n'-\ell)/2} \Big/ 2M(\ell!)^2$$

$$\times \sum_{K=0,1}^{\min(n-\ell,\ n'-\ell)} (-)^K B(n-\ell;K)B(n'-\ell;K)\frac{(K+2\ell+1)!}{K!\ (K+\ell+1)\ 2^{2\ell}}$$

$$\times \sum_{\lambda=0}^{j} \sum_{\lambda'=0}^{j'} \frac{BB(\sigma-\ell-K,\sigma'-\ell-K;\sigma+\sigma')}{\Gamma(\sigma)\ \Gamma(\sigma')\ 2^{\sigma+\sigma'}} (-)^{j-\ell+j'-\ell}\binom{j}{\lambda}\binom{j'}{\lambda'}, \tag{2.101}$$

where $\sigma = n+q-\ell$ and $\sigma' = n'+q'-\ell$. The equation applies only if n and n' are both even or both odd; otherwise the matrix element vanishes. The index K is taken over only even values if $n-\ell$ is even, odd values if $n-\ell$ is odd. In (2.101) we have set $j\equiv(n-\ell-k)/2$ and $j'\equiv(n'-\ell-k)/2$.

Although we have not computed Bethe-Salpeter Regge trajectories using the Schwinger variational principle, this is certainly possible, and has the advantage of allowing the trajectory to be computed in the elastic region.[20] The continuation of the potential integrals in ℓ is analogous to the method used with the Rayleigh-Ritz principle. On the other hand, we meet with some difficulty from the Green's function integrals. In particular the angular integral is proportional to

$$\int_0^\pi \frac{\cos^{n+n'-2\ell}\theta\ \sin^{2\ell+2}\theta\ d\theta}{A(x)\cos^2\theta + B(x,E)\cos\theta + C(x,E)} .$$

For integer (or half integer) ℓ we were able to evaluate this integral exactly, and we found that the resulting closed form expression was easily continued above threshold. When ℓ is general and the external masses

are equal [so $B(x,E) = 0$], the analytic continuation is also easy, since the above integral is expressible as a hypergeometric function whose continuation is standard. Such a continuation is doubtless possible even if the external masses are unequal; a judicious change of variables[21] would allow one to extract the $(x^2 - k^2)^{-1}$ singularity, leaving a well-behaved integral that could be evaluated numerically. We also note that the recursion formulas carry over to general ℓ with only minor alterations.

H. Example: One Trial Function

Before proceeding with a large-scale calculation, we can obtain some insight into the general form of the curves $\lambda(E^2)$ by the following simple model involving only one trial function. Assume the solution wave function to the eigenvalue problem is of the form of the lowest (for a given ℓ) zero-energy exact solution corresponding to exchange mass zero:

$$(\alpha R)^{n+1} K_1(\alpha R)\, Y_{n\ell m}(x), \qquad n = \ell. \tag{2.102}$$

We have inserted a free parameter α, which is unity for $M = 0$ but will be taken as a variational parameter.[20] Expanding the potential function about $M = 0$ and keeping the first three terms (it is easily seen that for <u>any</u> $M \neq 0$ an expansion to at least third order is necessary) leads to the Rayleigh-Ritz variational quotient:

$$\frac{(4n+10)}{(n+2)(n+3)}[\lambda] = \left[\frac{\xi^2 + \frac{2}{(n+3)}[\eta^2(n+1) - (n+2)]\xi + (1-\eta^2)^2}{\xi(n+1)^{-1} + A}\right] \tag{2.103}$$

where

$$A = \left[\frac{A(n)}{c} \frac{(n+2)(n+3)}{(2n+5)} \left(\ell n \frac{M}{2} + \gamma - \tfrac{1}{2} - \tfrac{1}{2}\ell n\xi\right)\right] M^2 , \tag{2.104}$$

$$A(n) = \frac{1}{2} \int_0^\infty \left[K_1(z)\right]^2 z^{2n+5} \ell nz \, dz, \qquad c = \int_0^\infty K_1(z)K_0(z) \, z^4 dz = 2/3,$$

$\xi = \alpha^2$, γ is Euler's constant, and $\eta = E/2$.

The variation condition

$$\frac{\delta[\lambda]}{\delta \xi} = 0 \tag{2.105}$$

is made difficult by the presence of $\ell n\alpha$ in A. In the following we will use for $\ell n\xi$ its value for the case $M = 0$, an approximation that will be checked for consistency at the close of the calculation. Condition (2.105) now implies

$$\xi = -A(n+1) + \left[\left((n+1)A\right)^2 - \frac{2A(n+1)}{(n+3)} \left((n+2) - \eta^2(n+1)\right) + (1-\eta^2)^2\right]^{\frac{1}{2}} . \tag{2.106}$$

As a check we set $M = 0$ and find

$$\alpha^2 = m^2(1 - \eta^2) , \tag{2.107}$$

$$\lambda_0(n) = (n+1)(n+2) - \left[\frac{2(n+1)(n+2)^2}{(2n+5)}\right] \eta^2 , \tag{2.108}$$

which is remarkably good fit to the exact solution up to $\eta \approx 0.8$ for $n = 0,1$. This very nearly linear character of λ vs η^2 is a

characteristic feature of ground-state solutions, and persists in the more general cases for which $M \neq 0$ and $m_1 \neq m_2$. Substitution of (2.107-8) into (2.103) gives

$$\left[\frac{(2n+5)}{(n+1)(n+2)(n+3)}\right]\lambda \approx (n+3)^{-1}\left[(n+2) - \eta^2(n+1)\right] - (n+1)A$$

$$+ \left[A^2(n+1)^2 - \frac{2A(n+1)}{(n+3)}\left((n+2) - \eta^2(n+1)\right) + (1-\eta^2)^2\right]^{\frac{1}{2}} \qquad (2.109)$$

If we take M to be 0.25 and calculate the constant A from the value of α given by (2.107), we find $\lambda(\eta=0) = 2.26$, compared with the true value $2.214 \cdots$. The resulting value of α is 1.11, compared with $\alpha_o = 1.0$, leads to an error in computation of A of only about 5%. As a further check, if $\eta^2 = 0.5$ we find $\lambda = 1.366$, which checks to three places on interpolation of Table II.

Equation (2.109) is particularly simple when $(-\eta^2)$ is large and positive (ghost region). Expanding the square root gives

$$[\lambda] = (-\eta^2)\left[\frac{2(n+2)^2(n+1)}{(2n+5)}\right] - \left[\frac{(n+1)(n-1)(n+2)}{(4n+10)}\right] M^2 \ln(-\eta^2) + O(\eta^o). \qquad (2.110)$$

The curves are nearly linear, with a slope dependent on ℓ, but independent of the exchange mass. The effect of $M \neq 0$ is to slightly steepen the slope.

Finally we would like to point out that Equation (2.109) gives an approximate expression for Regge trajectories below the elastic threshold.

III. NUMERICAL RESULTS

This section is broken into two parts, the first and more extensive dealing with bound states and the second with elastic scattering phase shifts.

A. Bound States

The BS equation can be written in a form which has exactly four nontrivial dimensionless quantities, for example,

$$m_1/m_T, \quad E/m_T, \quad M/m_T, \quad \lambda/m_T^2, \tag{3.1}$$

where $m_T = m_1 + m_2$. This enables us to fix the threshold energy at 2, all other thresholds being accessible through the scaling relation

$$\lambda(M, m_1, m_2, E) = \lambda(M\beta, m_1\beta, m_2\beta, E\beta)/\beta^2 \quad , \tag{3.2}$$

where $\beta = 2/m_T$. The eigenvalue parameter is most conveniently chosen as the coupling parameter λ, and is a function of three nontrivial parameters which we take to be E, m_1, and M. For calculational purposes, the choice M for the scaling mass would also be convenient, since then the same set of potential matrix elements could be used throughout.

By use of the variational methods described in Section II. Tables I-IX, constituting a "dictionary" of ground-state eigenvalues $\lambda(M, m_1, E)$, have been calculated. The parameter ranges contained in these tables are

$$0 < E < 2, \quad 1 < m_1 < 2, \quad 0 < M < 10 \text{ (if } \ell = 0), \quad M = 1 \text{ (if } \ell = 1).$$

This gives us the complete range of mass ratios of the external particles and all physical bound-state energies. The points near $E = 0$ were computed with the Rayleigh-Ritz quotient; for most other points the Schwinger principle was used.

Figure 5 is a graphical summary of Table V, and illustrates the basic behavior of λ with E^2. The slope is quite linear and independent of the mass difference in the region $E^2 \lesssim 3$, but becomes steeper as threshold is approached. That the slope is everywhere negative is intuitively clear, since less tightly bound states should require weaker coupling strengths in analogy to potential theory. Figure 6 demonstrates that this linear behavior persists into the "ghost" region $(E^2 < 0)$. The basic features of these curves are well approximated by the formulas (2.109) and (2.110), which resulted from calculations with a single trial function, provided that E is not too close to threshold and M is smaller than about 0.5.

In the limit $k^2 \ll m_1, m_2$ (near threshold) and $M \ll m_1, m_2$, the second term (G_c) in the Green's function reduction (2.58) contributes little, and the resulting BS equation becomes a Schrodinger equation with potential

$$V = \left[\frac{\lambda(m_1 + m_2)\pi}{m_1 m_2 E}\right] \frac{\exp(-Mr)}{r} \quad . \tag{3.3}$$

Figure 7 compares the BS solutions with those of the Schrödinger equation. We see that the Schrödinger curves lie consistently below those of the BS equation, although, of course, becoming close in the limit of M and E_B near zero. This is reasonable, since the term neglected in the Green's function is negative, so would tend to reduce $\langle G_o \rangle$ and to increase $\lambda(\sim\langle V^{-1}\rangle/\langle G\rangle)$. Another way of seeing this is to note that for $r \approx 0$ and $\tau = 0$ the Schrodinger Green's function behaves as $(1/r)$, compared with (ℓnr) for the BS equation, so the net effect is that the potential strength is overemphasized. A further common feature of the BS and Schrödinger curves is the threshold shape. Both approximately obey $(\lambda - \lambda_{thresh}) \propto (E - 2)^{\frac{1}{2}}$ for s-waves (the power is $\ell + \frac{1}{2}$ in general). The reason for the similarity lies in the fact that both Schrödinger and BS Green's functions have the same threshold branch cut in k^2.

The order of levels shown in Fig. 5, with the more nearly equal-mass states lying lower, can be intuitively understood as follows. For a short-range potential (i.e., large M) the extent of the wave function is roughly governed by $\exp(-m_2R)$ if $m_1 > m_2(= 2 - m_1)$, so larger m_1 corresponds to more extended wave functions. The corresponding particles feel a smaller average potential, and so will be less tightly bound. Figures 8 and 9 show that there are exceptions to this order if the potential is long-ranged. As M becomes small the levels cross, and eventually the order is exactly reversed.

Increasing E^2 causes the crossing points of Fig. 8 to occur closer to $M = 0$, and if M vanishes, the "crossing" point occurs at threshold (where $\lambda = 0$ or $\frac{1}{4}$).

The behavior of λ with exchange mass M is shown in Fig. 10. That increase in M leads to a decrease in binding is intuitively clear, since, asymptotically, if $R \approx 0$

$$V \sim 4\lambda/R^2 \quad \text{(independent of } M\text{)} \quad , \tag{3.4}$$

and if $R \approx \infty$

$$V \sim 4 \left[\frac{\pi M}{2 R}\right]^{\frac{1}{2}} \frac{\exp(-MR)}{R} \quad , \tag{3.5}$$

i.e., the tail of the exponential is cut off strongly as M grows, but the dominant behavior at the origin is unchanged. The roughly parabolic form of the $\lambda(M)$ curves is also a feature of the one-trial-function formulas of Section IH.

We might remark parenthetically here that the ϕ^3 spectrum is certainly not very physically reasonable when λ is large and positive, since the squares of the bound-state energies become negative. The equation may be modified in various (rather arbitrary) ways to force $E^2 > 0$. For example, we might include a factor $[E/(m_1 + m_2)]$ in the coupling parameter.[22] This factor cuts off the effective coupling strength as E gets near zero.

Both variational methods also provide approximate values for the excited-state eigenvalues, although with somewhat poorer accuracy. Without using matrices larger than 28 X 28 it has been possible to calculate with reasonable accuracy fifth or sixth excited levels. This does not, unfortunately, mean that the wave functions are very reliable for these levels; however, due in part to the variational principle, the eigenvalues clearly converge. The lowest three s states for an equal-mass system are shown in Fig. 11. The spectrum is real, and the states possess definite symmetry under τ-inversion. The $O(4)$ symmetry at $E = 0$ is apparent in the degeneracy of the 0^- and 1^+ states. Both wave functions are

$$f_1(R) \quad Y_{1\ell m}(x) \quad , \tag{3.6}$$

at this point, with $\ell = 0, 1$ for states $0^-, 1^+$. Since the $(0^+)^*$ and 0^- levels intersect (at $m_1 \approx 0.5$) and have opposite "τ parity," the perturbation argument in Section IIC suggests the possible mixing of these levels when the external masses are unequal. Figure 12 shows the same levels with $m_1 \neq m_2$. The upper pair of levels is seen to "collide" and become complex as expected. Figure 12 also shows the lowest 1^+ state, which lies very close to $(0^+)^*$. This level along with the lowest 0^+ level is only slightly affected by the mass alteration. In Fig. 11 the 1^+ and $(0^+)^*$ levels are also seen to cross, but naturally no mixing is possible here.

A comparison of exchange and nonexchange potentials (Fig. 13) confirms two features already suggested earlier. When $E = 0$, the exchange and nonexchange solutions are degenerate, a fact which is obvious due to the $O(4)$ symmetry at this point. Denoting by $\lambda^-(\lambda^+)$ solutions smoothly connected to the solutions for equal-mass systems for which the "time parity" is $+(-)$, we find a real spectrum with $\lambda^- < 0$ and $\lambda^+ > 0$. Finally we notice that the exchange solutions λ_x^+ lie lower than the nonexchange λ^+ for a given coupling constant, and in fact even lower than the equal mass solution. On the other hand $-(\lambda_x^-)$ lies higher than the corresponding equal-mass result.

We close this subsection with some examples of Regge trajectories computed by the methods of Section IIF. The three trajectories of Fig. 14 were calculated for m_1, m_2, and M equal to 1, but the method works for any combination of these masses. The coupling strength was chosen as 16.38 to force the highest trajectory through $\ell = 1$ at $E^2 = 0$. The trajectory labeled P (parent) is the highest-lying ("leading") trajectory and corresponds to the ground state with time-parity $P = +$. Trajectory P^* is simply the first excited level with $P = +$. This is intersected with the "daughter" trajectory D, which has $P = -$. If we now let $m_1 \neq m_2$, we expect these trajectories to mix at this point, thus leading to complex trajectories of the type already seen. A further feature is the spaceing $\ell = 1$ of the parent and daughter at $E^2 = 0$, and is a well-known result of the $O(4)$

symmetry. The corresponding exchange potential trajectories have a parent-daughter spacing of $\Delta \ell = 2$.

We have not computed trajectories above the elastic threshold with the Schwinger technique; however, note that the threshold behavior of these trajectories, being determined from the analytic properties of the Green's function matrix elements, must be the same in character as with the Schrödinger equation, since $G_{Schrödinger}$ and G_{BS} both have the same threshold branch cut, as was earlier noted.

B. Scattering States

1. Examples; "ϕ^3" phase shifts

We have calculated δ with the Schwinger variational principle for a large range of mass ratios and have found excellent convergence throughout. Since the scattering problem has one more free parameter than that for bound states, and since the difference introduced by unequal-mass kinematics is not great, we content ourselves with some examples. Figures 15 and 16 present the $\ell = 0$ and $\ell = 1$ phase shifts for external masses $m_1 = 1.5$, $m_2 = 0.5$. Comparison of these curves with the corresponding equal-mass curves (Figs. 3 and 4 of Ref. 5) show no qualitative difference and very little quantitative difference.

Due to the smoothness of these curves we might expect that a simple effective range formula might well describe the data. Figure 17 confirms this to a large extent; the curves are very well fitted with an effective range formula of the form

$$(k^{2\ell+1}/8\pi E)\cot\delta_\ell \approx a_\ell + b_\ell k^2 + c_\ell k^4 \quad , \tag{3.7}$$

for couplings below those needed for a bound state of angular momentum ℓ. Actually this formula is good even if there is a bound state, provided that the phase shift does not pass through π close to threshold (e.g., $\lambda = 5$, $\ell = 0$). If the simple pole at $\delta = \pi$ is removed the effective range formula may be good even in this region.

Figure 17 clearly shows the development of a p-wave resonance: the intersection of the effective range curve and the k^2 axis first occurs at $\lambda \approx 6$, $k^2 \approx 0.2$ and moves to threshold at $\lambda \approx 8$. For larger couplings we have a p-wave bound state.

2. Levinson's theorem

Levinson's theorem[23] is $\delta(E = E_{thres}, \lambda) = n_B \pi$, where n_B is the number of bound states and $\delta(E = 0, \lambda = 0) = 0$ and has not to our knowledge been proven for the BS equation. The theorem is checked by slowly turning on the coupling strength from zero and demanding (i) that for E not too close to threshold $\delta(E, \lambda)$ be a smooth function of λ, and (ii) that $\delta(E, 0) = 0$. If we then follow the phase shift to threshold at fixed λ, we numerically find that Levinson's theorem is valid, at least for potentials that bind only a small number of bound states.

One stipulation must, however, be made. When the scattering particles have equal mass, then it is possible to construct bound states whose wave functions (ψ_{odd}) have odd time parity. Since we can choose

μ_i such that the incident plane wave is independent of the relative time, such solutions cannot be excited in a scattering process. The corresponding scattering amplitude will not have a pole at the mass of such a particle (or, in other words, the residue of the pole is zero), and such a state will not be counted in the number n_B.

If we now allow the masses to become <u>unequal</u>, ψ_{odd} develops an even time-parity component through the odd part of the Green's function. Such a wave function can couple to the incident wave, so we would expect a pole to develop in the S matrix. Levinson's theorem would appear to require an extra jump of π as we increase λ through the zero binding value for the state. When we calculate phase shifts for unequal-mass scattering, however, we find that these states do not contribute to Levinson's theorem, nor do they appreciably affect the scattering phase shifts (Fig. 15).

