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PHYSICAL-REGION DISCONTINUITY EQUATION*

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ABSTRACT

A Cutkosky-type formula for the discontinuity around an arbitrary physical-region singularity is derived from precisely formulated S-matrix principles.

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I. INTRODUCTION

We shall derive the following result: The discontinuity of S around any physical-region singularity surface is given by a Cutkosky-type formula obtained by replacing each vertex of the corresponding diagram D by the associated (physical-region) S matrix, replacing the set of lines α joining each pair of vertices of D by a function S_α^{-1}, and integrating over all the (topologically inequivalent) mass-shell values of the variables corresponding to the intermediate lines. The function S_α^{-1} is defined by $S_\alpha S_\alpha^{-1} = I_\alpha$, where S_α is the restriction of S to the space corresponding to the set of lines α, and I_α is the corresponding restriction of unity.

This rule gives the discontinuity for S itself. The result for the connected part is obtained by retaining only connected graphs. Then the S occurring at each vertex is generally reduced to its connected part. However, there are some exceptions, so it is prudent to use the general formula.

The discontinuity formula state above is similar to the one obtained by Cutkosky.[1] However, his formula was incomplete because important questions concerning the sheet structure were not answered.[2] Also, his derivation depended on perturbation theory. The present results are derived within the mass-shell S-matrix framework and give the discontinuity in terms of the actual physical-region scattering functions. This confirms earlier indications[3,4] that the physical-region

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In Section II the results needed from earlier works are summarized. The discontinuity formula is derived in Section III by using an infinite series (mass-shell) expansion for S. Some properties of S_α^{-1} are discussed in Section IV.

A derivation not based on the infinite series for S is given in Section V, for the case of "leading singularities". A leading singularity is one such that the set of particles corresponding to the set of lines α joining any pair of vertices of D is a "leading set". A leading set of particles is a set that cannot make a transition to a set having a lower sum of rest masses. We hope to give later a derivation for the case of nonleading singularities that is not based on the infinite series for S.

In the final section our work is compared with other works in the field.

II. BASIC TOOLS

A. Cluster Decomposition

The S matrix is the transition matrix from "in" to "out". Linearity ensures that the transition matrix from "out" to "in" is S^{-1}. We do not use unitarity $(S^{-1} = S^{\dagger})$. [All that is used in S-matrix derivations of discontinuity equations are the cluster properties and $i\epsilon$ rules of S and S^{-1}: it is not important that S^{-1} be $S^{\dagger}$.]

The cluster decompositions of S and S^{-1} are conveniently represented by a diagram notation:[3] A box with a plus [minus] sign inside represents S $[S^{-1}]$; a bubble (i.e., circle) with a plus [minus] sign inside represents the connected part of S $[S^{-1}]$. The left side of each box or bubble is the origin of a set of leftward-directed lines, and the right side is the terminus of such a set. Each line j is associated with a physical-particle variable, which is a set (p_j, μ_j, t_j) consisting of a particle-type index t_j, a spin (magnetic) quantum number μ_j, and a real positive-energy mass-shell four-vector p_j.

The cluster decomposition of S $[S^{-1}]$ is represented by writing each plus [minus] box as a sum of columns of plus [minus] bubbles, the sum being over all topologically distinct ways that the lines originating and terminating on the box can be partitioned among bubbles of a column, with each bubble having at least one incoming and one outgoing line.

The connected parts of S and S^{-1} divided by the overall conservation delta function are the <u>scattering functions</u> S_c and S_c^- respectively.

B. Bubble Diagram Functions

The cluster decompositions of S and S^{-1} induce corresponding decompositions of quantities like SS^{-1}, $SS^{-1}S$, etc. The rule for computing such a product is to first draw all topologically distinct bubble diagrams B composed of the appropriate number of columns of the appropriately signed bubbles. The lines originating on the bubbles of one column terminate on those of the column standing to its left, if there is one. For each such B one constructs a corresponding function F^B by summing over all physical values of the variables (p_i, μ_i, t_i) for each internal line i, subject to the constraint that topologically equivalent contributions be counted only once. The function being calculated is precisely the sum of the functions F^B defined in this way.[3] [For fermions some signs must be considered.]

Two contributions to F^B are topologically equivalent if and only if the corresponding diagrams, with each line j identified by a corresponding variable (p_j, u_j, t_j), can be continuously distorted into each other with the external end points of the external lines held fixed. Each bubble is identified as to its column, and the distortions must leave each bubble in its own column. (Alternatively, one must keep all the "trivial" bubbles having only one incoming and one outgoing line. These bubbles are often omitted because they do not affect the value of the integral, except in this matter of counting.)

C. Macrocausality

Macroscopic particle phenomena has a characteristic space-time structure. If effects of long-range interactions and massless particles are ignored, then particles move along straight space-time trajectories except when they come close to other particles. A quantitative description of the phenomena is provided by the Newton-Einstein laws of motion. These laws assign to each particle j a momentum-energy vector p_j that is directed along its space-time trajectory, and that satisfies the mass-shell constraint $p_j^2 = m_j^2$. Momentum-energy is conserved, and is exchanged between particles only when they are close to each other; one imagines momentum-energy to be transmitted by a short-range interaction.

If one requires this space-time structure of macroscopic phenomena to emerge from S-matrix theory, in appropriate classical, macroscopic limits, and demands also that classical estimates based on short-range interactions should become valid in these limits, at least to order of magnitude, then certain physical-region analyticity properties follow. These include the cluster decomposition property described above, and also the properties described in the following two sections.

D. The Positive-α Rule

The first important consequence of the macrocausality condition is that the physical-region singularities of the scattering functions $S_c^{\pm}$ are confined to positive-α Landau surfaces[6] associated with connected diagrams.[7]

Landau surfaces are associated with Landau diagrams. A Landau diagram D is a diagram that represents a classical multiple-scattering process with point interactions. It consists of a set of leftward directed line segments j that meet at vertices v. Each line j is associated with a real momentum-energy vector p_j that satisfies the mass-shell constraint

$$p_j^2 - m_j^2 = 0 , \quad p_j^0 > 0 , \tag{2.1a}$$

where m_j is the mass of the particle associated with line j. Momentum-energy is conserved at each vertex v:

$$\sum_{\text{into } v} p_j - \sum_{\text{out of } v} p_j = 0 . \tag{2.1b}$$

The vector Δ_i from the space-time origin of internal line i of D to its space-time terminus must be directed along its momentum-energy: i.e., for some scalar α_i one has

$$\Delta_i = \alpha_i p_i . \tag{2.1c}$$

Finally, the sum of the space-time displacements Δ_i around any closed loop of internal lines of D must add to zero:

$$\sum_{\ell} \pm \Delta_i \equiv \sum_{\ell} \pm \alpha_i p_i = 0 . \tag{2.1d}$$

Here the $\pm$ sign is plus if the loop ℓ is directed along Δ_i and minus otherwise.

These equations express the constraints on the multiple-scattering diagram D imposed by classical relativistic particle mechanics. They are called the Landau equations. The Landau surface L(D) is the set of external $P \equiv (p_1, \cdots p_n)$ that are compatible with the Landau equations associated with diagram D. The trivial solution with all $\alpha_i = 0$ is not accepted.

Physical particles carry positive energy forward in time. The α_i must therefore be positive:

$$\alpha_i > 0 \quad . \tag{2.2}$$

The subset of L(D) that allows a solution of the Landau equations (2.1) subject to the positive-α condition (2.2) is denoted by $L^+(D)$, and is called a positive-α Landau surface. The positive-α rule says that the scattering functions $S_c^{\pm}(P)$ are analytic at all physical points not lying on the union of positive-α surfaces

$$L^+ \equiv \bigcup L^+(D) \quad . \tag{2.3}$$

The scattering functions $S_c^{\pm}$ are defined only by the mass shell $\mathcal{M}$, which is defined by the mass-shell constraints (2.1a) and the overall momentum-energy conservation law. Thus the ordinary definition of analyticity does not apply. The appropriate definition is given in Refs. 5, 7, and 8.

Certain general properties of the set L^+ are used in formulating the $i\epsilon$ rule. These are described now.

A given surface $L^+(D)$ generally coincides with the surfaces $L^+(\bar{D})$ of an infinite set of other diagrams $\bar{D}$. These arise in a trivial way: If a set of internal lines of D all originate at the same vertex v', and all terminate at the same vertex v'', then the Landau equation requires them all to be moving along together, relatively at rest. Thus they can undergo trivial forward scatterings upon each other without affecting the kinematic relations. Any number of these trivial forward scatterings can occur. This leads to an infinite set of diagrams $\bar{D}$ such that $L^+(\bar{D}) = L^+(D)$.

It is convenient to introduce diagrams that do not have these trivial forward scattering vertices. A <u>basic diagram</u> D_β is a Landau diagram that has no part that (i) is connected to the rest of the diagram at only two vertices, (ii) contains more than two vertices, and (iii) contains no external lines. Every $L^+(D)$ is confined to the $L^+(D_\beta)$ of some corresponding basic diagram D_β. Thus one can write

$$L^+ \equiv \bigcup L^+(D) = \bigcup L^+(D_\beta) \quad . \tag{2.3'}$$

Only a finite number of D_β have $L^+(D_\beta)$ that enter any bounded portion of the physical region.[9]

The representation of L^+ is further simplified by introducing "basic surfaces", defined as follows: Let $\mathcal{M}_0$ represent the part of the mass shell where two or more initial momentum-energy vectors p_j are parallel, or two or more final p_j are parallel. Then for any Landau diagram D the set $L_0^+(D)$ is that part of $L^+(D) - \mathcal{M}_0$ such that the Landau equation for $L^+(D)$ have no solution with any $\alpha_i = 0$.

It is clear that any point on $L^+(D) - \mathcal{M}_0$ that is not on $L_0{}^+(D)$ must lie on the $L_0{}^+(D')$ of a contraction D' of D constructed by contracting to points and removing from D the lines corresponding to $\alpha_i = 0$. Thus L^+ can be written as

$$L^+ = \bigcup L_0{}^+(D_\beta) + \mathcal{M}_0 . \qquad (2.3'')$$

The importance of this representation lies in the fact that $L_0{}^+(D_\beta)$ is a real codimension 1 analytic submanifold of the mass-shell $\mathcal{M}$.[8] That is, each point $\bar{P}$ of $L_0{}^+(D_\beta)$ has a mass-shell neighborhood $N(\bar{P})$ such that inside $N(\bar{P})$ the set $L_0{}^+(D_\beta)$ coincides with the set $f = 0$, where f is a real analytic function of the local real analytic coordinates of the mass shell at $\bar{P}$ (see eg. Refs. 7 or 8), and grad $f \equiv \nabla f$ is nonzero in $N(\bar{P})$.

The representation (2.3") shows that $(L^+ - \mathcal{M}_0)$ is the union of a set of codimension 1 real analytic submanifolds of $\mathcal{M}$, only a finite number of which enter any bounded portion of the physical region. Since $\mathcal{M}_0$ has codimension 3, the set L^+ has codimension 1. [The codimension of $\mathcal{S}$ plus the dimension of $\mathcal{S}$ is the dimension of imbedding space, here $3n + 4$.]

The positive-α rule says, therefore, that $S_c(P)$ is analytic at almost all physical points, and that the remaining set L^+ has, apart from the small set $\mathcal{M}_0$, a local representation as the zeros of a finite set of real analytic functions f_i each having nonzero gradient ∇f_i.

E. The $i\epsilon$ Rules

Macrocausality implies also that the scattering function S_c near any $\bar{P}$ of $L^+ - \mathcal{M}_0$ can be represented as the limit from any direction in the intersection of the upper-half planes $\text{Im}\, f_i > 0$ of the (unique) analytic continuation into this intersection of the function $S_c(P)$ defined on $L^+ - \mathcal{M}_0$. The functions f_i are the functions that define L^+ near $\bar{P}$, and their signs are fixed by the requirement that a formal increase of the masses associated with the internal lines of D by a common scale factor shifts $L_0^+(D)$ in the plus f direction. This sign is known to be independent of the particular diagram D that defines $L_0^+(D)$: all locally coincident surfaces $L_0^+(D)$ can be defined by the same function f. (Theorem 7 of Ref. 8)

This $i\epsilon$ rule for S_c is known as the plus $i\epsilon$ rule. The function S_c^- obeys the minus $i\epsilon$ rule, which is the same rule except that the upper-half planes $\text{Im}\, f_i > 0$ are replaced by lower-half planes $\text{Im}\, f_i < 0$.

These rules have content only at those points $\bar{P}$ of $L^+ - \mathcal{M}_0$ for which the appropriate half planes have a nonempty intersection that contains $\bar{P}$ on its boundary. This property is obviously satisfied for any $\bar{P}$ that lies on only one $L_0^+(D_\beta)$ [or only on several $L_0^+(D_\beta)$ that all locally coincide with one single one]. Such points comprise almost all of $L^+ - \mathcal{M}_0$, since the rest have codimension 2. Thus the $i\epsilon$ rules have content at almost all points of $L^+ - \mathcal{M}_0$.

It is important that the $i\epsilon$ rules have content also at a certain of the remaining points of $L^+ - \mathcal{M}_0$. It is known (Theorem 13, Ref. 8) that the intersection of the upper-half planes corresponding to $\bar{P}$ (on $L^+ - \mathcal{M}_0$) is nonempty, and contain $\bar{P}$ on its boundary, whenever all the D_β with $\bar{P} \in L_0^+(D_\beta)$ are contractions of some single D.

