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CONSPIRACY AND COUNTERCONSPIRACY IN EQUAL–MASS SCATTERING

J. H. Weis

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Submitted to Physical Review

UCRL-18397
Preprint

UNIVERSITY OF CALIFORNIA
Lawrence Radiation Laboratory
Berkeley, California

Contract No. W-7405-eng-48

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ABSTRACT

Group theory provides an elegant general solution of the equal-mass conspiracy conditions but leaves open the possibility of a " counterconspiracy" between a number of integrally spaced Lorentz poles. It is shown that factorization of Regge pole residues and coupling to channels of arbitrarily high spin imply that only one counterconspiracy (consisting of an infinite number of Regge poles) is possible in equal-mass scattering. It is also shown that if there is coupling to unequal-mass channels then even this counter-conspiracy is forbidden and the only possible solution is for the equal-mass channel residues to be those of a single Lorentz pole.

I. INTRODUCTION

An elegant solution to the equal-mass conspiracy conditions at $t = 0$ has been given by Toller[1] and Freedman and Wang[2] using group theoretical arguments. They find that the conditions are satisfied by a Lorentz (or Toller) pole which corresponds to an infinite sequence of integrally spaced Regge poles. This group theoretical approach suggests the plausible assumption that the scattering amplitude is given by a single Lorentz pole at $t = 0$ (for specified signature, internal quantum numbers, and trajectory intercept). However, there is the possibility of a "counterconspiracy" between a number of integrally spaced Lorentz poles. In fact there are some simple examples of counterconspiracies: if a Regge trajectory couples only to channel spin[3] zero (e.g., all particles spinless) no conspiracy is necessary or if it couples only to channel spin one only three trajectories are necessary.[4] In addition the single Lorentz pole hypothesis is much less plausible than the single Regge pole hypothesis, since all dynamics is invariant under $O(3)$ but $O(3,1)$ or $O(4)$ invariance applies at only one point in t and only for equal-mass channels. In this paper we consider the question of counterconspiracies and show that under some reasonable physical assumptions there is one and only one possible counterconspiracy.

More precisely, we assume: (i) angular momentum is conserved, (ii) there is only one Regge trajectory of given quantum numbers [including $\alpha(t=0)$] and its residue factorizes (no "accidental" crossing of trajectories), and (iii) the leading trajectory in a conspiracy couples to channels of arbitrarily high channel spin (for any channel spin s_o, there exists an equal-mass channel with channel spin $s \geq s_o$ to

which the trajectory couples at $t = 0$). We can then argue that there are just two possible conspiracies. One is that of a single Lorentz pole and the other is a new conspiracy which yields an s-channel amplitude that has only one helicity flip to all orders in the energy at $t = 0$. Therefore there is only one possible counterconspiracy. That a single Lorentz pole satisfies the conditions has been shown by Sciarrino and Toller[1,5] and Nakanishi,[6] who completed the proof of factorizability of the residues. We have not been able to give a general proof of the factorizability of the residues for the counterconspiracy, but we have checked it in many special cases.

If the leading trajectory also couples to an unequal-mass channel, as is to be usually expected on physical grounds, then using factorization we can show that only the single Lorentz pole conspiracy gives permissible equal-mass channel residues.[7] We thus come to the conclusion that the Lorentz pole conspiracy is expected to be the typical conspiracy and only in special cases (e.g., coupling only to equal-mass channels) are there other possibilities.

The assumption (iii) of coupling to channels of arbitrarily high channel spin is of course quite strong. It eliminates the examples of counterconspiracies for low spin channels mentioned above. However, it is not unreasonable, since one might assume that high spin quasi two-two amplitudes can be projected out of multiparticle amplitudes and then Reggeized in the usual manner. Also the assumption is quite appropriate for models with narrow resonances and infinitely rising trajectories.[8] In the final section some comments are made about the possibility of weakening this assumption without altering the conclusion.

In Section II we establish our notation. The conspiracy conditions

are formulated as angular momentum conservation in the s channel. We explicitly find their solution up to the order $s^{\alpha-3}$, where α is the $t = 0$ intercept of the leading trajectory,[9] i.e., for $n = 0,1,2,3$, where n is the conspirator number. There are only two possible solutions. In Section III we show that the residues of the leading trajectory and the first conspirator $(n = 0,1)$ uniquely determine all the other conspirator residues at $t = 0$. The two preceding results taken together prove that there are only two possible equal-mass conspiracies. In the course of the proof we derive an interesting property of the conspirator residues [see Eq. (22)]. In the Appendix the properties of the second conspiracy are discussed further. Section IV outlines the proof when there is parity conservation and time-reversal invariance. The unequal mass problem is treated briefly in Section V. We give a simple derivation of the most singular terms of the residues for arbitrary spins and point out that these agree with the more general results of Klein.[10] Then by considering the coupling of equal-mass to unequal-mass channels we show that only the Lorentz pole residues are allowed for the equal mass channel. In the final section we give a summary of the conclusions and make a few further comments.

The straightforward method employed below is less elegant than group theory, but we believe it gives additional insight into the way conspiracy conditions are satisfied when factorization plays an important role.

II. CONSPIRATOR RESIDUES FOR $n \leq 3$

We consider the s-channel process $1+2 \to 3+4$ where $m_1 = m_3$, $m_2 = m_4$, and $m_1, m_2 \neq 0$. The center-of-mass helicity amplitude is denoted by $H^s_{\lambda_3\lambda_4;\lambda_1\lambda_2}(s,t)$. At $t = 0$, rotational invariance about the direction of motion requires H^s to vanish unless $\lambda_1 - \lambda_2 = \lambda_3 - \lambda_4$; this is the conspiracy condition. Similarly the t-channel c.m. helicity amplitude is denoted $H^t_{\mu_3\mu_1;\mu_4\mu_2}(s,t)$.

At $t = 0$, the crossing relation gives[11]

$$H^s_{\lambda_3\lambda_4;\lambda_1\lambda_2}(s,0) = \sum_{\{\mu\}} d^{s_1}_{\mu_1\lambda_1}(\tfrac{\pi}{2})\, d^{s_2}_{\mu_2\lambda_2}(\tfrac{\pi}{2})\, d^{s_3}_{\mu_3\lambda_3}(\tfrac{\pi}{2})\, d^{s_4}_{\mu_4\lambda_4}(\tfrac{\pi}{2})\; H^t_{\mu_3\mu_1;\mu_4\mu_2}(s,0).$$

From the identity

$$d^{s_3}_{\mu_3\lambda_3}(\theta)\, d^{s_1}_{\mu_1\lambda_1}(\theta) = \sum_{s'} C(s_3,s_1,s';\mu_3,-\mu_1)C(s_3,s_1,s';\lambda_3,-\lambda_1)(-1)^{\mu_1-\lambda_1}$$

$$d^{s'}_{\mu_3-\mu_1,\lambda_3-\lambda_1}(\theta)$$

we see that, if we define

$$H^s_{s\lambda s'\lambda'}(s) = \sum_{\{\lambda\}} C(s_4s_2s;\lambda_4-\lambda_2)\; C\;(s_3s_1s';\lambda_3-\lambda_1)(-1)^{s_1-\lambda_1+s_2-\lambda_2}$$

$$H^s_{\lambda_3\lambda_4;\lambda_1\lambda_2}(s,0)$$

and

$$H^t_{s\mu s'\mu'}(s) = \sum_{\{\mu\}} C(s_4s_2s;\mu_4-\mu_2)C(s_3s_1s';\mu_3-\mu_1)(-1)^{s_1-\mu_1+s_2-\mu_2}$$

$$H^t_{\mu_3\mu_1;\mu_4\mu_2}(s,0)$$

the crossing relation takes the compact form,[10]

$$H^{s}_{s\lambda s'\lambda'}(s) = \sum_{\mu\mu'} d^{s}_{\mu\lambda}(\tfrac{\pi}{2})\, d^{s'}_{\mu'\lambda'}(\tfrac{\pi}{2})\, H^{t}_{s\mu s'\mu'}(s). \tag{1}$$

We will refer to s and s' as channel spins although they differ from the usual t-channel spins by the phase factors $(-1)^{s_1-\lambda_1}$ and $(-1)^{s_2-\lambda_2}$.

We neglect parity and signature until Section V; thus the contribution of a Regge trajectory with residue $\beta_{\mu_3\mu_1}\beta_{\mu_4\mu_2} = \beta_{\mu_3\mu_1\mu_4\mu_2}(t=0)$ and $\alpha = \alpha(t=0)$ is

$$H^{t(\mathrm{Regge})}_{\mu_3\mu_1;\mu_4\mu_2}(s,0) = \beta_{\mu_3\mu_1}\beta_{\mu_4\mu_2}\, e^{-\alpha-1}_{\mu_4-\mu_2,\mu_3-\mu_1}(z_t)\,, \tag{2}$$

where

$$z_t = \frac{2s + t - 2(m_1^2 + m_2^2)}{[(t - 4m_1^2)(t - 4m_2^2)]^{\frac{1}{2}}} \xrightarrow[t\to 0]{} - \frac{s - (m_1^2 + m_2^2)}{2m_1m_2}\,.$$

The e functions are those of Andrews and Gunson,[12]

$$e^{-\alpha-1}_{\mu\mu'}(z\pm i\epsilon) = \frac{e^{\pm\frac{i\pi}{2}(\mu-\mu')}}{2\Gamma(-2\alpha)}\,[\Gamma(\mu-\alpha)\,\Gamma(-\mu-\alpha)\,\Gamma\mu'-\alpha)\,\Gamma(-\mu'-\alpha)]^{\frac{1}{2}}\,.$$

$$\left(\frac{z-1}{2}\right)^{\alpha}\left(\frac{z+1}{z-1}\right)^{\frac{(\mu+\mu')}{2}} F(\mu-\alpha,\ \mu'-\alpha;\ -2\alpha;\ -\frac{2}{z-1})$$

for $z \geq 1$. The following convenient notations,

$$x \equiv \frac{2}{z-1} \sim -\frac{4m_1m_2}{s} + O(\frac{1}{s^2}) \qquad (z = z + i\epsilon \ \text{ since } \ t = t + i\epsilon),$$

$$\tilde{\beta}_{s\mu} = \sum_{\mu_2\mu_4} C(s_4s_2s;\mu_4-\mu_2)(-1)^{s_2-\mu_2}\, e^{i\frac{\pi}{2}\mu}\left[\frac{\Gamma(\mu-\alpha)\Gamma(-\mu-\alpha)}{2\Gamma(-2\alpha)}\right]^{\frac{1}{2}}\beta_{\mu_4\mu_2}\,,$$

$$\tilde{\beta}_{s'\mu'} = \sum_{\mu_1\mu_3} C(s_3s_1s';\mu_3-\mu_1)(-1)^{s_1-\mu_1}\, e^{-i\frac{\pi}{2}\mu'}\left[\frac{\Gamma(\mu'-\alpha)\Gamma(-\mu'-\alpha)}{2\Gamma(-2\alpha)}\right]^{\frac{1}{2}}\beta_{\mu_3\mu_1}\,, \tag{3}$$

enable us to write

$$H^{t\ (\mathrm{Regge})}_{s\mu s'\mu'} = \tilde{\beta}_{s\mu}\,\tilde{\beta}_{s'\mu'}\,x^{-\alpha}\,(1+x)^{\frac{\mu+\mu'}{2}}\,F(\mu-\alpha,\ \mu'-\alpha;\ -2\alpha;\ -x)\,. \quad (4)$$

The conspiracy condition is written as

$$H^{s}_{s\lambda s'\lambda'}(s) = C_{ss'\lambda}(x)\,x^{-\alpha}\,\delta_{\lambda\lambda'}\,, \quad (5)$$

where the $x^{-\alpha}$ has been taken out for convenience. As will soon be clear, we are led to consider the possibility of conspiring trajectories integrally spaced below the leading one. Thus we combine equations (1), (4), and (5) and introduce labels c, c' to indicate the channels to obtain

$$C^{cc'}_{ss'\lambda}(x)\,\delta_{\lambda\lambda'} = \sum_{n=0}^{\infty}\sum_{\mu\mu'} d^{s}_{\mu\lambda}(\tfrac{\pi}{2})\,d^{s'}_{\mu'\lambda'}(\tfrac{\pi}{2})\,\tilde{\beta}^{c(n)}_{s\mu}\,\tilde{\beta}^{c'(n)}_{s'\mu'} \quad (6)$$

$$x^{n}\,(1+x)^{\frac{\mu+\mu'}{2}}\,F(n-\alpha+\mu,\ n-\alpha+\mu',\ 2n-2\alpha;\ -x)\ ,$$

where α now denotes the intercept of the leading trajectory.[9] This is our basic equation. We have included only one trajectory and its possible conspirators on the right becuase it is clear that each trajectory plus conspirators must satisfy the conspiracy condition (5) separately, since they contribute different asymptotic powers to H^{s}.[13]