This situation is somewhat clarified by decomposing the wave function into its even and odd parts (in τ):

$$\psi = \alpha \psi_e + \beta \psi_o \quad . \tag{3.8}$$

By symmetry it follows that

$$(\psi_e, V \psi_o) = 0, \qquad (\psi_e, V \psi_e) = 1,$$

$$(\psi_o, V \psi_o) = 1, \qquad (\psi_o, V \emptyset) = 0,$$

$$(\psi_e V, \emptyset) = t, \tag{3.9}$$

where we have fixed the normalization of ψ_e and ψ_o arbitrarily. With this normalization the bound-state pole in ψ as a function of energy will appear in the coefficients α and β. The T matrix is $(\psi, V\,\phi)$ or simply αt.

Taking matrix elements of the BS equation and using relations (3.9), we find

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} t \\ 0 \end{pmatrix} + \begin{pmatrix} K_{ee} & K_{eo} \\ K_{oe} & K_{oo} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \tag{3.10}$$

where, for example

$$\begin{aligned} K_{eo} &= \int U_e(x)\, G(x-x')\, U_o(x')\, d^4x\, d^4x' \\ &= \int U_e(x)\, G_{odd}(x-x')\, U_o(x')\, d^4x\, d^4x' \\ &= -\int U_o(x)\, G(x-x')\, U_e(x')\, d^4x\, d^4x' \quad = -K_{oe}, \end{aligned} \tag{3.11}$$

and $U_{e,o} = V\,\psi_{e,o}$. The solution of Eq. (3.10) is

$$\alpha = -\,t(\lambda\, K_{oo} - 1)/\Delta \quad , \tag{3.12}$$

$$\beta = t(K_{eo})/\Delta \quad , \tag{3.13}$$

where

$$\Delta = (\lambda\, K_{ee} - 1)\,(\lambda\, K_{oo} - 1) + (\lambda\, K_{eo})^2 \quad . \tag{3.14}$$

If the bound particles have equal mass, then K_{eo} vanishes, so we find that $\lambda = K_{ee}^{-1}, K_{oo}^{-1}$ correspond to zeros of Δ, hence appear to be bound-state poles of the T matrix (i.e., poles of α). The pole at $\lambda = K_{oo}^{-1}$, however, is an illusion, since the numerator of (3.11) has a simple zero at the same point, which cancels the pole. If we now allow the masses to become slightly unequal the cancellation does not occur, and the T matrix is left with two poles and a zero. For small K_{eo} and (as is usual) $K_{ee} > K_{oo}$ we find

$$(\lambda_{pole} K_{oo} - 1) \approx - [K_{eo}^2/(K_{ee} - K_{oo})] \quad ,$$

$$(\lambda_{zero} K_{oo} - 1) = 0 \quad , \tag{3.15}$$

so $\lambda_{pole} < \lambda_{zero}$. The positions of the singularities are shown schematically in Fig. 18. At a fixed value of λ it is seen that the zero lies between the bound-state pole and the physical scattering region. This explains why the nearby bound-state pole did not have a large effect on the phase shift. Also the absence of a jump of the phase shift by π is not surprising, since the relevant number in most proofs of Levinson's theorem is the <u>difference</u> of the numbers of zeros and poles, which is unchanged when the masses become unequal.[24]

3. <u>Example: "ϕ^4" phase shifts</u>

The interaction $\lambda_s \phi^4$ gives rise to a ladder whose rungs consist of a pair of particles. The corresponding potential is the

product of two propagators of the same argument, namely

$$V(R) \sim \lambda_s \left[\frac{M\ K_1(MR)}{R}\right]^2 \quad .$$

If $R \approx 0$, we see that $V(R) \sim (\lambda_s/R^4)$, which is as singular as the centrifugal term in $L(p)$. As a consequence the potential strength λ_s appears in the indicial equation. As is shown by Bastai et al.,[10] there is a finite region about $\lambda_s = 0$ for which all four roots are real, with two larger than zero and two smaller than 0, as was the case for the eigenvalue problem. On the other hand, if λ_s exceeds a certain value (depending on and the sign of λ_s) at least two roots become complex. This leads to behavior at the origin of the form

$$\psi \sim R^{\gamma_R + i\gamma_I} = R^{\gamma_R}\ [\cos(\gamma_I\ \ell n\ R) + i\ \sin(\gamma_I\ \ell n\ R)] \quad , \tag{3.15}$$

thus the solution has an infinite number of nodes in any neighborhood of $R = 0$, a nonphysical result.

Fixing λ_s in the "allowable" region, we have chosen radial basis functions

$$\left[c_1 R^{\gamma_1} + c_2 R^{\gamma_2}\right] R^k \exp(-\alpha\ R) \quad , \tag{3.16}$$

and where γ_1 and γ_2 are the acceptable roots of the indicial

equation. In order to find bound states, however, we have had to include an additional ϕ^3 interaction (for s waves, we are not even allowed to have $\lambda_s > 0$). Our numerical results were in excellent agreement with the exact special case of Bastai et al.

It was also decided to introduce a cutoff at small distances; this allowed us to use any coupling strength desired. The form chosen was

$$V_{cutoff}(R) = V(R) \left[\frac{R}{R + \Lambda}\right]^2 . \tag{3.17}$$

This turn has the advantage of being relativistically invariant and giving V the same behavior at the origin as the ϕ^3 potential. When the coupling strength is large enough that the singular problem is in the "continuum" range, we expect that the result is sensitive to Λ, a suspicion confirmed by direct calculation (Fig. 19). Fixing Λ arbitrarily at 0.5, we have calculated $\delta(k^2)$ for several values of λ (Fig. 20). The behavior of δ implies that a bound state exists when $\lambda \gtrsim 0.5$, since at this point the phase shift abruptly jumps to π. These curves are very similar to the ϕ^3 results, and can be fitted rather well by effective range formulas up to $k^2 \approx 0.5$.

We note in passing that this potential, being positive definite, cannot produce a negative s-wave effective range such as seems to be observed in π-π $I = 0$ scattering. Such negative effective ranges could be obtained, however, by the addition of a repulsive long-range interaction.

IV. TWO-CHANNEL CALCULATIONS

We can formally generalize the BS equation to systems of N coupled two-body channels by a simple matrix technique similar to the matrix N/D method of Bjorken.[25] Furthermore, by interpreting Eq. (2.71) as a matrix equation in the channel indices, we will be able to apply the Schwinger variational principle to the two-channel problem with only minor modifications.

We define channel "i" to have particles of mass $m_1^{\,i}$, $m_2^{\,i}$ with momenta defined by $(k_i^{\,2} + m_1^{\,i2})^{\frac{1}{2}} + (k_i^{\,2} + m_2^{\,i2})^{\frac{1}{2}} = E$. The free two-body propagator for channel i is denoted by $G_i(x-x')$. The potentials $V_{ij}(x)$ are illustrated in Fig. 21. We will need the matrices (given here for N = 2 for simplicity):

$$\left(P_k(x)\right) = \begin{pmatrix} e^{i(\underset{\sim}{k}_1\cdot\underset{\sim}{x})} & 0 \\ 0 & e^{i(\underset{\sim}{k}_2\cdot\underset{\sim}{x})} \end{pmatrix}, \tag{4.1}$$

$$\left(E(\tau)\right) = \begin{pmatrix} e^{-\nu_1\tau} & 0 \\ 0 & e^{-\nu_2\tau} \end{pmatrix}, \tag{4.2}$$

$$(\rho_E) = (\tilde{\rho}_E)(\theta_E) = \begin{pmatrix} k_1/8\pi E & 0 \\ 0 & k_2/8\pi E \end{pmatrix}\begin{pmatrix} \theta(k_1^{\,2}) & 0 \\ 0 & \theta(k_2^{\,2}) \end{pmatrix}, \tag{4.3}$$

$$\left(G(x,x')\right) = \begin{pmatrix} G_1(x,x') & 0 \\ 0 & G_2(x,x') \end{pmatrix} = E(\tau)\, G(x,x')\, E(-\tau') \quad , \tag{4.4}$$

$$\left(V(x)\right) = \begin{pmatrix} V_{11}(x) & V_{12}(x) \\ V_{21}(x) & V_{22}(x) \end{pmatrix} \quad , \tag{4.5}$$

and

$$\left(V'(x)\right) = \left(E(-\tau)\right)\left(V(x)\right)\left(E(\tau)\right) \quad . \tag{4.6}$$

In the above, $\nu_i = \mu_2\, \omega_1^{\,i} - \mu_1\, \omega_2^{\,i}$ and $k_1 \equiv |\underset{\sim}{k}_1|$. The BS wave function is denoted by

$$\left(\psi_k(x)\right) = \begin{pmatrix} \psi_1^{\,1}(x) & \psi_1^{\,2}(x) \\ \psi_2^{\,1}(x) & \psi_2^{\,2}(x) \end{pmatrix} , \tag{4.7}$$

where the superscript labels the incident beam channel and the subscript the channel consisting of the "legs" of the wave function.

The wave function satisfies the equation

$$\left(\psi_k(x)\right) = \left(E(\tau)\right)\left(P_k(x)\right) + \int \left(G'(x,x')\right)\left(V(x')\right)\left(\left(\psi_k(x')\right)\right) d^4x' , \tag{4.8}$$

which can be rewritten as

$$\left(\psi_k'(x)\right) = \left(P_k(x)\right) + \int \left(G'(x,x')\right)\left(V'(x)\right)\left(\psi_k'(x')\right) d^4x' \quad , \tag{4.9}$$

where

$$\left(\psi_k'(x)\right) = \left(E(-\tau)\right)\left((\psi_k(x)\right) \quad . \tag{4.10}$$

We will drop the primes on ψ', G', and V', understanding that they are always present.

We next present the generalization of Section IVB of Ref. 5 to multichannel systems. The main points of interest are the definition of the scattering amplitude and the proof of unitarity below all three-body thresholds. To facilitate comparison to Ref. 5 we use the same format. From Eq. 7 and the asymptotic form of the Green's function we see:

(A) $$\left(\psi_k(x)\right) = \left(P_k(x)\right) + \int \left(G(x,x')\right)\left(V(x')\right)\left(\psi_k(x')\right) d^4x' \quad .$$

(B) $$\left(f(\underset{\sim}{k}' \leftarrow \underset{\sim}{k})\right) = (8\pi E)^{-1} \int \left(P_{k'}(-x)\right)\left(V(x)\right)\left((\psi_k(x)\right) d^4x \quad .$$

(C) $$\left(\psi_{-k}(-x,\tau)\right) = \left(\psi_k(x,\tau)\right) \quad [\text{use } G(x,\tau;\, x',\tau') = G(x',\tau;\, x,\tau')] \; .$$

(D) $$\left(G(x,x')\right) = \left(G(-x',-x)\right) \quad .$$

(E) $$(1/2i)\left\{\left(G(x,x')\right) - \left(G(x,x')\right)^*\right\} = \int \frac{d\Omega_{\hat{k}''}}{4\pi}\left(P_{k''}(x)\right)(\rho_E)\left(P_{k''}(x')\right)^* .$$

Note that below the $\underline{i}$th threshold G_i becomes real, and this is taken care of on the right by the Θ functions.

(F) $V(x)$ is real and invariant under $\underset{\sim}{x} \to -\underset{\sim}{x}$; we also have

$$\left(V(x)\right) = \left(V(-x)\right)^T \quad .$$

(G) If the mass ratios are the same in each channel then $\left(V(x)\right)$ is symmetric, since $\nu_1 - \nu_2 = 0$.

Because of (D) and (F), we have the adjoint equation

$$\text{(H)} \quad \left(\psi_{k'}(-x)\right)^T = \left(P_{k'}(x)\right)^T + \int \left(\psi_{k'}(-x')\right)^T \left(V(x')\right)\left(G(x',x)\right) d^4x' \quad ,$$

and the corresponding form for the scattering amplitude,

$$\text{(I)} \quad \left(f(\underset{\sim}{k}' \leftarrow \underset{\sim}{k})\right) = (8\pi E)^{-1} \int \left(\psi_{k'}(-x)\right)^T \left(V(x)\right)\left(P_k(x)\right) d^4x$$

(use A, B, and H to prove this relation). Note that the potential is not transposed in these expressions. From (B), (C), and (I) we have

$$\text{(J)} \quad \left(f(\underset{\sim}{k}' \to \underset{\sim}{k})\right) = \left(f(-\underset{\sim}{k}' \to -\underset{\sim}{k})\right) \quad ,$$

$$\text{(K)} \quad \left(f(\underset{\sim}{k}' \to \underset{\sim}{k})\right) = \left(f(\underset{\sim}{k} \leftarrow \underset{\sim}{k}')\right)^T \quad .$$

To prove unitarity:

(i) multiply (A) by $\left(\psi_{k'}(-x)\right)^\dagger \left(V(x)\right)$ from the left and integrate over x,

(ii) multiply the complex conjugate of (H) by $\left(V(x)\right)\left((\psi_k(x)\right)$ from the right and integrate over x,

(iii) subtract the results of 1 and 2, and use (B), (I), and (J) to get

$$(1/2i)\left\{\left(f(\underset{\sim}{k}' \leftarrow \underset{\sim}{k})\right) - \left(f(\underset{\sim}{k}' \leftarrow \underset{\sim}{k})\right)^*\right\} =$$

$$\iint \left(\psi_{-k'}(-x')\right)^\dagger \left(V(x')\right) \operatorname{Im}\left(G(x,x')\right)\left(V(x)\right)\left(\psi_k(x)\right) d^4x\, d^4x' \quad .$$

Finally, by using (E) we reduce the right-hand side to

$$\text{(L)} \quad 8\pi E \int \frac{d\Omega_{k''}}{4\pi} \left(f(\underset{\sim}{k}' \leftarrow \underset{\sim}{k}'')\right)(\rho_E)\left(f(\underset{\sim}{k}'' \leftarrow \underset{\sim}{k})\right) \quad ,$$

which is the standard result. Using (K), we could also write

$$\left(f(\underset{\sim}{k}' \leftarrow \underset{\sim}{k})\right)^* = \left(f(\underset{\sim}{k} \leftarrow \underset{\sim}{k}')\right)^\dagger = \left(f(\underset{\sim}{k}' \leftarrow \underset{\sim}{k})\right)^\ddagger \quad ,$$

where $\dagger$ stands for conjugation and <u>matrix</u> transpose, and $\ddagger$ also includes the transpose of $\underset{\sim}{k}$ and $\underset{\sim}{k}'$.

The T matrix is defined (on shell) by

$$\text{(M)} \quad (T_{\underset{\sim}{k}'\underset{\sim}{k}}) = 8\pi E \left(f(\underset{\sim}{k}' \leftarrow \underset{\sim}{k})\right) \quad ,$$

and in partial-wave form becomes

$$\text{(N)} \quad (T_{\underset{\sim}{k}'\underset{\sim}{k}}) = \sum_\ell (2\ell + 1)\, P_\ell(\hat{\underset{\sim}{k}}' \cdot \hat{\underset{\sim}{k}}) \left(T_\ell(E)\right) \quad .$$

Unitarity now reads

$$\text{(O)} \quad (1/2i)\left\{\left(T_\ell(E)\right) - \left(T_\ell(E)\right)^\dagger\right\} = \left(T_\ell(E)\right)^\dagger \rho_E \left(T_\ell(E)\right) \quad .$$

As in the single-channel problem, we define

$$\left(U(x)\right) \equiv \left(V(x)\right)\left(\psi(x)\right) \quad \text{and} \quad \left(\bar{U}(x)\right) \equiv \left(U(-x)\right) \quad . \tag{4.11}$$

It follows that

$$\left(\psi(-x)\right)^T \left(V(x)\right) = \left\{\left(V(-x)\right)\left(\psi(-x)\right)\right\}^T = \left(\bar{U}(x)\right)^T \quad , \tag{4.12}$$

and so we can construct a variational principle for the scattering amplitude of the form (schematically)

$$8\pi E\left(f(\underset{\sim}{k}' \leftarrow \underset{\sim}{k})\right) = \int (\bar{P}_{k'})^T (U_k) + \int (\bar{U}_{k'})(P_k)$$

$$+ \iint (\bar{U}_{k'})^T (G)(U_k) - \int (\bar{U}_{k'})^T (V^{-1})(U_k) \quad . \tag{4.13}$$

Equations (A) and (H) assure us that this expression is stationary.

As an example, let us assume that $\nu_1 = \nu_2$ to remove the explicit dependence of V_{12} on τ. We further assume that the potential is

$$(V_{11}/\lambda_1) = (V_{22}/\lambda_2) = (V_{12}/\lambda_3) = V \quad , \tag{4.14}$$

where

$$V(x) = (4\ M/R)\ K_1(M\ R) \quad , \tag{4.15}$$

as would be the case if the exchanged meson were always the same particle. An example of a ladder generated by this model is shown in Fig. 21.

As was earlier the case, we can reduce the double integral of the Green's function to a single integral by use of the Fourier transform,

$$\tilde{U}_{ik}{}^{j}(p) = \int d^4x \, e^{-ip\cdot x} \, U_{ik}{}^{j}(x) \quad . \tag{4.16}$$

This leads to the variational expressions (suppressing the k, k' subscripts)

$$[T_{11}] = \tilde{U}_1{}^1(\underset{\sim}{k}_1',0) + \tilde{U}_1{}^{1'}(\underset{\sim}{k}_1,0) + \int [\tilde{U}_1{}^{1'}(p) \, G_1(p) \, \tilde{U}_1{}^1(p) + \tilde{U}_2{}^{1'}(p) \, G_2(p) \, \tilde{U}_2{}^1(p)] \frac{d^4p}{(2\pi)^4}$$

$$+ \int [\lambda_2 \bar{U}_1{}^{1'}(x) \, V^{-1}(x) \, U_1{}^1(x) - \lambda_3 \, \bar{U}_1{}^{1'}(x) \, V^{-1}(x) \, U_2{}^1(x)$$

$$- \lambda_3 \, \bar{U}_2{}^{1'}(x) \, V^{-1}(x) \, U_1{}^1(x) + \lambda_1 \, \bar{U}_2{}^{1'}(x) \, V^{-1}(x) \, U_2{}^1(x)] d^4x \quad , \tag{4.17}$$

with similar expressions for T_{12}, T_{21}, and T_{22}. Each component of U is expanded in partial waves:

$$U_1^{\,1}(x) = \sum_{\ell} (2\ell + 1)i^{\ell}\, u_{\ell 1}^{\,1}(x)\, P_{\ell}(\hat{\underset{\sim}{x}}\cdot\hat{\underset{\sim}{k}}_1) \quad ,$$

$$U_1^{\,1'}(x) = \sum_{\ell} (2\ell + 1)i^{\ell}\, u_{\ell 1}^{\,1}(x)\, P_{\ell}(\hat{\underset{\sim}{x}}\cdot\hat{\underset{\sim}{k}}_1') \quad ,$$

$$U_1^{\,2'}(x) = \sum_{\ell} (2\ell + 1)i^{\ell}\, u_{\ell 1}^{\,2}(x)\, P_{\ell}(\hat{\underset{\sim}{x}}\cdot\hat{\underset{\sim}{k}}_2') \quad , \tag{4.18}$$

$$\vdots$$

where $u_{\ell j}^{\,i}(x)$ is a function of $|\underset{\sim}{x}|$ and τ only.