There are, however, some points $\bar{P}$ of $L^+ - \mathcal{M}_0$ such that the intersections of the various upper-half planes associated with $\bar{P}$ are empty near $\bar{P}$. The scattering function S_c cannot be represented near such a $\bar{P}$ as the limit of a single analytic function. To cope with such points we shall introduce in the next section an independence property, which says, in effect, that singularities associated with unrelated diagrams are independent. This will allow the $i\epsilon$ rule to be applied at all points of $L^+ - \mathcal{M}_0$.

Full technical details concerning the $i\epsilon$ rules are given in Refs. 7 and 8. The intersection of the upper-half planes at $\bar{P}$ is defined, in effect, as the set of mass shell variations δ that satisfy $\mathrm{Im}\, \delta \cdot \nabla f_i(\bar{G}) > 0$, where G is a set of local real analytic coordinates at $\bar{P}$, and $\bar{G} = G(\bar{P})$. (See also Ref. 10)

The basic tool in the analysis of physical-region singularities is a theorem that extends the positive-α and $i\epsilon$ rules to all bubble diagram functions. This theorem is described next.

F. Fundamental Theorem[11,12]

1. Assumptions of Theorem

(a) Positive-α Rule. The physical-region singularities of the scattering functions S_c and S_c^- are confined to the union L^+ of positive-α Landau surfaces.

(b) Independence Property. Each point $\bar{P}$ of $L^+ - \mathcal{M}_0$ has a real mass-shell neighborhood $N(\bar{P})$ such that $S_c^{\pm}(P)$ in $N(\bar{P}) - L^+$ decomposes into a finite sum of terms, one for each basic diagram D_β for which $L^+(D_\beta)$ contains $\bar{P}$. The singularities of the term of $S_c^{\pm}$ associated with D_β are confined to

$$\hat{L}^{\pm}(D_\beta) \equiv L^{\pm}(D_\beta) \cup [\cup L^{\pm}(D'_\beta)] \tag{2.4}$$

where D'_β is any contraction of D_β. Each term obeys a corresponding $i\epsilon$ rule, as is described next. [The justification of the independence property is given in Section G.]

(c) The $i\epsilon$ Rules. The individual terms of S_c and S_c^- described in the independence property obey the plus and minus $i\epsilon$ rules, respectively. The upper- and lower-half planes for each term are specified by the singularity surfaces occurring in that term alone.

(d) Technical Assumption. The singularities at $\mathcal{M}_0$ are not too pathological. [This assumption is discussed in Subsection 3.]

2. Conclusions of Theorem

Let B be any connected bubble diagram. Let F^B be the corresponding bubble diagram function. Define

$$F_c^B(P) \equiv F^B(P)/\delta^4(\sum_{in} p - \sum_{out} p) . \tag{2.5}$$

Then the following properties hold:

(a) Generalized Positive-α Rule. The physical-region singularities of F_c^B are confined to the union of the Landau surfaces $L^\sigma(D_B)$.

A D_B is a Landau diagram constructed by inserting a connected basic Landau diagram D_b for each bubble b of B, with the incoming and outgoing lines of D_b identified in a one-to-one fashion with the incoming and outgoing lines of b, respectively. The surface $L^\sigma(D_B)$ is the part of $L(D_B)$ that is compatible with the Landau equations of $L(D_B)$, subject to the constraint that each line i of D_B that is an internal line of some D_b must have an α_i that satisfies

$$\alpha_i \, \sigma_b \geqslant 0 \; , \tag{2.6}$$

where σ_b is the sign of b. The (original) lines of B itself, which are external lines of various D_b, have no sign constraint.

(b) Generalized Independence Property. Each point $\overline{P}$ of $\bigcup L^\sigma(D_B) - \mathcal{M}_0$ has a real mass-shell neighborhood $N(\overline{P})$ such that $F_c{}^B$ decomposes on $N(\overline{P}) - \bigcup L^\sigma(D_B)$ into a finite sum of terms one for each D_B for which $L^\sigma(D_B)$ contains $\overline{P}$. The singularities of the term associated with a given D_B are confined to

$$\hat{L}^\sigma(D_B) = L^\sigma(D_B) \bigcup \left[\bigcup L^\sigma(D_B') \right] \tag{2.7}$$

where the D_B' are contractions of lines of D_B that are internal lines of some D_b.

(c) Generalized iє Rule. The functions $F_c{}^B(P)$ obey a rule that is completely analogous to the plus iє rule, except that the upper-half planes at $\overline{P}$ are now defined by using, instead of $f = f(P)$, the functions

$$\sigma_{\bar{P}}(P) \equiv \sum \alpha_i(\bar{P}) \, [p_i(P) - p_i(\bar{P})] \quad . \tag{2.8}$$

There is one such function for each solution at $\bar{P}$ of the Landau equations of $L^{\sigma}(D_B)$. The $\alpha_i(\bar{P})$ and $p_i(\bar{P})$ are the parameters of the internal lines of D_B corresponding to the solution at $\bar{P}$. The $p_i(P)$ is any set of internal p_i satisfying the conservation law constraints of D_B at P. [The function $\sigma_{\bar{P}}(P)$ will not depend on the particular choice of the $p_i(P)$, because of the Landau loop equation.]

The ordinary $i\epsilon$ rules connect the physical-region scattering functions in different sectors of $\mathcal{M} - L^+$. Similarly, the generalized $i\epsilon$ rules connect the "physical-region" functions F_c^B in different sectors of $\mathcal{M} - \bigcup L^{\sigma}(D_B)$. The physical-region functions F^B are defined as integrals over the physical-region scattering functions. These are the functions F^B that occur in the decomposition of the functions SS^{-1}, $SS^{-1}S$, etc.

It may, of course, be possible to continue F_c^B from some given sector of $\mathcal{M} - \bigcup L^{\sigma}(D_B)$ by following different alternative paths around some $L^{\sigma}(D_B) - \mathcal{M}_0$. The generalized $i\epsilon$ rule asserts that it definitely is possible to continue through the intersection of the upper planes defined by (2.8), provided the intersection of these upper-half planes is nonempty arbitrarily close to $\bar{P}$, and that moreover the function arrived at on the other side of $L^{\sigma}(D_B) - \mathcal{M}_0$ will then be precisely the physical-region function F_c^B. Also, an integral over the physical-region function F_c^B can be represented by an integral over a contour distorted infinitesimally away from $\overline{P} \in \bigcup L^\sigma(D_B)$ and into the intersection of the upper half planes at $\overline{P}$.

By F_c^B we shall, unless otherwise stated, always mean the physical-region F_c^B, not some analytic continuation of it; the only continuations considered are the infinitesimal ones specified by the general $i\epsilon$ rules, unless otherwise stated.

The generalized $i\epsilon$ rule has content at $\overline{P}$ of $L^\sigma(D_B) - \mathcal{M}_0$ only if the various upper-half planes at $\overline{P}$ have a nonempty intersection at $\overline{P}$ [i.e., only if there is a $(3n - 4)$ dimensional variation δ in $\mathcal{M}$ satisfying $\text{Im}\, \delta \cdot \nabla\, \sigma_{\overline{P}}(\overline{P}) > 0$ for all $\sigma_{\overline{P}}(P)$ associated with $L^\sigma(D_B)$.] If this intersection is empty at $\overline{P}$, then no continuation past $L^\sigma(D_B)$ is assured at $\overline{P}$.

There are some important points $\overline{P}$ of $L^\sigma(D_B)$ for which the intersection of the upper half planes is obviously empty. In particular, every point of $L^\sigma\big(D(B)\big)$ has this property.

The diagram $D(B)$ is the particular D_B obtained by replacing each bubble b of B by a point vertex. Since no line of $D(B)$ comes from inside any bubble, there are no constraints on the signs of the $\alpha_i(\overline{P})$. Thus the reversal of all these signs will give another solution. This solution will have the signs of all the functions $\sigma_{\overline{P}}(P)$ reversed. Thus the positions of all upper-half planes will be reversed. Thus the intersection of the upper half plances at $\overline{P}$ will be empty, and the $i\epsilon$ rule will be without content there.

This failure of the $i\epsilon$ analyticity property at points of $L^\sigma\big(D(B)\big)$ plays a crucial role in what follows. It is related to the breakdown of the definition of F^B at these points. The function F^B is defined as an integral that contains, in effect, a conservation-law delta function for each bubble b of B, and a mass-shell delta function for each internal line i of B. A product of delta functions under an integral sign is defined as follows: one transforms to a new set of variables that contains the argument g_j of each delta function as an independent variable, and then omits the integrations on these variables. This definition fails at $\overline{P}$ (i.e., the Jacobian becomes infinite) if the gradients ∇g_j are linearly dependent at $\overline{P}$.

These linear dependence relations turn out to be precisely the Landau loop equations corresponding to $D(B)$. Since the mass-shell and conservation-law constraints are also satisfied, the equations that define the points where F^B is ill-defined are just the Landau equations for $D(B)$, and the corresponding set of points P is the Landau surface $L\big(D(B)\big) \equiv L^\sigma\big(D(B)\big)$.

The function F^B generally does not continue into itself around points of $L\big(D(B)\big)$. That is, F^B in different sectors of $\mathcal{M} - L\big(D(B)\big)$ near $\overline{P}$ of $L\big(D(B)\big)$ are generally not parts of a single analytic function. In fact, the function F^B is obviously identically zero at points of $\mathcal{M}$ where it is not possible to satisfy simultaneously the various mass-shell and conservation-law constraints associated with B. The boundary of this region lies in $L\big(D(B)\big)$. Furthermore, every point of $L^+\big(D(B)\big)$ lies on this boundary. Thus F^B can never continue into itself around $L^+\big(D(B)\big)$, unless it is identically zero.

The portion of $\mathcal{M}$ where it is possible to satisfy all the mass-shell and conservation-law constraints of B is called the <u>physical region</u> of B. According to the above remarks, the physical-region F^B is nonzero only in physical region of B. Moreover, $L^+\big(D(B)\big)$ lies on the boundary of this region. The sign conventions on the functions f_i are such that the physical region of B near $\bar{P}$ of $L_0^+\big(D(B)\big)$ is either confined to $L_0^+\big(D(B)\big)$ or lies on the positive-f side of it.[10] That is, F^B is identically zero on the negative-f side of $L_0^+\big(D(B)\big)$.

The above-mentioned fact is important in the derivation of the discontinuity formula. It ensures that <u>all</u> the terms in the discontinuity formula vanish on the negative-f side of the singularity surface $L_0^+(D_\beta)$ in question. The "principal term" of the discontinuity formula, which is the one such that each vertex v of D_β corresponds to the connected part of the corresponding S, will have its physical region bounded by $L_0^+(D_\beta)$. Generally speaking, the physical regions of the nonprincipal terms will not extend to $L_0^+(D_\beta)$ because of the extra constraints imposed by the extra conservation laws. Thus the nonprincipal terms will generally not contribute to the discontinuity around $L_0^+(D_\beta)$. But if the physical region of some nonprincipal term does reach $L_0^+(D_\beta)$, then this term will contribute to the discontinuity around $L_0^+(D_\beta)$.

3. The Technical Assumption

The macrocausality condition does not rule out singularities at $\mathcal{M}_0$. The proof of the theorem requires, however, that the singularities at $\mathcal{M}_0$ be not too pathological. It is known from the boundedness property $S_c[\phi_1, \cdots \phi_n] \leq ||\phi_1|| \cdots ||\phi||$, which follows from linearity and the probability interpretation, that the integrals defining F^B do not diverge at $\mathcal{M}_0$. An additional requirement is that the integrals defining the derivative of F^B also be well defined at $\mathcal{M}_0$.

G. Maximal Analyticity

This principle is that $S^{\pm 1}(P)$ has only those singularities that are required by general principles. The full content of this principle, as it applies to physical-region points, is the independence property (b): singularities violating this property are not required to be present, hence they are required to be absent.

The point is this. The positive-α rule and the $i\epsilon$ rules impose certain constraints on the allowed singularities. But they do not require any singularity actually to be present in S_c or S_c^-. On the other hand, the cluster properties of S and S^{-1}, by themselves, actually require the scattering functions to have singularities.

These arise as follows. Suppose one expresses identities such as $SS^{-1} = I$, $S = SS^{-1}S$, or $S = SS^{-1}SS^{-1}S$ etc. in the form of bubble diagram equations,

$$\sum_{B \in \mathcal{B}'} F^B = \sum_{B \in \mathcal{B}''} F^B , \tag{2.8}$$

where $\mathcal{B}'$ and $\mathcal{B}''$ are classes of bubble diagrams. Then the assumption that the S_c and S_c^- are all singularity free gives contradictions: certain terms of (2.8) will have explicit singularities that cannot be cancelled by any other singularities. Thus the cluster properties of S and S^{-1} definitely require some of the scattering functions to have singularities.

The above argument does not show precisely which singularities are required in S_c and S_c^-. However, it can be extended to do just that. In particular, the various identities (2.8), which follow simply from the cluster properties of S and S^{-1}, supplemented by the conclusions of the fundamental theorem, permit the derivation of a formula for the discontinuity around each physical-region singularity allowed by the positive-α rule. This formula shows that each allowed singularity is also required: i.e., it has a nonzero discontinuity. These required singularities are apparently compatible with the independence property. Thus we have an apparently self-consistent singularity structure that has no singularities that violate the independence property. Thus no singularity that violates this property is required. Then maximal analyticity says none is allowed. Hence the independence property must hold.

