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NOTES ON $\pi\pi$ SCATTERING
II. SUM RULES AND THRESHOLD BEHAVIOR

Joel Yellin

January 2, 1969

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NOTES ON $\pi\pi$ SCATTERING. II.

SUM RULES AND THRESHOLD BEHAVIOR*

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January 2, 1969

These notes are detailed background material for a set of lectures given at Lawrence Radiation Laboratory in the fall of 1968. This is the second of three parts. The first and third parts are contained in UCRL-18637 and UCRL-18665, respectively. The last volume deals mainly with J-plane phenomena, and was prepared in collaboration with D. Sivers.

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IV. $\pi\pi$ SCATTERING ... PCAC AND CURRENT ALGEBRA[1]

We use the notation of Section I, and first summarize the relevant results. In the notation of (1.6) we define the Adler point:

$$x_A = (\mu^2,\mu^2,\mu^2;\ 0,\mu^2,\mu^2,\mu^2) \quad ; \tag{4.1a}$$

the Weinberg point:

$$x_W = (\mu^2,0,\mu^2;\ 0,\mu^2,0,\mu^2) \quad ; \tag{4.1b}$$

the symmetry point:

$$x_S = (4/3\ \mu^2,\ 4/3\ \mu^2,\ 4/3\ \mu^2;\ \mu^2,\ \mu^2,\ \mu^2,\ \mu^2) \quad ; \tag{4.1c}$$

the threshold point:

$$x_T = (4\mu^2,0,0;\ \mu^2,\mu^2,\mu^2,\mu^2) \quad . \tag{4.1d}$$

In terms of the amplitudes of Section I.B., the $I = 0,2$, s wave scattering lengths are given by

$$a_I = -A_I(x_T)/2\mu \quad , \tag{4.2}$$

and the combinations of interest to us will be

$$R = a_0/a_2 \quad , \tag{3.83}$$

and

$$L = \frac{1}{6}\,(2a_0 - 5a_2) \quad . \tag{3.82}$$

The three important relations we will use are:

(A) The Adler consistency condition:

$$A(x_A) = B(x_A) = C(x_A) = 0 \quad . \tag{4.3}$$

(B) The low energy theorem

$$d/d\nu \; A_1^{\,t}(x)\Big|_{x=x_w} = -\frac{1}{8\pi f_\pi^{\,2}} \qquad [\nu \equiv \tfrac{1}{2}(s-u)] \quad . \tag{4.4}$$

(C) The σ model relation

$$A_2^{\,t}(x_w) = 0 \quad . \tag{4.5}$$

Relation (A) follows from PCAC alone, while (B) follows from the $SU(2) \otimes SU(2)$ algebra of axial charges,[2] and (C) follows from the assumption that the commutator

$$[D^i, Q^{A^j}] \propto \delta_{ij} \quad , \tag{4.6}$$

transforms like an isoscalar.[3] [In (4.6), D^i is the isospin component of the divergence of the axial current density $\underset{\sim}{J}_\mu^{\,A}$, and Q^{A^j} is the $\underline{j}$th component of the axial charge, $\underset{\sim}{Q}_A(t) = \int d^3x \; \underset{\sim}{J}_0^{\,A}$.]

IV. A. The Adler $\pi\pi$ Sum Rule[2]

We now turn (4.4) into a sum rule for $A_1^t(x)$. Defining $\nu = \frac{1}{2}(s - u)$, and suppressing for the moment any dependence on p_i^2, we assume $A_1^t(\nu, t)$, which is crossing odd, satisfies the unsubtracted dispersion relation

$$A_1^t(\nu, t) = \frac{2\nu}{\pi} \int_{\nu_0}^{\infty} \frac{d\nu' \, D_1^t(\nu', t)}{\nu'^2 - \nu^2} , \tag{4.10}$$

where the dispersion integral has been folded over in the usual manner, using the odd crossing property of A_1^t, and where D_1^t is the discontinuity in ν of A_1^t.