The functions u_{ℓ} are then expanded in trial functions $\phi_i(x)$, which we choose to be the same as in Section IID:

$$u_1^{\,1}(x) = \sum_i c_i\, \phi_i(x), \qquad u_1^{\,2}(x) = \sum_i c_i'\, \phi_i(x) \quad ,$$

$$u_2^{\,1}(x) = \sum_i d_i\, \phi_i(x), \qquad u_2^{\,2}(x) = \sum_i d_i'\, \phi_i(x) \tag{4.19}$$

where $i = 1,2,\cdots,N$. Substituting (4.19) into (4.17) then leads to algebraic equations of the form

$$[T_{\ell 11}] = 2\sum_i c_i\, \tilde{u}_i(k_1,0) + \sum_{i,j}\{c_i\, c_j\, (G_1)_{ij}$$

$$+ d_i\, d_j (G_2)_{ij} + \lambda_2[c_i\, c_j - \lambda_3(c_i\, d_j + d_i\, c_j)$$

$$+ \lambda_1\, d_i\, d_j](V^{-1})_{ij}\} \quad , \tag{4.20}$$

with similar expressions for the other components of T. The conditions that these expressions can be stationary are $(d[T_{\ell 11}])/(dc_i) = 0, \cdots,$ and lead to the matrix equation

$$(\tilde{u}) + (G)(C) - (V)(C) = 0 \quad , \tag{4.21}$$

where (G) and (V) are 2N X 2N dimensional matrices defined by

$$(G) = \begin{pmatrix} (G_1) & 0 \\ 0 & (G_2) \end{pmatrix}, \quad (G_k)_{ij} = \int \tilde{u}_i(p)\, G_k(p)\, \tilde{u}_k(p)\, \frac{d^4p}{(2\pi)^4} \quad , \tag{4.22}$$

$$(V) = \begin{pmatrix} \lambda_2(V^{-1}) & -\lambda_3(V^{-1}) \\ -\lambda_3(V^{-1}) & \lambda_1(V^{-1}) \end{pmatrix}, (V^{-1})_{ij} = \int u_i(-x)\, V^{-1}(x)\, u_j(x)\, d^4x \ , \tag{4.23}$$

and $(\tilde{u})$ and (C) are 2 X 2N dimensional matrices given by

$$(C) = \begin{pmatrix} (c) & (c') \\ (d) & (d') \end{pmatrix} \quad , \qquad (c) = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{pmatrix} \quad , \cdots \tag{4.24}$$

$$(\tilde{u}) = \begin{pmatrix} [\tilde{u}(k_1,0)] & 0 \\ 0 & [\tilde{u}(k_2,0)] \end{pmatrix}, \quad [\tilde{u}(k_1,0)] = \begin{pmatrix} \tilde{u}_1 \\ \tilde{u}_2 \\ \vdots \\ \tilde{u}_N \end{pmatrix} \ , \ \cdots \tag{4.25}$$

The T matrix is then reconstructed by Eq. 4.19:

$$T_{\ell 11} = \sum_i c_i \, \tilde{u}_i(k_1,0) \qquad \text{(the other terms cancel by 4.21)},$$

$$T_{\ell 12} = \sum_i d_i \, \tilde{u}_i(k_2,0) \quad ,$$

$$T_{\ell 21} = \sum_i c_i' \, \tilde{u}_i(k_1,0) \quad ,$$

$$T_{\ell 22} = \sum_i d_i' \, \tilde{u}_i(k_2,0) \quad . \tag{4.26}$$

As in the one-channel calculation we have found it convenient to deal with the K matrix instead of the T matrix. The foregoing equations are converted to stationary expressions for the K matrix by simply interpreting the Green's function integral as a principal-value integral. The only tricky point is that we will need the matrix element for the Green's function with the higher threshold in the region between elastic thresholds. The analytic continuation of the principal-value integral is achieved by noting that, above both thresholds,

$$P \int [\tilde{u}(p)]^2 \, G(p) \, \frac{d^4p}{(2\pi)^4} = \int [\tilde{u}(p)]^2 \, G(p) \, \frac{d^4p}{(2\pi)^4}$$

$$- \; i[\tilde{u}(k,0)]^2 \, \frac{k}{8\pi \, E} \quad . \tag{4.27}$$

The continuation to below the threshold is found by simply continuing the two terms on the right.

In terms of the K matrix, which is given (on mass shell) by

$$(K_\ell) = (T_\ell)\,[(1) + i(\tilde{\rho})\,(T_\ell)]^{-1} \quad , \tag{4.28}$$

the unitarity relation (L) is equivalent to the statement: (K_ℓ) is a Hermitian matrix. Furthermore, Eq. (K) can be used to prove that (T_ℓ) [and hence (K_ℓ)] is a symmetric matrix. This proves that (K_ℓ) is a real, symmetric matrix.

The procedure which we follow is to evaluate (K_ℓ) through the variational principle just described, and then to reconstruct (T_ℓ), using definition (P). The resulting (T_ℓ) is then used to compute the phase shifts δ_i and the mixing angle ϵ [or inelasticity parameter $\eta \equiv \cos(2\epsilon)$]. These are defined by

$$(S_\ell) = (1) + 2i(\rho^{\frac{1}{2}})(T_\ell)(\rho^{\frac{1}{2}}) = \begin{pmatrix} \cos(2\epsilon)e^{2i\delta_1} & i\,\sin(2\epsilon)e^{i(\delta_1+\delta_2)} \\ i\,\sin(2\epsilon)e^{i(\delta_1+\delta_2)} & \cos(2\epsilon)e^{2i\delta_2} \end{pmatrix}. \tag{4.29}$$

Figure 22 shows a typical set of scattering phase shifts and the corresponding inelasticity parameter. Both channel coupling strengths are below that needed for forming a bound state. As the coupling between the channels is slowly turned on, a threshold cusp is seen to

develop. If the second channel coupling (λ_2) is increased, we find that a bound state can be formed at the second threshold (provided that $\lambda_3 = 0$). This is reflected in a jump of π in δ_2. Further increase of λ_2 causes the bound state to move between the two thresholds. The lower channel phase shift δ_1 is unchanged provided that $\lambda_3 = 0$. If we now coupled the channels, the "bound state" can now decay into channel 1, and δ_1 is dramatically altered. This behavior is illustrated in Fig. 23. For $\lambda_3 = 0.1$ we see that the phase shift rapidly rises by π as E passes the "bound state" energy $E \approx 2.28$. This would give rise to a spike in the cross section at this energy; its narrowness indicates a long-lived resonance and is characteristic of a weak coupling between channels. As λ_3 is increased, δ_1 rises less rapidly and the resonance becomes broader. When λ_3 increases to 1.0 we see that the pseudo bound state moves below the lowest threshold and becomes absolutely stable. We further note that Levinson's theorem remains valid in this example if we use the form $\delta_1(k_1=0) + \delta_2(k_2=0) = n_{bound}\,\pi$, where we have chosen $\delta_i \to 0$ as $\lambda_i \to 0$ if $k^2 > 0$.

V. SUMMARY AND OUTLOOK

From a numerical point of view, perhaps the most interesting conclusion is that the Schwinger and Rayleigh-Ritz variational principles as formulated in Refs. 4 and 5 can be extended without loss of convergence properties to the cases of unequal-mass systems and systems with two two-body channels. Provided care is taken concerning the choice of trial functions, the cases of general ℓ and marginally singular potentials also present few problems. One rather striking result is the relatively weak dependence of the level structure and phase shifts on the mass ratio of the external particles provided that M is not too small and (for excited-state spectra) the binding energy is not too great. The similarity of phase shifts for equal- and unequal-mass systems was particularly surprising for coupling strengths of a size such as to barely bind a 0^- state, since this state couples to the scattering amplitude only in the unequal-mass case. The one major difference between equal- and unequal-mass spectra was the appearance of complex levels in the latter case.

Only the first few steps have been taken to understand the Bethe-Salpeter equation; a tremendous amount needs to be done before any deep understanding of the equation can be achieved. For example, what is the effect of adding higher-order terms to the kernel? How does one add the direct-channel poles to the kernel without encountering renormalization problems? How does one calculate the scattering amplitude at arbitrary energies (in practice!)? etc. Even more difficult

(and promising) appear to be the recent formulations of the BS equation in a fully crossing-symmetric form. But how does one solve such equations?

VI. ACKNOWLEDGMENTS

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APPENDIX I.

First we summarize some properties of the angular functions. A set of basis functions in θ is

$$Z_{n\ell}(\cos\theta) = \left\{2^{2\ell+1}(n+1)(n-\ell)!/\pi\Gamma(n+\ell+2)\ \left[\Gamma(\ell+1)\right]\right\}^{\frac{1}{2}}$$

$$\times\ \sin^{\ell}\theta\ C^{\ell+1}_{n-\ell}(\cos\theta) \quad \text{for} \quad n-\ell = 0,1,2,\cdots. \tag{A1.2}$$

The square root in (A1.2) will be denoted by $N_{n\ell}$. These functions satisfy the orthogonality relation

$$\int_0^{\pi} d\theta\ \sin^2\theta\ Z_{n\ell}(\cos\theta)\ Z_{n'\ell}\cos\theta) = \delta_{nn'}\ , \tag{A1.3}$$

and are solutions of the Laplace equation

$$\Box\, g_n(R)\ Y_{n\ell m}(x) = Y_{n\ell m}(x)\left(\frac{\partial^2}{\partial R^2} + \frac{3}{R}\ \frac{\partial}{\partial R} - \frac{n(n+2)}{R^2}\right)\ g_n(R)\ . \tag{A1.4}$$

In (A1.4) we have set $Y_{n\ell m}(x) = Z_{n\ell}(\cos\theta)\ Y_{\ell m}(\vartheta,\varphi)$ if ℓ is integral; however, (A1.2-3) are valid whether or not ℓ is integral, provided that $n-\ell$ is.

We now summarize the effect of $\Box$ and $\partial/\partial\tau$ on the basis functions. For notational convenience we take $\phi_k = R^k\exp(-\sigma R)$ and $\phi_{kn} = R^{n+k}K_k(R)$. Then Eq. (A1.4) implies

$$\Box\,(\phi_k\ Z_{n\ell}) = [T_1\phi_k + T_2\phi_{k-1} + T_3(k,n)\phi_{k-2}]\ \ Z_{n\ell}\ , \tag{A1.5a}$$

$$\Box\,(\phi_{kn}\ Z_{n'\ell}) = [S_1(n,n')\phi_{kn-2} + S_2(n)\phi_{k-\ell n} + \phi_{k-2,n+2}]\ Z_{n'\ell}\ , \tag{A1.5b}$$

where

$$T_1 = \alpha^2, \quad T_2(k) = -(2k+3), \quad T_3(k,n) = k^2 + 2k - n(n+2)$$

and

$$S_1(n,n') = n(n+2) - n'(n'+2), \qquad S_2(n) = -2(n+2).$$

The effect of $ip_4(=\partial/\partial\tau)$ is calculated from the recursion formulas for the Gegenbauer polynomials and definition (A1.2), from which follow

$$(\partial/\partial\tau)[\phi_k Z_{n\ell}] = [-\alpha\phi_k + (k+n+2)\phi_{k-1}] Z_{n-1\ell}A_{n\ell}$$

$$+ [-\alpha\phi_k + (k-n)\phi_{k-1}] Z_{n+1\ell}A_{n+1\ell} \qquad \text{(A1.6a)}$$

and

$$(\partial/\partial\tau)[\phi_{kn}Z_{n'\ell}] = [(n+n'+2)\phi_{kn-1} - \phi_{k-1,n+1}] Z_{n'-1\ell}A_{n'\ell}$$

$$+ [(n-n')\phi_{kn-1} - \phi_{k-1,n+1}] Z_{n'+1\ell}A_{n'+1\ell}, \qquad \text{(A1.6b)}$$

where $A_{00} = 0$ and $A_{n\ell} \doteq [(n-\ell)(n+\ell+1)/4n(n+1)]^{\frac{1}{2}}$.

From (A1.5) and (A1.6) the effect of the operator L on a basis function is easily computed.

APPENDIX II.

As an example of the inclusion of an internal symmetry to the BS equation, we consider the case in which the external particles are identical and possess isospin $1[\phi_i]$ and the exchanged particle has isospin 0, 1, or 2 $[\sigma_{ij}]$. The Lagrangian density can be written as a sum of the terms

$$\mathcal{L}^I(x) = g^I \sum_{i,j,k,\ell} \phi_i(x)\phi_j(x)\, P^{ij}_{k\ell}(I)\, \sigma_{k\ell}(x) .$$

The projection operator $P(I)$ for isospin I is defined by

$$P^{ij}_{k\ell}(0) = (1/3)\, \delta_{ij}\, \delta_{k\ell} ,$$

$$P^{ij}_{k\ell}(1) = (1/2)\, (\delta_{ki}\, \delta_{\ell j} - \delta_{\ell i}\, \delta_{kj}) ,$$

$$P^{ij}_{k\ell}(2) = (1/2)\, (\delta_{ki}\, \delta_{\ell j} + \delta_{\ell i}\, \delta_{kj}) - P^{ij}_{k\ell}(0) .$$

These operators satisfy the symmetry properties

$$P^{ji}_{k\ell}(I) = (-)^I\, P^{ij}_{k\ell}(I) ,$$

$$P^{\ell j}_{ki}(I) = \sum_{I'} \beta_{II'}\, P^{ij}_{k\ell}(I')$$

where (β) is the standard crossing matrix

$$\begin{pmatrix} 1/3 & 1 & 5/3 \\ 1/3 & -1/2 & -5/6 \\ 1/3 & 1/2 & 1/6 \end{pmatrix} ,$$

hence we find that I' exchange results in a BS equation for a system

of isospin I of the form

$$\psi^{I}(x) = \lambda\beta_{II'}\,[1 + (-)^{\ell+I}] \int G_o(x-x')\, V(x')\, \psi^{I}(x')\, d^4x'.$$

The factor $[1 + (-)^{\ell+I}]$ results from the sum of the direct and exchange graphs.

APPENDIX III. Analytic Continuation by Variable Change.

We have seen that the analytic continuation of an integral $I(k)$, where k is a parameter, into the region in which the defining integral diverges can often be achieved by the use of recursion formulas (see for example Eq. (2.87)). Another very powerful way of extracting singularities is illustrated by considering the integral $\int_0^\infty \exp(a\,x)\,dx$, which converges to $(-1/a)$ if $a<0$, and diverges if $a>0$. The change of variable $y = \exp(a\,x)$ leads to $(-1/a)\int_0^1 dy$. The integral now converges for all a, as the change of variable has extracted the pole $(-1/a)$ which was causing the divergence. The preceding example becomes non-trivial if we had instead $\int_0^\infty f(x)\,dx$ where f is an entire function which has the asymptotic behavior $f(x) \underset{x\to\infty}{\sim} \exp(a\,x)$; the same variable change extracts the pole at the origin and leaves a convergent integral.

As a second example consider the integral

$$\int_0^\pi d\theta \left[(x^2 - k^2)^2 + f(\theta)\right]^{-1} \qquad (C.1)$$

where $f(\theta) = (\theta - \theta_0)^2\, g(\theta)$, $0<\theta_0<\pi$ and $g(\theta)$ being non-zero in the interval $(0, \pi)$. An integral of this form was encountered earlier (Green's function matrix element), however, we were able to evaluate the integral by contour methods. If the renormalization correction is applied to the denominator, $f(\theta)$ becomes quite complicated, and it is no longer easy to find a closed form expression which can be analytically continued to energies above threshold. If we let

$$\theta - \theta_o = \left[g(\theta_o)\right]^{-\frac{1}{2}} (x^2 - k^2) \tan \xi \qquad (C.2)$$

$$\bar{g}(\xi) = g(\theta)$$

then the integral becomes

$$(x^2 - k^2)^{-1} \; g(\theta_o)^{-\frac{1}{2}} \int_L^U d\xi \left\{\cos^2\xi + \sin^2\xi \left[\bar{g}(\xi)/\bar{g}(0)\right]\right\}^{-1} \qquad (C.3)$$

where

$$L = \tan^{-1}\left(\frac{-\theta_o \, g(\theta_o)^{\frac{1}{2}}}{x^2 - k^2}\right) \quad \text{and} \quad U = \tan^{-1}\left(\frac{(\pi - \theta_o)\, g(\theta_o)^{\frac{1}{2}}}{x^2 - k^2}\right)$$

The resulting integral can now be evaluated numerically. This procedure has removed the pole $1/(x^2 - k^2)$ but does not extract the branch cut which is present in the exact result.

FOOTNOTES AND REFERENCES

* This work was done under auspices of the United States Atomic Energy Commission.

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Values and Slopes of Daughter Trajectories for the Bethe-Salpeter Equation," University of California at Santa Barbara, preprint. In their article λ and E are fixed and real, and the orbital angular momentum is allowed to become complex.

10. For bubble exchange diagrams arising from ϕ^4 interactions the indicial equation is a function of the potential strength. The resulting values of γ_n are nonintegral, and may in fact become complex if the potential is too strong (in analogy to the Dirac equation with Coulomb potential). Our method has been applied only for potential strengths small enough that all four roots of the indicial equation are real. Again only two solutions are finite at the origin. See A. Bastai, L. Bertocchi, S. Fubini, G. Furlan, and M. Tonin, Nuovo Cimento 30, 1512 (1963).

11. This set of trial functions is used for expansion of ψ. When we deal with the Schwinger variational principle, the function $V\psi$ will be expanded (Section E), in which case the index k takes on values $-1, 0, 1, 2, \cdots$.

12. M. J. Levine, J. Wright, and J. A. Tjon, Phys. Rev. 157, 1416 (1967).

13. This theorem is proved for potential theory in Roger C. Newton, Scattering Theory of Waves and Particles (McGraw Hill Book Co., New York, 1966), p. 326.

14. R. W. Haymaker, Phys. Rev. 165, 1790 (1968).

15. For the Fourier transform of $G(x)$ to exist, we must first un-Wick-rotate or go to an energy below the elastic threshold so that $G(x)$ decays in all directions.

16. We have followed the general approach outlined in Ref. 13, p. 298, for potential theory.

17. The function $G_R(x)$ is the analytic continuation of the function defined in the scattering region by Eq. 2.5. Below threshold it is not simply the real part of $G(x)$.