We turn now to the derivation of the discontinuity formula. It will be convenient to assign to each internal line i of each Landau diagram D a sign σ_i that determines the sign of α_i in the corresponding Landau equations:

$$\alpha_i \sigma_i \geqslant 0 .$$

A diagram that has all $\sigma_i = +1$ is called a positive-α diagram and is denoted by D^+. Thus

$$L(D^+) \equiv L^+(D) .$$

III. ITERATIVE SOLUTION

A. Expansion of S

Introducing $R^{\pm} \equiv S^{\pm 1} - 1$, we obtain

$$R^{+} + R^{-} + R^{+}R^{-} = 0 \quad . \tag{3.1}$$

The formal iterative solution for R^{+} gives

$$R^{+} = \sum_{n=1}^{\infty} (-1)^{n} (R^{-})^{n} \quad . \tag{3.2}$$

Each factor R^{-} is represented by a sum of columns of minus bubbles, the sum being over all topologically different ways of joining a column of bubbles to the external lines. However, at least one bubble of each column must be nontrivial. [Trivial bubbles are those with just one incoming line and just one outgoing line.]

In the assessment of topological equivalence one considers the bubbles to be confined to particular columns.[3] This means that the three terms shown in Fig. 1 must all be counted.

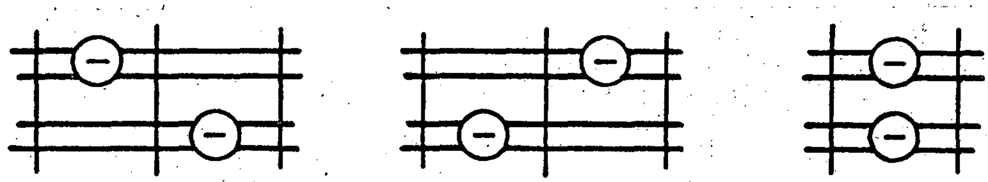


Fig. 1. Three contributions to the expansion of a four-line S. The vertical lines show the separation into factors R^{-}. Trivial bubbles have been omitted,since they do not alter the function.

The first two factors have coefficients $(-1)^2 = 1$ in (3.2), whereas the last has coefficient (-1). Thus there is a cancellation and only one term survives.

This result is general: In the expansion (3.2) one needs to count only one of any set of topologically equivalent contributions, where in the assessment of topological equivalence one now disregards both trivial bubbles and the separation of bubbles into columns. The sign of the single surviving term is $(-1)^n$, where n is the number of (nontrivial) minus bubbles of the term.

The bubbles b of the original B are partially ordered by the ordering of the columns in which they lie. If the column identification of the bubbles is removed then the bubbles are partially ordered only by the requirement that all lines be directed from right to left. For each such partially ordered B^- there remains, after the cancellations, precisely one term F^{B^-}. Thus if the unit contribution is added back to give $S = 1 + R^+$, one obtains[13]

$$S = \sum_{B^-} (-1)^n F^{B^-} . \tag{3.2'}$$

The sum is over all topologically different partially ordered bubble diagrams B^- having only nontrivial minus bubbles, and n is the number of bubbles of B^-.[13]

The expansion (3.2') contains in an implicit form an expression for the discontinuities. As one moves across a positive-α threshold, new terms appear in (3.2'). If mixed-α singularities (i.e., singularities corresponding to solutions of Landau equations that require α's of both signs) can be ignored (see Section VI below)and if only one positive-α surface is relevant, then the discontinuity is just the sum of these new terms. This is because any term in (3.2') that is present below the threshold will, by virtue of the Fundamental Theorem, continue around any singularity at threshold via the minus $i\epsilon$ rule. This leaves the new terms as the discontinuity. The problem of calculating the discontinuity is then to identify the infinite number of terms that appear in (3.2') as one crosses the threshold, and to combine them into a useful form. The following sections are, in effect, devoted to that end.

B. A Fundamental Identity

Let α be some set of incoming lines of S. A minus bubble in the expansion (3.2') of S will be called an α bubble if and only if all the incoming lines of that bubble belong to the set α. We define S^{α} to be the subset of the expansion (3.2') consisting of all terms having no α bubble. Thus for each term of S^{α} each line in the set α either ends at a minus bubble that has some incoming line not belonging to α, or it touches no minus bubble at all, and is therefore an "unscattered" line (i.e. it is both incoming and outgoing).

It is convenient to represent S^{α} by the diagram shown in Fig. 2.

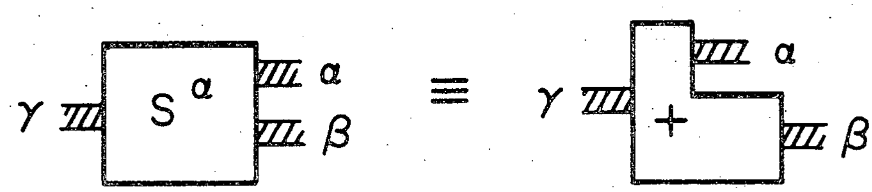
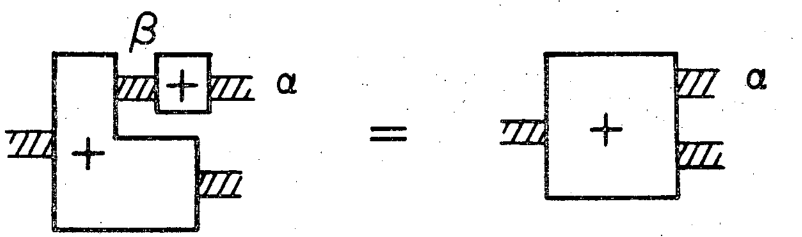


Fig. 2. Diagrammatic representation of S^{α}. The shaded strips represent arbitrary sets of external lines.

The diagram on the right of Fig. 2 is to be regarded as a representation of a partial sum of terms of the expansion (3.2'). The missing section indicates the absence of all terms having an α bubble.

With this notation a fundamental identity is this:



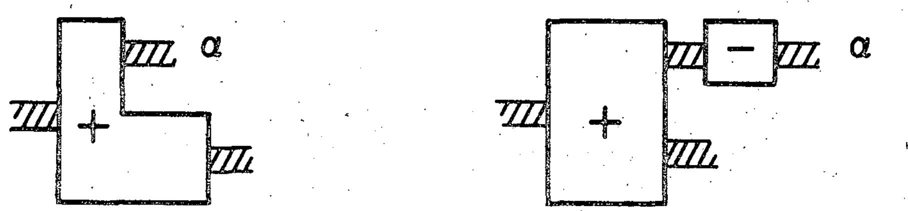
(3.3)

This equation expresses the fact that if one attaches to S^{β} the set obtained from the expansion of the small plus box, and sums over β, then one obtains the full expansion (3.2') of S. In particular, all the terms with α bubbles are reinstated, and each one only once.

To prove (3.3) the concept of a <u>cut</u> is useful. The lines of the $D(B^-)$ corresponding to any B^- are drawn running from right to left. A <u>flow line</u> is a continuous curve in D that runs from the extreme right to the extreme left. It consists of an ordered sequence of line segments L_j of D. A <u>cut</u> is a set of lines that contains at most one line L_j of any flow line. The set of flow lines <u>defined</u> by a cut is the set of all flow lines that contain a line contained in the cut. <u>Equivalent cuts</u> are cuts that define identical sets of flow lines. A line ℓ_1 <u>lies left</u> of ℓ_2 if and only if ℓ_1 lies left of ℓ_2 on some flow line. A cut C_1 <u>lies left</u> of a cut C_2 if and only if C_1 is equivalent to C_2, at least one line of C_1 lies left of some line of C_2, and no line of C_2 lies left of any line of C_1. A <u>leftmost</u> cut is a cut such that no cut lies left of it.[13,14]

In (3.3) the cut β is the leftmost cut equivalent to α. That no cut lies left of it follows from the definition of S^{β}. For each fixed β the terms of (3.2') give, independently, all terms of S^β on the left of β and all terms of the small plus box $S_{\beta\alpha}$ on the right.

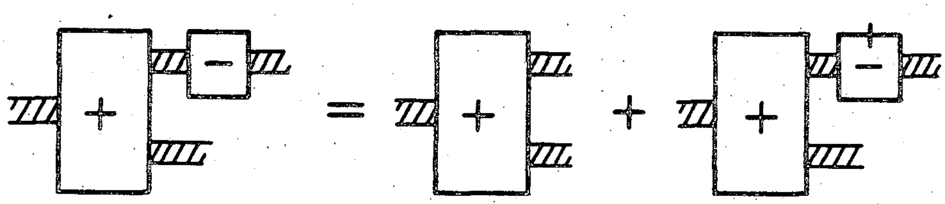
Multiplication of (3.3) by a small minus box on the right gives



(3.3')

The fact that the combination on the right is equivalent to a sum of bubble diagram functions F^B corresponding to B's having no α bubbles was shown earlier in Ref. 15. There only finite operations were used and the sum was over a finite number of terms. [Both plus and minus bubbles occurred in the B's representing the terms of that finite expression.]

The validity of (3.3') can be seen directly from the expansion (3.2'). If this expansion is substituted into both terms of the right side of



(3.4)

where the slashed box is R^-, one finds an exact cancellation of all terms having an α bubble: Each bubble diagram B^- that has precisely one

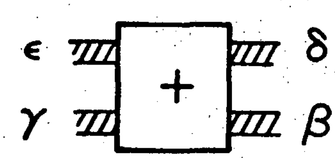
α bubble appears precisely twice on the right, and these two terms have opposite signs. Each term having precisely two α bubbles appears four times, twice with a plus sign and twice with minus sign. Each term having precisely $n > 0$ α bubbles appears 2^n times, half with plus and half with minus signs. However, each term with no α bubbles appear only once, and in the first term. This confirms (3.3') and gives an independent confirmation of (3.3).

C. Leading Normal Threshold Formula

Using the identity just obtained, one easily derives the normal threshold formula obtained earlier[15] without using infinite series.

In the expansion (3.2') of

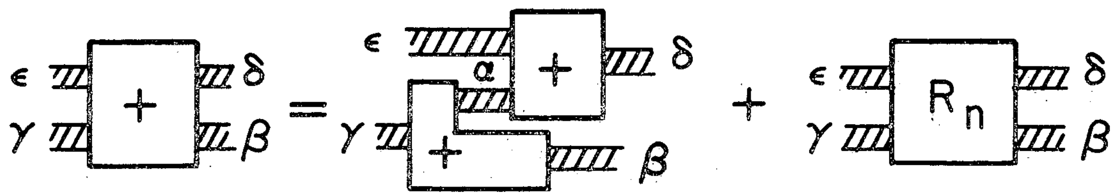
$$S =$$



(3.5)

some terms will have a cut C such that all the flow lines through this cut begin in δ and end in γ, and such that the removal of the lines of this cut separates S into two disjoint parts, one containing ϵ and δ, the other containing γ and β. Let the sum of terms having no such (empty or nonempty) cut C be called R_n.

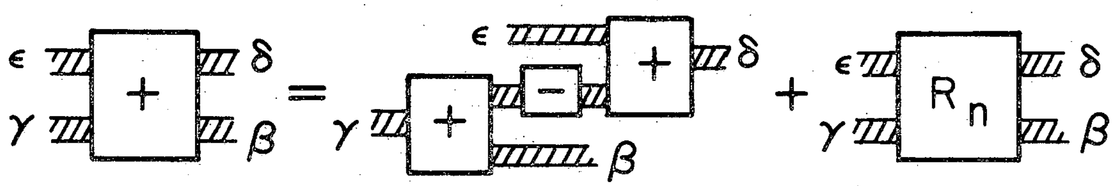
A term having such a cut C may have several. All these must be equivalent, since each defines precisely the set of all flow lines that begin at δ and end at γ. Let the leftmost of these cuts be labelled α. Then the separation of the terms of the expansion of (3.5) into terms having, or not having, a cut C gives



(3.6)

Each term in the expansion of the left side either has no cut C, and hence belongs to R_n, or has a leftmost cut α, and appears precisely once in the first term on the right of (3.6).

Insertion of (3.3') into (3.6) gives



(3.6')

This formula is essentially the same as that derived (laboriously) in Ref. 15, by means of finite methods. There the plus boxes were the actual S matrices (rather than their infinite-series expansion) and R_n was a certain <u>finite</u> sum of bubble diagram functions F^B having just the property that defines R_n: i.e., no B corresponding to a term of the sum R_n has a D_B having point vertices for all minus bubbles that supports a cut C of the kind described.

The important property of R_n is that it contains no B having a D_B that contracts to any positive-α normal threshold diagram D_n^+ of the form indicated in Fig. 3. [D_B is defined in Section II H.]

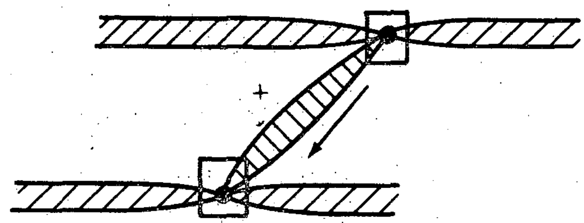
$D_n^+ \equiv$ 

Fig. 3. The positive-α normal threshold diagram D_n^+. The + sign indicates that the σ_i of all lines of the set of lines between the two vertices are plus one. The arrow indicates that all lines have the direction indicated. D_n^- is defined by the same diagram with minus in place of plus. The boxes around the vertices indicate that it is not necessary that the vertices within them be single points; a point within a box can represent several disconnected point vertices.