To turn (4.10) into a sum rule, we note (1.14) says

$$\begin{aligned} A_1^t &= -\frac{1}{6}(5A_2^s + 2A_0^s) + \frac{1}{2} A_1^s \\ &= A(\pi^+\pi^- \to \pi^+\pi^-) - A(\pi^+\pi^+ \to \pi^+\pi^+) . \end{aligned} \tag{4.11}$$

Using (4.2) in (4.1) and going to the s channel threshold x_T,

$$\begin{aligned} &-\frac{1}{6}[5A_2^s(x_T) - 2A_0^s(x_T)] + \frac{1}{2} A_1^s(x_T) \\ &= -2\mu L = \frac{4\mu^2}{\pi} \int_{2\mu^2}^{\infty} \frac{d\nu}{\nu^2 - 4\mu^4}[D^{+-}(\nu, 0) - D^{++}(\nu, 0)] , \end{aligned} \tag{4.12}$$

where $D^{ab} = \mathrm{Disc}_\nu A(\pi^a\pi^b \to \pi^a\pi^b)$, and where we have used the fact that $A_1^s(x_T) = 0$ since it contains p waves or higher, and also taken $\nu = 2\mu^2$ at threshold for physical pions.

Since $t = 0$ at x_T, the unitarity relation (1.27) tells us

$$D(\nu, 0) = -\frac{1}{16\pi} R^{\frac{1}{2}}(s, p_1^2, p_2^2)\sigma_{TOT}(\nu) \quad , \tag{4.13}$$

where

$$R = s^2 + (p_1^2)^2 + (p_2^2)^2 - 2sp_1^2 - 2sp_2^2 - 2p_1^2 p_2^2 \quad . \tag{4.14}$$

For physical pions, $\nu = s - 2\mu^2$ at $t = 0$, and

$$R = s(s - 4\mu^2) = \nu^2 - 4\mu^4 \quad . \tag{4.15}$$

If one initial and one final pion have zero mass - as at x_W - $\nu = s - \mu^2$ at $t = 0$, and

$$R = (s - \mu^2)^2 = \nu^2 \quad . \tag{4.16}$$

If all 4 pions have zero mass, $\nu = s$ at $t = 0$, and

$$R = s^2 = \nu^2 \quad . \tag{4.17}$$

[In the PCAC-current algebra game, <u>internal</u> pions are supposed to keep their physical masses, so $\nu_0 = 2\mu^2$ in (4.10).][2]

Using (4.15) in (4.12), we get[4]

$$L = \frac{\mu}{8\pi^2} \int_{2\mu^2}^{\infty} \frac{d\nu}{(\nu^2 - 4\mu^4)^{\frac{1}{2}}} [\sigma^{+-}(\nu) - \sigma^{++}(\nu)] \quad . \tag{4.18}$$

To use (4.4) we will have to go to x_W, at which point $\nu = 0$, and both sides of (4.1) vanish. Since $A_1^s(x)/\nu\big|_{x=x_W} = 0$, we define

$$\chi = \frac{1}{6\nu}\left[2A_0^{\,s}(x) - 5A_2^{\,s}(x)\right]\Big|_{x=x_W} , \tag{4.19}$$

and using (4.16), (4.12) and (4.18) take the forms

$$\chi = -\frac{1}{8\pi^2}\int_{2\mu^2}^{\infty}\frac{d\nu}{\nu^2}\left(D_+^{\,-}(\nu, 0) - D_+^{\,+}(\nu, 0)\right)$$

$$= \frac{1}{8\pi^2}\int_{4\mu^2}^{\infty}\frac{ds}{s-\mu^2}\left[\sigma_+^{\,-}(s) - \sigma_+^{\,+}(s)\right] , \tag{4.20}$$

where $\sigma_b^{\,a}$ is now the cross section for a zero mass π^a on a physical π^b.