18. This calculation was first suggested to me by D. Z. Freedman (Institute for Advanced Study, Princeton, New Jersey), private communication. A calculation of trajectories using the general method outlined here has recently appeared (with a simplified potential) in R. Cutkosky and B. B. Deo, Phys. Rev. Letters 19, 1256 (1967).

19. Since the asymptotic behavior of the trajectory function $\ell(E)$ is known [see, for example, B. W. Lee and R. F. Sawyer, Phys. Rev. 127, 2266, 2274 (1962)], and since the threshold behavior is the same as encountered in Schrödinger theory [see (2.58)], it may be possible to extrapolate our solutions to the region above threshold by means of Pade approximants. Such a procedure has been successful with Schrödinger theory. (R. W. Haymaker, University of California, Santa Barbara, private communication.)

20. We have earlier presented the Schrödinger analog of such a calculation: Phys. Rev. 164, 1991 (1967).

21. Appendix III gives an indication of how such variable changes can be used to extract singularities.

22. R. H. Dalitz, Symmetries and the Strong Interactions, in Proceedings of the XIIIth International Conference on High Energy Physics, 1966

(University of California Press, Berkeley, California, 1967).

23. This form of Levinson's theorem [N. Levinson, Kgl. Danske Videnskab. Selskab Mat.-Fys. Medd. 25, No. 9 (1949)] was suggested to me by Professor C. Zemach.

24. In the N/D formalism the difference of the number of (CDD) poles and zeros of the D function only is counted. The correlation between the N/D and our results then suggests that the zero of α corresponds to a CDD pole, since if the zero were included in the N function it would not contribute to Levinson's theorem.

25. D. J. Bjorken, Phys. Rev. Letters 4, 473 (1960).

FIGURE CAPTIONS

Fig. 1. Ladder Series. The exchange (b) differs from the nonexchange (a) ladder by the switching of particles 1 and 2 between successive rungs.

Fig. 2. Spectrum at $E = 0$. For each n there is a sequence of degenerate "daughters" differing only in ℓ value. If $E \neq 0$ this degeneracy is destroyed.

Fig. 3. Model Collision. This is the type of collision between two real eigenvalues as a function of $m_1 (= 2 - m_2)$ predicted by Eq. 2.38.

Fig. 4. Computed Collision. The parameters are $M = 1.0$, $E = 1.4$.

Fig. 5. Ground-State Eigenvalues $(\ell = 0)$. The exchange mass is 1. The curves are labelled by $m_1 (= 2 - m_2)$.

Fig. 6. Ghosts. The eigenvalues pass smoothly into the region $E^2 < 0$ as the coupling increases.

Fig. 7. Schrodinger vs. Bethe-Salpeter. The Schrödinger points were calculated with potential (3.3). Note that the curves approach each other only if both binding energy and exchange mass are small.

Fig. 8. Effect of Unequal Mass. Relative orientation of the levels for different external masses $m_1 = (2 - m_2)$ as a function of exchange mass M. Note the reversal of order as M becomes small. The energy is fixed at 1.4. As E approaches threshold, the crossing points approach $M = 0$.

Fig. 9ab. Effect of Unequal Mass. A different representation of the trends seen in Fig. 8. Here E is zero. The $M = 0$, $m_1 = 2$ point is from Ref. 3.

Fig. 10. Variation with Exchange Mass. Curves are labelled by energy of system. Subscript e means $m_1 = m_2 = 1$ and subscript u means $m_1 = 0$, $m_2 = 2.0$.

Fig. 11. Excited Levels I. These curves represent the lowest four levels of the equal-mass $(m_1 = m_2 = M = 1)$ Bethe-Salpeter equation. Note: (a) the 0^-, 1^+ degeneracy at zero energy, (b) the 0^-, $(0^+)^*$ degeneracy at $E^2 = 0.5$, and (c) the cusplike behavior at $E^2 = 4$ for 0^+ and $(0^+)^*$ levels (but not for the 0^- level).

Fig. 12. Excited Levels II. This figure is the analog of Fig. 11 for unequal-mass kinematics $(m_1 = 0.4,\ m_2 = 1.6,\ M = 1)$. Note the mixing of the 0^- and $(0^+)^*$ levels. The upper graph gives $\mathrm{Im}\,\lambda$ for the mixed levels. The 1^+ state has been omitted.

Fig. 13. Excited Levels III. The effect of the exchange potential (curves with subscript X) are shown here. Note that the spectrum is real if an exchange potential is used. The parameters are $m_1 = 1.5$, $m_2 = 0.5$, and $M = 1.0$.

Fig. 14. Regge Trajectories. The trajectories shown are for equal-mass systems $(m = 1)$. The coupling has been fixed to place the parent (P) through $\ell = 1$ at zero energy.

Fig. 15. Scattering Phase Shifts I. ϕ^3 phase shifts for s waves are given here for $m_1 = 1.5$, $m_2 = 0.5$, and $M = 1.0$. The parameter is λ, the coupling strength. Bound states occur at 0.8222 (0^+), 6.772 (0^{+*}) and 12.49 (0^-).

Fig. 16. Scattering Phase Shifts II. This is the $\ell = 1$ analog of Fig. 15.

Fig. 17. Effective Range Curves. Eq. 3.7 is plotted vs. k^2 for $\ell = 1$, $m_1 = 1.5$, $m_2 = 0.5$, and $M = 1.0$. Note the roughly parabolic form.

Fig. 18. Pole and Zero of T. Matrix.

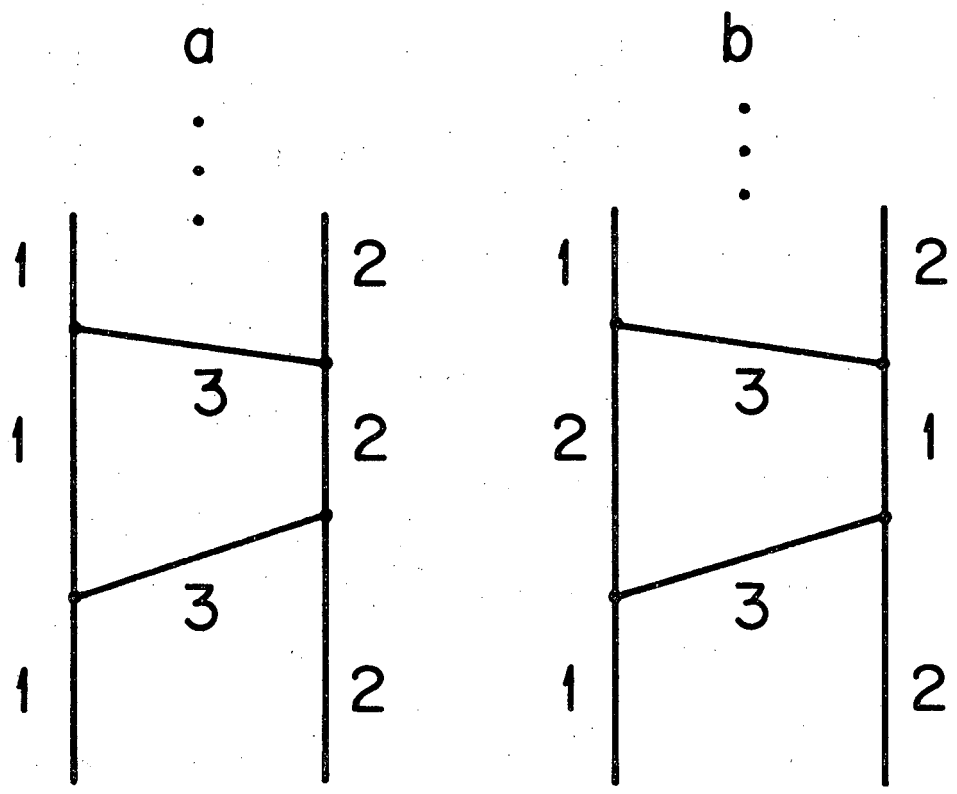
Fig. 19. Scattering Phase Shifts III. Effect of variation of cutoff. For $\lambda > 0$ the problem without cutoff has a continuous spectrum so we expect (and find) large variation with cutoff.

Fig. 20. Scattering Phase Shifts IV. S-wave equal mass ($m = 1$), ϕ^4 phase shifts with cutoff parameter $\Lambda = 0.5$ are given.

Fig. 21. Ladder Generated by Eq. 4.8.

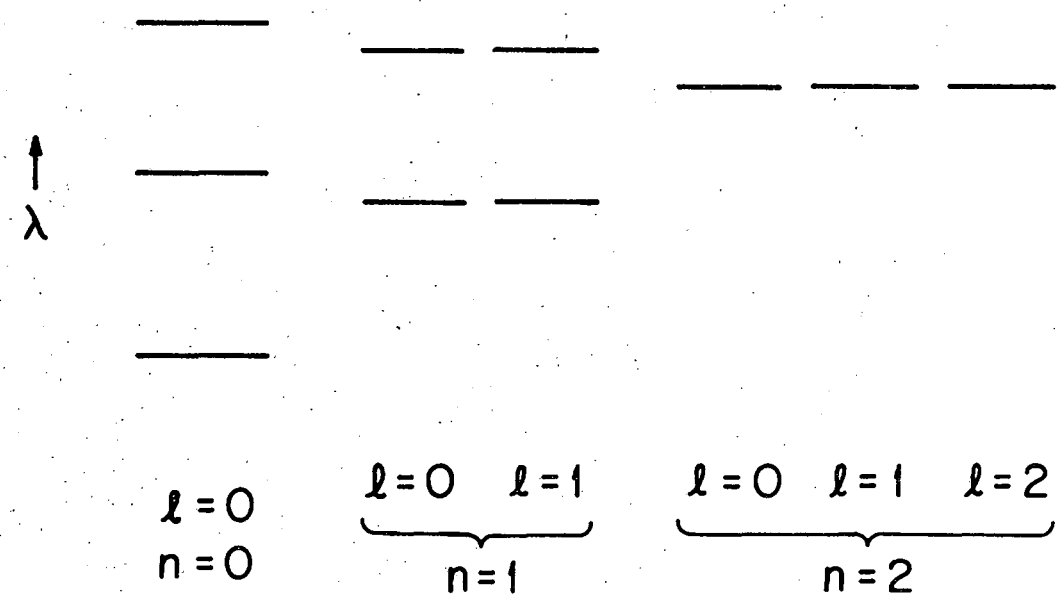
Fig. 22. Scattering Phase Shifts, Two Channel I. Channel thresholds are 2.0 and 2.4. Each channel has equal-mass system. The parameters are $\ell = 0$, $\lambda_1 = 0.7$, $\lambda_2 = 1.0$. The subscript gives channel number and the argument of δ is λ_3, the coupling strength between channels.

Fig. 23. Scattering Phase Shifts, Two Channel II. Same as Fig. 20 except $\lambda_2 = 2.5$. This places a bound state in the uncoupled ($\lambda_3 = 0$) system at $E \approx 2.28$.