The first term on the left of (3.6') vanishes below the leading normal threshold associated with diagrams of the form D_n^+. The second term on the left has, by construction, no positive-α singularity corresponding to any diagram that contracts to any diagram of the form D_n^+. If mixed-α singularities (i.e., singularities associated with solutions of Landau equations that involves α_i of both signs) can be ignored (see Section VI) and if the only diagrams D^+ giving surfaces $L(D^+)$ through a point $\overline{P}$ are those that contract to a diagram of the form

D_n^+, then the only singularities of R_n at $\overline{P}$ are those associated with diagrams that contract to D_n^-. The function R_n must then, by virtue of the Fundamental Theorem, continue into itself via a minus $i\epsilon$ rule around the threshold. It is consequently the continuation of S from the region just below threshold to the region underneath the cut starting at threshold. The first term on the right of (3.6') is thus just the discontinuity around the normal threshold.

D. A Generalized Identity

The function S^α is the set of terms of (3.2') such that no cut lies left of the cut α.

Let the mass M_α of a set of lines α be the sum of rest masses of the lines α. Let α' denote a cut that lies left of α and also satisfies $M_{\alpha'} \geqslant M_\alpha$. Let $S^{\alpha'}$ be the subset of (3.2') that has no α'.

Let $P(\alpha)$ be the projection function that is zero or one according to whether the set of lines β on which it acts satisfies $M_\beta < M_\alpha$ or $M_\beta \geqslant M_\alpha$. Let $S_\alpha = P_\alpha S P_\alpha$. That is, S_α is S if both incoming and outgoing lines have mass $\geqslant M_\alpha$, but it is zero otherwise. Then near the α threshold one obtains the following generalization of (3.3): for any S with a (sub) set of incoming lines α

$$S^{\alpha'} S_\alpha = S , \tag{3.7}$$

where, in complete analogy to (3.3), S_α acts between the sets α and α'. [The proof is essentially the same as for (3.3); the nearness to threshold ensures that the leftmost cut α' is unique.[13]]

From (3.7) one obtains, as the generalization of (3.3')

$$S^{\alpha'} = S\, S_\alpha^{-1} \quad , \tag{3.8}$$

where S_α^{-1} is the inverse of S_α:

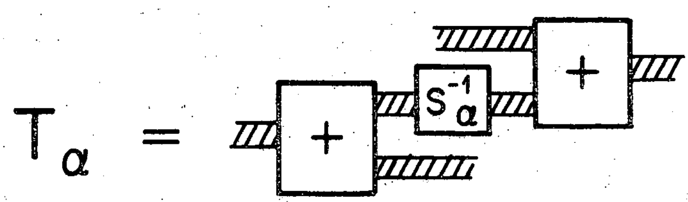
$$S_\alpha\, S_\alpha^{-1} = P_\alpha \quad . \tag{3.9}$$

[This definition of S_α^{-1} is slightly more general than the one given in the introduction; it covers also the special case when two different sets of communicating particles have the same sum of rest masses.]

E. General Normal Threshold Formula

Consider the expansion (3.2') of S of (3.5). Let α be a cut of the type described below (3.5) with the additional condition that M_α be equal to or greater than some fixed sum of rest masses.

The arguments leading to (3.6) are now repeated, but now with R_α containing the terms having no cut α. One then obtains for the discontinuity around the α normal threshold the formula



(3.10)

This result is the same as that obtained by finite methods in Ref. 15, except that there M_α was required to be less than the lowest communicating four-particle threshold. This limitation is here removed.

F. General Physical-Region Discontinuity Formula

Essentially the same argument gives the general discontinuity formula described in the introduction.

Consider some basic positive-α diagram D_β^+. Let α label the sets of lines connecting the various pairs of vertices of D_β^+. Let the mass of a set of lines be the sum of the rest masses of these lines, and let M_α be the mass of α.

A bubble diagram B is said to contain D_β^+ if and only if D(B) contains D_β^+. [D(B) is the diagram obtained by shrinking the bubbles of B to points.] A D contains D_β^+ if and only if it has a set of mutually disjoint cuts C_α, one corresponding to each of the sets α of D_β^+. The cut C_α corresponding to the set α must be a cut that consists of positively signed lines having mass M_α. Moreover, the cutting of all the lines of all these sets C_α must divide D into a set of N mutually disjoint parts, one corresponding to each of the N vertices of D_β^+. The part of D corresponding to the nth vertex of D_β^+ must contain the appropriate end points (leading or trailing) of the appropriate lines of the appropriate sets, as prescribed by $\epsilon_{\alpha n}$. [$\epsilon_{\alpha n}$ is the common sign of the ϵ_{in} of D_β^+ for i in α.] The connectedness of the part n of D is irrelevant; as in Fig. 3 it can be either connected or disconnected.

A B excludes D_β^+ if and only if no D_B contains D_β^+. [D_B is defined in Section II H. Notice that "contain" and "exclude" are opposites provided all the bubbles of B are minus bubbles.]

The important properties of these two classes are these: First, any sum T of F^{B}'s over B's that contain D_β^+ must vanish outside the physical region of D_β^+, and hence on the negative-f side of $L(D_\beta^+)$ [see Section II F]. Second, any sum R of F^{B}'s over B's that exclude D_β^+ must, by virtue of the Fundamental Theorem, have a minus $i\epsilon$ continuation into itself past $\overline{P}$ of $L(D_\beta^+)$, provided $\overline{P}$ lies on no $L(D^+)$ except those such that D^+ contains D_β^+, and provided R has no mixed-α singularities at $\overline{P}$. It follows that a separation of S in two terms T and R that contain and exclude D_β^+, respectively, exhibits T as the discontinuity around any such $\overline{P}$ of $L(D_\beta^+)$.

Consider any B^- that contains D_β^+. Then $D(B^-)$, which is the diagram obtained by replacing each (minus) bubble of B^- by a point vertex, must have some set of cuts C_α corresponding to the sets α of D_β^+. A cut <u>strongly equivalent</u> to C_α is a cut that is equivalent to C_α and has the same mass. Any C_α may be replaced by any cut strongly equivalent to it without destroying its correspondence to α of D_β^+.

The result just stated is proved in Appendix C. It is assumed there, and in what follows, that the point $\overline{P}$ under consideration lies on $L(D_\beta^+)$, and lies on no $L(D^+)$ unless D^+ contains D_β^+.

The Landau equations for D_β^+ at $\overline{P}$ require the momentum-energy vectors of all the lines in a given set α of D_β^+ to have a common direction d_α. It also is assumed in Appendix C, and in what follows, that these directions d_α are all different, for the $\overline{P}$ under consideration.

Consider now the structure $\tilde{T}$ obtained by replacing each vertex of D_β^+ by the expansion (3.2') of the S corresponding to that vertex. Delete from the expansion of each S all terms corresponding to diagrams having some cut that is strongly equivalent to, and stands left of, the cut corresponding to any set α of incoming lines of that S.

This structure T contains every term B^- in the expansion (3.2') of S that contains D_β^+: For any such term there must be a set of cuts C_α that correspond to the various α of D_β^+. Consider the leftmost cuts C_α' strongly equivalent to these. These C_α' separate B^- into parts that correspond to the vertices of D_β^+. The part corresponding to the nth vertex will be some term in the expansion (3.2') of the S corresponding to that vertex. And it will be one of the terms that is retained in the construction of T.

Thus any term in the expansion (3.2') of S that contains D_β^+ will be some term in the structure T. And any term in the structure T evidently contains D_β^+, and is a term of (3.2').

It remains to show that each term of (3.2') that contains D_β^+ is contained precisely once in T. If this is true then the remainder R will exclude D_β^+, and the desired separation of S will be achieved.

Each term in (3.2') that contains D_β^+ will be contained precisely once in T provided any B^- that contains a set of leftmost cuts C_α' corresponding to the α of D_β^+ contains precisely one such set: for every such set of cuts C_α' in B^- this term is contained precisely once in T.[13] Thus we must show that each B^- that has a set of leftmost C_α' corresponding to the α of D_β^+ has precisely one such set.

Suppose for some B^- there are two sets of leftmost cuts C_α' that correspond to the α of D_β^+. The function F^{B^-} will vanish in an infinitesimal neighborhood of $\overline{P}$ unless the constraints of B^- allow the p_i''s corresponding to the lines of each of these sets of C_α''s to assume the (unique) values $p_i(\overline{P})$ that solve the Landau equations of D_β^+ at $\overline{P}$.

Consider a reduced diagram $\overline{D}^+$ that contains only those lines of $D(B^-)$ that lie on one or the other of the two sets C_α'. Since the Landau equations at $\overline{P}$ must be satisfied for the lines coming from each of the sets C_α' separately, they must be satisfied for the whole diagram $\overline{D}^+$: $\overline{P}$ must lie on $L(\overline{D}^+)$ if B^- is to contribute near $\overline{P}$.

The conditions on D_β^+ for there to be a $\overline{D}^+$ that contains D_β^+ in two essentially different ways, as above, are very stringent. For example, the leading vertex of D_β^+ that expands into more than a single vertex of $\overline{D}^+$ must have a set of outgoing lines that represent particles that can decay into the particles represented by another set of outgoing lines of that vertex. (See Fig. 7) This places strong conditions on the momenta p_j associated with these lines, and hence stringent conditions on $\overline{P}$. We call "redundancy conditions" these conditions on $\overline{P}$ that must be satisfied if D_β^+ is to be contained in several essentially different ways in some D^+.

Our conclusion then is this: Suppose the following conditions are satisfied:

1) $\overline{P}$ lies on $L(D_\beta^+)$ and on no $L(D^+)$ unless D^+ contains D_β^+.
2) The directions d_α of the p_j of the various sets of lines α of D_β, as defined by the Landau equations of D_β^+ at $\overline{P}$, are all different.
3) The redundancy conditions on D_β^+ are not satisfied at $\overline{P}$.
4) The remainder $R = S - T$ has no mixed-α singularities at $\overline{P}$ (see Section VI).

Then the discontinuity of S around $L(D_\beta^+)$ at $\overline{P}$ is given by the rules described at the beginning of the paper, where the diagram D is just D_β^+. Notice that condition (1) ensures that $\overline{P}$ lies on the codimension 1 surface $L_0(D_\beta^+)$ [see Section II D].

The disconnected parts of S have, of course, conservation law delta function factors. The discontinuities associated with these parts are calculated in the natural way, by taking the discontinuity corresponding to a path that encircles the singularity surface $L_0(D_\beta^+)$ while remaining in the manifold defined by the appropriate conservation law delta functions.

We believe the discontinuity formula for S itself, rather than its connected part, will be the more useful in practice, because in any applications based on unitarity (or on other physical conditions) it is the full S, rather than its connected part, that is relevant. One lesson we have learned from our work is that general results for multiparticle processes are hard to derive from unitarity if one separates out the disconnected parts before the final stage.

The derivation given in this section is based on the infinite series expansion for S. However, all infinite series are eliminated from the final result. This suggests that the results should be derivable directly from the equation $SS^{-1} = I$ that generated the infinite series. This has been done in many special cases.[3,4,15] In Section V we derive the result for all "leading" singularities, without using infinite series.

The expansion of (3.2') for S has an infinite number of terms, one for each diagram D^+. An interesting finite expression is obtained by grouping together the contributions corresponding to different structures s. A structure s corresponds to the <u>class</u> of basic diagrams D_β^+ that differ only by the masses associated with the various sets of lines α. That is, the masses of the particles that pass between the two vertices specified by α are not restricted; they are allowed to be anything.

This grouping of terms gives

$$S = \sum_s S_s \quad . \tag{3.11}$$

The expression for S_s is obtained by replacing each vertex of the structure diagram by a minus bubble, and each set of lines α by the entire S matrix acting between the two corresponding minus bubbles.

This expansion (3.11) for S is something like a Feynman expansion, but with the following important differences:

1) It is strictly mass-shell and physical-region.

2) Only a finite number of terms contribute at any finite energy.

3) Each propagator is the entire physical S-matrix.

4) Each vertex is a minus bubble.

This system of exact integral equations appears to be interesting, but their exploitation is not our present aim.

IV. PROPERTIES OF S_α^{-1}

The function S_α^{-1} is the inverse of $S_\alpha = P_\alpha S P_\alpha$, where P_α is the projection on configurations of communicating particles having a sum of rest masses greater than or equal to the mass M_α associated with the lines α of some Landau diagram. The equation for S_α^{-1} has a formally Fredholm structure. In the case that M_α lies below the lowest four-particle threshold (for communicating particles) the equation for S_α^{-1} has been converted to strict Fredholm form.[3] This has not yet been done in the general case.

The function S_α^{-1} can be expressed in terms of S and S^{-1} and their continuations. To obtain these expressions introduce first the definitions

$$R_\alpha^- = S_\alpha^{-1} - I_\alpha \tag{4.1}$$

and

$$R_\alpha = S_\alpha - I_\alpha \quad . \tag{4.2}$$

These satisfy

$$R_\alpha^- + R_\alpha + R_\alpha^- R_\alpha = 0. \tag{4.3}$$

Both R_α and R_α^- are restricted to the space allowed by $P_\alpha \equiv I_\alpha$. The function R_α^- is the restriction to this space of the $\bar{R}_\alpha^-$ defined by

$$\bar{R}_\alpha^- + R + \bar{R}_\alpha^- P_\alpha R = 0 \quad . \tag{4.4}$$

[The projection of (4.4) on α is just (4.3).]