If we assume the error in extrapolating from x_W to x_T is small

$$\chi \cong L/\mu . \tag{4.21}$$

Adler's result, (4.4), gives, using the Goldberger-Treiman relation,

$$-\chi \cong \left(\frac{g_r^{\,2}}{4\pi}\right)\left(\frac{\mu}{M}\right)^2 \frac{1}{2g_A^{\,2}\mu^2} \cong \frac{15}{50}\cdot\frac{1}{3\mu^2} \cong \frac{1}{10\mu^2} , \tag{4.22}$$

where M is the nucleon mass, g_r is the renormalized πNN coupling constant, and g_A is the $q^2 = 0$ limit of the axial current form factor which multiplies $\gamma_\mu\gamma_5$ in $\langle N|J_\mu^{\,A}|N\rangle$.

If one tries to saturate (4.18) with the $\rho(1^-)$ and an $\epsilon(0^+)$ state, one gets[5]

$$\frac{1}{8\pi f_\pi^2} \cong -\chi = \frac{g_{\rho\pi\pi}^2}{4\pi\, m_\rho^2} + \frac{g_{\epsilon\pi\pi}^2}{16\pi} , \tag{4.23}$$

where the ρ and ϵ widths are

$$\Gamma_{\rho\pi\pi} = \left(\frac{g_{\rho\pi\pi}^2}{4\pi}\right)\left(\frac{m_\rho}{12}\right)\left[1 - \frac{4\mu^2}{m_\rho^2}\right]^{3/2} \tag{4.24}$$

and,

$$\Gamma_{\epsilon\pi\pi} = \left(\frac{g_{\epsilon\pi\pi}^2}{4\pi}\right) \cdot \frac{3m_\epsilon^3}{32} \left[1 - \frac{4\mu^2}{m_\epsilon^2}\right]^{\frac{1}{2}} . \tag{4.25}$$

IV. B. The I = 2 Superconvergent Sum Rule

We want now to write a sum rule for $A_2^t(x)$, which is crossing even. The analog of (4.10) is then

$$A_2^t(\nu, t) = \frac{2}{\pi}\int_{\nu_0}^{\infty} \frac{\nu' d\nu'}{\nu'^2 - \nu^2} D_2^t(\nu', t) \quad . \tag{4.30}$$

For large ν, we will assume

$$A_2^t(\nu, 0) \xrightarrow[\nu\to\infty]{} c\nu^w \quad , \tag{4.31}$$

where c is a constant and where $w < -2$. Inserting (4.31) in (4.30),

$$\lim_{\nu\to\infty} \nu^{-w} A_2^t(\nu, 0) = c = -\frac{2}{\pi}\nu^{-w-2}\int_{\nu_0}^{\infty} \nu d\nu\, D_2^t(\nu, 0) \quad , \tag{4.32}$$

so that $c = 0$ for $w < -2$, and we have the (superconvergent) sum rule

$$\int_{\nu_0}^{\infty} \nu d\nu\, D_2^t(\nu, 0) = 0 \quad . \tag{4.33}$$

If we saturate (4.33) with ρ and ϵ, we get

$$(1 - \mu^2/m_\rho^2)4g_{\rho\pi\pi}^2 - (m_\epsilon^2 - \mu^2)g_{\epsilon\pi\pi}^2 = 0 \quad . \tag{4.34}$$

Experimentally,[6]

$$\frac{g_{\rho\pi\pi}^{2}}{4\pi m_{\rho}^{2}} \approx + \frac{1}{16\pi f_{\pi}^{2}} \quad , \tag{4.35}$$

so that (4.23) becomes

$$g_{\rho\pi\pi}^{2}/m_{\rho}^{2} \approx g_{\epsilon\pi\pi}^{2}/4 \quad . \tag{4.36}$$

Using (4.36) in (4.34), and neglecting μ^2,

$$m_{\rho}^{2} \approx m_{\epsilon}^{2} \quad . \tag{4.37}$$

The last two relations, on insertion into (4.24) and (4.25) give

$$\Gamma_{\rho\pi\pi}/\Gamma_{\epsilon\pi\pi} \approx 2/9 \quad . \tag{4.38}$$

All this is equivalent to the approximate statement

$$D_2^{t}(\nu, 0) \propto \left(P_0(z_s) - P_1(z_s)\right) \quad , \tag{4.39}$$

(times Im part of Breit-Wigner form).