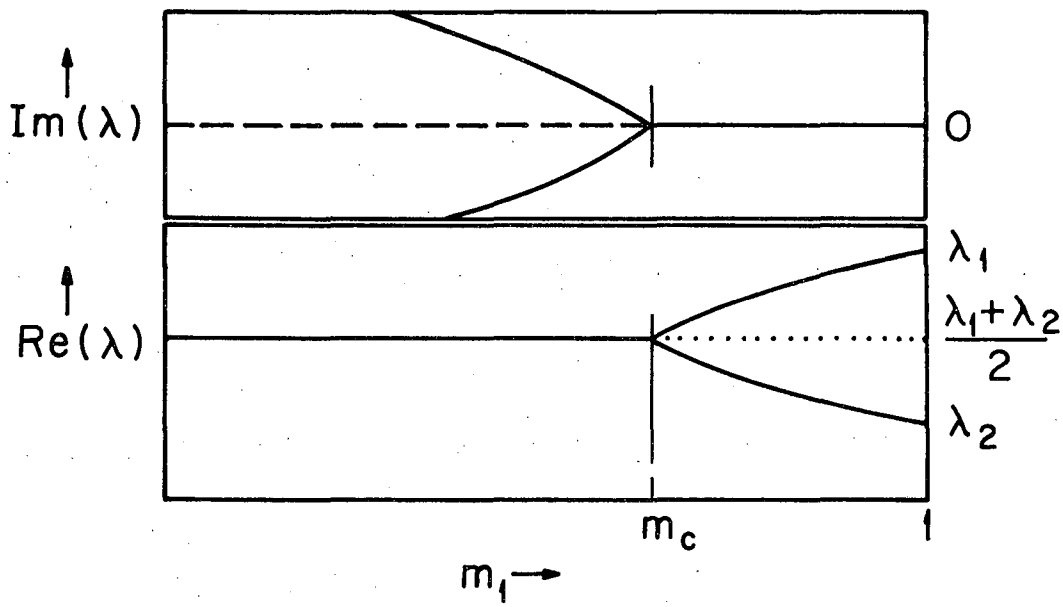
XBL685-2573

Fig. 1



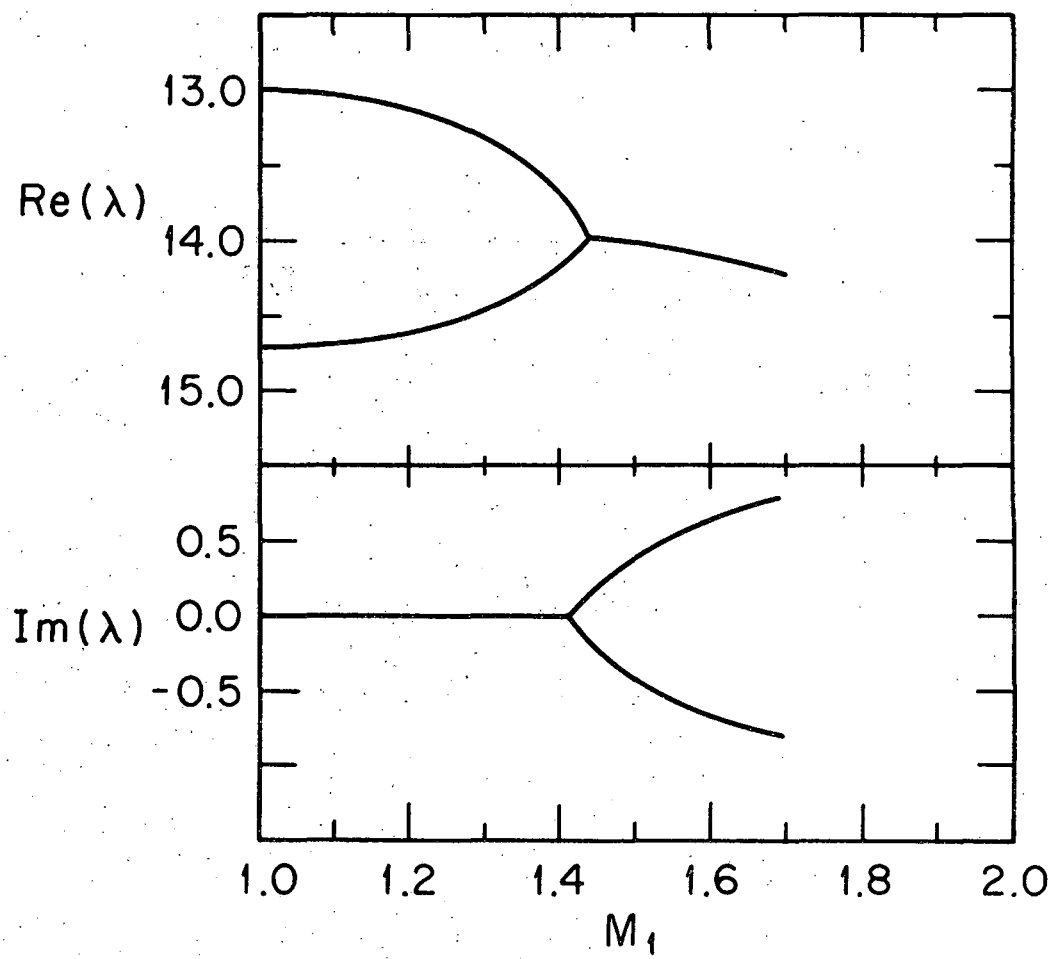
XBL685-2574

Fig. 2



XBL685-2575

Fig. 3



XBL685-2576

Fig. 4

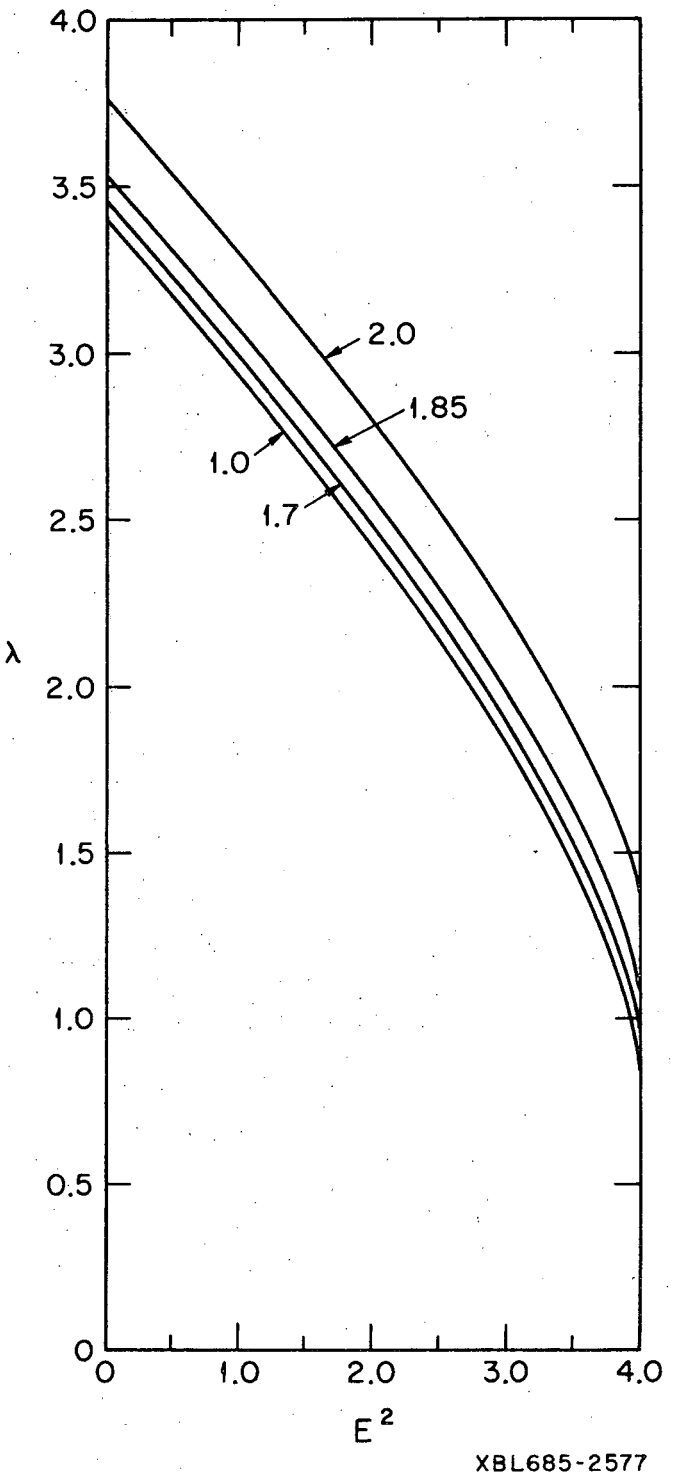
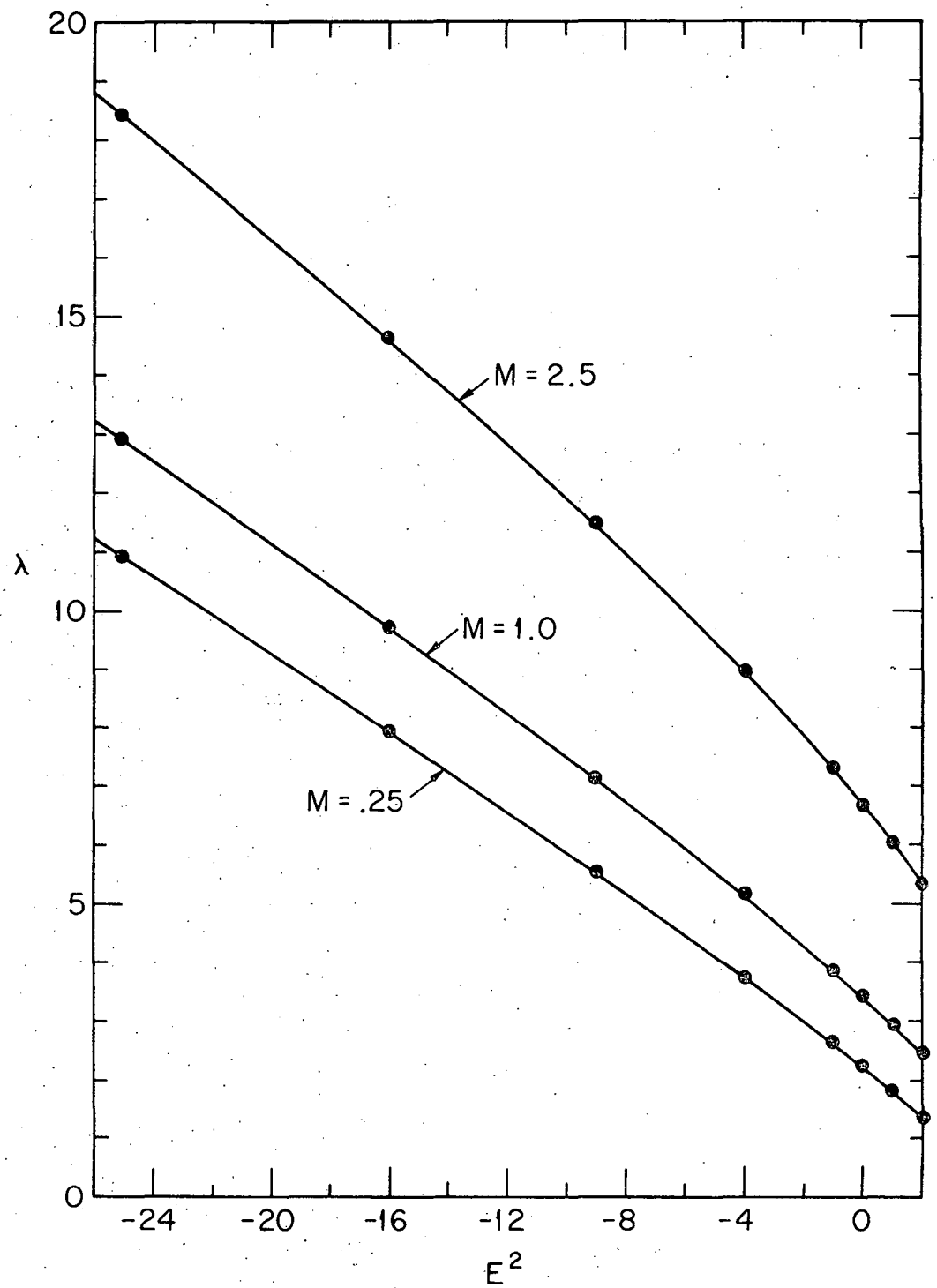


Fig. 5



XBL685-2578

Fig. 6

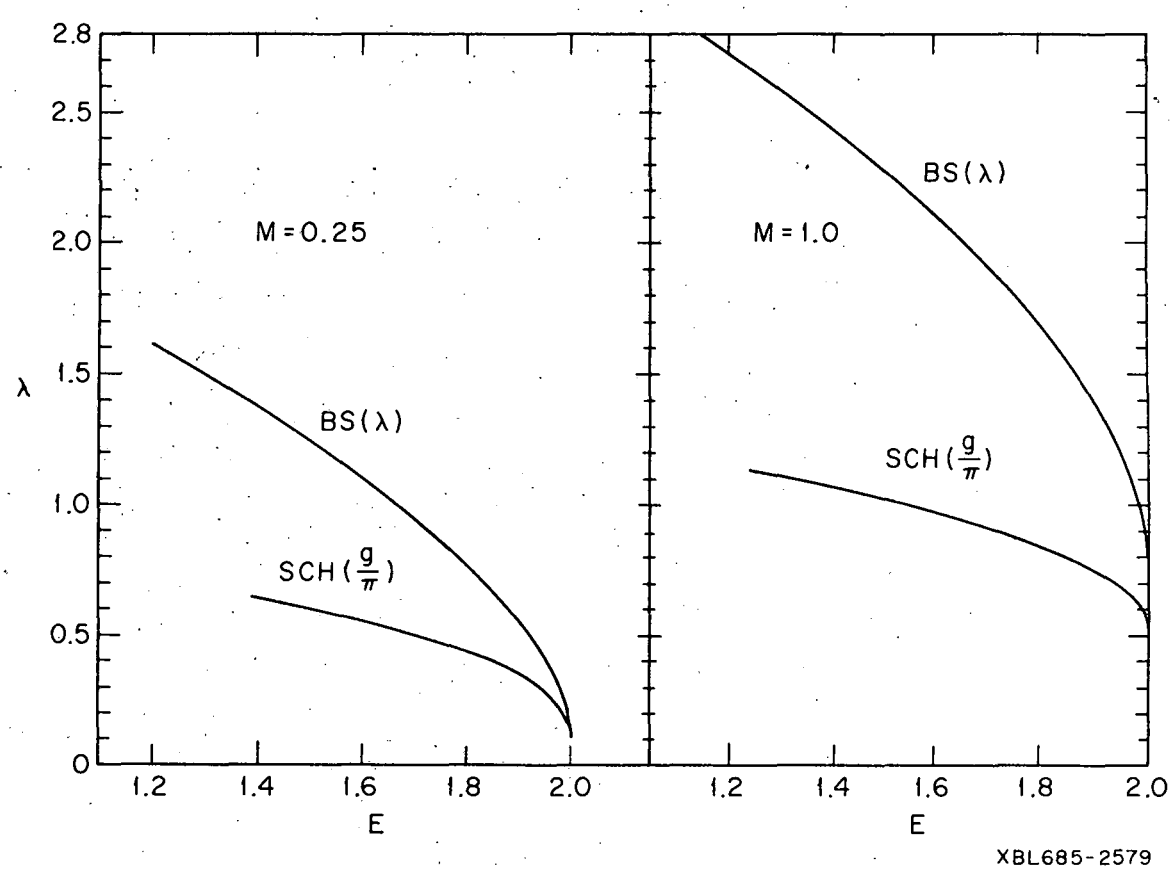
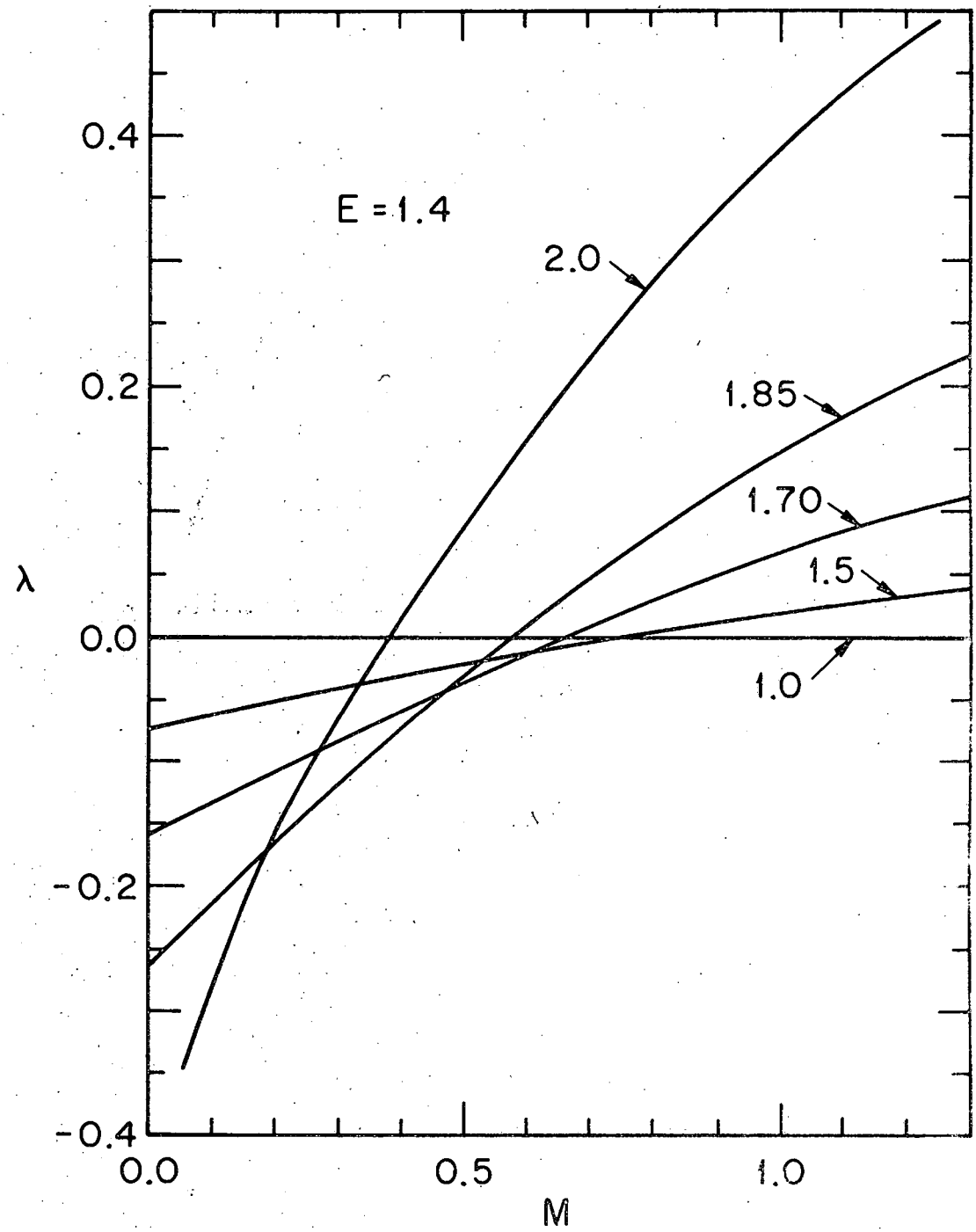

2.8
2.5
2.0
1.5
1.0
0.5
0
λ
M = 0.25
BS(λ)
SCH(g/π)
M = 1.0
BS(λ)
SCH(g/π)
1.2
1.4
1.6
1.8
2.0
E
1.2
1.4
1.6
1.8
2.0
E
XBL685-2579

Fig. 7



XBL685-2580

Fig. 8

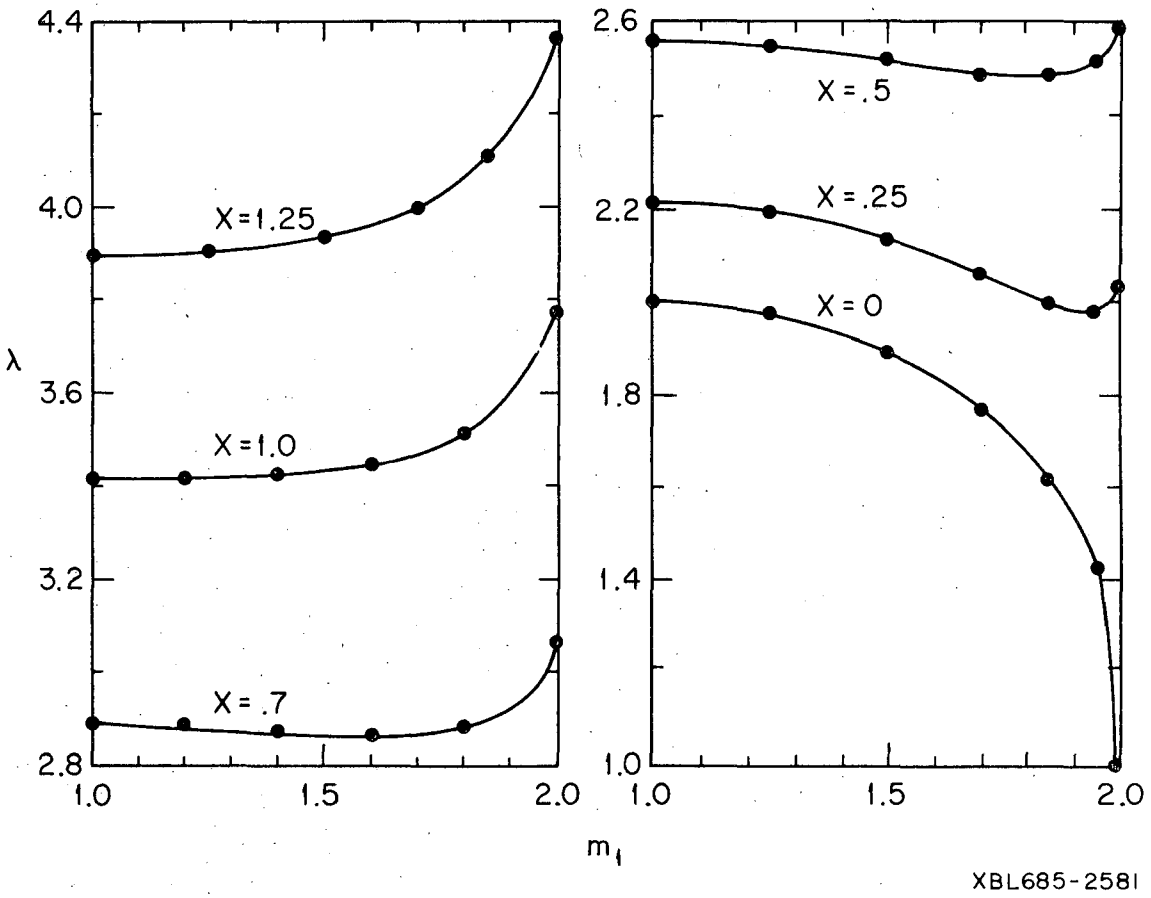
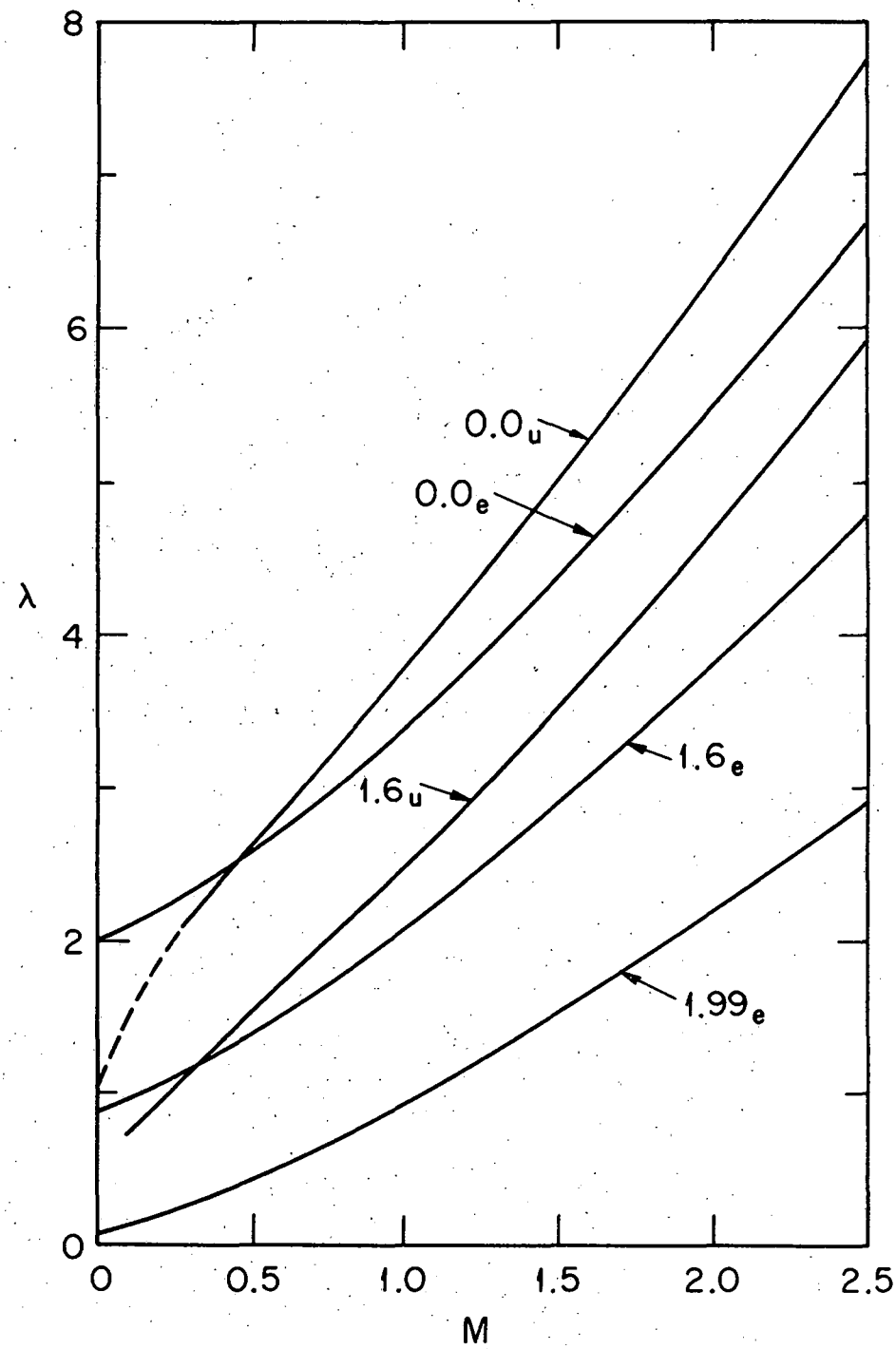