We now try to satisfy (6) power by power in x. We write

$$C^{cc'}_{ss'\lambda}(x) = \sum_{k=0}^{\infty} C^{cc'(k)}_{ss'\lambda}\,x^{k}. \quad (7)$$

Comparing coefficients of x^{0} gives

$$C^{cc'(0)}_{ss'\lambda}\,\delta_{\lambda\lambda'} = \sum_{\mu\mu'} d^{s}_{\mu\lambda}(\tfrac{\pi}{2})\,\tilde{\beta}^{c'(0)}_{s\mu}\,d^{s'}_{\mu'\lambda'}(\tfrac{\pi}{2})\,\tilde{\beta}^{c(0)}_{s'\mu'}\ .$$

The rhs shows explicitly that the contribution of a Regge trajectory factorizes to leading order.[14] The rhs thus is a rank 1 matrix in λ

and λ'. Since the lhs is diagonal we conclude at most one diagonal element is nonzero. We call this element M. M always has a significance with this definition and it will be shown below that in the Lorentz Pole solution it is the usual M quantum number. Since we also have factorization in the channel types c, c' we conclude:

$$C^{cc'(0)}_{ss'\lambda} = \gamma_s^{\ c}\ \gamma_{s'}^{\ c'}\ \delta_{\lambda M}, \tag{8}$$

which leads directly to

$$\tilde{\beta}^{c(0)}_{s\mu} = d^{s}_{\mu M}(\tfrac{\pi}{2})\ \gamma_s^{\ c}\ , \qquad \tilde{\beta}^{c'(0)}_{s'\mu'} = d^{s'}_{\mu' M}(\tfrac{\pi}{2})\ \gamma_{s'}^{\ c'}\ . \tag{9}$$

The reduced residues $\gamma_s^{\ c}$ for channel and channel spin (c, s) along with M characterize completely the coupling of the leading trajectory and, as will be seen, all the conspirators. The assumption (iii) can now be restated as: For every s_o, there is a channel c and spin $s \geq s_o$ with $\gamma_s^{\ c} \neq 0$. From (8) it is of course clear that $\gamma_s^{\ c} = 0$ if $s < |M|$.

We now compare coefficients of x^1 in (6). For future reference we quote

$$(1+x)^{\frac{\mu+\mu'}{2}}\ F(\mu-\alpha,\ \mu'-\alpha;\ -2\alpha;\ -x) =$$

$$1 + \frac{1}{2}\left(\frac{\mu\mu'}{\alpha} + \alpha\right) x + \left(\frac{\mu^2\mu'^2}{2(2\alpha)(2\alpha-1)} - \frac{\mu^2+\mu'^2}{8(2\alpha-1)} + \frac{(\alpha-1)}{2(2\alpha)}\mu\mu' + \frac{\alpha(\alpha-1)^2}{4(2\alpha-1)}\right) x^2$$

$$+ O(x^3)\ . \tag{10}$$

Using (6), (7), (9), and (10) and choosing (c, s) and (c', s') so $\gamma_s^{\ c}$, $\gamma_{s'}^{\ c'} \neq 0$:

$$C^{cc'(1)}_{ss'\lambda}\ \delta_{\lambda\lambda'} = \sum_{\mu\mu'} d^{s}_{\mu\lambda}(\tfrac{\pi}{2})\ d^{s'}_{\mu'\lambda'}(\tfrac{\pi}{2}) \left\{ d^{s}_{\mu M}(\tfrac{\pi}{2})\ \gamma_s^{\ c}\ d^{s'}_{\mu' M}(\tfrac{\pi}{2})\ \gamma_{s'}^{\ c'} \left[\frac{1}{2}\left(\frac{\mu\mu'}{\alpha} + \alpha\right)\right] + \tilde{\beta}^{c(1)}_{s\mu}\tilde{\beta}^{c'(1)}_{s'\mu'} \right\}\ .$$

From the identity

$$\sum_{\mu} d^{s}_{\mu\lambda}(\tfrac{\pi}{2})\ (\mu)^{k}\ d^{s}_{\mu M}(\tfrac{\pi}{2}) = \langle s\lambda|R_{y}^{-1}(\tfrac{\pi}{2})\ (J_{z})^{k}\ R_{y}(\tfrac{\pi}{2})|sM\rangle =$$

$$\langle s\lambda|\ (-J_{x})^{k}|sM\rangle = \langle s\lambda|\ -\left(\frac{J_{+}+J_{-}}{2}\right)^{k}\ |sM\rangle \qquad (11)$$

it follows that [15]

$$C^{cc'(1)}_{ss'\lambda}\ \delta_{\lambda\lambda'} = \left[\frac{1}{2\alpha}\ \gamma_{s}^{\ c}\ \gamma_{s}^{\ c'}\ \langle s\lambda|\left(\frac{J_{+}+J_{-}}{2}\right)|sM\rangle\ \langle s'\lambda'|\left(\frac{J_{+}+J_{-}}{2}\right)|s'M\rangle\right.$$

$$\left. + \frac{\alpha}{2}\ \gamma_{s}^{\ c}\ \gamma_{s'}^{\ c'}\ \delta_{\lambda M}\ \delta_{\lambda' M}\right] + \left[\left(\sum_{\mu} d^{s}_{\mu\lambda}(\tfrac{\pi}{2})\ \beta^{c(1)}_{s\mu}\right)\right.$$

$$\left.\left(\sum_{\mu'} d^{s'}_{\mu'\lambda'}\ \tilde{\beta}^{c'(1)}_{s'\mu'}\right)\right] . \qquad (12)$$

By assumption (ii), (c,s) and (c',s') can be chosen such that $s,\ s' \geqq \max|M \pm 1|$ and hence the leading trajectory gives an off-diagonal contribution which must be canceled by a conspirator. The contribution of the parent is written as

$$\langle s\lambda|\left(\frac{J_{+}}{2}\right)^{\lambda-M}|sM\rangle\ \langle s'\lambda'|\left(\frac{J_{+}}{2}\right)^{\lambda'-M}|sM\rangle\ B^{(1)}_{s\lambda s'\lambda'}\ \gamma_{s}^{\ c}\ \gamma_{s'}^{\ c'} \quad \text{for} \quad \lambda,\lambda' \geqq M, \qquad (13)$$

and similarly with a J_{-} for λ or $\lambda' < M$. This removes the common factors. Therefore

$$B^{(1)}_{s\lambda s'\lambda'} = \begin{array}{c} \\ \lambda=M+1 \\ \lambda=M \\ \lambda=M-1 \end{array} \begin{array}{c} \begin{array}{ccc} \lambda'=M+1 & \lambda'=M & \lambda'=M-1 \end{array} \\ \begin{pmatrix} \frac{1}{2\alpha} & 0 & \frac{1}{2\alpha} \\ 0 & \frac{\alpha}{2} & 0 \\ \frac{1}{2\alpha} & 0 & \frac{1}{2\alpha} \end{pmatrix} \end{array} .$$

The first conspirator contribution must differ from the negative of this by the diagonal matrix $c^{(1)}\delta_{\lambda\lambda'}$. Therefore using the factorizability of the first conspirator contribution:

$$\sum_{\mu} d^{s}_{\mu\lambda}(\tfrac{\pi}{2})\, \beta^{c(1)}_{s\mu} = \frac{1}{(-2\alpha)^{\frac{1}{2}}} \langle s\lambda| \left\{ b_{+}\left(\frac{J_{+}}{2}\right) + b_{-}\left(\frac{J_{-}}{2}\right)\right\} |sM\rangle \quad \gamma_{s}^{\ c} ,$$

$$\sum_{\mu'} d^{s}_{\mu'\lambda'}(\tfrac{\pi}{2})\, \beta^{c'(1)}_{s'\mu'} = \frac{1}{(-2\alpha)^{\frac{1}{2}}} \langle s'\lambda'| \left\{ b_{+}'\left(\frac{J_{+}}{2}\right) + b_{-}'\left(\frac{J_{-}}{2}\right)\right\} |s'M\rangle \quad \gamma_{s'}^{\ c'} ,$$

with $b_{+}b_{-}' = b_{-}b_{+}' = 1$. Or

$$\tilde{\beta}^{c(1)}_{s\mu} = \frac{1}{(-2\alpha)^{\frac{1}{2}}} \langle s\mu|\, R_{y}(\tfrac{\pi}{2}) \left\{ b_{+}\left(\frac{J_{+}}{2}\right) + b_{-}\left(\frac{J_{-}}{2}\right)\right\} |sM\rangle \quad \gamma_{s}^{\ c}, \qquad (14)$$

and similarly for $\tilde{\beta}^{c'(1)}_{s'\mu'}$ with $b_{\pm}'$.

The preceeding arguments prove that (14) holds for all (c,s) with $\gamma_{s}^{\ c} \neq 0$ and $s \geq \max|M\pm 1|$. We now consider a (c,s) with $\gamma_{s}^{\ c} \neq 0$ and $s < \max|M\pm 1|$. We can choose $\gamma_{s'}^{\ c'} \neq 0$ and $s' \geq \max|M\pm 1|$. It is clear that the residues for the channel (c',s') given in (14) and the relationship (12) determine the residues for the channel (c,s). We can write the residues for (c,s) in the form (14) by using the fact that either $J_{+}|sM\rangle$ or $J_{-}|sM\rangle$ is zero (both are zero for $s = M = 0$ - an example of the well-known fact that only alternate conspirators couple to spinless channels). Finally we consider a (c_{o},s_{o}) with $\gamma_{s_{o}}^{\ c_{o}} = 0$. We show that $\tilde{\beta}^{c_{o}(1)}_{s_{o}\mu} = 0$ and therefore (14) holds for this case too. By considering coupling of (c_{o},s_{o}) to itself and repeating the arguments leading to (9), we see that $\tilde{\beta}^{c_{o}(1)}_{s_{o}\mu} = d^{s_{o}}_{\mu M_{o}}(\tfrac{\pi}{2}) g_{s_{o}}^{\ c_{o}}$ since $\tilde{\beta}^{c_{o}(0)}_{s_{o}\mu} = 0$. However, we see that this would give a contribution to H^{s} for $\lambda \neq \lambda'$ when coupled to a channel (c,s)

with $s \geqq \max|M\pm 1|$. Therefore $g_{s_o}^{c_o}$ is zero as required. We have therefore shown that (14) holds for all (c,s). The preceeding argument can be used again for $n = 2,3,\cdots$ and therefore fortunately will not need to be repeated.

We now compare coefficients of x^2 and find further conditions on $b_\pm$, $b_\pm'$ as well as all the second conspirator residues. We insert (9), (10), and (14) into (6) and use the identity (11). $B^{(2)}_{s\lambda s'\lambda'}$ is defined as the contribution to x^2 of the parent and first conspirator with the factors taken out as in (13). We find:

$$
B^{(2)}_{\lambda\lambda'} =
\begin{array}{c|ccccc}
 & \lambda'=M+2 & \lambda'=M+1 & \lambda'=M & \lambda'=M-1 & \lambda=M-2 \\
\hline
\lambda=M+2 & \frac{1}{2(2\alpha)(2\alpha-1)} - \frac{b_+b_+'}{2(2\alpha)(2\alpha-1)} & 0 & -\frac{1}{8(2\alpha-1)} + \frac{u'+v'}{2(2\alpha)(2\alpha-1)} - \frac{b_+b_-'u' + b_+b_+'v'}{2(2\alpha)(\alpha-1)} & 0 & \frac{1}{2(2\alpha-1)(2\alpha-2)} \\
\lambda=M+1 & 0 & \frac{\alpha-1}{4\alpha}(1-b_+b_+') & 0 & 0 & 0 \\
\lambda=M & -\frac{1}{8(2\alpha-1)} + \frac{u+v}{2(2\alpha)(2\alpha-1)} - \frac{b_+b_+'u + b_+b_+'v}{2(2\alpha)(\alpha-1)} & 0 & \frac{\alpha(\alpha-1)^2}{4(2\alpha-1)} - \frac{(u+v+u'+v')}{8(2\alpha-1)} + \frac{(u+v)(u'+v')}{2(2\alpha)(2\alpha-1)} - \frac{(b_-u + b_+v)(b_-'u' + b_+'v')}{2(2\alpha)(2\alpha-1)} & 0 & -\frac{1}{8(2\alpha-1)} + \frac{u+v}{2(2\alpha)(2\alpha-1)} - \frac{b_-b_-'u + b_+b_-'v}{2(2\alpha)(\alpha-1)} \\
\lambda=M-1 & 0 & 0 & 0 & \frac{\alpha-1}{4\alpha}(1-b_-b_-') & 0 \\
\lambda=M-2 & \frac{1}{2(2\alpha-1)(2\alpha-2)} & 0 & -\frac{1}{8(2\alpha-1)} + \frac{u'+v'}{2(2\alpha)(2\alpha-1)} - \frac{b_-b_-'u' + b_-b_+'v'}{2(2\alpha)(\alpha-1)} & 0 & \frac{1}{2(2\alpha)(2\alpha-1)} - \frac{b_-b_-'}{2(2\alpha)(\alpha-1)}
\end{array}
$$

where $u = \langle sM| \frac{J_+J_-}{4} |sM\rangle = \frac{1}{4}(s^2 + s - M^2 + M)$,

$$v = \langle sM| \frac{J_-J_+}{4} |sM\rangle = \frac{1}{4}(s^2 + s - M^2 - M),$$

and similarly for u', v' in terms of s'.