Define the quantity $\overline{R}^{+}_{\alpha}$ by

$$\overline{R}^{+}_{\alpha} + R^{-} + R^{-} Q_{\alpha} \overline{R}^{+}_{\alpha} = 0, \tag{4.5}$$

where $Q_{\alpha} + P_{\alpha} = I$ and $R^{-} = S^{-1} - 1$. The restriction of $\overline{R}^{+}_{\alpha}$ to the space allowed by Q_{α} is called R^{+}_{α}:

$$R^{+}_{\alpha} \equiv Q_{\alpha} \overline{R}^{+}_{\alpha} Q_{\alpha} \quad . \tag{4.6}$$

It satisfies

$$R^{+}_{\alpha} + Q_{\alpha} R^{-} Q_{\alpha} + Q_{\alpha} R^{-} R^{+}_{\alpha} = 0 \, . \tag{4.5'}$$

Below the α threshold the Q_{α} are irrelevant and R^{+}_{α} can be identified with $Q_{\alpha} R Q_{\alpha}$. We showed in Ref. 3 that R^{+}_{α} evaluated just above the α threshold coincides with the continuation of $Q_{\alpha} R Q_{\alpha}$ from the physical region lying just below the α threshold, the continuation being via the minus $i\epsilon$ rule. We also established a number of interesting relationships between $\overline{R}^{+}_{\alpha}$ and $\overline{R}^{-}_{\alpha}$, such as

$$\overline{R}^{+}_{\alpha} = -\overline{R}^{-}_{\alpha} \tag{4.8}$$

and

$$S_{\alpha}^{-1} = P_{\alpha} S^{-1} P_{\alpha} - P_{\alpha} S^{-1} Q_{\alpha} S^{-1} P_{\alpha} - P_{\alpha} S^{-1} R^{+}_{\alpha} S^{-1} P_{\alpha} \quad . \tag{4.9}$$

This latter equation (C.12 of Ref. 3) allows S_{α}^{-1} to be expressed in terms of S^{-1} and the continuation of $Q_{\alpha} R Q_{\alpha}$ to underneath the α cut.

In Ref. 3 the results just described were derived only for energies lying below the lowest four-particle threshold of the channel in question. However, they hold also in general, at least in our iterative framework. To see this one can first consider R_α^+ to be defined to be the sum of all terms of the expansion (3.2) that contain no direct channel α cut. That is, R_α^+ is the sum of all terms of expansion (3.2) that exclude the direct channel normal threshold structure diagram D_α^+, where α specifies a certain sum of rest masses. In this case our general expansion of S according to D_α^+ gives [see (3.10)]

$$S = S\, S_\alpha^{-1}\, S + R_\alpha^+ + Q_\alpha \quad . \tag{4.10}$$

Multiplication on the left by S^{-1} gives

$$I = S_\alpha^{-1} S + S^{-1} R_\alpha^+ + S^{-1} Q_\alpha \quad . \tag{4.11}$$

Recalling that

$$S_\alpha^{-1} = P_\alpha\, S_\alpha^{-1}\, P_\alpha \tag{4.12}$$

and noting that

$$R_\alpha^+ = Q_\alpha\, R_\alpha^+\, Q_\alpha \quad , \tag{4.13}$$

we obtain by left multiplication of (4.11) by Q_α the original definition of (4.5') of R_α^+.

Left and right multiplication of (4.11) by P_α gives the defining equation for S_α^{-1}. Left multiplication of (4.11) by P_α and right multiplication by $S^{-1} P_\alpha$ gives (4.9). Equation (4.8) can be derived in the same way as in Ref. 3. [See (5.18) and Appendix C of Ref. 3.]

The above argument shows that the quantity R_α^+ defined by (4.5') is equal to the sum of all terms (but Q_α) of the expansion (3.2') of S that exclude D_α^+, and that it is accordingly, the continuation of $Q_\alpha R Q_\alpha$ to underneath the cut starting at the α threshold.

It is surprising that the R_α^+ defined by (4.5') is the continuation of $Q_\alpha R Q_\alpha$ to underneath the α cut. For many terms of iterative solution to (4.5') do contain D_α^+. However, a detailed examination shows that each such term of $Q_\alpha R^- Q_\alpha + Q_\alpha R^- R_\alpha^+$ is cancelled by an identical term with opposite sign.

This cancellation allows the results of Ref. 15 to be extended without essential change to the regions above the lowest four-particle channel threshold, except that the justification of some steps by Fredholm theory is no longer supplied. We expect it could be supplied by the same sort of arguments that were given in Ref. 3 for the two- and three-particle intermediate states.

V. INDUCTIVE SOLUTION

This section contains an alternative derivation of the discontinuity around "leading" singularities. This derivation does not rely on the infinite series expansion for S, but is based instead on the results of Ref. 15. The point $\bar{P}$ is as above.

The principal results of Ref. 15 are these: (i) over any bounded domain S can be converted by a finite number of applications of $SS^{-1} = I$ to the form $T[D_n^+] + R[D_n^+]$, where $T[D_n^+]$ is the first term on the right of (3.6'), and $R[D_n^+]$ is a certain finite sum of bubble diagram functions F^B, each corresponding to a B that excludes the normal threshold diagram D_n^+ of Fig. 3. (ii) The quantity Σ on the right of (3.3') can be similarly converted to a finite sum Σ' of F^B's, each corresponding to a B that has no cut $\alpha' \neq \alpha$ that is equivalent to α.

The discontinuity around any leading singularity can be derived by repeated application of these two results. To do this, first select a leading vertex V of D_β^+ [i.e., all incoming lines of V are incoming lines of D_β^+]. Let $D_n^+(V)$ be the D_n^+ obtained by contracting all internal lines of D_β^+ but those that are outgoing lines of V. Then any B that excludes $D_n^+(V)$ will exclude also D_β^+. Thus the second term on the right of

$$S = T[D_n^+(V)] + R[D_n^+(V)] \quad , \tag{5.1}$$

consists of terms that exclude D_β^+.

The first term on the right of (5.1) has the form of the first term on the right of (3.6'). The part Σ of this term that is the right-hand side of (3.3') can be converted by means of (ii) to a sum Σ' of F^B's, each corresponding to a B that has no $\alpha' \neq \alpha$ equivalent to α. This gives the alternative form

$$S = T' + R[D_n^{\;+}(V)] \quad . \tag{5.2}$$

Let D' be any $D_{T'}$ that contains $D_\beta^{\;+}$, with $\overline{P}$ on $\overline{L}(D')$.

Let C_V be the sum of the leftmost cuts C_α' of D' that correspond to the sets α that begin at V of $D_\beta^{\;+}$. Property (ii), together with the requirement that the sets α be leading sets, entails that any C_V in D' consist precisely of the set of lines Γ of T' that run out of the right-hand plus box and into Σ'. That is, property (ii) requires any C_V to lie to the right of Σ', and the condition that the various sets α be leading sets rules out the possibility that C_V lies inside the plus box. (i.e., the kinematic constraints at $\overline{P}$ do not allow the particles in different leading sets α to come together again after leaving V. See Appendix C.)

Thus any C_V in D' must consist of precisely the lines Γ. Let $\{p_i(\overline{P})\}$ be the $\{p_i\}$ of the unique[8] solution of the Landau equations of $D_\beta^{\;+}$ at $\overline{P}$. Then the only part of the integral over the lines of Γ that contributes to the singularity at $\overline{P}$ associated with $D_B^{\;+}$ comes from the region near the points where the p_i of Γ assume the values $p_i(\overline{P})$: the other parts of the integral do not allow the Landau equations of $D_\beta^{\;+}$ to be satisfied at $\overline{P}$.

Let the lines of Γ be divided into sets Γ_α, one for each of the sets C_α' of C_V, such that near the point $p_i = p_i(\overline{P})$ the set Γ_α contains the lines contained in C_α'. Then $T \equiv T[D_n^+]$ can be separated into three terms:

$$T = T^a + T^b + T^c \quad . \tag{5.3}$$

The term T^a consists of those terms of T such that some minus bubble of T connects lines from different sets Γ_α. The remaining terms have no minus bubble connecting these sets, and the separation into sets Γ_α of the set Γ induces a corresponding separation into sets Γ_α' of the set of lines Γ' that emerge from the minus box and enter the left-hand plus box. Let this plus box be written as $T[\hat{D}_\beta^+] + R[\hat{D}_\beta^+]$, where $\hat{D}_\beta^+$ is the diagram obtained by removing V from D_β^+. The two corresponding terms of T are called T^b and T^c, respectively. Then T^b is the desired $T[D_\beta^+]$.

We proceed by induction on the number of vertices of D_β^+. Thus $T[\hat{D}_\beta^+]$ is assumed to have the form described in the introduction, and $R[\hat{D}_\beta^+]$ is assumed to have no singularities corresponding to diagrams D^+ that contain $\hat{D}_\beta^+$. The analogous property must then be derived for D_β^+.

In this section we shall accept <u>an extended independence property</u> that asserts that in any equation $G = 0$ derived from unitarity (or $S\bar{S}^{-1} = I$) the <u>net</u> singularity corresponding to any basic diagram D_β^+ is zero. That is, the various singularities corresponding to any one D_β^+ cancel among themselves. This is what one would naturally expect; the singularities corresponding to different basic diagrams should generally have different analytic characters and would not be expected to cancel against each other, even if they could coincide.

This assumption simplifies the present proof, but is not actually necessary, as is discussed in Section VI.

The work of Ref. 15 that gives property (ii) can be extended to show that $T^b \equiv T[D_\beta^{\ +}]$ can be converted to a form $T^{b'}$ that has the same property as T': any cut C_V must lie in Γ.

Consider, then, the identity

$$T' - T^{b'} = T^a + T^b \quad . \tag{5.4}$$

Multiplication on the right by the inverse of the right-hand plus box gives

$$F' = F \quad . \tag{5.5}$$

The equality of the two sides of this equation is a consequence of unitarity (or $SS^{-1} = I$).

The function F' has the property of Σ': any cut C_V must lie in Γ. The function F has the opposite property: no cut C_V can lie in Γ. We conclude that F' has no net singularity corresponding to C_V in Γ. But then $T' - T^{b'} = T - T^b$ can have no singularity corresponding to $D_\beta^{\ +}$. This property holds true also for $S - T$ [see (5.1)]. Thus it must hold for their sum

$$S - T^b \equiv R^b = R(D_\beta^+) \quad .$$

This completes the induction proof.

VI. DISCUSSION OF ASSUMPTIONS

The assumptions used in our derivation of the discontinuity formula are these: First, there are some general assumptions embodied in the cluster decomposition principle, the positive-α rule (which says that the singularities of S_c and S_c^- are confined to positive-α Landau surface) and the $i\epsilon$ rule. These general assumptions are consequences of the macrocausality requirement, as was discussed in Section II. Second, there are the independence property and the technical assumption, which are needed for the Fundamental Theorem. The independence property is the full content in this work of maximal analyticity. We plan to discuss the technical assumption elsewhere.

A third set of assumptions are special conditions on the point $\overline{P}$. In the first place, $\overline{P}$ is required to lie on $L(D_\beta^+)$, but on no $L(D^+)$ unless D^+ contains D_β^+. Second, the directions d_α of the momentum-energy vectors corresponding to different sets α of internal lines of D_β^+ at $\overline{P}$ are required to be all different. And third, $\overline{P}$ is required to be such that at $\overline{P}$ no $\overline{D}^+$ contains D_β^+ in two essentially different ways. These conditions on $\overline{P}$ are to ensure that positive-α singularities associated with diagrams other than D_β^+ do not contribute at $\overline{P}$, and that those associated with D_β^+ contribute precisely once.

The discontinuities at points $\overline{P}$ where these conditions on $\overline{P}$ fail can be calculated by making use of the independence property. Suppose for example that $\overline{P}$ lies on $L(D^+)$ for some D^+ that does not contain D_β^+. The diagram D^+ can be assumed to be basic. Then

$\overline{P}$ must lie also on $L(\overline{D}_\beta^{\,+})$, where the basic diagram $\overline{D}_\beta^{\,+}$ is a contraction of D^+. (One contracts out the lines of D^+ that correspond to $\alpha_i = 0$.) The independence property then ensures that the singularities at $\overline{P}$ associated with the $D_\beta^{\,+}$ and $\overline{D}_\beta^{\,+}$ are independent (i.e., additive) unless there is some $\hat{D}_\beta^{\,+}$ that contains both $D_\beta^{\,+}$ and $\overline{D}_\beta^{\,+}$, with $\overline{P}$ on $L(\hat{D}_\beta^{\,+})$. Since the Landau equations for $L(D_\beta^{\,+})$ and $L(\overline{D}_\beta^{\,+})$ are both satisfied at $\overline{P}$, this point must lie also on $L(\hat{D}_\beta^{\,+})$. If $\overline{P}$ lies on $L(D^+)$ for no other basic diagram D^+, then one can classify all basic diagrams $\tilde{D}_\beta^{\,+}$ such that $\overline{P}$ lies on $L(\tilde{D}_\beta^{\,+})$ according to whether $\tilde{D}_\beta^{\,+}$ contains just $D_\beta^{\,+}$, just $\overline{D}_\beta^{\,+}$, or both (and hence also $\hat{D}_\beta^{\,+}$). The terms corresponding to the last case would be counted in both $T[D_\beta^{\,+}]$ and $T[\overline{D}_\beta^{\,+}]$. But they are also the terms included in $T[\hat{D}_\beta^{\,+}]$. Thus the discontinuity is
$T[D_\beta^{\,+}] + T[\overline{D}_\beta^{\,+}] - T[\hat{D}_\beta^{\,+}]$.

In this case $\overline{P}$ lies on both $L(D_\beta^{\,+})$ and $L(\overline{D}_\beta^{\,+})$, and the above discontinuity is the difference between the function in the physical region of $D_\beta^{\,+}$ and its continuation around both $L(D_\beta^{\,+})$ and $L(\overline{D}_\beta^{\,+})$, where the continuation moves first through the plus $i\epsilon$ region associated with $\hat{D}_\beta^{\,+}$, and then through the corresponding minus $i\epsilon$ region.