4.4
4.0
3.6
3.2
2.8
λ
X=1.25
X=1.0
X=.7
1.0
1.5
2.0
2.6
2.2
1.8
1.4
1.0
X=.5
X=.25
X=0
1.0
1.5
2.0
m_1
XBL685-2581

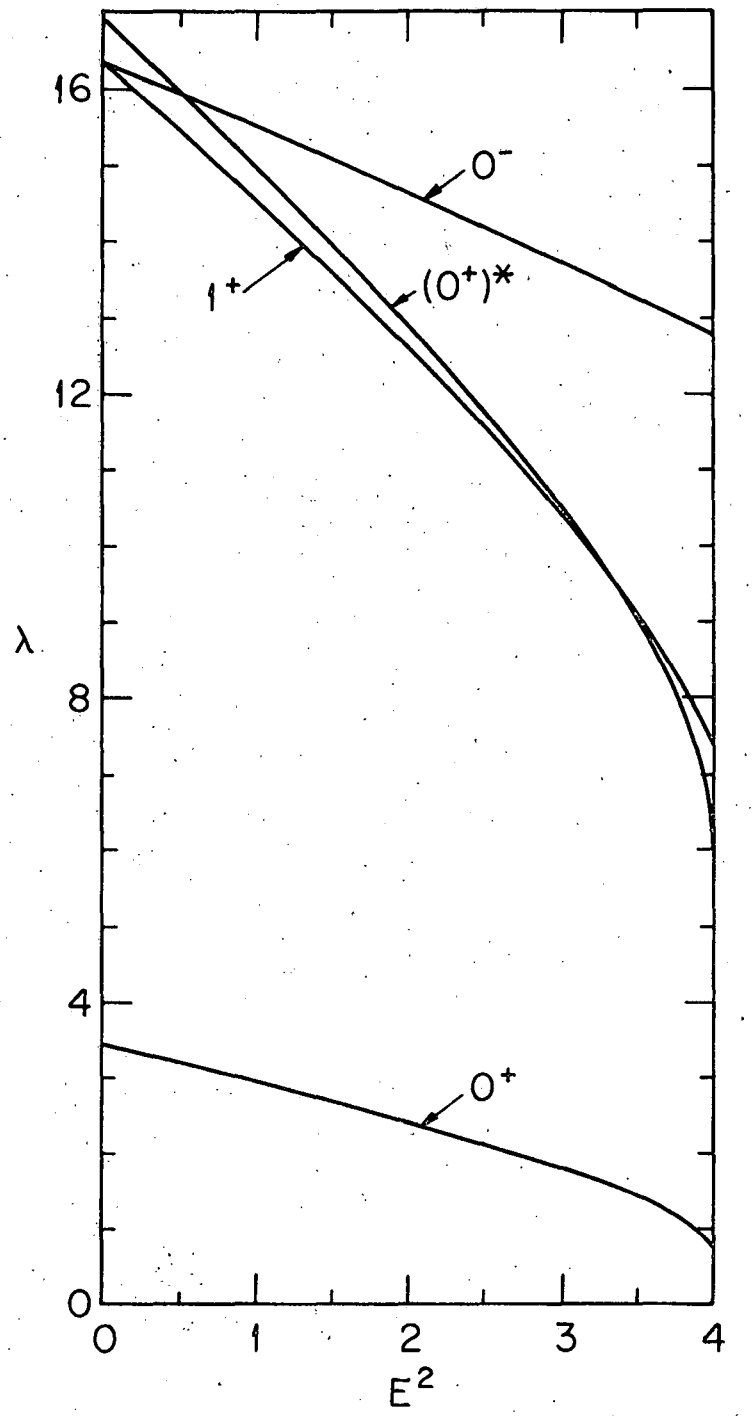
Fig. 9

a. b.



XBL685-2582

Fig. 10



XBL685-2583

Fig. 11

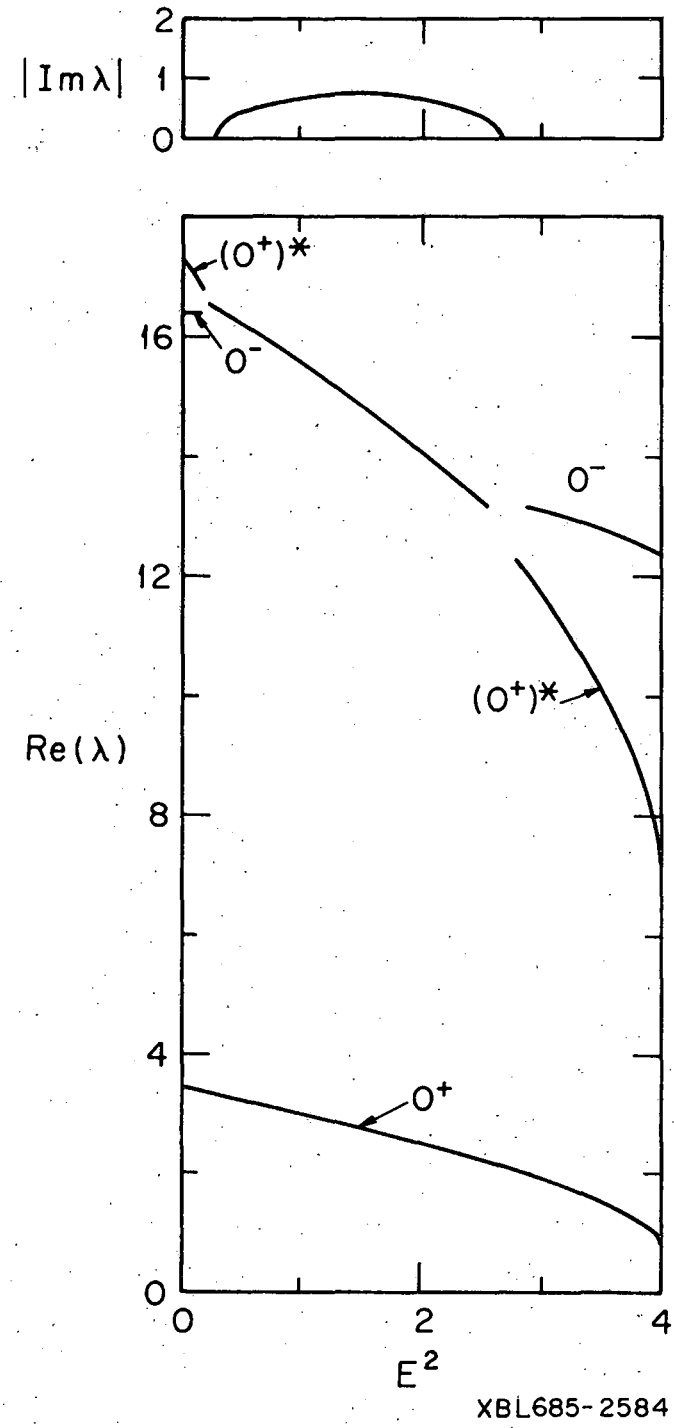


Fig. 12

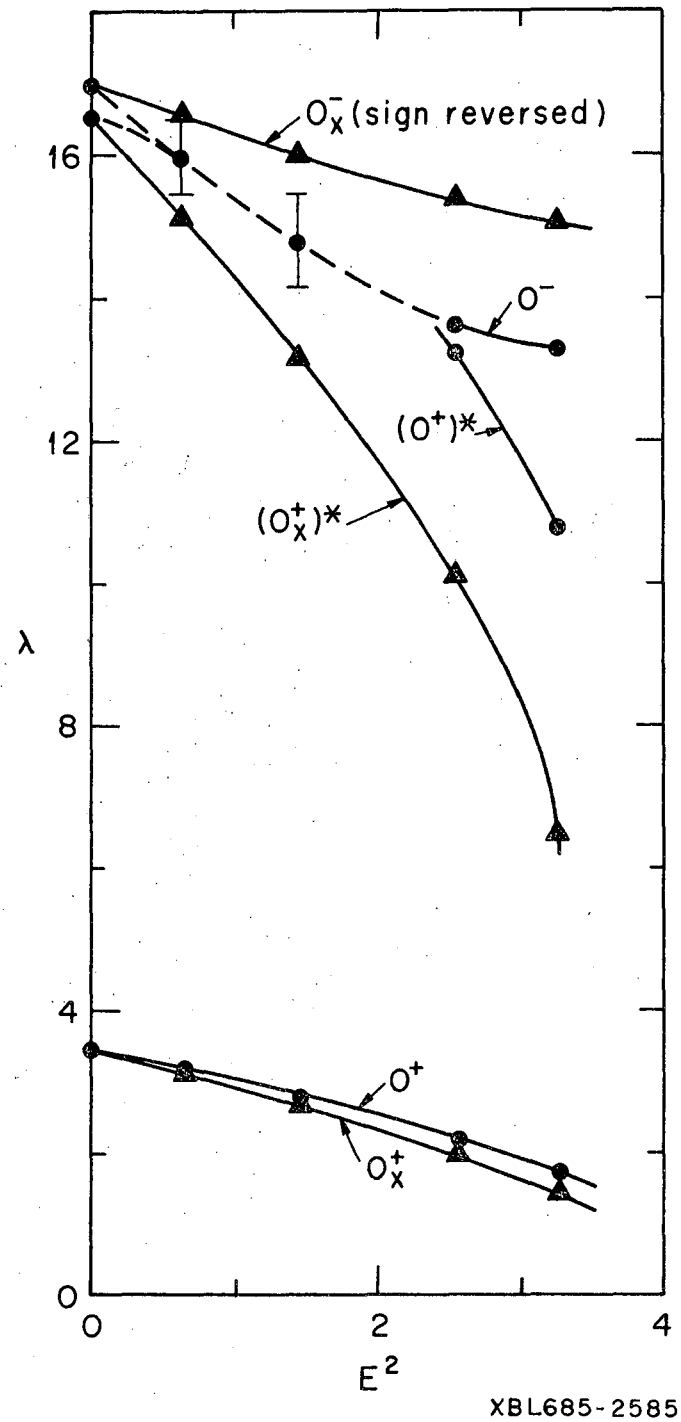


Fig. 13

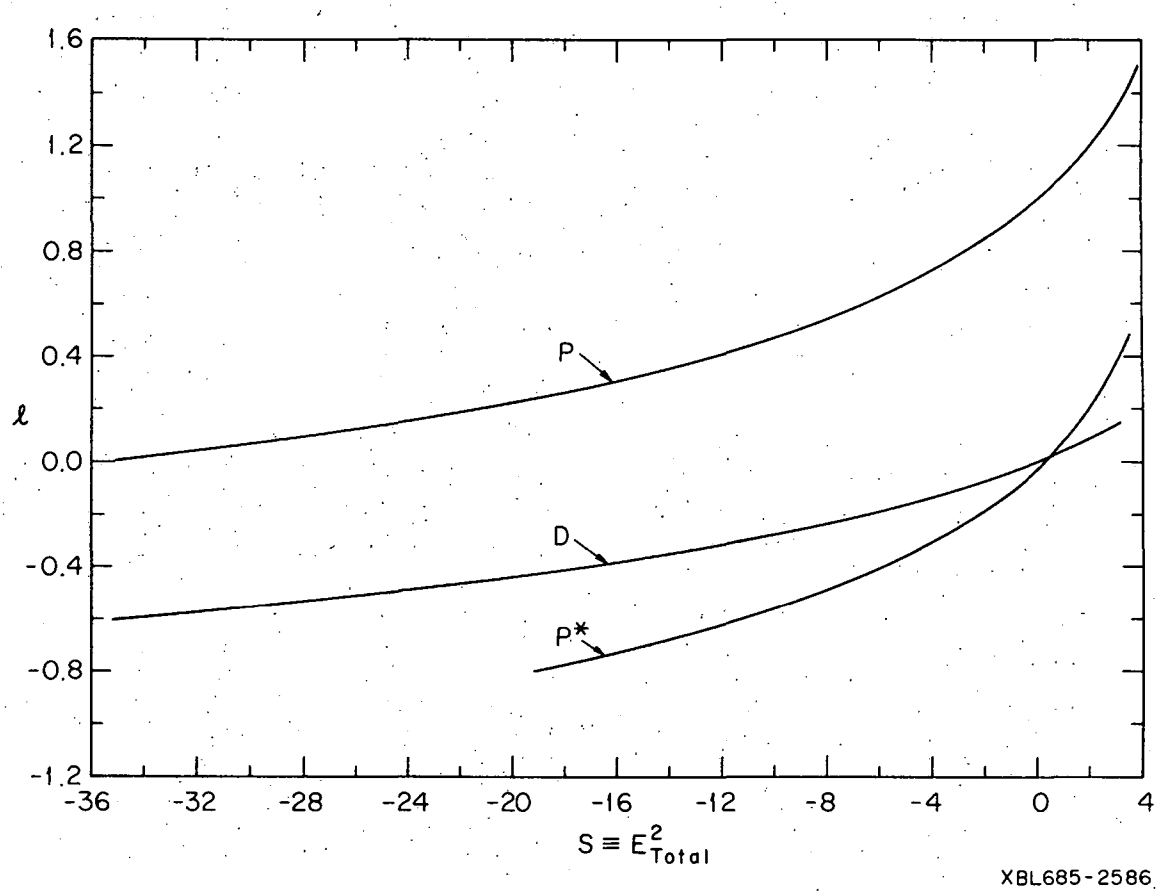

1.6
1.2
0.8
0.4
0.0
-0.4
-0.8
-1.2
ℓ
P
D
P*
-36
-32
-28
-24
-20
-16
-12
-8
-4
0
4
S ≡ E²Total
XBL685-2586

Fig. 14

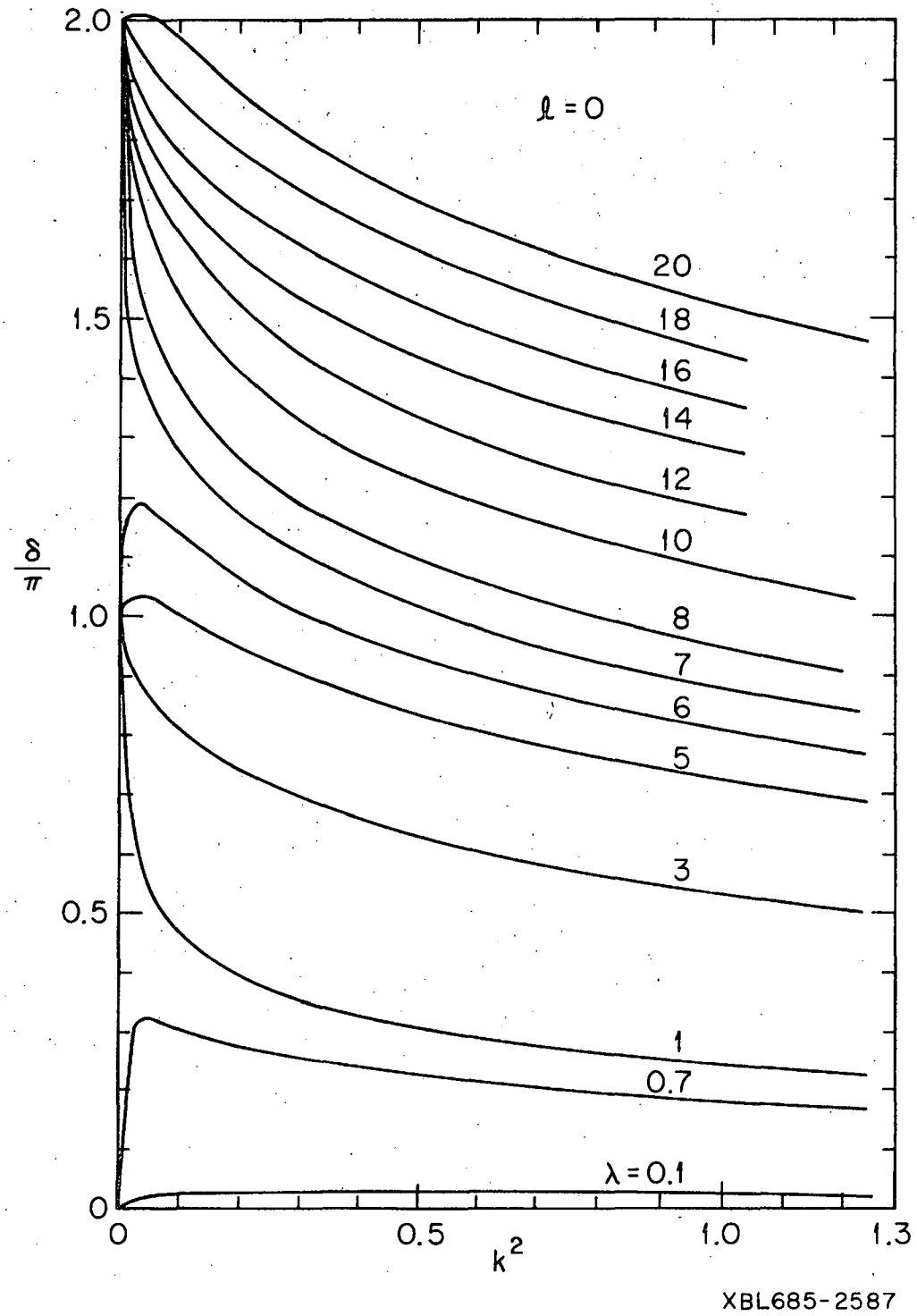

2.0
1.5
1.0
0.5
0
δ/π
ℓ = 0
20
18
16
14
12
10
8
7
6
5
3
1
0.7
λ = 0.1
0
0.5
1.0
1.3
k²
XBL685-2587