By assumption (iii) we can choose (c,s) and (c',s') with $\gamma_s{}^c$, $\gamma_{s'}{}^c \neq 0$ and $s, s' \geq \max |M\pm 2|$. Then $B^{(2)}_{\lambda\lambda'}$ must differ from a factorizable matrix (the $n = 2$ contribution) by a diagonal matrix. This gives one condition;

$$B^{(2)}_{M+2,M} \; B^{(2)}_{M\,M-2} = B^{(2)}_{M+2\;M-2} \; (B^{(2)}_{MM} - C^{(2)}_{ss'M}) = B^{(2)}_{M\,M+2} \; B^{(2)}_{M-2\;M},$$

since $B^{(2)}_{M+2\;M-2} = B^{(2)}_{M-2\;M+2}$. Substituting the explicit values gives an equation relating $b_\pm$, $b_\pm'$. After some algebra during which the s and s' dependence factors out we find the two solutions [16]

$$\frac{b_+}{b_-} = \frac{b_+'}{b_-'} = \begin{cases} \frac{M+\alpha}{M-\alpha} \\ 1 \end{cases} . \tag{15}$$

Reinserting this we can obtain the second conspirator residues using factorization. As before the cases $s < \max |M\pm 2|$ and (c_o, s_o) with $\gamma_{s_o}{}^{c_o} = 0$ are then considered and the residues for all channels obtained. They are given in Table I . $K^{M\alpha(n)}_{s\mu}$ is the "kinematic residue" and

$$\tilde{\beta}^{c(n)}_{s\mu} = K^{M\alpha(n)}_{s\mu} \; \gamma_s{}^c \; ; \quad \tilde{\beta}^{c'(n)}_{s'\mu'} = K^{m\alpha(n)}_{s'\mu'} \; \gamma_s{}^c . \tag{16}$$

A symmetric factorization has been chosen. This can be done because a change $\tilde{\beta} \to f\tilde{\beta}$, $\tilde{\beta}' \to \frac{1}{f}\tilde{\beta}'$ is always permissible, since as yet the requirements of the discrete symmetries have not been invoked. The solution $\frac{b_+}{b_-} = \frac{M+\alpha}{M-\alpha}$ can be identified with a Lorentz pole by comparing with Eq. (A17) of Sciarrino and Toller.[5]

The above procedure can be extended to lower conspirators step by step. The algebra becomes extremely complicated, however. We have done it for $n = 3$ and find that for both solutions (15) it is possible to cancel the off-diagonal contributions of $n=0,1,2$ by using a third conspirator with factorizable residue. The results are given in the table. The Lorentz pole residues agree with those given by Akyeampong, Boyce, and Rashid.[17]

For future reference we state the s-channel amplitude to order x^3 which results from the second solution in (15),

$$H^s_{s\lambda s'\lambda'} = \delta_{\lambda M}\,\delta_{\lambda' M}\,x^{-\alpha}\left[1 + \frac{2\alpha}{2^2}x + \frac{(2\alpha-1)(2\alpha-2)}{2\cdot 2^4}x^2 + \frac{(2\alpha-2)(2\alpha-3)(2\alpha-4)}{2\cdot 3\cdot 2^6}x^3 + O(x^4)\right], \tag{17}$$

and notice that it is pure helicity flip M. In contrast, a Lorentz pole has helicity flips from $M+n$ to $M-n$ in order x^n, as can be seen from (6) and (11) and the next section, Eq. (22).

III. UNIQUENESS OF THE TWO CONSPIRACIES

We now prove that under the assumptions (i) - (iii) above the residues for $n = 0$ and 1 uniquely determine[18] those for all n if they are consistent with the assumptions. In the previous section we have shown that the $n = 0,1$ residues uniquely determine the $n = 2$ residue [the $b_\pm$ must be given by (15) if they are to be consistent with the assumptions]. To complete the proof by induction we show that, given the residues for $n = 0,1,2,\cdots,N-1$, the $n = N$ residue is uniquely determined.

By examining Eq. (16) in order x^N, we see the $n = N$ residue is given by $B^{(N)}_{s\lambda s'\lambda'}$ (the contribution of the first $N-1$ trajectories) and the unspecified diagonal matrix $C^{cc'(N)}_{ss'\lambda}\ \delta_{\lambda\lambda'}$:

$$R^{c(N)}_{s\lambda}\ R^{c'(N)}_{s'\lambda'} = C^{cc'(N)}_{ss'\lambda}\ \delta_{\lambda\lambda'} - B^{(N)}_{s\lambda s'\lambda'}\ \gamma_s{}^c\ \gamma_{s'}{}^{c'}\ , \tag{18}$$

where

$$R^{c(n)}_{s\lambda} = \sum_\mu d^s_{\mu\lambda}(\tfrac{\pi}{2})\ \tilde{\beta}^{c(n)}_{s\mu}\ , \qquad R^{c'(n)}_{s'\lambda'} = \sum_{\mu'} d^{s'}_{\mu'\lambda'}(\tfrac{\pi}{2})\ \tilde{\beta}^{c'(n)}_{s'\mu'}. \tag{19}$$

One can try to solve for the residues by taking

$$\frac{R^{c(N)}_{s\lambda}}{R^{c(N)}_{s\lambda_o}} = \frac{B^{(N)}_{s\lambda s'\lambda_o'}}{B^{(N)}_{s\lambda_o s'\lambda_o'}}\ , \tag{20a}$$

where $\lambda_o \neq \lambda_o' \neq \lambda$. This will give the ratios of all the $R^{c(N)}_{s\lambda}$ except $R^{c(N)}_{s\lambda_o'}$. To obtain it we could take

$$\frac{R^{c(N)}_{s\lambda_o'}}{R^{c(N)}_{s\lambda_1}} = \frac{B^{(N)}_{s\lambda_o' s'\lambda_1'}}{B^{(N)}_{s\lambda_1 s'\lambda_1'}}\ , \tag{20b}$$

where $\lambda_o' \neq \lambda_1' \neq \lambda_1$. This will work if and only if λ_o, λ_o', λ_1, λ_1'

can be found with

$$B^{(N)}_{s\lambda_o s'\lambda_o'}\ ,\ B^{(N)}_{s\lambda_1 s'\lambda_1'} \neq 0. \tag{21}$$

We will show the choice $\lambda_o = M-N$, $\lambda_o' = M+N$, $\lambda_1 = M-N+2$, $\lambda_1' = M-N$ works. Once the residues for one (s,c) have been determined uniquely, all other residues are determined uniquely from the $B^{(N)}$ matrices by factorization.

In the proof we will need the fact that the "crossed residue" $R^{c(n)}_{s\lambda}$ has contributions only for $\max\{M-n, -s\} \leqq \lambda \leqq \min\{M+n, s\}$. From the table this is clear for $n = 0,1,2,3$. We prove it by induction at the same time as the main assertion. Therefore

$$\tilde{\beta}^{c(n)}_{s\mu} = \langle s\mu|\ R_y(\tfrac{\pi}{2}) \left\{ b_n^{(n)} \left(\frac{J_+}{2}\right)^n + b_{n-1}^{(n)} \left(\frac{J_+}{2}\right)^{n-1} + \cdots + b_{-n}^{(n)} \left(\frac{J_-}{2}\right)^n \right\} |sM\rangle\ \gamma_s^{\ c}.$$

Also $b^{(n)}_{n-k}$ for k odd vanish. This is true for $n = 0,1,2,3$, can be seen from Table I, and is proved below by induction. Therefore

$$\tilde{\beta}^{c(n)}_{s\mu} = \langle s\mu|R \left\{ b_n^{(n)} \left(\frac{J_+}{2}\right)^n + b_{n-2}^{(n)} \left(\frac{J_+}{2}\right)^{n-2} + \cdots + b_{-n+2}^{(n)} \left(\frac{J_-}{2}\right)^{n-2} + b_{-n}^{(n)} \left(\frac{J_-}{2}\right)^n \right\} |sM\rangle$$
$$\times\ \gamma_s^{\ c}\ . \tag{22}$$

Being more precise, we now use assumption (iii) to chose (c,s) and (c',s') with $\gamma_s^{\ c}$, $\gamma_{s'}^{\ c'} \neq 0$ and $s, s' \geqq \max|M\pm N|$. The contribution of the $\underline{n}$th trajectory $(n \leqq N-1)$ to order x^N depends on the term of order x^{N-n} in

$$(1+x)^{\frac{\mu+\mu'}{2}}\ F(n-\alpha+\mu,\ n-\alpha+\mu';\ 2n-2\alpha;\ -x). \tag{23}$$

It follows that the highest powers of μ and μ' occurring are $(\mu)^{N-n}$ and $(\mu')^{N-n}$. From (11) and the induction hypothesis (22), these give a contribution

$$B^{(N)}_{s\lambda s'\lambda'} \propto \sum_{\mu\mu'} \left(d^{s}_{\mu\lambda}(\tfrac{\pi}{2})(\mu)^{N-n} \langle s\mu | R \left\{ b_n^{(n)} \left(\tfrac{J_+}{2}\right)^n + \cdots + b_{-n}^{(n)} \left(\tfrac{J_-}{2}\right)^n \right\} | sM\rangle \right)$$

$$\times \left(d^{s'}_{\mu'\lambda'}(\tfrac{\pi}{2})(\mu')^{N-n} \langle s'\mu' | R \left\{ b'^{(n)}_n \left(\tfrac{J_+}{2}\right)^n + \cdots + b'^{(n)}_{-n} \left(\tfrac{J_-}{2}\right)^n \right\} | s'M\rangle \right)$$

$$\propto \left(\langle s\lambda | R \left\{ \left(\tfrac{J_+ + J_-}{2}\right)^{N-n} \left[b_n^{(n)} \left(\tfrac{J_+}{2}\right)^n + \cdots + b_{-n}^{(n)} \left(\tfrac{J_-}{2}\right)^n \right] \right\} | sM\rangle \right)$$

$$\left(\langle s'\lambda' | R \left\{ \left(\tfrac{J_+ + J_-}{2}\right)^{N-n} \left[b'^{(n)}_n \left(\tfrac{J_+}{2}\right)^n + \cdots + b'^{(n)}_{-n} \left(\tfrac{J_-}{2}\right)^n \right] \right\} | s'M\rangle \right)$$

$$= 0 \quad \text{unless} \quad M-N \leqq \lambda,\ \lambda' \leqq M+N.$$

Since (23) is unchanged under $\mu \longleftrightarrow -\mu$ and $\mu' \longleftrightarrow -\mu'$ we see the total power of $(\mu^k)(\mu')^\ell$ in (23) must be even. Thus fixing $\lambda' = M \pm n$ we see

$$B^{(N)}_{s\lambda s'\, M\pm n} = 0 \quad \text{unless} \quad \lambda - (M \pm n) \text{ is even.}$$

Thus if (21) holds, (22) is true by induction.

Now consider $B^{(N)}_{s\, M-N,\ s'\, M+N}$ [as usual the factor $\langle s\ M-N | \left(\frac{J_-}{2}\right)^N | sM\rangle \times \langle s'\ M+N | \left(\frac{J_+}{2}\right)^N | sM\rangle$ has been taken out - see (13)]. It is clear from (11) and (22) that only a term $(\mu\mu' x)^{N-n}$ from (23) will give a contribution for the <u>n</u>th trjectory.