More general cases are treated similarly, by using the general principle of inclusion and exclusion [see Appendix D of Ref. 15]. The same sort of considerations apply also to cases where one or both of the other two conditions on $\overline{P}$ fail: again one uses the independence property together with the principle of inclusion and exclusion to isolate the relevant set of terms.

The final assumption is that $R = S - T$ has no mixed-α singularities at $\overline{P}$.

We now argue that the sum on the right of $S = R + T$ should have no net mixed-α singularities. Since the quantity S on the left has singularities only on positive-α Landau surfaces, the only possible net mixed-α singularities on the right are those that happen to lie exactly on top of positive-α surfaces.

It is conceivable that these particular mixed-α would not cancel out, like all the others must, but it seems unlikely. In the first place the physical arguments (macrocausality) that imply that the singularities of S are confined to positive-α surfaces correlate these singularities to positive-α diagrams. Thus it would be unnatural for them to arise mathematically from other diagrams, which just happen to give the same Landau surfaces.[16] In the second place, the mixed-α singularities that happen to lie on positive-α surfaces are intimately related via hierarchy effects to the mixed-α singularities that do not lie on positive-α surfaces. It seems unlikely that the latter could all vanish identically without the former vanishing also.

On the basis of these arguments we shall accept the proposition that in any equation of the form $S = X$ derived from $SS^{-1} = I$ the mixed-α singularities of the bubble diagram functions that comprise the right-hand side exactly cancel out (in the physical region). This will be our basic assumption about mixed-α singularities. It may be possible to derive it by some inductive argument, but we do not attempt this here.

On the basis of this assumption we can confirm the absence of the mixed-α singularities in $R = S - T$ by confirming it rather for T.

The only lines of T that can be minus lines are the lines of the cuts C_α. By virtue of energy conservation the momenta of all these lines are fixed at precisely the value defined by the Landau equations of D_β^+ at $\bar{P}$. [The Landau equations define the unique way of achieving the boundary point of the physical region of D_β^+. See Section II F.]

Any mixed-α D_T such that $\bar{P}$ lies on $L(D_T)$ is a member of a continuum of such D_T. This continuum is generated by adding to the solution of the Landau equations corresponding to $\bar{P}$ on $L(D_T)$ a real multiple of the solution corresponding to $\bar{P}$ on $L(D_\beta^+)$. If the real multiple is sufficiently large and positive, then the mixed-α D_T is converted to a D_T^+, because all the lines corresponding to the C_α are eventually made positive. Thus any point $\bar{P}$ on $L(D_\beta^+)$ that lies on the $L(D_T)$ of a mixed-α D_T must lie also on $L(D_T^+)$ for a continuum of $D_T^+ \neq D_\beta^+$, where D_T^+ contains D_β^+.

This shows that T can have no mixed-α singularities at simple points of $L(D_\beta^+)$, which are points that correspond to just one D_β.

At the nonsimple points $\bar{P}$ of $L(D_\beta^+)$ that lie on $L(D_T^+)$ for the continuum of $D_T^+ \neq D_\beta^+$ the meaning of our assumption about mixed-α singularities must be clarified. For we have to consider diagrams that can be continuously shifted from mixed-α to positive-α status. The correspondence between singularities and diagrams then becomes ambiguous. At these points of $L(D_\beta^+)$, where these flexible diagrams could give mixed-α singularities to T, we interpret our assumption that all mixed-α singularities of $T + R$ cancel to mean that the only net mixed-α singularities of R are those associated with the same flexible diagrams that give the possible mixed-α singularities of T.

With this interpretation we can show that the mixed-α singularities of R that might occur at these special points would not, in any case, upset our proof. The point is that contributions to R associated with these flexible diagrams must have minus $i\epsilon$ continuations past the surface $L(D_\beta^+)$. This is because the construction of R ensures that these contributions can occur only if the minus lines of the (flexible) diagram come from inside minus bubbles. But then the proof of the Fundamental Theorem shows that the continuation past the surface $L(D_\beta^+)$ will follow the minus $i\epsilon$ rule, due to the presence of these necessarily minus lines. But then the proof of the discontinuity formula would go through even at these very special points at which the flexible diagrams give singularities.

In Section V an extra assumption (extended independence) was used to simplify the argument. To avoid the assumption one need modify the proof only slightly. First the function $R[D_\beta^+]$ is considered to be decomposed (using the ordinary independence property) according to basic positive-α diagrams $\tilde{D}_\beta^+$ [this decomposition is unambiguous]. Then the assumption of the induction argument is that all terms corresponding to diagrams $\tilde{D}_\beta^+$ that contain $\hat{D}_\beta^+$ vanish from $R[\hat{D}_\beta^+]$. The analogous property must then be proved for $R[D_\beta^+]$.

The proof proceeds as before, but one now decomposes also the two sides of $F' = F$ according to basic positive-α diagrams. Only the terms that can contribute to the final D_β^+ need be considered (see below). But the singularity surfaces bounding the supports of these terms are not the same on the two sides of $F' = F$. Thus these terms must vanish. But then $T - T^b$ has no terms corresponding to D_β^+. Nor does $S - T$. Thus neither does their sum $S - T^b = R^b$.

[The condition that $\overline{P}$ lies on no $L(D^+)$ for any D^+ not containing D_β^+ implies that one need consider only terms that contribute to the final D_β^+. For if any other diagrams could exactly compensate for the missing term in F', then this term also would give an unallowed D^+.]

The argument given above in effect justifies the extended independence property, in the context in which it was used.

The present work generalized the results obtained earlier by ourselves[3,13] and by the Cambridge group.[4] We now contrast our methods and results with theirs.

Regarding final results our discontinuity formula covers all physical-region singularities whereas their general result covers only the case of simple diagrams. (In simple diagrams each set α consists of just one line.) They have obtained results also for certain special nonsimple diagrams, and are working toward the general result.

Some theorems in the early part of their work are somewhat similar to our Fundamental Theorem. However, the treatment of technical details is considerably different in the two works.

Our basic procedure is quite different from that of the Cambridge group. Their approach is in a way more general, since they first derive general formulas for discontinuities of integrals in terms of the discontinuities of their integrands. Then they use these results to show that for singularities associated with simple diagrams the Cutkosky discontinuity formula is consistent with unitarity. Finally they show, by means of an inductive procedure, that no other solution is possible: if the Cutkosky formula is valid for all simple diagrams up to a certain order of complexity, then it must hold also for diagrams of the next order of complexity, provided singularities corresponding to nonsimple diagrams can be ignored.

Their procedure, then, is first to make a detailed general analysis of discontinuity formulas and then to introduce these results into unitarity, which is used in only a limited way.

Our procedure is the reverse. The manipulations involved in our approach are purely topological and involve multiple applications of unitarity (or more accurately the cluster properties of S and S^{-1}). These topological manipulations give equations $S = R[D_\beta^+] + T[D_\beta^+]$, where the topological characteristics of the terms on the right guarantee that $R[D_\beta^+]$ is the continuation of S around $L(D_\beta^+)$ via the minus $i\epsilon$ rule, and hence that $T[D_\beta^+]$ is the discontinuity. Analyticity is used only at the last stage, and thus complications connected with distortions of contours are avoided.

This procedure is more special, in that it refers to the particular problem at hand. But it yields a variety of strict identities[15] that can be used in other contexts. These identities are consequences of the cluster properties alone and are purely topological in nature; analyticity is not involved.

The assumptions needed in the two approaches are, with one important exception, essentially the same. In particular, the independence and boundedness properties are needed in both methods.[11] And the considerations involving the special conditions on $\overline{P}$ are essentially the same.

The one important difference is that the Cambridge group does not assume that the singularities of S and S^{-1} are confined to positive-α surfaces: their aim is to derive this result. On the other hand, they do assume the $i\epsilon$ rules, for positive-α points, and also certain similar rules at mixed-α points. Our viewpoint is that these strong $i\epsilon$ requirements should not be imposed ad hoc, but must be justified. We justify the $i\epsilon$ rules on the basis of macrocausality, and get the positive-α rule at the same time. Alternatively, one might justify the $i\epsilon$ rules on the basis of self consistency, but one should then also prove uniqueness.

Appendix A. The Independence Property and the Fundamental Theorem

The Fundamental Theorem quoted in Section II H has slightly weaker assumptions and slightly stronger conclusions than the theorems proved in Ref. 12. In this Appendix we discuss these assumptions, and show how the proof of Ref. 12 can be extended to give the theorem quoted in Section II F.

One technical detail should be mentioned first. What is proved in Ref. 7 is that S_c (or S_c^-) considered as a distribution can be represented as the limit of the analytic function. That is, this representation is shown to be valid when one is calculating the average of S_c (or S_c^-) over a Schwartz test function. But what is needed to prove the structure theorems is something slightly different. One needs to evaluate products of different S_c's and S_c^-'s with one another.

In the proof of the structure theorems each of these functions S_c and S_c^- was considered to be a limit of the analytic functions described above, and their products were defined, for certain fixed real values of the external (unintegrated) momenta, by performing the appropriate integration over internal momenta along a multidimensional contour that remains in the region of analyticity of all the relevant functions S_c and S_c^-. This contour is such that it can be shifted (staying in the analyticity domain) to a position arbitrarily close to the real physical region. By virtue of the (multidimensional) Cauchy theorem such a shift does not alter the value of the integral.

For any fixed real value of the (external) variables K of $F^B(K)$ the integrations occurring in the definition of F^B were assumed to be given by the above rule, provided the relevant domains of analyticity of the various functions S_c and S_c^- overlap in such a way that the required contour through the intersection of the analyticity domains infinitesimally removed from the real physical region exists. The function $F^B(K)$ was shown to be analytic at such values of K, and the rule for continuing the thus defined function $F^B(K)$ around any singularity at real K was derived.

This rule defining the integrals in $F^B(K)$ was used to evaluate the terms of SS^{-1}, $SS^{-1}S$, etc. If one considers the S matrix to be defined basically in terms of limits of analytic functions, then this definition of the meaning of the SS^{-1}, $SS^{-1}S$, etc. is the reasonable one. However, if one starts with S and S^{-1} considered to be operators in a Hilbert space, then this rule for defining their products must be justified. The required justification is given at the end of this appendix.

It was asserted in Section II F that the independence properties of S_c and S_c^- lead to analogous properties of the bubble diagram functions F^B. The point is that the proofs of the structure theorems show that the singularities of F^B corresponding to any basic diagram D_β^+ arise from singularities of the bubbles b of B that are associated with the parts $D_{\beta b}^+$ of D_β^+ that lie in b, when D_β^+ is regarded as a D_B^+. These parts $D_{\beta b}^+$ must be basic diagrams, if D_β^+ is. Now by virtue of the independence property of S_c the singularities of b associated with different basic daigrams $D_{\beta b}^+$ are independent. If any one specific $D_{\beta b}^+$ is inserted into each b of B then one specific D_B^+ is formed. This contracts to some unique basic $D_{B\beta}^+$. It thus follows that the singularities of F^B corresponding to different basic diagrams D_β^+ must arise from independent singularities of at least one b of B, and must therefore be independent.

The independence property can, alternatively, be derived from macrocausality at almost all points of the surface of singularities L^+. However, there is then the problem of extending the property to those rare (Type II) points at which this argument breaks down.

The independence property is not included among the assumptions mentioned by the Cambridge group.[4] This omission is connected to their somewhat relaxed way of specifying the precise conditions under which their basic theorems are valid. If one wishes to formulate their theorems precisely, in forms strong enough to do the job, then the independence property or something similar seems required. Following their philosophy one might try to justify the independence property by an inductive procedure: the independence property for complex basic diagrams might be shown to follow from that of the simpler ones. However, an inductive procedure for proving independence would involve an artificial assumption that the singularities can be "ordered", and that one can proceed by stages, completely ignoring "higher order" singularities at each stage. But since the discontinuity associated with any D_β^+ is, in effect, a some of contributions corresponding to diagrams that are <u>more complex</u> than D_β^+, a justification of independence based on "hierarchy" is subject to question. In the procedure we adopt no ordering is invoked, and there is never any "temporary neglecting" of certain singularities. Also, the full content of maximal analyticity is explicitly stated.

The Second and Third Structure Theorems given in Ref. 12 are specifically restricted to simple points of the Landau surfaces $L(D_B)$. That is, it is assumed that the point $\overline{P}$ corresponds to a <u>unique</u> basic diagram. This assumption is needed because the arguments cover only the case where there is only one constraint (3.7) (of Ref. 12). Now suppose there are many such constraints. The question is whether there is a set of variations δh_f of the Feynman loop parameters that keeps all the $\delta p_j^2 = 0$ and all the $\delta\sigma > 0$. (Such a set of variations would shift the contour simultaneously into the domain of analyticity of all the bubble functions, while maintaining all the mass shell and conservation law constraints.)

To solve this problem consider the following lemma:

<u>Lemma A</u> For any set of real numbers η_{ba} the system of equations

$$\sigma_b = \sum_a \eta_{ba}\, \delta_a, \quad . \quad \sigma_b > 0, \tag{A.1}$$

has a solution δ_a if an only if the system of equations

$$\sum \alpha_b \eta_{ba} = 0, \quad \alpha_b > 0 \quad , \tag{A.2}$$

has no solution.

<u>Proof</u> Suppose (A.1) has a solution. Insertion of this solution into (A.2) gives a contradiction. Thus (A.2) can have no solution. Conversely, suppose (A.2) has no solution. Then the space X spanned by positive linear combinations of the vectors $\overline{\eta}_b$ with components η_{ba} is convex. Then there exists some vector δ that has positive inner product with every vector of X. This vector solves (A.1), and the lemma is proved.