Fig. 15

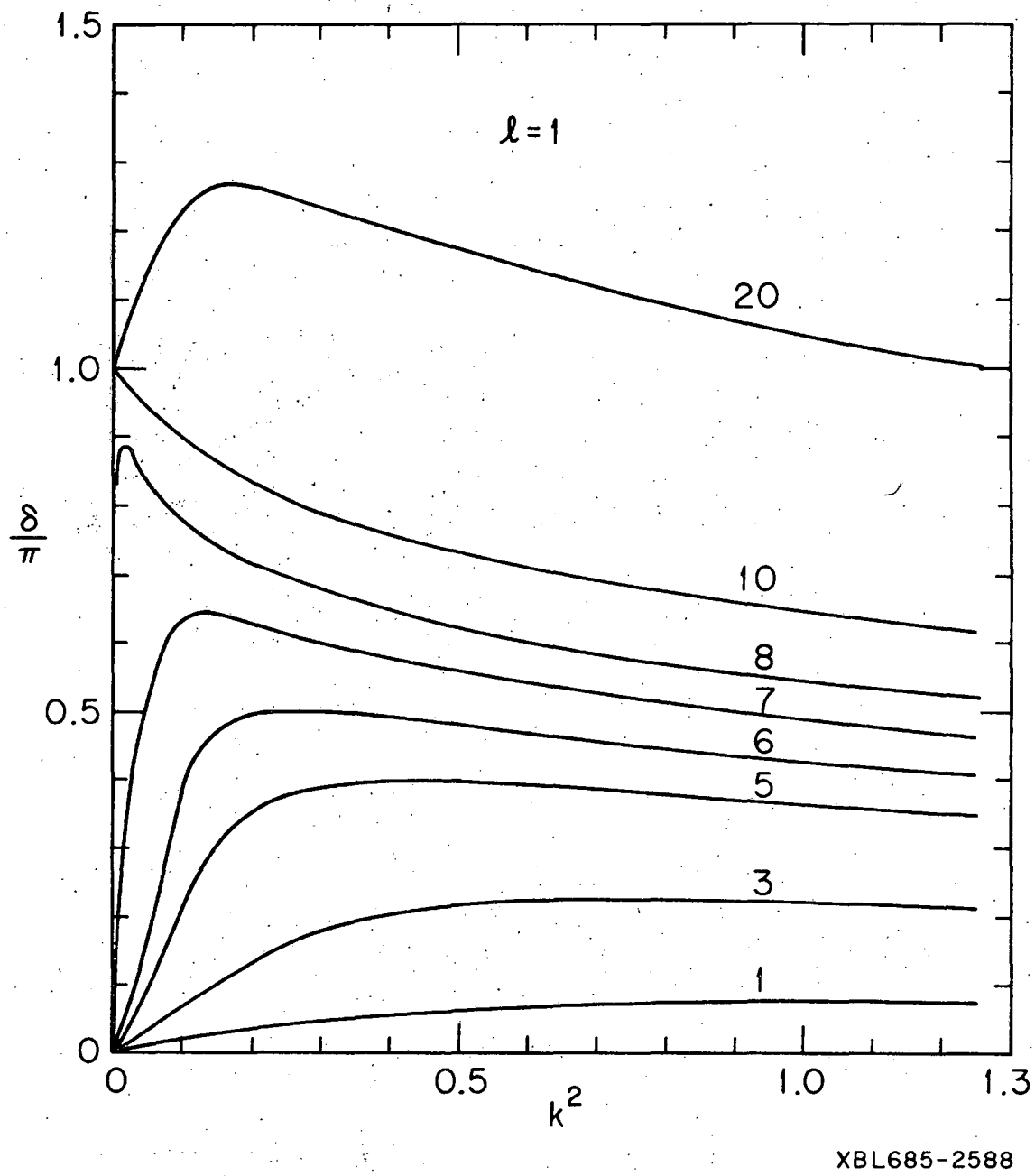

1.5
1.0
0.5
0
$\frac{\delta}{\pi}$
$\ell = 1$
20
10
8
7
6
5
3
1
0
0.5
1.0
1.3
k^2
XBL685-2588

Fig. 16

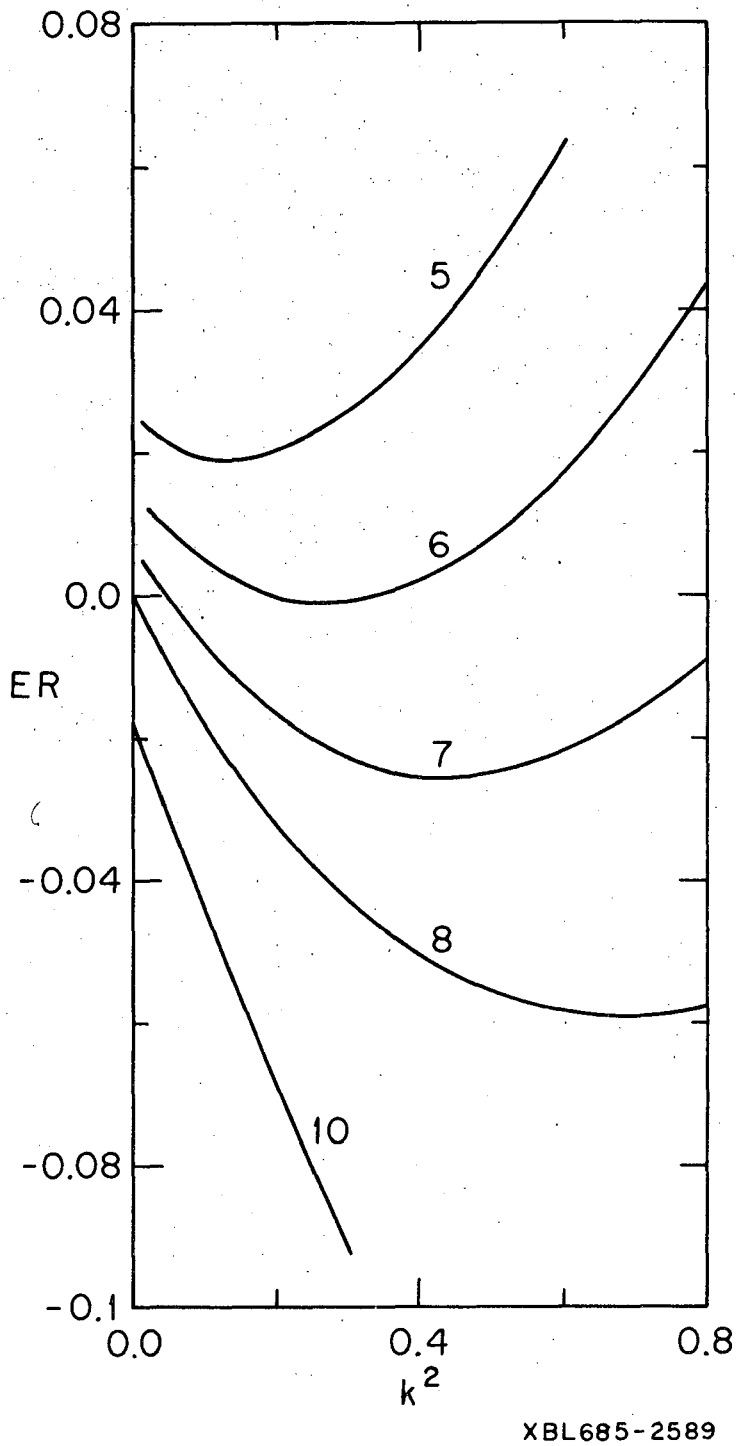
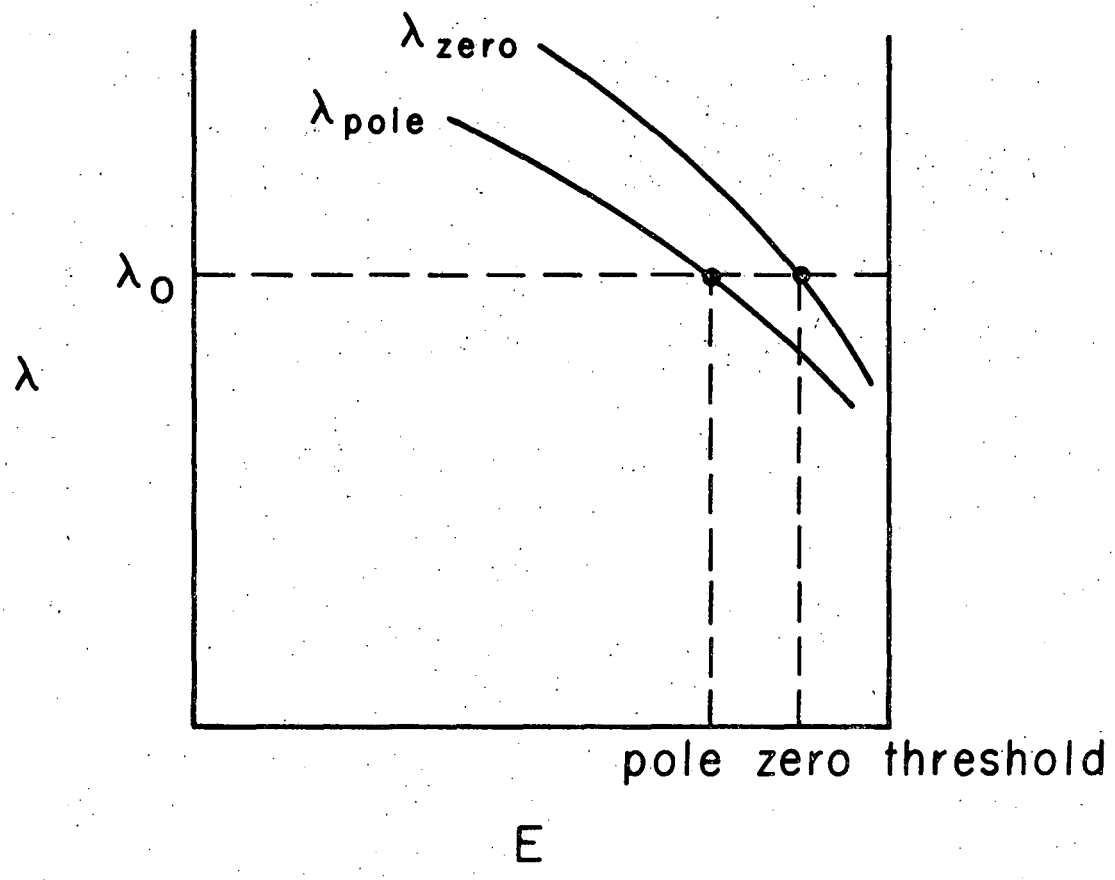