The coefficient of this term is easily seen to be

$$\frac{(-1)^{N-n}\, \Gamma(2n - 2\alpha)}{(N-n)!\ \Gamma(N + n - 2\alpha)} .$$

Thus

$$-b^{(N)}_{-N}\, b'^{(N)}_N = B^{(N)}_{s\, M-N,\ s'\, M+N} = \sum_{n=0}^{N-1} \frac{(-1)^{N-n}\, \Gamma(2n-2\alpha)}{(N-n)!\ \Gamma(N + n - 2\alpha)}\, b^{(n)}_{-n}\, b'^{(n)}_n .$$

It is not difficult to iterate this equation to find

$$B^{(N)}_{s\ M-N,\ s'\ M+N} = -\frac{\Gamma(N-2\alpha-1)}{N!\ \Gamma(2N-2\alpha-1)}\ , \tag{24}$$

which is nonzero. Therefore

$$b^{(N)}_{-N}\ b'^{(N)}_{N} = b^{(N)}_{N}\ b'^{(N)}_{-N} = \frac{1}{N!}\ \frac{\Gamma(N-2\alpha-1)}{\Gamma(2N-2\alpha-1)}\ . \tag{25}$$

Now consider $B^{(N)}_{s\ M-N+2,\ s'\ M-N} = -b^{(N)}_{-N+2}\ b'^{(N)}_{-N}$. From (25) both $b^{(N)}_{-N}$ and $b'^{(N)}_{N}$ are nonzero. Thus $B^{(N)}_{s\ M-N+2,\ s'\ M-N}$ is nonzero if $B^{(N)}_{s\ M-N+2,\ s'M+N} = -b^{(N)}_{-N+2}\ b'^{(N)}_{N}$ is nonzero and we consider the latter. The coefficient of the term x^{N-n} in (23) is in powers of $\mu + \mu'$:

$$\left[\frac{(-1)^{N-n}}{(N-n)!}\ \frac{\Gamma(2n-2\alpha)}{\Gamma(N+n-2\alpha)}\left(\mu^{N-n} + \frac{(N-n)(N-n-1)(2n+N-3\alpha-2)}{12}\ \mu^{N-n-2}\right)\right](\mu')^{N-n}$$

$$+ \text{ lower powers.} \tag{26}$$

Only the terms given in (26) and the last two terms in (22) contribute to the element of interest:

$$B^{(N)}_{s\ M-N+2,\ s'\ M+n}\ \langle s\ M-N+2\ |\left(\frac{J_-}{2}\right)^{N-2}|sM\rangle$$

$$= \sum_{n=0}^{N-1} \frac{(-1)^{N-n}\ \Gamma(2n-2\alpha)}{(N-n)!\ \Gamma(N+n-2\alpha)}\ \langle s\ M-N+2\ |\left\{\left(\frac{J_+ + J_-}{2}\right)^{N-n}\right.$$

$$\left. + \frac{(N-n)(N-n-1)(2n+N-3\alpha-2)}{12}\left(\frac{J_+ + J_-}{2}\right)^{N-n-2}\right\} \times \left\{b^{(n)}_{-n+2}\left(\frac{J_-}{2}\right)^{n-2}\right.$$

$$\left. + b^{(n)}_{-n}\left(\frac{J_-}{2}\right)^{n}\right\}\ |sM\rangle\ b'^{(n)}_{n}\ .$$

After some claculations we obtain

$$B^{(N)}_{s\ M-N+2\ s'\ M+N} = \sum_{n=0}^{N-1} \frac{(-1)^{N-n}\ \Gamma(2n-2\alpha)}{(N-n)!\ \Gamma(N+n-2\alpha)}$$

$$\times\left\{b^{(n)}_{-n+2} + \frac{(N-n)}{4}\left[s^2 + s - (M+2-n)(M+2-N) + (N-n-1)(n-1)\right]b^{(n)}_{-n}\right\}b'^{(n)}_{n}$$

$$= -b^{(N)}_{-N+2}\, b'^{(N)}_{N} \quad \equiv \quad -\left\{Y^{(N)}\,(s^2+s) + Z^{(N)}\right\} \tag{27}$$

Since $b^{(n)}_{-n}\, b'^{(n)}_{n}$ is already known, [Eq. (25), (27)] can be solved for $b^{(N)}_{-N+2}\, b'^{(N)}_{N}$ by iteration. Actually it is sufficient to find the coefficient $Y^{(N)}$ of (s^2+s) in $B^{(N)}_{s\ M-N+2\ s'\ M+N}$. If it is nonzero there is at most one positive s_o for which $B^{(N)}_{s\ M-N+2\ s'\ M+N}$ vanishes and we can choose $s \neq s_o$. Taking the coefficient of s^2+s on both sides of (27):

$$-Y^{(N)} = \sum_{n=0}^{N-1} \frac{(-1)^{N-n}\,\Gamma(2n-2\alpha)}{(N-n)!\,\Gamma(N+n-2\alpha)} \left\{Y^{(n)} + \frac{(N-n)}{4}\, b^{(n)}_{-n}\, b'^{(n)}_{n}\right\} . \tag{28}$$

It is not difficult to iterate this equation using (15) and (25) to obtain

$$Y^{(N)} = \begin{cases} \dfrac{\Gamma(N-2\alpha-1)}{4(N-1)!\,\Gamma(2N-2\alpha-1)} \quad \dfrac{(M-N+1+\alpha)}{(M-\alpha)} \\[2ex] \dfrac{\Gamma(N-2\alpha-1)}{4(N-1)!\,\Gamma(2N-2\alpha-1)} \end{cases} \tag{29}$$

corresponding to the two choices in (15). Thus $Y^{(N)}$ does not vanish and equations (20a,b) can be used to determine the ratios of the residues.

For $s \leq |M \pm N|$ and $\gamma_s^{\ c} = 0$, we can use arguments exactly as in Section II. Therefore all the residues for $n = N$ are uniquely determined[18] and the assertion is true by induction. Note that N must be > 1 for the above to apply since for $N = 1$, $\lambda_o = \lambda_1$ and Eqs. (20a) and (20b) become the same and the method fails. This was also seen in Section II. Thus the $n = 0,1$ residues must both be specified.

Since the $n = 0$ and 1 residues of the first solution agree with the Lorentz pole residues and a single Lorentz pole is known to be a possible conspiracy, the first solution must be a <u>single</u> Lorentz pole. The second solution, if it exists, is a new one (if one wishes it can be regarded as

an infinite counter-conspiracy of Lorentz Poles). We will call it a Class II conspiracy and a Lorentz Pole a Class I conspiracy.[19] Under the stated assumptions the possible conspiracies are characterized completely by α, γ_s^c, M, and the Class I or II. The Class II conspiracy is discussed further in the Appendix.

IV. PARITY AND TIME REVERSAL

We first state our conventions for parity-conserving amplitudes and Reggeization.[20] Let H be a c.m. helicity amplitude. Then

$$H_{\lambda_a\lambda_b;\lambda_c\lambda_d}(z) = \sum_{J=m}^{\infty} (2J+1)\, a^J_{\lambda_a\lambda_b;\lambda_c\lambda_d}\, d^J_{\lambda\lambda'}(z)\,, \quad (m = \max\{|\lambda|, |\lambda'|\}), \tag{30}$$

where z is the cosine of the c.m. scattering angle and $\lambda = \lambda_c - \lambda_d$, $\lambda' = \lambda_a - \lambda_b$. Partial-wave amplitudes of definite parity ($P = \eta(-1)^{J-v}$, $\eta = \pm 1$) are defined by

$$a^{JP}_{\lambda_a\lambda_b;\lambda_c\lambda_d} = a^J_{\lambda_a\lambda_b;\lambda_c\lambda_d} + \eta\, \eta_a\eta_b\, (-1)^{s_a+s_b-v}\, a^J_{-\lambda_a-\lambda_b;\lambda_c\lambda_d}\,, \tag{31}$$

where $v = \{0$ if $2(s_a+s_b)$ is even and $\frac{1}{2}$ if $2(s_a+s_b)$ is odd$\}$ and the parities η_i are taken to be real $[\eta_{\bar{i}} = (-1)^{2s_i}\eta_i]$. We define the signatured amplitudes $[\tau = (-1)^{J-v}$ for J integral$]$ and continue a^J to complex J. Then

$$a^{JP\tau}_{\lambda_a\lambda_b;\lambda_c\lambda_d} = \sigma\, \eta_a\, \eta_b\, (-1)^{s_a+s_b-v}\, a^{JP\tau}_{-\lambda_a-\lambda_b,\lambda_c\lambda_d}\,, \tag{32}$$

where $\sigma = P\tau$. Therefore at a pole in the complex J plane

$$a^{JP\tau}_{\lambda_a\lambda_b;\lambda_c\lambda_d} \approx \frac{\beta^{P\tau}_{\lambda_a\lambda_b}\ \beta'^{P\tau}_{\lambda_c\lambda_d}}{J - \alpha(t)}$$

we have

$$\beta^{P\tau}_{\lambda_a\lambda_b} = \sigma\, \eta_a\eta_b\, (-1)^{s_a+s_b-v}\, \beta^{P\tau}_{-\lambda_a-\lambda_b}\,. \tag{33}$$

The Mandelstam-Sommerfeld-Watson transform then gives

$$H^{(\mathrm{Regge})}_{\lambda_a\lambda_b;\lambda_c\lambda_d}(z) = \frac{-(2\alpha+1)}{2}\left(1 + \tau\, e^{-i\pi(\alpha-v)}\right) \beta^{P\tau}_{\lambda_a\lambda_b}\, \beta^{P\tau}_{\lambda_c\lambda_d}\, e^{-\alpha-1}_{\lambda\lambda'}(z) \tag{34}$$

Using time-reversal invariance in the t channel we have chosen β' to be the same function of τ' and the other variables as β. The residues in (43) are actually the residues of the $O(2,1)$ partial-wave amplitude $f^{JP}_{\lambda_a\lambda_b;\lambda_c\lambda_d}$.[1] The definite-parity amplitude f^{JP} is related to f^J by (31) and

$$f^J_{\lambda_a\lambda_b;\lambda_c\lambda_d} = \frac{1}{2}\int_1^\infty dz\, H_{\lambda_a\lambda_b;\lambda_c\lambda_d}(z)\, d^J_{\lambda\lambda'}(z) = (-1)^{\lambda-\lambda'}\, f^{-J-1}_{\lambda_a\lambda_b;\lambda_c\lambda_d}$$

The implications of parity conservation for conspiracies are obtained by using the basic Eqs. (33) and (34). Parity conservation in the s channel,

$$H^s_{\lambda_3\lambda_4;\lambda_1\lambda_2}(s,\ t=0) = \frac{\eta_3\eta_4(-1)^{s_3+s_4}}{\eta_1\eta_2(-1)^{s_1+s_2}}\,(-1)^{\lambda-\lambda'}\, H^s_{-\lambda_3-\lambda_4,-\lambda_1-\lambda_2}(s,\ t=0),$$

gives

$$H^s_{s\lambda s'\lambda'}(s) = \frac{\eta_3\eta_4}{\eta_1\eta_2}\,(-1)^{s+s'}\, H^s_{s-\lambda\ s'-\lambda'}(s), \tag{35}$$

since $\lambda = \lambda'$. This implies that

$$C^{cc'(k)}_{ss'\lambda}(x) = \frac{\eta_3\eta_4}{\eta_1\eta_2}\,(-1)^{s+s'}\, C^{cc'(k)}_{ss'-\lambda}. \tag{36}$$

To preserve the symmetry of the residues we now define

$$\tilde{\beta}^{cP\tau}_{s\mu} = \sum_{\mu_2\mu_4} C(s_4\ s_2\ s;\ \mu_4-\mu_2)\,(-1)^{s_2-\mu_2} e^{i\frac{\pi}{2}\mu}\left[\frac{\Gamma(\mu-\alpha)\Gamma(-\mu-\alpha)}{2\Gamma(-2\alpha)}\right]^{\frac{1}{2}} \beta^{cP\tau}_{\mu_1\mu_2} \tag{37}$$

for both (c,s) and (c',s'). From (33) it follows that

$$\tilde{\beta}^{cP\tau}_{s\mu} = \sigma\,\eta_2\eta_4\, e^{i\pi v}\,(-1)^{s+\mu}\,\tilde{\beta}^{cP\tau}_{s-\mu}, \tag{38}$$

and similarly for c' with $\eta_2\eta_4 \to \eta_{\bar{1}}\,\eta_{\bar{3}}$. The "crossed residues" have the property

$$R_{s\lambda}^{cP\tau} = \sum_{\mu} d_{\mu\lambda}^{s}(\tfrac{\pi}{2})\, \tilde{\beta}_{s\mu}^{cP\tau} = \sigma \begin{Bmatrix} \eta_2 & \eta_4 \\ \eta_{\bar{1}} & \eta_{\bar{3}} \end{Bmatrix} e^{i\pi v}(-1)^{s+\lambda}\, R_{s-\lambda}^{cP\tau}, \quad (39)$$

and finally the leading-order contribution of a trajectory is $\sigma\ (-1)^{\lambda'+v}\eta_{\bar{1}}\ \eta_{\bar{3}}\ R_{s\lambda}\ R_{s'\lambda'}$.

Now the arguments of Sections II and III must be repeated in the light of Eqs. (36) and (38). There is the possibility of four leading trajectories with different quantum numbers P and τ. However, since the products of the residues $\beta_{\mu_1\mu_3}^{P\tau(n)}\beta_{\mu_2\mu_4}^{P\tau(n)}$ are relatively real and the signature factors $[1+\tau e^{-i\pi(\alpha-v)}]$ are relatively complex for opposite signature, trajectories of opposite signature can be treated separately. Also the signature of the nth conspirator must be $(-1)^n\tau$ to give the same phase. Thus the contribution to order x^0 in (6) comes from at most two trajectories and is at most of rank two. From (36), for $M \neq 0$ $C^{(1)}$ is of rank two so there must be two leading trajectories of opposite parity. For $M = 0$ there need only be one, and (39) shows in fact only one parity can contribute, since $\lambda = 0$.