A slight generalization is

<u>Lemma A'</u> For any sets of real numbers η_{ba} and λ_{ca} the system of equations

$$\sigma_b = \sum_a \eta_{ba} \delta_a \quad , \quad \sigma_b > 0 \quad , \tag{A.3a}$$

$$0 = \sum_a \lambda_{ca} \delta_a \quad , \tag{A.3b}$$

has a solution δ_a if and only if the system of equations

$$\sum_b \alpha_b \eta_{ba} + \sum_c \beta_c \lambda_{ca} = 0 \quad , \quad \alpha_b > 0 \quad , \tag{A.4}$$

has no solution.

<u>Proof</u> If (A.3) has a solution then (A.4) can clearly have none. Conversely, if (A.4) has no solution then the space X of positive

linear combinations of the $\bar{\eta}_b$ must be convex and must contain no vector in the linear space Y spanned by the $\bar{\lambda}_c$. Thus the orthogonal complement $X^{\perp}$ of X must have dimension at least that of Y. Moreover, $X^{\perp}$ cannot be contained in $Y^{\perp}$, for then X would contain vectors in Y. Thus if Y is non null then there must be a nonzero vector that lies in $X^{\perp}$ but not in $Y^{\perp}$. The sum of a multiple of this vector with the vector in X satisfying (A.3a) (found in Lemma A) solves (A.3), and the lemma is proved.

Lemma A$'$ is precisely what is needed to extend the Second and Third Structure Theorems to nonsimple points.

It was mentioned at the beginning of this appendix that the integrations occurring in the definitions of the bubble diagram functions $F^B(K)$ were defined to be along contours displaced infinitesimally from the physical region into the simultaneous analyticity domain of all the occurring functions S_c and S_c^-, provided the real K was such that such a contour exists. The proofs of the structure theorems show that such contours do exist for most real K, that the $F^B(K)$ is analytic at such points, and that $F^B(K)$ continues analytically around the remaining real points K via paths defined by certain rules.

It is reasonable to define the integrations in the way indicated. But if one begins with the idea that S and S^{-1} are operators in a Hilbert space then this rule must be justified. The problem is that macrocausality gives the analytic representation for S_c and S_c^- considered as distributions, rather than as operators.

It is not known whether this representation is valid for operators. However, we now show that the functions F^B considered as products of operators restricted to the space of Schwartz test functions can be defined by performing the integrations along the distorted contours described above.

Let H_p, H_q, and H_k be three Hilbert spaces of square integrable functions of the multidimensional variables p, q, and k respectively. Let $A:H_q \to H_p$ and $B:H_p \to H_k$ be two bounded operators. Let $\varphi(q)$, $\chi(p)$, and $\psi(k)$ be Schwartz test functions of compact support. Suppose for sufficiently small supports we know that

$$(\chi, A\,\varphi) = \lim_{\epsilon \to 0} \int dp\, dq\, \chi^*(p)\, A_\epsilon(p, q)\, \varphi(q)$$

and

$$(B\,\psi, \chi) = \lim_{\epsilon \to 0} \int d\,k\, dp\, \psi^*(k)\, B_\epsilon(k, p)\, \chi(p)$$

where $A_\epsilon(p, q) = A(p + i\epsilon_p, q + i\epsilon_q)$, and $\epsilon = (\epsilon_p, \epsilon_q)$ is a vector of fixed direction lying in a certain open convex cone (which can depend on the small supports of χ and φ), and similarly for $B_\epsilon(k, p)$. The function $A(p + i\epsilon_p, q + i\epsilon_q)$ is supposed to be analytic when p and q are in the supports of χ and φ, respectively, and ϵ is in the cone, and similarly for B.

[The functions A and B have certain energy-momentum delta functions as factors. The analyticity discussed above is for the

the factor that multiplies these delta functions, as described in detail in Refs. 7 and 8. We shall not explicitly write down the delta function factors, but we will use the fact that the conservation laws entail that $A\varphi$ and $B\psi$ have compact supports if φ and ψ do. That is, the region of integration is a compact "cycle"--it has no boundaries. (See Ref. 12)

Consider fixed φ and ψ of small compact supports. Let χ_i be a finite set of Schwartz test functions such that $\sum \chi_i = I$ on the compact p space. Suppose the χ_i can be chosen so that the corresponding domains of analyticity of A and B overlap, in the sense that there is a contour $\mathcal{C}$ defined by $\epsilon(p)$ such that $A\big(p + i\epsilon(p), q\big)$ is in the domain of analyticity corresponding to χ_i and φ whenever p and q are in the supports of χ_i and φ, respectively, and similarly for B. We wish to show that

$$(B\psi, A\varphi) = \int dp\, dq\, dk\, \psi^*(k)\, B\big(k, p + i\epsilon(p)\big) \times A\big(p + i\epsilon(p), q\big)\, \varphi(q) .$$

That is, we wish to show that the operator product $B^\dagger A$, acting between the Schwartz test functions ψ and φ can be represented by an integral over the fixed contour $\mathcal{C}$. The contour $\mathcal{C}$ is displaced by a finite amount from the real axis, but the assumption is that it can be shifted to arbitrarily close to the real region, staying always in the cones of analyticity.

It is sufficient for our purposes to consider only a special class of functions χ_i. These will be functions formed by taking products of functions in the individual variables of p. Furthermore the functions in each individual variable will be unity except at distance less than $\lambda > 0$ from the ends of its supports. The function in the support and at distance less than λ from the left end of the support will be given by the function

$$f_\lambda(x) = e^{-x^{-1/5}} \left(e^{-x^{-1/5}} + e^{-(\lambda-x)^{-1/5}} \right)^{-1}$$

$$= 1 - e^{-(\lambda-x)^{-1/5}} \left(e^{-x^{-1/5}} + e^{-(\lambda-x)^{-1/5}} \right)^{-1} .$$

The right end will be given by the analogous function. The virtue of these functions is first that they are easily combined to give functions that add to unity, and second that they are analytic except at zero and λ, and approach their values at these points exponentially from any direction in the cut (along their support) plane.

Consider now the integral on the right of

$$(\chi_i, A\varphi) = \lim_{\epsilon \to 0} \int dp\ \chi_i(p)\ A(p + i\epsilon_p, q)\ \varphi(q) .$$

Because of the analyticity properties of χ_i one can perform the $\lim \epsilon \to 0$ by, instead of shifting the entire contour down to the real axis, merely extending the contour in the surfaces $\mathrm{Re}\ z = x = 0$ and $x = \lambda$ along the direction of ϵ into $\epsilon = 0$. This follows from a distortion of the multidimension contour.

Macrocausality guarantees that the functions corresponding to A and B grow no faster than some inverse power of $|\epsilon|$ as $\epsilon \to 0$ inside the cone of analyticity. The exponential fall off of X_i at $X = 0$ then guarantees that the limit $\epsilon \to 0$ can actually be taken; one can extend the contour right down to the physical region. At $X = \lambda$ the contour also can be extended to $\epsilon = 0$, for the same reason, provided one combines the parts coming from the two sides of $x = \lambda$. [On one side one has the X_i of the form of $f_\lambda(x)$, while on the other side one has $X_i = 1$. The difference falls of exponentially as $\epsilon \to 0$ on the surface $\mathrm{Re}\; z = \lambda$.]

One observes now that the contributions from these strips at $\mathrm{Re}\; z = x = 0$ and λ are exactly cancelled by the contributions from the neighboring X_i. Thus if one adds contributions from many different neighboring X_i the contour of integration is free to move about in the domain of analyticity except for the parts corresponding to the outer boundary strips associated with $X = 0$ and $X = \lambda$.

That is, in our original form the ϵ were required to be constant over each domain X_i (and generally a different constant for different X_i) but we have now converted this to a single continuous contour $\mathcal{C}$ that varies smoothly over the union of the supports. In our case where the union of the X_i cover the entire compact cycle in p space the contour $\mathcal{C}$ never descends to the real axis, but remains always in the domain of analyticity.

The above results apply equally if all the X_i are replaced by $X_i\, e^{ipu}$. Thus the Fourier transform

$$F(u) = (e^{ipu}, A\varphi)$$

is given by

$$F(u) = \int_{\mathcal{C}} dp\, dq\, e^{iup}\, A(p, q)\, \varphi(q) \quad .$$

Similarly, one has

$$G(-u) = (B\,\psi,\ e^{-ipu})$$

$$= \int_{\mathcal{C}'} dk\, dp\, \psi(k)\, B(k, p)\, e^{-ipu} \quad .$$

Because A and B are bounded operators these Fourier transforms are well defined, and one can write (up to factors of 2π)

$$(B\psi, A\varphi) = \int du\, G(-u)\, F(u) \quad .$$

The integrand in the expressions for F and G are analytic is p. That is, the integration region in p space can be divided into small regions in which local coordinates can be introduced. And, in each region the variables corresponding to conserved energy-momentum are introduced as coordinates and then eliminated by the delta functions, leaving A and B analytic in the remaining (local) coordinates on the contour.

The function $G(-u)\ F(u)$ is infinitely differentiable (because of the compact supports in p space) and it falls off rapidly (faster than any polynomial) in all directions. The rapid fall off is due in part to the infinite differentiability of the $\varphi(q)$ and $\psi(k)$ [which are brought in by the elimination of delta functions] and in part to the analyticity properties of A and B in the remaining (local) coordinates. The A and B are analytic in some common cone C in the local coordinates, and they grow no faster than some inverse power of $|\epsilon|$ on approach to the real physical region. Thus the argument of Chapter IV C.a of Ref. 7 shows that $F(u)$ and $G(u)$ fall off rapidly uniformly in the complement of the polar cone C^+. The boundedness of $F(u)$ and $G(-u)$ follows from the boundedness of A and B Because of the different sign of the arguments of $F(u)$ and $G(-u)$ the intersection of the complements of the two effective polar cones C^+ is empty. Thus $G(-u)\,F(u)$ falls off rapidly in all directions.

This rapid fall off implies that

$$(B\psi, A\varphi) = \lim_{\eta_i \to 0} \int du\, e^{-\sum |u_i| \eta_i}\, G(-u)\, F(u)$$

where the right-hand side is analytic in η_i. Because of the compactness of the p-space region of integration the order of the integrations can be inverted, for sufficiently large η_i. The u integration then gives a sum of products of poles of the form $(p_i - p_i' \pm i\eta_i)^{-1}$. Taking the limit $\eta_i \to 0$ then gives, after some algebra, the desired form. The main point is that as one lets the $\eta_i \to 0$ certain poles cross the fixed contours $\mathcal{C}$ and/or $\mathcal{C}'$ and effectively reduce them to a single contour.

The methods used above can be extended to show the various other properties entailed by the assertion that the analytic representation extends in the natural way from distributions to products of bounded operators considered as distributions. In particular, the result described above carries over to products of many operators, and to the case where the q and k must also be shifted. In this latter case one wants to show that if (for sufficiently small supports of φ and ψ) there is a cone C of analyticity in (q, k) such that for each point in this cone one can find a contour over the internal variables that remains always in the domain of analyticity [and hence that the product of the functions $B^{+}A = H$ is analytic in $(q, k) = z$]. Then $(\psi, H\,\varphi)$ can be represented as

$$\lim_{\eta\to 0} \int H(z + i\eta) \quad \Omega(z)\, dz \quad ,$$

where $\Omega = \psi\,\varphi$, and $i\eta$ is in the cone C. The proof goes precisely as before with H and Ω replacing $B^{+}\psi$ and $A\varphi$. The fall off of $\tilde{\Omega}(u)$ is now due to the infinite differentiability of $\Omega(z)$.

Appendix B. Supplementary Notes

Page 21, line 16

A proof of (3.2') by induction is easy. Suppose each term of (3.2') corresponding to a diagram B' having n nontrivial bubbles gives correctly the sum of the corresponding terms of (3.2). Let B be a diagram with n + 1 nontrivial bubbles. Select from among these a bubble b all incoming lines of which are also incoming lines of B. Let the removal of b from B give B'. Let α be the incoming lines of B' identified with the outgoing lines of b. Consider the various terms t' in (3.2) that sum to give the term of (3.2') corresponding to B'. From each such t' we construct 2m + 1 terms t of (3.2) that correspond to B, where m is the number of columns of t' lying to the right of the first nontrivial bubble b' of B' reached by the incoming lines α of B'. These 2m + 1 terms are constructed by placing b either in one of m columns that lie to the right of b', or in a new column (containing only b) that stands just to the left of any of these columns, or in a new column (containing only b) that stands just to the right of the first column of t'. The m + 1 terms t involving a new column will all have one new minus sign, whereas the m terms not involving a new column will not have an extra minus sign Aside from these signs all the terms are equal, and equal to the operator product of F^b with the $F^{B'}$ corresponding to the particular term t' of (3.2). Thus the sum of the 2m + 1 terms t is just minus one times the operator product of F^b with this $F^{B'}$. Summing over all terms t' of (3.2) corresponding to this B', one obtains all the terms t of (3.2) corresponding to B. Since the same operator $-F^b$ is applied to each term one obtains by induction the term of (3.2') corresponding to B.

An alternative proof of (3.2') that makes use of (3.1) is as follows: Suppose (3.2') has been shown to hold for terms corresponding to bubble diagrams having up to $n - 1$ nontrivial minus bubbles. Substitute (3.2') into the second term of the right-hand side of the equation $R^+ = -R^- - R^+R^-$, and consider the contributions to the right-hand side corresponding to a bubble diagram B_n, where the subscript n indicates the number of nontrivial minus bubbles. The contributions to $-R^+R^-$ correspond to some B_n of the product form $B_j^+B_k^-$ (so that the outgoing lines of B_k^- are identical with the ingoing lines of B_j^+) where B_k^- consists of a column of k nontrivial minus bubbles and of unscattered lines and where $j + k = n$, with j and k no less than 1. Let i be the number of initial bubbles of B_n where an initial bubble is a nontrivial bubble whose incoming lines are all external. All bubbles of B_k^- are initial bubbles.