Fig. 17



XBL685-2728

Fig. 18

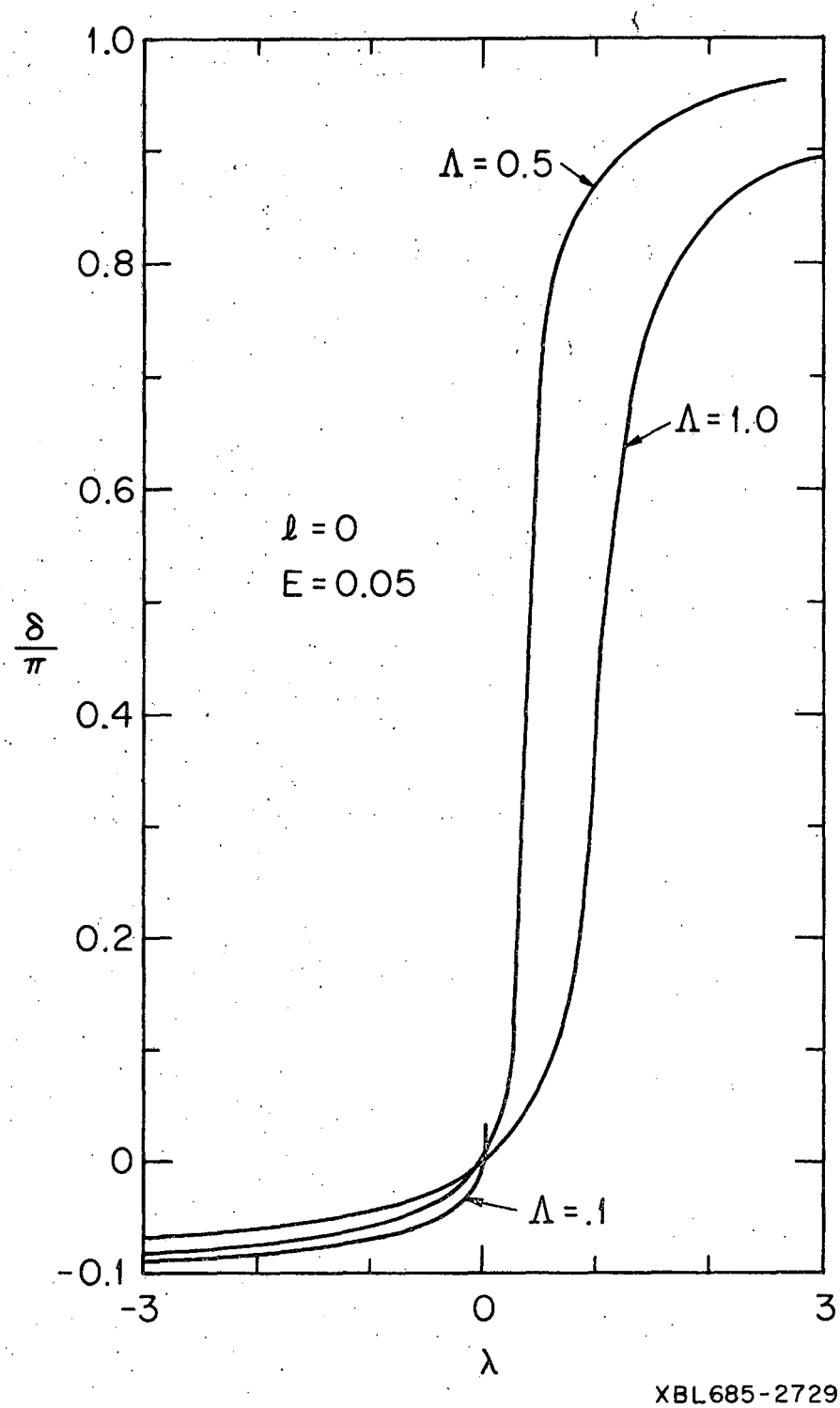


Fig. 19

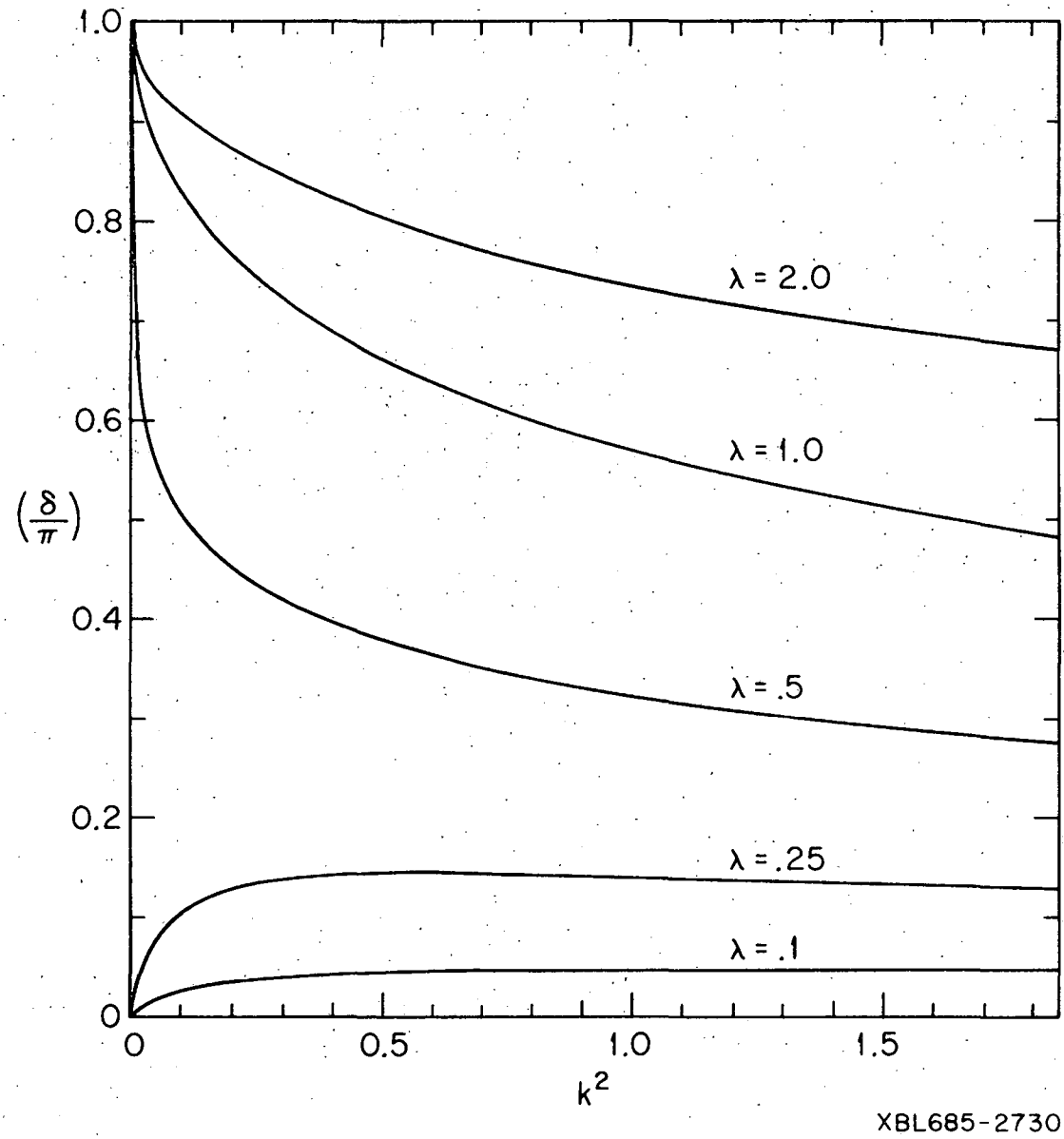
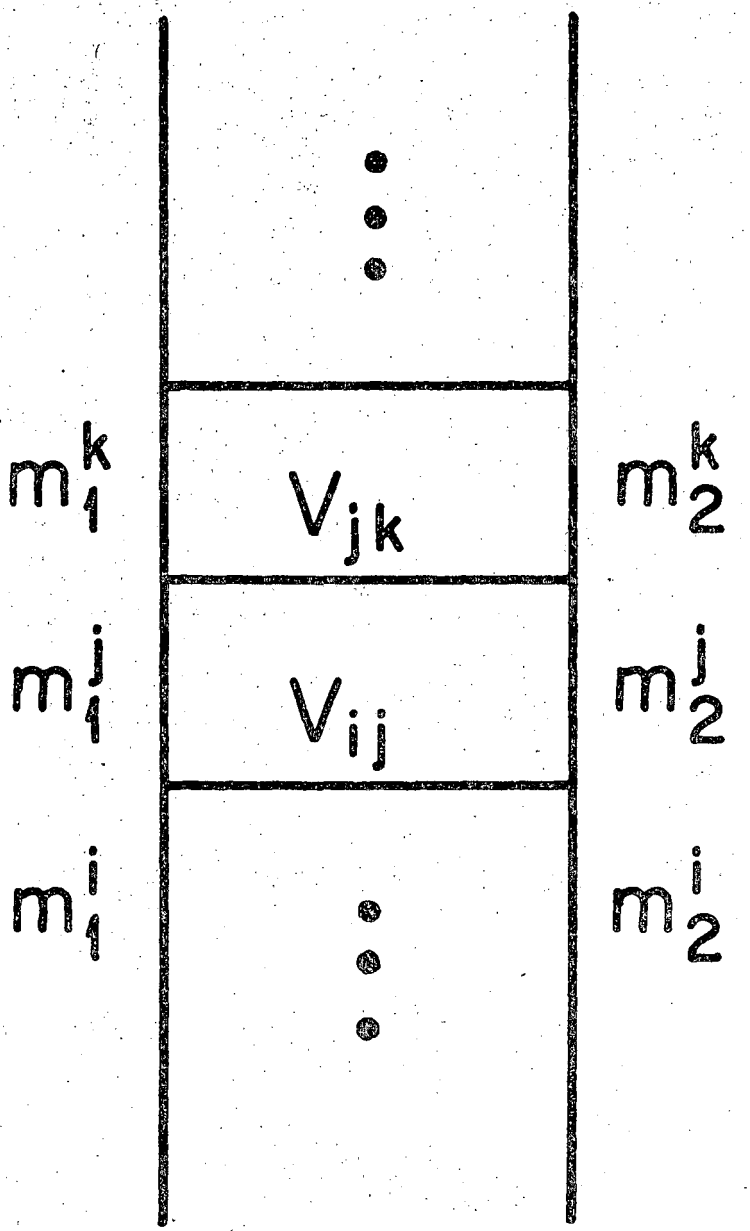
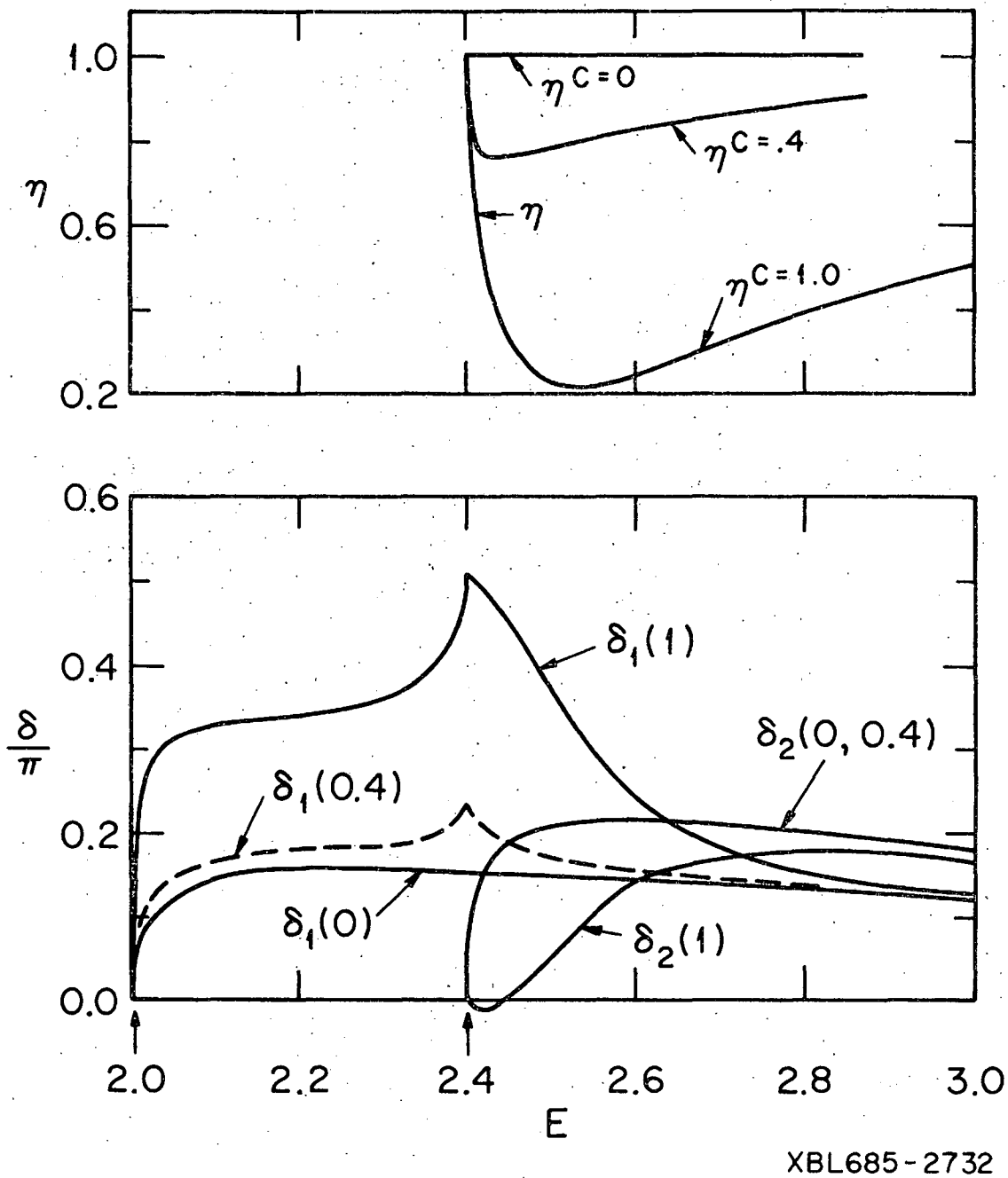

1.0
0.8
0.6
0.4
0.2
0
$\left(\frac{\delta}{\pi}\right)$
λ = 2.0
λ = 1.0
λ = .5
λ = .25
λ = .1
0
0.5
1.0
1.5
k^2
XBL685-2730

Fig. 20



XBL685-2731

Fig. 21



XBL685-2732

Fig. 22

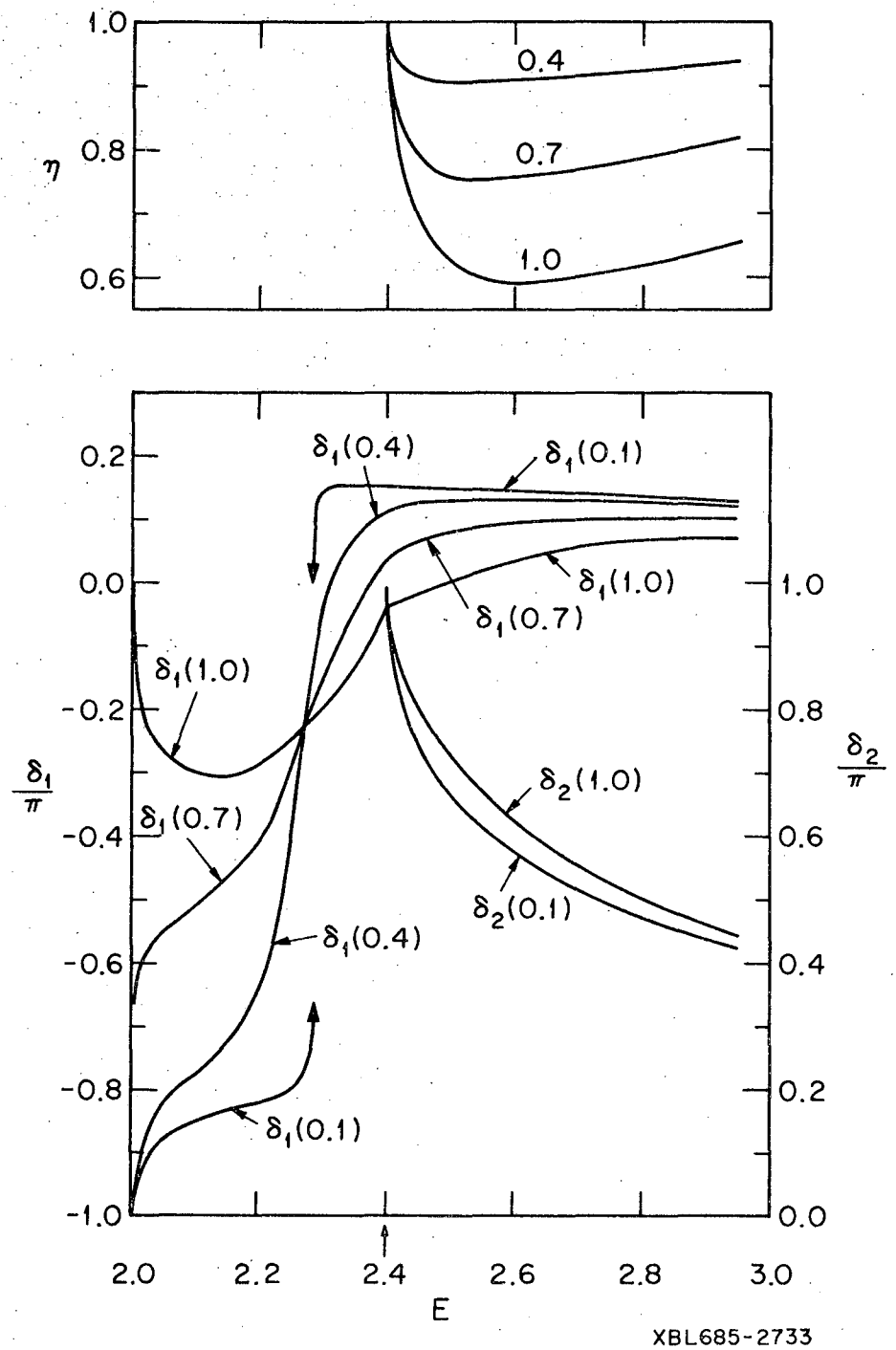

1.0
0.8
0.6
η
0.4
0.7
1.0
0.2
0.0
-0.2
-0.4
-0.6
-0.8
-1.0
δ1/π
δ1(0.4)
δ1(0.1)
δ1(1.0)
δ1(0.7)
δ1(1.0)
δ1(0.7)
δ1(0.4)
δ1(0.1)
δ2(1.0)
δ2(0.1)
1.0
0.8
0.6
0.4
0.2
0.0
δ2/π
2.0
2.2
2.4
2.6
2.8
3.0
E
XBL685-2733

Fig. 23

Table I. Coupling strength with exchange mass = 0.125.

	m_1				
E	1.0	1.5	1.7	1.85	2.0
0.8	1.816	1.731	1.640	1.540	
1.2	1.482	1.409	1.332	1.248	
1.6	0.983	0.931	0.876	0.816	0.788
1.8	0.646	0.611	0.574	0.532	
1.9	0.436	0.412	0.387	0.36_2	
1.95	0.30_8	0.29_1	0.27_5	0.25_9	0.28_2
1.99	0.16_6	0.16_0	0.15_3	0.14_8	0.19
1.9999	0.08_5	0.08_6	0.09_0	0.09_2	0.15_5

Table II. Coupling strength with exchange mass = 0.25.

	m_1				
E	1.0	1.5	1.7	1.85	2.00
0.0	2.214	2.139	2.065	2.00_0	2.0_4
0.6	2.067_5	1.99_7	1.927	1.86	1.91
0.8	1.952	1.885	1.819	1.76	1.81
1.0	1.802	1.739	1.677	1.62	1.6_8
1.2	1.614	1.557	1.502	1.4_6	1.5_3
1.4	1.385	1.336	1.288	1.24_8	1.27_7
1.6	1.108	1.068	1.031	1.00	1.0_5
1.7	0.94_7	0.912	0.881	0.857	0.902
1.8	0.764	0.737	0.713	0.696	0.748_6
1.9	0.54_7	0.530	0.515	0.508	0.572
1.95	0.411_2	0.401	0.393	0.392	0.466
1.99	0.25_2	0.253_4	0.253_9	0.262	0.35_6

Table III. Coupling strength with exchange mass = 0.5.

	m_1				
E	1.0	1.5	1.7	1.85	2.00
0.0	2.566	2.525	2.496	2.49_2	2.6_1
0.6	2.414	2.37_5	2.34_9	2.3_4	2.4_6
0.8	2.293	2.25_8	2.234	2.2_3	2.3_5
1.0	2.136	2.104	2.083	2.0_8	2.2_0
1.2	1.940	1.911_5	1.894	1.9_0	2.0
1.4	1.966	1.676	1.663	1.67	1.8_0
1.6	1.405	1.388	1.382	1.39_5	1.5_2
1.7	1.233	1.220	1.2167	1.2334	1.354
1.8	1.035	1.027	1.0283	1.049	1.176
1.9	0.795	0.7938	0.8010	0.8273	0.966
1.95	0.6412	0.6443	0.6559	0.6864	0.838
1.99	0.458_3	0.4676	0.485_1	0.5218	0.702_5

Table IV. Coupling strength with exchange mass = 0.7.

	m_1				
E	1.0	1.5	1.7	1.85	2.00
0.0	2.888				3.0_7
0.6	2.729_7	2.715	2.717	2.749_3	2.9_2
0.8	2.604_4	2.591_7	2.595	2.628_2	2.7_9
1.0	2.440_4	2.429_9	2.435	2.4969_9	2.6_3
1.2	2.234_7	2.227_0	2.235	2.271_4	2.4_4
1.4	1.981_7	1.977_7	1.989	2.027_8	2.2_0
1.6	1.671	1.671_7	1.686_7	1.729_3	1.9_6
1.7	1.486_7	1.490_4	1.508_1	1.552_9	1.72_0
1.8	1.274	1.281_6	1.302_5	1.350_1	1.52_4
1.9	1.013_6	1.025_8	1.051_1	1.103_1	1.291_4
1.95	0.843_7	0.8594_5	0.888_0	0.943_6	1.114_6
1.99	0.638_5	0.659_1	0.692_6	0.754_0	0.995

Table V. Coupling strength with exchange mass = 1.0.

	m_1				
E	1.0	1.5	1.7	1.85	2.00
0.0	3.418_6				3.7_7
0.6	3.249	3.266_5	3.305	3.380	3.6_1
0.8	3.115	3.133_5	3.1730	3.248_6	3.4_8
1.0	2.939	2.958_8	2.9994	3.076	3.3_1
1.2	2.718	2.739	2.780_9	2.858	3.0_8
1.4	2.44_4	2.467_2	2.51_1	2.590	2.8_3
1.6	2.104	2.131	2.177	2.258	2.4_9
1.7	1.901	1.930	1.977	2.060	2.287
1.8	1.665	1.696	1.746	1.831	2.067
1.9	1.371_4	1.405_3	1.458_3	1.547	1.802_5
1.95	1.176_7	1.213_2	1.268_8	1.361_6	1.637_5
1.99	0.937_3	0.977_5	1.037_2	1.136_8	1.461_3

Table VI. Coupling strength with exchange mass = 1.25.

	m_1				
E	1.0	1.5	1.7	1.85	2.00
0.0	3.896	3.937	4.00_3	4.11_0	4.3_6
0.6	3.717	3.7_5	3.834	3.933	4.1_9
0.8	3.575	3.617	3.68_4	3.79_0	4.0_5
1.0	3.388	3.43_1	3.49_9	3.60_1	3.8_7
1.2	3.152	3.19_7	3.2_6	3.37	3.6_4
1.4	2.870	2.90_5	2.97_4	3.08_3	3.3_5
1.6	2.495	2.54_2	2.61_3	2.7_2	3.0_0
1.7	2.275	2.3242	2.396_5	2.509	2.78_2
1.8	2.018	2.068_9	2.142_7	2.258	2.54_2
1.9	1.696	1.7487	1.825_3	1.945	2.252
1.95	1.480_0	1.534_9	1.613_8	1.737_5	2.07_0
1.99	1.211	1.2696	1.352_6	1.484_5	1.875

Table VII. Coupling strength with exchange mass = 2.5.

	m_1				
E	1.0	1.5	1.7	1.85	2.00
0.0	6.68_5	6.84	7.02	7.2_6	7.6_8
0.6	6.45_3	6.60_5	6.79	7.0_2	7.4_6
1.0	6.02_0	6.17	6.36	6.6_0	7.0_3
1.2	5.70_5	5.86	6.04	6.2_0	6.7_2
1.4	5.3_1	5.4_7	5.6_5	5.8_9	6.3
1.6	4.8	4.9_6	5.1_5	5.3_9	5.9
1.7	4.532	4.650	4.841	5.09_2	5.572
1.8	4.163	4.282	4.752	4.73_2	5.237
1.9	3.687	3.808	4.005	4.273	4.824
1.95	3.359	3.481	3.683	3.962	4.562
1.99	2.937	3.062	3.273	3.571	4.278

Table VIII. Coupling strength with exchange mass = 10.0.

	m_1				
E	1.0	1.5	1.7	1.85	2.00
0.6	33.91	34.71	35.57	36.54	37.97
0.8	33.39	34.19	35.06	36.04	37.50
1.0	32.69	33.50	34.38	35.37	36.86
1.2	31.78	32.60	33.49	34.51	36.045
1.4	30.60	31.43	32.35	33.40	35.00
1.7	28.05	28.93	29.90	31.04	32.83
1.8	26.82_5	27.72	28.73_5	29.92	31.84_5
1.9	25.16	26.10	27.17	28.41	30.59
1.95	23.95	24.92	26.04	27.40	29.78
1.99	22.31	23.33	24.53	26.03	28.88

Table IX. Coupling strength with angular momentum = 1 and exchange mass = 1.

	m_1				
E	1.0	1.5	1.7	1.85	2.00
0.0	16.3_8	17.0_2	17.7_8	18.7_0	20.0_8
0.8	15.2_3	15.8_6	16.6_3	17.5_6	18.9_4
1.2	13.7_3	14.3_5	15.1_2	16.0_6	17.4_7
1.6	11.4_2	12.0_5	12.8_3	13.8_2	15.2_9
1.8	9.81_5	10.45_6	11.2_5	12.25_6	13.8_4
1.9	8.80	9.4_5	10.2_7	11.3_1	13.0
1.9999	7.37_9	8.0_8	8.9_7	10.1_0	12.0

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