The unique solution[18] satisfying the requirements is easily found to be

$$R_{s\lambda}^{c+\tau(0)} = \frac{1}{(2\alpha+1)^{\frac{1}{2}}}\left[\delta_{\lambda M} + \tau\begin{Bmatrix} \eta_2 & \eta_4 \\ \eta_{\bar{1}} & \eta_{\bar{3}} \end{Bmatrix} e^{-i\pi v}\ (-1)^{s+M}\ \delta_{\lambda-M}\right]\gamma_s^{\ c}$$

$$R_{s\lambda}^{c-\tau(0)} = \frac{i}{(2\alpha+1)^{\frac{1}{2}}}\left[\delta_{\lambda M} - \tau\begin{Bmatrix} \eta_2 & \eta_4 \\ \eta_{\bar{1}} & \eta_{\bar{3}} \end{Bmatrix} e^{-i\pi v}\ (-1)^{s+M}\ \delta_{\lambda-M}\right]\gamma_s^{\ c}, \quad (40)$$

where by convention M is now taken to be positive and $\gamma_s^{\ c}$ is pure real or pure imaginary. This solution is like a superposition of the previous

solutions for M and $-M$ with the residues arranged so the nondiagonal contributions in λ, λ' cancel between the trajectories of opposite parity.

As before, the contribution of the leading trajectories does not satisfy the conspiracy condition in order x^1 and conspirators must be introduced. The analysis proceeds as before. It is not difficult to see that the following set of residues provides a solution,

$$R_{s\lambda}^{cP\tau_n(n)} = \frac{e^{i\frac{\pi}{4}(1-P)}}{[2(\alpha-n)+1]^{\frac{1}{2}}} \left[R_{s\lambda}^{M(n)} + \sigma\, \eta_2\eta_4\, e^{-i\pi v}(-1)^{s+M} \begin{Bmatrix} 1 \\ (-1)^n \end{Bmatrix} R_{s\lambda}^{-M(n)} \right] \gamma_s \,, \tag{41}$$

where

$$\begin{Bmatrix} 1 \\ (-1)^n \end{Bmatrix} = \begin{Bmatrix} \text{Class I} \\ \text{Class II} \end{Bmatrix},$$

$\tau_n = \tau(-1)^n$, and $R_{s\lambda}^{M(n)}$ is the crossed residue in the case without parity conservation. Also

$$B_{s\lambda s'\lambda'}^{(N)} = 2\eta_{\bar{1}}\,\eta_{\bar{3}}\,(-1)^{M+v}\left[B_{s\lambda s'\lambda'}^{M(N)} + \frac{\eta_3\eta_4}{\eta_1\eta_2}\,(-1)^{s+s'}\, B_{s\lambda s'\lambda'}^{-M(N)} \right]. \tag{42}$$

From (41) and the symmetry properties (A9ab) we obtain for either class I or class II,

$$\tilde{\beta}_{s\mu}^{cP\tau_n(n)} = \frac{e^{i\frac{\pi}{4}(1-P)}}{[2(\alpha-n)+1]^{\frac{1}{2}}} \left[1 + \sigma \begin{Bmatrix} \eta_2 & \eta_4 \\ \eta_{\bar{1}} & \eta_{\bar{3}} \end{Bmatrix} e^{i\pi v}\,(-1)^{\mu+M+n} \right] K_{s\mu}^{M\alpha(n)}\, \gamma_s, \tag{43}$$

with $\tau_n = (-1)^n\tau$ (compare with Ref. 1). That (45) satisfies the conspiracy condition can be shown by induction on n by using the symmetry properties (A9ab) and the results of the previous sections that

$$R_{s\lambda}^{M(n)}\, R_{s'\lambda'}^{M(n)} = -\, B_{s\lambda s'\lambda'}^{M(n)} + \text{diagonal matrix.}$$

It remains to show that the solution (42) is unique. First we discuss $M \neq 0$. For half integral M it is easy to show uniqueness by induction. In each step we reduce the problem to the one without parity conservation and use the results of Secs. II and III. In Fig. 1, the possible nonzero elements of $B^{(N)}$ are shown. As a result of Eq. (22) the contributions from $B^{M(N)}$ and $B^{-M(N)}$ never coincide. Therefore if, for example, we take $\lambda,\ \lambda' = M-N,\ M-N+2, \cdots, M+N-2,\ M+N$,

$$(-1)^N \left[R_{s\lambda}^{\sigma=+1\ (N)}\, R_{s'\lambda'}^{\sigma=+1\ (N)} - R_{s\lambda}^{\sigma=-1\ (N)}\, R_{s'\lambda'}^{\sigma=-1\ (N)} \right]$$

$$= 2\, B_{s\lambda s'\lambda'}^{M(N)}\, \gamma_s^{\ c}\, \gamma_{s'}^{\ c'} + \eta_{\bar{1}}\, \eta_{\bar{3}}\, (-1)^{M+v}\, C_{ss'\lambda}^{cc'(N)}\, \delta_{\lambda\lambda'}$$

and

$$(-1)^N \left[R_{s-\lambda}^{\sigma=+1\ (N)}\, R_{s'\lambda'}^{\sigma=+1\ (N)} - R_{s-\lambda}^{\sigma=-1\ (N)}\, R_{s'\lambda'}^{\sigma=-1\ (N)} \right] = 0\ .$$

Using (39) and adding or subtracting the equations yields

$$R_{s\lambda}^{P\tau(N)}\, R_{s'\lambda'}^{P\tau(N)} = \sigma(-1)^N\, B_{s\lambda s'\lambda'}^{M(N)}\, \gamma_s^{\ c}\, \gamma_{s'}^{\ c'} + \frac{\eta_{\bar{1}}\, \eta_{\bar{3}}\, (-1)^{M+v}}{2}\, C_{ss'\lambda}^{cc'(N)}\, \delta_{\lambda\lambda'}.$$

This is of the same type as considered in the no-parity case and all the previous arguments can be applied to find there are only two possibilities for $R_{s\lambda}^{P\tau(N)}$. Taking $-\lambda,\ -\lambda'$ in the above range yields a similar equation with B^{M} replaced by B^{-M}. Requiring the property (39) gives the result (47). For λ not in the above range or $s \leq |M \pm N|$ or $\gamma_s^{\ c} = 0$, arguments just like those in Section II can be used.

For integral M the proof is complicated by the fact that the

contributions from B^M and B^{-M} overlap as shown in Fig. 2. A slightly different method, not given here, must be used. Special care must be exercised for $M = 1$ to be sure the result analogous to Eq. (15) still follows.

For $M = 0$, it follows from (41) that only one of the two possible conspirators actually couples. For Class I (Lorentz pole), σ is the same for all conspirators (the parity alternates), whereas for Class II, σ alternates (the parity is the same). The symmetry properties (A9ab) thus have a striking consequence in this case. For $M = 0$ parity doublets are not required and one would expect that their occurrence would therefore be "accidental." If we thus assume there is no doubling it is easy to show (43) gives the unique solution. We have not been able to exclude the possibility of "accidental" parity doubling, but we believe only the following type of doubling can occur. For Class II it is possible to add to the conspiracy α_o, P_o, τ_o another Class II conspiracy $\alpha_1 = \alpha_o - (2k-1)$, $-P_o$, $-\tau_o$ (k integral). There is no interference, since the parities of trajectories of the same α are opposite. There is not an analogous possibility for Class I, because the σ of the leading member of the second conspiracy would have to be opposite to that of the first conspiracy and therefore by (39) the residue of this leading member would have to vanish.

V. COUPLING TO UNEQUAL MASS CHANNELS

We now consider the additional constraints that factorization imposes on the equal-mass channel residues when the leading trajectory couples to an unequal-mass channel. We first determine the residues at the unequal mass vertex from unequal-unequal-mass scattering and then determine the equal-mass residues from factorization and equal-unequal-mass scattering. In both cases we appeal to the results of Freedman and Wang.[21] We consider an amplitude free of kinematic singularities, expand in an asymptotic series in s for fixed $t \neq 0$, and demand that the coefficient of each power of s be nonsingular at $t = 0$. Only the potentially most singular term for each power of s is considered, and therefore only the leading t behavior of the residues is determined. To begin with we ignore signature, parity, and time reversal.

First consider the t-channel reaction

$$m_1 + m_2 \rightarrow m_1 + m_2 \quad ; \quad m_2 > m_1 .$$

Then

$$z_t = \frac{2ts + [t - (m_1 + m_2)^2][t - (m_1 - m_2)^2]}{[t - (m_1 + m_2)^2][t - (m_1 - m_2)^2]} = 1 + \frac{2ts}{\mathcal{T}^2} ,$$

where

$$\mathcal{T} = \left\{[t - (m_1 + m_2)^2][t - (m_1 - m_2)^2]\right\}^{\frac{1}{2}} . \tag{44}$$

As usual we define

$$x \equiv \frac{2}{z_t - 1} = \frac{\mathcal{T}^2}{ts} , \tag{45}$$

The t-channel momenta are given by

$$p_{12} = \frac{\mathcal{T}}{2t^{\frac{1}{2}}} . \tag{46}$$

Finally we remove the threshold behavior by defining,

$$\bar{\beta}^{(n)}_{s\mu} = (2p_{12})^{-\alpha+n}\,\beta_{s\mu}\,, \quad \bar{\beta}^{(n)}_{s'\mu'} = (2p_{12})^{-\alpha+n}\,\beta_{s'\mu'} \tag{47}$$

where the $\tilde{\beta}$ are given by (3). The t-channel helicity amplitude is then given by

$$H^{t}_{s\mu s'\mu'} = \sum_{n} \bar{\beta}^{(n)}_{s\mu}\,\bar{\beta}^{(n)}_{s'\mu'}\; s^{\alpha-n}\,(1+x)^{\frac{\mu+\mu'}{2}}\,F(\mu-\alpha+n,\ \mu'-\alpha+n,\ 2n-2\alpha,\ -x). \tag{48}$$

In this case the crossing relation

$$H^{s}_{\lambda_3\lambda_4;\lambda_1\lambda_2} = \sum_{\{\mu\}} d^{s_1}_{\mu_1\lambda_1}(\chi_1)\, d^{s_2}_{\mu_2\lambda_2}(\chi_2)\, d^{s_3}_{\mu_3\lambda_3}(\chi_3)\, d^{s_4}_{\mu_4\lambda_4}(\chi_4)\, H^{t}_{\mu_3\mu_1;\mu_4\mu_2} \tag{49}$$

does not take the simple form (1). However,

$$\cos\chi_1 = \frac{-s(t+m_2^{\,2}-m_1^{\,2})}{[s(s-4m_2^{\,2})]^{\frac{1}{2}}\,\mathcal{T}} \xrightarrow[s\to\infty]{} \frac{-t+m_1^{\,2}-m_2^{\,2}}{\mathcal{T}} \xrightarrow[t\to 0]{} -1$$

$$\cos\chi_3 \xrightarrow[\substack{s\to\infty\\ t\to 0}]{} -1\,, \tag{50}$$

$$\cos\chi_{2,4} \xrightarrow[\substack{s\to\infty\\ t\to 0}]{} 1\,.$$

When $s\to\infty$, the s-channel boundary approaches $t = 0$, so for the leading asymptotic power of H^s we have

$$H^{s}_{s\lambda s'\lambda'} \xrightarrow[s\to\infty]{} C_{ss'\lambda}\;\delta_{\lambda\lambda'}\;s^{\alpha} \tag{51}$$

$$= \bar{\beta}^{(0)}_{s\lambda}\,\bar{\beta}^{(0)}_{s'-\lambda'}\;(-1)^{s'-\lambda'}\;s^{\alpha}\,.$$

By the same arguments given in Sec. II it follows that we can write[22]

$$\bar{\beta}^{(0)}_{s\mu} = \delta_{\mu M}\;\gamma_s\,, \qquad \bar{\beta}^{(0)}_{s'\mu'} = \delta_{\mu'-M}\,\gamma_{s'}\,, \tag{52}$$

where M is the s-channel helicity flip in leading order.

Since $\sin\frac{\theta_s}{2}$ and $\cos\frac{\theta_s}{2}$ have no singularities at $t = 0$, H^s itself must have no singularities at $t = 0$. From (50) it is clear that the crossing angles have no singularities at $t = 0$, and therefore H^t must have no singularities at $t = 0$. Because $x = \frac{\mathcal{D}^2}{ts}$, the contribution of the leading trajectory will have singularities at $t = 0$ for lower powers of s. These singularities are canceled by conspirator trajectories with singular residues.[21]

The $n = 0,1$ contributions to (48) give for $\mu = M$, $\mu' = -M$:

$$H^t_{sMs'-M} \sim s^\alpha \left(1 - \frac{(M-\alpha)(-M-\alpha)}{(-2\alpha)} \frac{\mathcal{D}^2}{ts}\right) + \bar{\beta}^{(1)}_{sM}\, \bar{\beta}^{(1)}_{s'-M}\, s^{\alpha-1} .$$

Therefore,

$$\bar{\beta}^{(1)}_{s\mu} = f\, i \left[\frac{M^2 - \alpha^2}{-2\alpha}\right]^{\frac{1}{2}} \frac{\mathcal{D}}{t^{\frac{1}{2}}}\, \delta_{\mu M}\, \gamma_s$$

$$\bar{\beta}^{(1)}_{s'\mu'} = \frac{1}{f}\, i \left[\frac{M^2 - \alpha^2}{-2\alpha}\right]^{\frac{1}{2}} \frac{\mathcal{D}}{t^{\frac{1}{2}}}\, \delta_{\mu'-M}\, \gamma_{s'} . \tag{53}$$

The scaling factor f is arbitrary at present but is fixed by the equal-unequal-mass case since we have already fixed the scaling factor in the equal mass residue.