Suppose at first that B_n does not consists of a single column of nontrivial minus bubbles and unscattered lines. Then all contributions to $-R^- - R^+R^-$ having n nontrivial minus bubbles come from $-R^+R^-$ only and must correspond to bubble diagrams $B_n = B_j^+B_k^-$ where $k = 1,2,\cdots,i$ with $i < n$. There are $2^i - 1$ different ways of constructing B_n all of which give contributions to $-R^+R^-$ having the value $\pm F^{B_n}$. These add up to

$$-F^{B_n}\left[(-1)^{n-1}\binom{i}{1} + (-1)^{n-2}\binom{i}{2} + \cdots + (-1)^{n-i}\binom{i}{i}\right] = (-1)^n F^{B_n} .$$

Suppose next that B_n does consist of a bubble diagram topologically equivalent to a column of n nontrivial minus bubbles and of unscattered lines so that $i = n$. Then the reasoning just given still applies but now the last term in the above sum is missing because $k < n = i$, and also $-R^-$ in $-R^- - R^+R^-$ now gives a contribution $-F^{B_n}$. Since $-F^{B_n}$ is equal to $-(-1)^{n-i}\binom{i}{i}F^{B_n}$ when $i = n$, we get the same answer as before. Thus, expansion (3.2') is verified.

<u>Page 21, last line</u>

As an example of the meaning of topological equivalence consider the bubble diagram of Fig. 4.

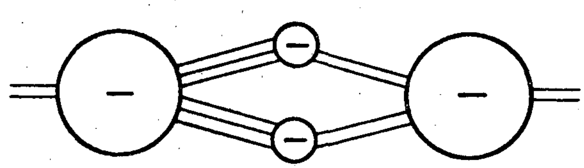


Fig. 4. A bubble diagram B

Certain contributions to F^B will correspond to the case where all the internal lines correspond to the same type of particle. If one simply integrated without respecting the requirement of topological independence then one would get a contribution that would be too large by a factor of 2! 2! 2! 3! 3! . The two 3!'s come from the triples of lines on the left of the two intermediate bubbles. Two of the 2!'s come from the pairs of lines on the right of these bubbles. The other 2! comes from the topological equivalence of the upper and lower intermediate bubbles.

Page 24, last line minus 2

The definitions of equivalent cuts and of left most cuts are illustrated in Fig. 5.

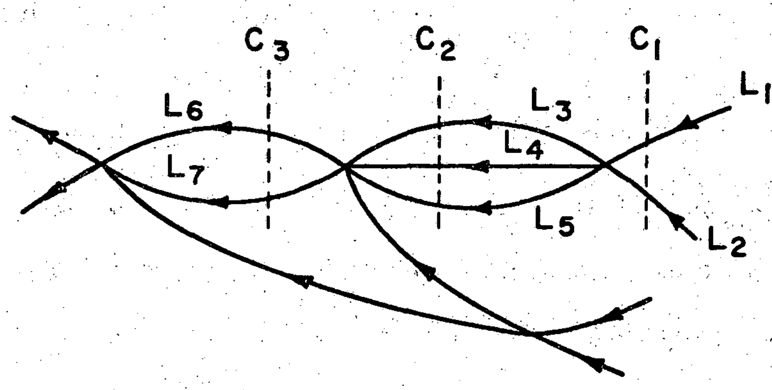


Fig. 5. The cuts $C_1 = (L_1, L_2)$ and $C_2 = (L_3, L_4, L_5)$ are equivalent. C_2 is a leftmost cut. $C_3 = (L_6, L_7)$ is not equivalent to C_1 or C_2.

Page 29, last line

The uniqueness, near the α threshold, of the leftmost cut equivalent to a cut C_α plays an important role in the arguments. At some finite distance above threshold this uniqueness may fail, as the following diagram shows.

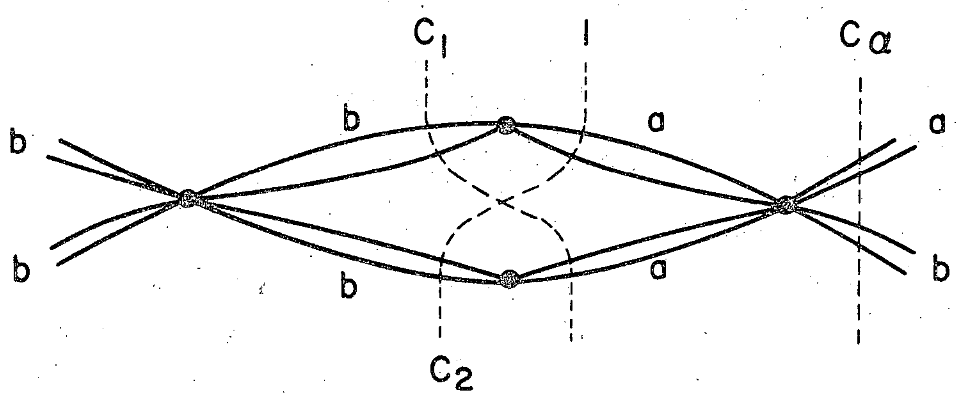


Fig. 6 A diagram with two leftmost cuts equivalent to C_α. We take $M_a < M_b$ Throughout this work it is assumed that the mass values of the stable particles have no accumulation points. It is then easy to see that the leftmost cut is unique in some finite neighborhood of the α threshold.

<u>Page 34, line 2</u>

For any set of leftmost cuts C_α' in B^- corresponding to the sets α of D_β^+ there is a mapping Γ of $D(B^-)$ onto D_β^+. Each such Γ defines a set of parts $\Gamma^{-1}V$ of $D(B^-)$ [and hence of B^-] corresponding to the V of D_β^+. Each such Γ defines, in fact, precisely one way that B^- is realized as a term of T.

An example of a B^- that contains a D_β^+ in two distinct ways is shown in Fig. 7.

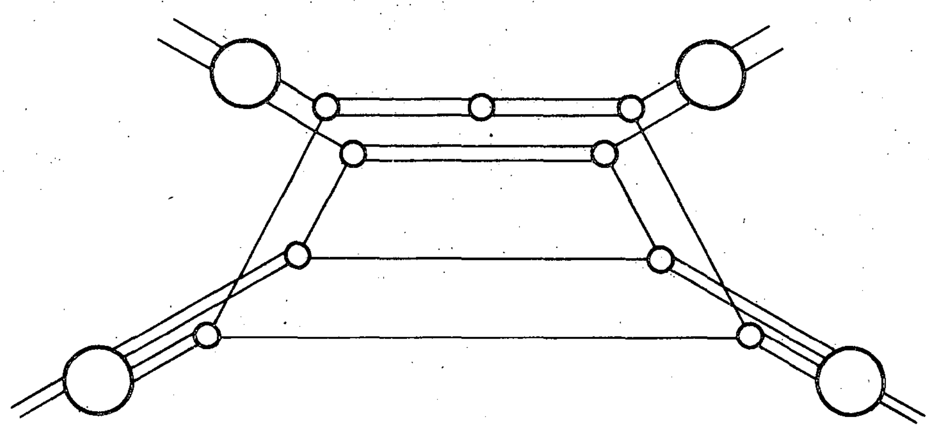


Fig. 7. A bubble diagram B that contains a certain D_β^+ in two essentially different ways. This D_β^+ is shown in Fig. 8.

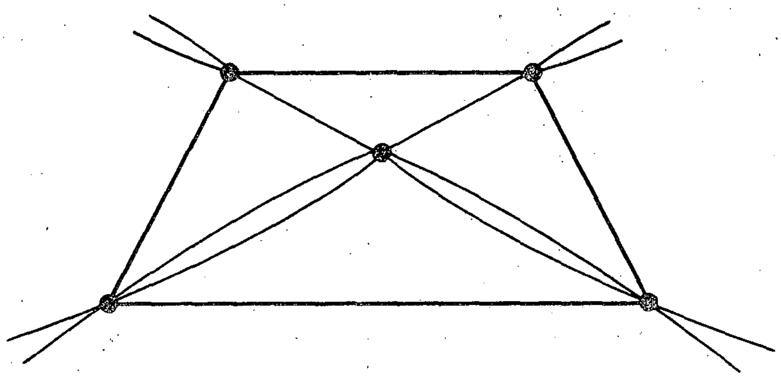


Fig. 8. A D_β^+ that is contained in two essentially different ways in the B of Fig. 7.

Appendix C. Strongly Equivalent Cuts

In this appendix we show that any cut C_α corresponding to α of D_β^+ can be replaced by the leftmost cut C_α' strongly equivalent to it without destroying its correspondence to α of D_β^+.

The condition that B^- contain D_β^+ is equivalent to the condition that there is a continuous mapping $\Gamma : D(B^-) \to D_\beta^+$ that maps $D(B^-)$ onto D_β^+. The external lines of $D(B^-)$ must map onto the external lines of D_β^+ identified with them. The lines of the cuts $C_\alpha = \Gamma^{-1}\alpha$ are in one to one correspondence with the lines of α. The inverse image $\Gamma^{-1}V$ of vertex V of D_β^+ is the part of $D(B^-)$ that corresponds to V.

The point $\bar{P}$ is assumed to satisfy the following conditions:

(1) $\bar{P}$ lies on $L(D_\beta^+)$.

(2) $\bar{P}$ lies on $L(D^+)$ only if D^+ contains D_β^+.

(3) The solution of the Landau equation of D_β^+ at $\bar{P}$ defines momentum-energy vectors p_j such that no line of any set α of D_β^+ has its p_j parallel to that of any line of any other set α of D_β^+. As before, α runs over pairs of vertices of D_β^+, and specifies the set of lines L_j running between that pair of vertices.

We make use of one important kinematic result: If $D(B^-)$ contains D_β^+, then the equations of energy-momentum and mass constraint alone require that if the external lines of $D(B^-)$ have the $\bar{p}_j$'s defined by $\bar{P}$, then the unique values of the p_j's of the lines of $C_\alpha = \Gamma^{-1}\alpha$, subject to the conservation law and mass-shell constraints on these lines, are those defined by the Landau equations of D_β^+ at $\overline{P}$. This result is closely connected to the fact that $L(D_\beta^+)$ lies on the boundary of the physical region of D_β^+, and is proved in the same way.[10,3]

The arguments in the text are purely topological. In this appendix we make use also of the kinematic requirement just described. That is, we shall require that the contribution to the integral corresponding to B^- actually satisfy the energy-momentum conservation laws required at $\overline{P}$. By considering a sufficiently small neighborhood of $\overline{P}$ the internal p_j can be confined to an arbitrarily small neighborhood of the values required at $\overline{P}$. Thus we can consider the p_j of the lines of the various sets C_α to be in a small neighborhood of the values defined by the Landau equations.

At $\overline{P}$ the momentum-energy vectors of the various lines corresponding to any single C_α are all parallel, by virtue of the Landau equations. In some particular Lorentz frame they are all at rest. Consider any C_α' strongly equivalent to C_α. Since C_α' and C_α define the same set of flow lines their total energy momentum is the same. Since the total rest masses are also equal, the lines of C_α' must also all correspond to particles at rest, in this particular frame.

We now prove the following result: If C_α' is strongly equivalent to C_α, and lies left of it, then C_α' lies in $\Gamma^{-1} V$, where V is the vertex of D_β^+ upon which the set α terminates.

Let β label the various outgoing sets of lines of V and let $C_\beta = \Gamma\beta$. The momentum-energy vectors of the lines of C_β are, by assumption, not parallel to those of C_α. Thus no line of C_α' can coincide with any line of any C_β. Thus C_α' must either lie completely within Γ^{-1} V, or there is a part of $D(B^-)$ that consists of a set of paths that begin with certain lines of the sets C_β and end with certain lines of C_α'. Let this part of $D(B^-)$ be called Q. We wish to show that Q is necessarily empty; i.e., that C_α' lies in Γ^{-1} V.

Consider ΓQ, the image of Q in D_β^+. The energy-momentum conservation requirements at $\overline{P}$ can be satisfied only if the lines of ΓQ carry the momentum-energy prescribed by the Landau equations, as already noted. But if the energy-momentum vectors are as prescribed by the Landau equations then the vectors $\alpha_i p_i = \Delta x_i$ can be interpreted as spacetime displacements: these displacements must fit together to give a classical-multiple scattering process. But then the arguments of Ref. 9 immediately rule out the possibility that Q is nonempty. For the initial particles of ΓQ all start at the common vertex V, and they diverge from that point. It is then not possible that they transform by multiple scattering into a set of particles all relatively at rest, without allowing extra particles that come in from outside (i.e., that do not start at V). But interactions with extra incoming particles that do not start at V is incompatible with the condition that C_α' be strongly equivalent to C_α.

Thus ΓQ must be empty and C_α' must therefore lie completely in $\Gamma^{-1} V$.

But if C_α' lies completely in $\Gamma^{-1} V$ then it can be used in place of C_α in making the correspondence of $D(B^-)$ to D_β^+: the topological structure is not altered by replacing C_α by the leftmost cut C_α' that is strongly equivalent to it. This is the result that we need. A slight alternation of the argument shows that C_α can be replaced by any cut strongly equivalent to it without disrupting the correspondence to α of D_β^+.

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