It is straightforward to write down the general condition for the most singular term in the coefficient of $s^{\alpha-N}$,

$$\sum_{n=0}^{N} \frac{(-1)^{N-n}}{(N-n)!} \frac{\Gamma(M-\alpha+N+1)}{\Gamma(M-\alpha+n+1)} \frac{\Gamma(-M-\alpha+N+1)}{\Gamma(-M-\alpha+n+1)} \frac{\Gamma(2n-2\alpha)}{\Gamma(N+n-2\alpha)} \left(\frac{\mathcal{D}^2}{t}\right)^n \bar{\beta}^{(n)}_{sM}\, \bar{\beta}^{(n)}_{s'-M} = 0.$$

Iteration leads to a solution much like (24),

$$\bar{\beta}^{(N)}_{sM}\, \bar{\beta}^{(N)}_{s'-M} = \frac{\Gamma(N-2\alpha-1)\, \Gamma(M-\alpha+N)\, \Gamma(-M-\alpha+N)}{N!\, \Gamma(2N-2\alpha-1)\Gamma(M-\alpha)\, \Gamma(-M-\alpha)} \left(\frac{\mathcal{D}^2}{t}\right)^N \gamma_s\, \gamma_{s'} .$$

This agrees with the result of Klein.[10] The inclusion of signature, parity,

and time reversal is straightforward with use of (38). The result is

$$\tilde{\beta}_{s\mu}^{P\tau_n(n)} = \frac{e^{i\frac{\pi}{4}(1-P)\epsilon}}{[2(\alpha-n)+1]^{\frac{1}{2}}}\left[\frac{(-1)^n\,\Gamma(n-2\alpha-1)}{n!\;\Gamma(2n-2\alpha-1)}\;\frac{\Gamma(M-\alpha+n)}{\Gamma(M-\alpha)}\;\frac{\Gamma(-M-\alpha+n)}{\Gamma(-M-\alpha)}\right]^{\frac{1}{2}} \tag{54}$$

$$\times\left(\frac{\mathcal{J}^2}{t}\right)^{\alpha/2}\left[\delta_{\mu M} + \sigma\begin{Bmatrix}\eta_2 & \eta_4\\ \eta_{\bar{1}} & \eta_{\bar{3}}\end{Bmatrix} e^{-i\pi v}\,(-1)^{s+M}\,\delta_{\mu-M}\right]\gamma_s\,,$$

where $\tilde{\beta}$ is given by (37) for both s and s'. The phases are chosen to be consistent with (52) and $\epsilon = \mp 1$ for $m_{4,3} \gtrless m_{2,1}$.

We now consider the t-channel process

$$m + m \to m_1 + m_2\,.$$

Then

$$z_t = \frac{t^{\frac{1}{2}}\,[2s + t - (2m^2 + m_1^2 + m_2^2)]}{\mathcal{J}N} \xrightarrow[t\to 0]{} \frac{t^{\frac{1}{2}}}{\mathcal{J}N}\,(2s - \Sigma),$$

where $N = (t - 4m^2)^{\frac{1}{2}}$, $\Sigma = 2m^2 + m_1^2 + m_2^2$. Also

$$x \equiv \frac{2}{z_t - 1} = \frac{\mathcal{J}N}{t^{\frac{1}{2}}s\left[1 - \frac{1}{2s}\left(\Sigma + \frac{\mathcal{J}N}{t^{\frac{1}{2}}}\right)\right]} \xrightarrow[\substack{s\to\infty\\ t\to 0}]{} \frac{\mathcal{J}N}{t^{\frac{1}{2}}s}\left(1 + \frac{\mathcal{J}N}{2t^{\frac{1}{2}}s} + \cdots\right), \tag{55}$$

where only the most singular terms are kept. Finally

$$p = \frac{(t-4m^2)^{\frac{1}{2}}}{2} = \frac{N}{2}\,.$$

The behavior of the crossing angles is rather different in this case,

$$\cos\chi_1 = \frac{-(s + m_2^2 - m^2)(t + m_2^2 - m_1^2) - 2m_2^2(m_1^2 - m_2^2)}{\left\{[s - (m + m_2)^2][s - (m - m_2)^2]\right\}^{\frac{1}{2}}\mathcal{J}}$$

$$\xrightarrow[s\to\infty]{} \frac{-t + m_1^2 - m_2^2}{\mathcal{J}} \xrightarrow[t\to 0]{} -1\,,$$

$$\cos\chi_3 \xrightarrow[\substack{s\to\infty\\ t\to 0}]{} -1\,,$$

$$\cos\chi_2 = \frac{(s + m^2 - m_2^2)t - 2m^2(m_1^2 - m_2^2)}{\left\{[s - (m + m_2)^2][s - (m - m_2)^2]\right\}^{\frac{1}{2}}t^{\frac{1}{2}}N}$$

$$\xrightarrow[s\to\infty]{} \frac{st + 2m^2\mathcal{J}}{2t^{\frac{1}{2}}N} = \frac{t^{\frac{1}{2}}}{N} + \left(\frac{2m^2}{N^2}\right)\left(\frac{\mathcal{J}N}{t^{\frac{1}{2}}s}\right) \xrightarrow[t\to 0]{} -\frac{\mathcal{J}N}{2t^{\frac{1}{2}}s} ,$$

$$\cos \chi_4 \xrightarrow[\substack{s\to\infty \\ t\to 0}]{} -\frac{\mathcal{J}N}{2t^{\frac{1}{2}}s} .$$

As before, since $\sin\frac{\theta_s}{2}$ and $\cos\frac{\theta_s}{2}$ are nonsingular at $t = 0$, H^s is nonsingular at $t = 0$. The crossing angles χ_1 and χ_3 are nonsingular at $t = 0$ but χ_2 and χ_4 are singular. There we require that

$$\sum_{\mu_2\mu_4} d^{s_2}_{\mu_2\lambda_2}(\chi_2)\, d^{s_4}_{\mu_4\lambda_4}(\chi_4)\, H^t_{\mu_3\mu_1;\mu_4\mu_2}$$

be nonsingular at $t = 0$. This is our formulation of the conspiracy conditions. As we will consider only the terms s^{α} and $t^{-\frac{1}{2}}\, s^{\alpha-1}$ we can take $\chi_2 = \chi_4 = \chi$ and use the method of Sec. II to combine the two d functions. We therefore simply require that

$$\sum_{\mu} d^s_{\mu\lambda}(\chi)\, H^t_{s\mu,\, s'-M}$$

be nonsingular (n.s.). Considering the contribution for $n = 0,1$ gives, using (9), (52) and (53),

$$\sum_{\mu} d^s_{\mu\lambda}(\chi) \left\{ d^s_{\mu M}(\tfrac{\pi}{2})\gamma_s\gamma_{s'}\left(\frac{t^{\frac{1}{2}}x}{\mathcal{J}N}\right)^{-\alpha} (1+x)^{\frac{\mu-M}{2}} \left(1 - \frac{(\mu-\alpha)(-M-\alpha)}{-2\alpha}\, x\right) \right.$$
$$\left. + \bar{\beta}^{(1)}_{s\mu} \left[\frac{i}{f}\left(\frac{\mu^2-\alpha^2}{-2\alpha}\right)^{\frac{1}{2}} \frac{\mathcal{J}}{t^{\frac{1}{2}}}\right] \gamma_{s'}\, s^{\alpha-1} \right. = \text{n.s.} \qquad (56)$$

The $\bar{\beta}$ for the equal-mass channel are defined analogously to (47). An argument just like that leading to (52) shows M for both channels must be the same.

Keeping only the most singular terms,

$$\sin\left(\frac{\pi}{2} - \chi\right) = \cos\chi = \frac{\mathcal{J}N}{2t^{\frac{1}{2}}s} ,$$

so we have

$$d^s_{\mu\lambda}(\chi) = \langle s\mu|e^{-i\chi J_y}|s\lambda\rangle = \langle s\mu|e^{-i\frac{\pi}{2}J_y}\, e^{i(\frac{\pi}{2}-\chi)J_y}|s\lambda\rangle$$

$$\approx \langle s\mu|e^{-i\frac{\pi}{2}J_y}\left(1 - \frac{i}{2}\frac{\mathcal{J}N}{t^{\frac{1}{2}}s}\, J_y\right)|s\lambda\rangle$$

$$= \langle s\lambda|\left(1 + \frac{i\mathcal{J}N}{2t^{\frac{1}{2}}s}\, J_y\right)e^{i\frac{\pi}{2}J_y}\,|\,s\mu\rangle$$

Substituting this in (53) gives

$$\sum_\mu \langle s\lambda|\left(1 + \frac{i}{2}\frac{\mathcal{J}N}{t^{\frac{1}{2}}s}\, J_y\right)e^{i\frac{\pi}{2}J_y}|s\mu\rangle\left(1 - \frac{\mu M}{2\alpha}\frac{\mathcal{J}N}{t^{\frac{1}{2}}s}\right)\langle s\mu|e^{-i\frac{\pi}{2}J_y}|sM\rangle s^\alpha \gamma_s \gamma_{s'}$$

$$+\left[\sum_\mu d^s_{\mu\lambda}(\tfrac{\pi}{2})\bar{\beta}^{(1)}_{s\mu}\right]\left[\frac{i}{f}\left(\frac{M^2-\alpha^2}{-2\alpha}\right)^{\frac{1}{2}}\right]\frac{\mathcal{J}}{t^{\frac{1}{2}}s}\;\gamma_{s'}\, s^\alpha$$

$$= \gamma_s\,\gamma_{s'}\,\delta_{\lambda M}\, s^\alpha + \left\{\langle s\lambda|\left(\frac{M}{2\alpha}\, J_x + \frac{i}{2}J_y\right)|sM\rangle\gamma_s + \frac{1}{N}\,\bar{R}^{(1)}_{s\lambda}\right.$$

$$\left.\times\quad\left[\frac{i}{f}\left(\frac{M^2-\alpha^2}{-2\alpha}\right)^{\frac{1}{2}}\right]\right\}\frac{\mathcal{J}N}{t^{\frac{1}{2}}s}\gamma_{s'}\, s^\alpha$$

where (11) and (19) are used in the last step. Therefore

$$\bar{R}^{(1)}_{s\lambda} = -\frac{i}{f}\left(\frac{-2\alpha}{M^2-\alpha^2}\right)^{\frac{1}{2}}\langle s\lambda|\left(\frac{M}{2\alpha}\, J_x + \frac{i}{2}\, J_y\right)|sM\rangle\, N\,\gamma_s$$

$$= -\frac{i}{f}\left(\frac{1}{(-2\alpha)(M^2-\alpha^2)}\right)^{\frac{1}{2}}\langle s\lambda|(M+\alpha)\left(\frac{J_+}{2}\right) + (M-\alpha)\left(\frac{J_-}{2}\right)|sM\rangle\, N\,\gamma_s\ . \tag{57}$$

Comparing with Table I one sees that this is precisely the form of the Lorentz pole residue and the required choice for f is $f = -i$. It is easy to verify this result when the symmetries are included using (43), (54), and (57).

Since, as shown in Sec. III, the n=0 and n=1 equal mass residues determine all the others under our assumptions, we conclude that only the

Class I or Lorentz pole equal mass conspiracy is permitted to couple to unequal-mass channels. We believe that from (54) and analyticity in equal-unequal-mass scattering it should be possible to show that all equal-mass conspirator residues must be those of a single Lorentz pole without the use of assumption (iii).[7]

Finally we remark that for zero-mass channels $(m_2 = m_4 = 0)$ the residues are

$$\beta^{P\tau(n)}_{\mu_4\mu_2} = e^{i\frac{\pi}{4}(1-P)} \left[\delta_{\nu M} + \sigma\, \eta_2\, \eta_4\, (-1)^{s_2+s_4-v}\, \delta_{\nu -M} \right] t^{\frac{a}{2}}\, \gamma^{(n)}, \qquad (58)$$

where $\nu = -\mu_4 - \mu_2$. From (58) we observe that only the leading trajectory couples at $t = 0$ in Compton scattering.

VI. SUMMARY AND DISCUSSION

Under the reasonable physical assumptions stated in Sec. I we find that there are only two possible ways to satisfy the equal-mass conspiracy relations. Both of these conspiracies are characterized by the "quantum numbers" M and τ (signature) - and P (parity) if $M = 0$. The first or Class I conspiracy is a single Lorentz pole. The second or Class II conspiracy is new and has the property that the s-channel amplitude is pure helicity flip M to all orders in s; it can be regarded as an infinite counterconspiracy of Lorentz poles. We also show that if there is coupling to an unequal-mass channel only the single Lorentz pole solution is permitted in the equal-mass channel. On physical grounds it is therefore reasonable to regard as mathematical accidents our Class II equal-mass conspiracy and the known equal-mass conspiracies for low channel spins (e.g., the Volkov-Gribov conspiracy).

Our assumption (iii) of coupling to arbitrarily high channel spins is quite strong, but we conjecture that the same equal-mass results follow if the leading trajectory couples to only two channels of channel spin $|M| + 2$ or greater. This appears plausible to us because if $B^{(N)}_{s\lambda s'\lambda'}$ is restricted to such s and s' there are still enough conditions to determine the residues uniquely except for special values of α when too many of the $B^{(N)}_{s\lambda s'\lambda'}$ might vanish. Of course, if there is coupling to unequal mass channels the stronger constraints imposed by analyticity in equal-unequal-mass scattering imply a single Lorentz pole and such considerations are unnecessary.

In conclusion we point out that the definition of M as the s-channel helicity flip in leading order[10,22,23] provides a consistent interpretation for all mass configurations and Eqs. (43), (54), and (58) give the leading t behavior of the residues in all cases: equal-, unequal-, and zero-mass channels.

ACKNOWLEDGMENTS

It is a pleasure to thank Professor Stanley Mandelstam for suggesting this problem and for his unfailing advice and encouragement. I would also like to thank Professor Stanley A. Klein for discussions on the material in Sec. V.

APPENDIX. PROPERTIES OF THE CLASS II CONSPIRACY

The second conspiracy in (15) will now be discussed further. The form (17) of the contribution to H^s is very suggestive and leads us to conjecture:

$$H^s_{s\lambda s'\lambda'} = \delta_{\lambda M}\,\delta_{\lambda' M}\,x^{-\alpha}\sum_{k=0}^{\infty}\frac{1}{k!}\,\frac{\Gamma(2k-2\alpha-1)}{\Gamma(k-2\alpha-1)}\left(-\frac{x}{4}\right)^k \gamma_s{}^c\,\gamma_{s'}{}^{c'} \qquad \text{(A1)}$$

$$= \delta_{\lambda M}\,\delta_{\lambda' M}\,x^{-\alpha}\,F(-\alpha,\ -\alpha-\tfrac{1}{2};\ -2\alpha-1;\ -x)\,\gamma_s{}^c\,\gamma_{s'}{}^{c'}$$

$$= \delta_{\lambda M}\,\delta_{\lambda' M}\,\frac{x^{-\alpha}}{(1+x)^{\frac{1}{2}}}\left\{\frac{1}{2}\left[1+(1+x)^{\frac{1}{2}}\right]\right\}^{2(\alpha+1)}\gamma_s{}^c\,\gamma_{s'}{}^{c'}$$

Also Table I leads us to conjecture

$$\tilde{\beta}^{c(n)}_{s\mu} = K^{M\alpha(n)}_{s\mu}\,\gamma^c_s = (-1)^n\,\tilde{P}^{(n)}(\alpha,\mu)\,d^s_{\mu M}(\tfrac{\pi}{2})\,\gamma_s{}^c\ , \qquad \text{(A2)}$$

where $\tilde{P}^{(n)}$ is a polynomial of degree n in μ. This is also suggested by (A1), which indicates symmetry under $J_+ \longleftrightarrow J_-$ and also leads to (A2). Inserting (A1) and (A2) into (6) yields for the conjectured second conspiracy

$$F(-\alpha,-\alpha-\tfrac{1}{2};\ -2\alpha-1,\ -x) = \sum_{n=0}^{\infty}\tilde{P}^{(n)}(\alpha,\mu)\,\tilde{P}^{(n)}(\alpha,\mu')\,x^n(1+x)^{\frac{\mu+\mu'}{2}}$$

$$\times\,F(n-\alpha+\mu,\ n-\alpha+\mu';\ 2n-2\alpha;\ -x)\ , \qquad \text{(A3)}$$

This equation can be iterated to obtain

$$P^{(n)}(\alpha,\mu)\,P^{(n)}(\alpha,\mu') = \sum_{k=0}^{n}\sum_{\ell=0}^{n-k}\frac{(-1)^{n-k-\ell}\,n!}{k!\,\ell!\,(n-k-\ell)!\,4^{n-k-\ell}}$$

$$\times\left\{\frac{\Gamma(2n-2\alpha-1-k)\,\Gamma(2n-2\alpha-2k-2\ell-1)\,\Gamma\left(-\frac{\mu+\mu'}{2}+1\right)}{\Gamma(n-2\alpha-1)\,\Gamma(n-2\alpha-k-\ell-1)\,\Gamma\left(-\frac{\mu+\mu'}{2}+1-\ell\right)}\right. \qquad \text{(A4)}$$

$$\left.\times\,\frac{\Gamma(n-\alpha+\mu)\,\Gamma(n-\alpha+\mu')}{\Gamma(n-\alpha+\mu-k)\,\Gamma(n-\alpha+\mu'-k)}\right\}\ ,$$

where $\tilde{P}^{(n)}(\alpha,\mu) = \left[\frac{\Gamma(n-2\alpha-1)}{n!\ \Gamma(2n-2\alpha-1)}\right]^{\frac{1}{2}} P^{(n)}(\alpha,\mu)$.

If the solution factorizes as required, $P^{(n)}(\alpha,\mu)$ can be obtained by taking the coefficient of $(\mu')^n$ in (A4),

$$P^{(n)}(\alpha,\mu) = \sum_{k=0}^{n} \frac{n!(2)^{k-n}}{k!(n-k)!} \frac{\Gamma(2n-2\alpha-k-1)}{\Gamma(n-2\alpha-1)} \frac{\Gamma(n-\alpha+\mu)}{\Gamma(n-\alpha+\mu-k)} : \tag{A5}$$

$P^{(n)}(\alpha,\mu)$ is thus a polynomial in μ and α of at most degree n in each. We can use the integral representation of the beta function twice to convert this into an expression useful for small μ rather than small n.

$$P^{(n)}(\alpha,\mu) = \begin{cases} \dfrac{(-1)^{\frac{n}{2}}}{2^{n-2\mu+1}} \dfrac{\Gamma(2n-2\alpha-2\mu)}{\Gamma(n-2\alpha-1)} \displaystyle\sum_{k=0}^{[\mu]} \dfrac{(-1)^k(2\mu)!}{(2k)!\ (2\mu-2k)!} \dfrac{\Gamma(\frac{n+1}{2}+k)}{\Gamma(\frac{1}{2})} \dfrac{\Gamma(\frac{n}{2}-\frac{1}{2}-\alpha+\mu-k)}{\Gamma(n+\frac{1}{2}-\alpha-\mu)}, & n \text{ even} \quad (A6) \\[2ex] \dfrac{(-1)^{\frac{n-1}{2}}}{2^{n-2\mu+1}} \dfrac{\Gamma(2n-2\alpha-2\mu)}{\Gamma(n-2\alpha-1)} \displaystyle\sum_{k=0}^{[\mu-\frac{1}{2}]} \dfrac{(-1)^k(2\mu)!}{(2k+1)!(2\mu-2k-1)} \dfrac{\Gamma(\frac{n}{2}+1+k)}{\Gamma(\frac{1}{2})} \dfrac{\Gamma(\frac{n}{2}-1-\alpha+\mu-k)}{\Gamma(n+\frac{1}{2}-\alpha-\mu)}, & n \text{ odd} \quad (A7) \end{cases}$$

for $\mu \geq 0$, and $P^{(n)}(\alpha,\mu) = (-1)^n P^{(n)}(\alpha,-\mu)$.

We have not been able to give a general proof that the residues

$$\tilde{\beta}^{c(n)}_{s\mu} = \left\{ \sum_{k=0}^{n} \frac{(-1)^n(-2)^{k-n}}{k!\ (n-k)!} \left[\frac{n!}{\Gamma(2n-2\alpha-1)\Gamma(n-2\alpha-1)}\right]^{\frac{1}{2}} \frac{\Gamma(2n-2\alpha-k-1)\Gamma(n-\alpha+\mu)}{\Gamma(n-\alpha+\mu-k)} \right\} \times d^{s}_{\mu M}(\tfrac{\pi}{2})\ \gamma_s^{\ c} \tag{A8}$$

when inserted into (6) satisfy the conspiracy condition or, alternatively, Eq. (A1) is a possible form of H^s and thus (A4) and (A5) are consistent for all μ,μ'. The following special cases have been checked. The conjecture

has been checked for $n \leq 4$ with the method of Sec. II. Since $P^{(n)}$ is a polynomial in μ, the consistency of (A4) and (A5) can be checked by comparing expressions for special values of μ, μ'. Taking $\mu = k + \alpha - 1 - n$, $\mu' = k' + \alpha - 1 - n$ introduces considerable simplification and we have verified the consistency for k and $k' = 0,1,2,3$ and <u>all</u> n and α. This constitutes another complete proof for $n \leq 3$ and checks a large number of values for $n > 3$. For $s = s' = M = 0$, the first conspirator residue vanishes and thus from Sec. III both conspiracies are identical. In this case (A6) gives

$$P^{(n)}(\alpha,0) = \begin{cases} (-1)^{\frac{n}{2}} \dfrac{\Gamma(\frac{n+1}{2})\,\Gamma(n-\alpha)}{\Gamma(\frac{1}{2})\,\Gamma(\frac{n}{2}-\alpha)} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

and (A4) agrees. These residues agree with the Lorentz pole residues given by Akyeampong, Boyce, and Rashid.[17] We thus have confidence that the second solution is a possible conspiracy.

In conclusion we give the symmetry properties of the kinematic residues. For the Class I conspiracy (Lorentz pole),

$$K_{s\mu}^{M\alpha(n)} = (-1)^{s-M+n}\, K_{s-\mu}^{M\alpha(n)} = (-1)^{s+\mu+n}\, K_{s\mu}^{-M\alpha(n)} . \tag{A9a}$$

These follow from the results of Sciarrino and Toller[5] when the relationship between our K and their V and W is taken into account. For the Class II conspiracy (pure helicity flip M),

$$K_{s\mu}^{M\alpha(n)} = (-1)^{s-M+n}\, K_{s-\mu}^{M\alpha(n)} = (-1)^{s+\mu}\, K_{s\mu}^{-M\alpha(n)} . \tag{A9b}$$

These follow trivially from Eqs. (A2) and (A7). Thus the two conspiracies can be distinguished by the behavior of the kinematic residues under $M \to -M$. This leads to some characteristic differences when parity is included.

FOOTNOTES AND REFERENCES

* This work was supported in part by the United States Atomic Energy Commission.

+ National Science Foundation Predoctoral Fellow.

1. M. Toller, Nuovo Cimento 53, 671 (1968).

2. D. Z. Freedman and J.-M. Wang, Phys. Rev. 160, 1560 (1967).

3. The precise definition of channel spin is given in Sec. II. Channel always refers to various t-channels unless otherwise stated.

4. D. V. Volkov and V. N. Gribov, Soviet Physics-JETP 17, 720 (1963).

5. A. Sciarrino and M. Toller, J. Math. Phys. 8, 1252 (1967).

6. N. Nakanishi, Kyoto preprint, 1968.

7. After completion of this work we received a MIT preprint of J. R. Bronzan and C. E. Jones in which they use similar techniques to derive this result in the spinless case for all conspirators.

8. See, for example, S. Mandelstam, Phys. Rev. 166, 1539 (1968).

9. As usual, in order to avoid singularities, α is assumed not to be integral or half-integral.

10. S. A. Klein, Claremont Colleges preprints, 1968.

See also K. M. Bitar and G. L. Tindle, Berkeley preprint, 1968.

11. T. L. Trueman and G. C. Wick, Ann. Phys. 26, 322 (1964). Our conventions are those of L. L. Wang, Phys. Rev. 142, 1187 (1966).

12. M. Andrews and J. Gunson, J. Math. Phys. 5, 1391 (1964).

13. Of course trajectories integrally spaced below α must be included among the conspirators of α by the factorizability assumption.

14. G. C. Fox and E. Leader, Phys. Rev. Letters 18, 628 (1967).

15. The minus sign in front of the operators can be dropped, since the sum of the powers of μ and μ' is always even.

16. As an aside, we point out that if there is coupling only to $s = s' = 1$, the conspiracy conditions can be satisfied with $M = 0$ and

$$(b_+ + b_-)(b_+ + b_-') = -4\frac{(\alpha-1)^2}{(2\alpha-1)} \quad , \quad (b_+ - b_-)(b_+' - b_-') = -4\frac{\alpha(\alpha-1)}{2\alpha-1} ,$$

with all other conspirator residues vanishing. This corresponds to Volkov and Gribov's first conspiracy (Ref. 5). Their second solution has $M = +1$ and $M = -1$ parts and parity must be included for a proper discussion.

17. D. A. Akyeampong, J. F. Boyce, and M. A. Rashid, Nuovo Cimento 53A, 737 (1968), Eq. (25).

18. To within a transformation $\beta \to f\beta$, $\beta' \to \frac{1}{f}\beta'$ common to all residues of a given n. Such a transformation has no physical significance.

19. Class I and Class II are not to be confused with Type I and Type II of Freedman and Wang (Ref. 2).

20. Phase conventions are those of M. Jacob and G. C. Wick, Ann. Phys. 7, 404 (1959), and Andrews and Gunson (Ref. 12). Reggeization closely follows P. D. B. Collins and E. J. Squires, Regge Poles in Particle Physics (Springer Tracts in Modern Physics, Vol. 45, 1968) and J. F. Boyce, J. Math. Phys. 8, 625 (1967).

21. D. Z. Freedman and J.-M. Wang, Phys. Rev. 153, 1596 (1967).

22. This was first pointed out to the author by S. Mandelstam, personal communication, 1968.

23. G. Cosenza, A. Sciarrino, and M. Toller, Istituto "G. Marconi," Nota Interna, No. 158 (1968).

Table I: Kinematic Residues. $[R = R_y(\frac{\pi}{2})]$

Class I. Lorentz Pole.

$$K_{s\mu}^{M\alpha(0)} = \langle s\mu|R|sM\rangle = d^s_{\mu M}(\tfrac{\pi}{2})$$

$$K_{s\mu}^{M\alpha(1)} = \frac{1}{(-2\alpha)^{\frac{1}{2}}} \langle s\mu|R\left\{\frac{1}{(M^2-\alpha^2)^{\frac{1}{2}}}\left[(M+\alpha)\left(\frac{J_+}{2}\right) + (M-\alpha)\left(\frac{J_-}{2}\right)\right]\right\}|sM\rangle$$

$$K_{s\mu}^{M\alpha(2)} = \frac{1}{[2(2\alpha-1)(2\alpha-2)]^{\frac{1}{2}}} \langle s\mu\Big| R \left\{\frac{1}{[(M^2-\alpha^2)(M^2-(\alpha-1)^2]^{\frac{1}{2}}}\left[(M+\alpha)(M+\alpha-1)\left(\frac{J_+}{2}\right)^2\right.\right.$$

$$\left.\left. + \frac{M^2-(\alpha-1)^2}{2}(s^2+s-M^2-\alpha)\, I + (M-\alpha)(M-\alpha+1)\left(\frac{J_-}{2}\right)^2\right]\right\}\Big|sM\rangle$$

$$K_{s\mu}^{M\alpha(3)} = \frac{1}{[-6(2\alpha-2)(2\alpha-3)(2\alpha-4)]^{\frac{1}{2}}} \langle s\mu\Big| R \left\{\frac{1}{[(M^2-\alpha^2)(M^2-(\alpha-1)^2)(M^2-(\alpha-2)^2]^{\frac{1}{2}}}\right.$$

$$\times\left[(M+\alpha)(M+\alpha-1)(M+\alpha-2)\left(\frac{J_+}{2}\right)^3 + \frac{3}{4}(M^2-(\alpha-2)^2)(M+\alpha-1)(s^2+s-M^2-2\alpha+M)\left(\frac{J_+}{2}\right)\right.$$

$$\left.\left. + \frac{3}{4}(M^2-(\alpha-2)^2)(M-\alpha+1)(s^2+s-M^2-2\alpha-M)\left(\frac{J_-}{2}\right)+(M-\alpha)(M-\alpha+1)(M-\alpha+2)\left(\frac{J_-}{2}\right)^3\right]\right\}\Big|sM\rangle$$

Class II. Pure Helicity Flip M.

$$K_{s\mu}^{M\alpha(0)} = d^s_{\mu M}(\tfrac{\pi}{2})$$

$$K_{s\mu}^{M\alpha(1)} = \frac{1}{(-2\alpha)^{\frac{1}{2}}}\langle s\mu\Big| R\left\{\left(\frac{J_+}{2}\right) + \left(\frac{J_-}{2}\right)\right\}\Big|sM\rangle\ \gamma_s{}^c = \frac{1}{(-2\alpha)^{\frac{1}{2}}}\,\mu\, d^s_{\mu M}(\tfrac{\pi}{2})$$

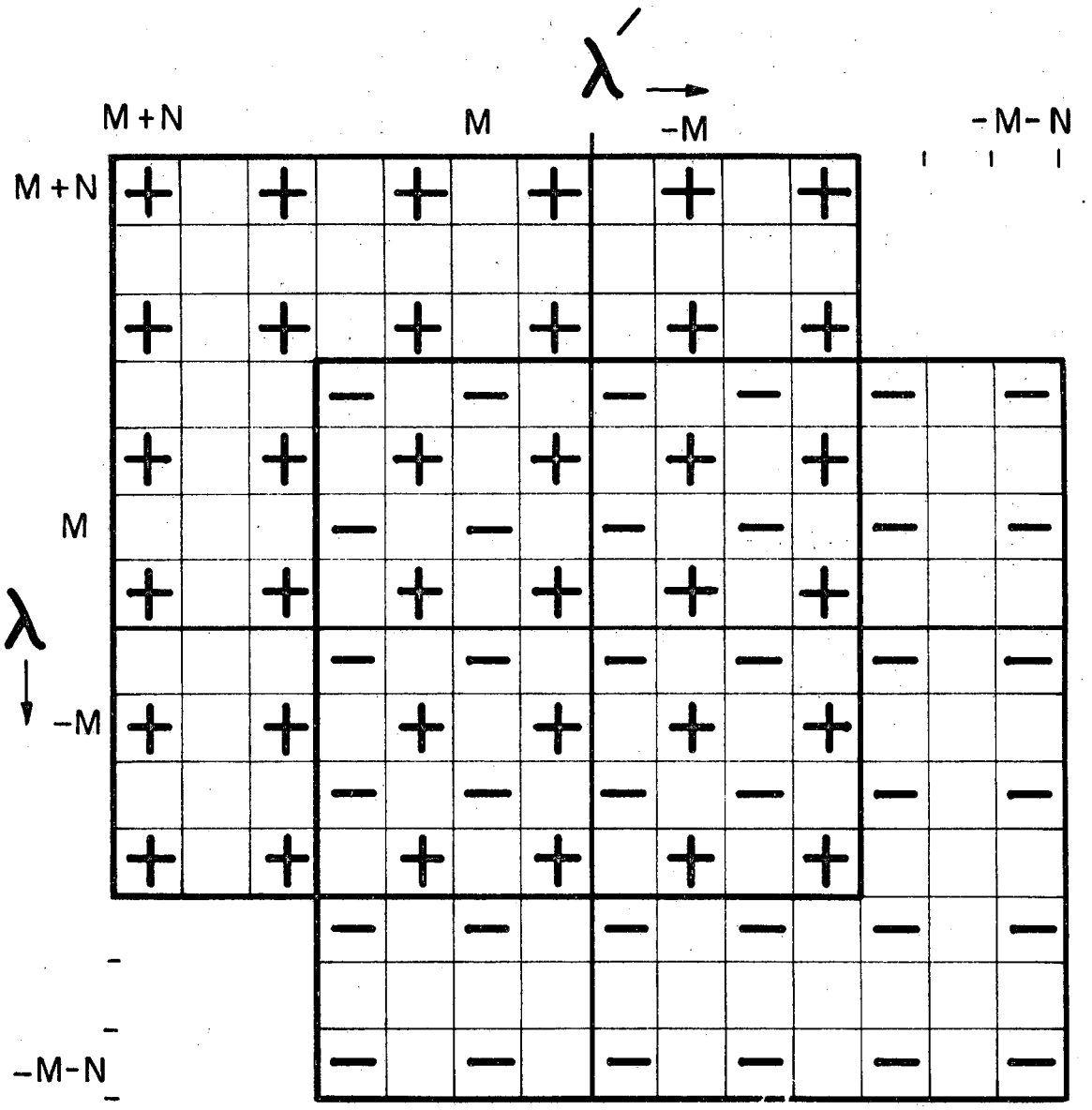
$$K_{s\mu}^{M\alpha(2)} = \frac{1}{[2(2\alpha-1)(2\alpha-2)]^{\frac{1}{2}}}\left(\mu^2 + \frac{\alpha-1}{2}\right)\ d^s_{\mu M}(\tfrac{\pi}{2})$$

$$K_{s\mu}^{M\alpha(3)} = \frac{1}{[-6(2\alpha-2)(2\alpha-3)(2\alpha-4)]^{\frac{1}{2}}}\left(\mu^3 + \frac{3\alpha-5}{2}\mu\right)\ d^s_{\mu M}(\tfrac{\pi}{2})$$

FIGURE CAPTIONS

Figure 1. $B^{(N)}_{s\lambda s'\lambda'}$. M half integral (N odd). + = from $B^{M(N)}$, - = from $B^{-M(N)}$.

Figure 2. $B^{(N)}_{s\lambda s'\lambda'}$. M integral (N odd). + = from $B^{M(N)}$, - = from $B^{-M(N)}$, o = from both.



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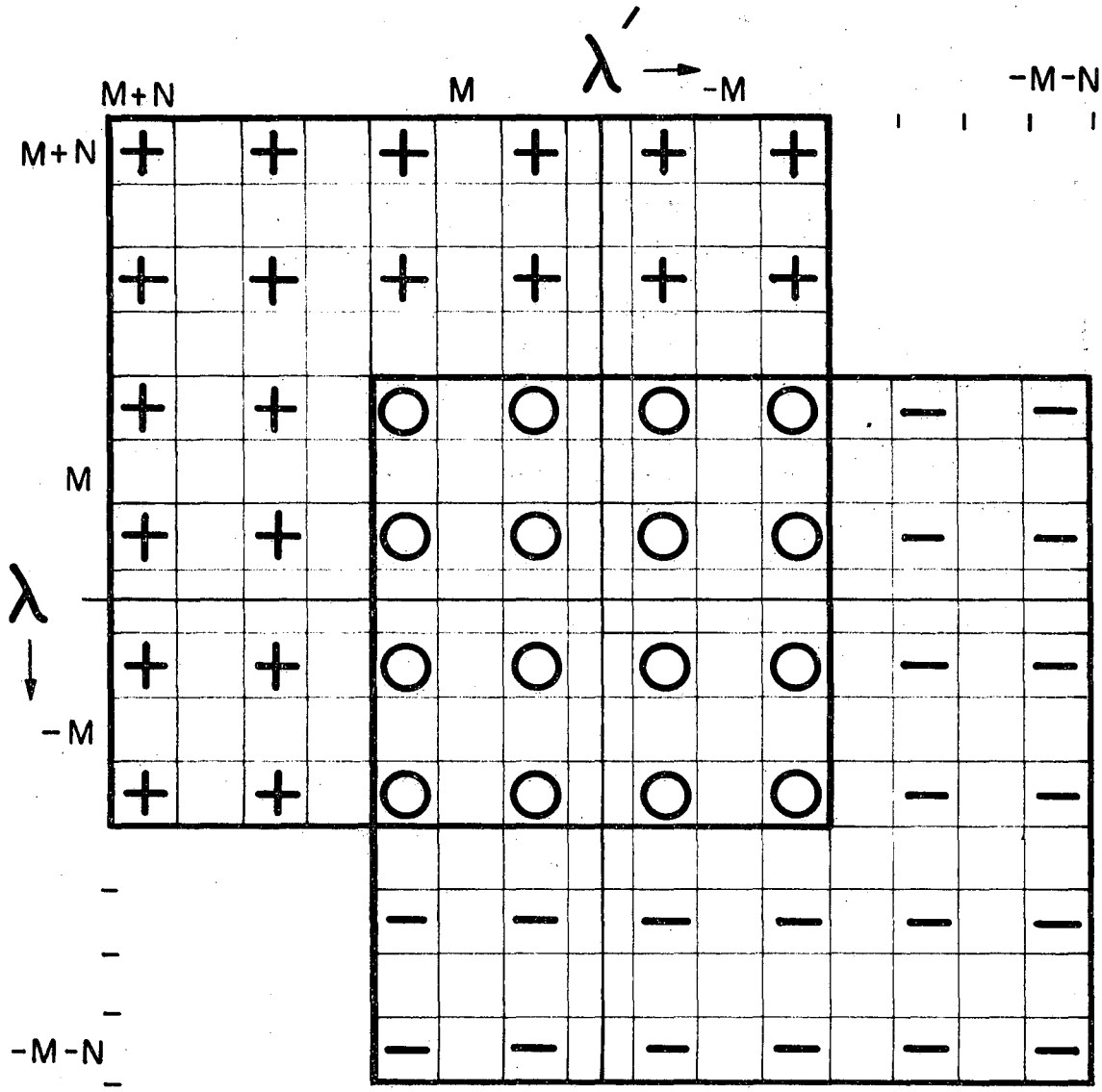

XBL688-3541

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