We will compare (4.39) with the $I_t = 2$ discontinuity in the Veneziano model for $\pi\pi$ scattering, below.

IV.C. Scattering Length Relations

We have already mentioned the current algebra low energy theorem

$$L = \frac{1}{6}(2a_0 - 5a_2) \cong \frac{1}{10\mu} . \tag{4.50}$$

In this section we will focus our attention on the quantity $R = a_0/a_2$. Following Weinberg,[3] we expand the amplitude to first order, about $x = 0$,

$$A(x) \cong a + b(t + u) + cs , \tag{4.51a}$$

$$B(x) \cong a + b(s + u) + ct , \tag{4.51b}$$

$$C(x) \cong a + b(s + t) + cu . \tag{4.51c}$$

Linear terms in the p_i^2 cannot appear here because of the symmetry properties of the amplitude and the relation (1.7).

Using the three relations (4.3, 4.4, 4.5) we can immediately compute R. Equations (4.3) - (4.5) give, using (4.51),

$$a + 2\mu^2 b + c = 0 , \tag{4.52a}$$

$$2(b - c) = -\frac{1}{8\pi f_\pi^2} = \chi , \tag{4.52b}$$

$$a + \mu^2(b + c) = 0 . \tag{4.53c}$$

The solution of these equations is

$$b = 0; \quad a = -c\mu^2 = -\mu^2/8\pi f_\pi^2 \tag{4.54}$$

and

$$R = \frac{3A(x_T) + B(x_T) + C(x_T)}{B(x_T) + C(x_T)} = \frac{5a + 12\mu^2 c + 8\mu^2 b}{2a + 8\mu^2 b} = -\frac{7}{2} , \tag{4.55}$$

so that (4.50) now gives

$$a_0 \cong \frac{7}{4} L \cong \frac{7}{40\mu} ; \qquad a_2 \cong -\frac{1}{20\mu} . \tag{4.56}$$

V. SUM RULES IN THE VENEZIANO MODEL

We now return to the model in which the t channel amplitude can be written [cf (2.5)],

$$\mathbf{x}^t \equiv \begin{pmatrix} A_0{}^t \\ A_1{}^t \\ A_2{}^t \end{pmatrix} = g \begin{pmatrix} -\frac{1}{2}F_0[\alpha(s),\alpha(u)] + \frac{3}{2}F_0[\alpha(t),\alpha(s)] + \frac{3}{2}F_0[\alpha(t),\alpha(u)] \\ F_0[\alpha(t),\alpha(u)] - F_0[\alpha(t),\alpha(s)] \\ F_0[\alpha(s),\alpha(u)] \end{pmatrix} \tag{5.1}$$

where, as before,

$$F_0(x,y) = \frac{\Gamma(1-x)\,\Gamma(1-y)}{\Gamma(1-x-y)} .$$

We introduce the variables

$$\tau = 1 - x - y , \tag{5.2a}$$

$$\eta = \frac{1}{2}(x - y) . \tag{5.2b}$$

If $a = 1/2$, $b = 1\ \mathrm{GeV}^{-2}$, $\mu = 0$; $\tau = t$, and $\eta = \nu = \frac{1}{2}(s - u)$, in $F_0[\alpha(s),\alpha(u)]$ Fig. 5.1 shows the Mandelstam structure of $F_0[\alpha(s),\alpha(u)]$ in this event.

We will formulate sum rules at fixed τ, and we therefore are interested here in the discontinuity in η. To complete our set of variables, we introduce w, where

$$w + x + y = \frac{3}{2}; \quad w - \tau = \frac{1}{2} . \tag{5.3}$$

As will be shown in Part III, $F_0(x,y)$ can be expanded in partial fractions as follows:[7]

$$F_0(x,y) = \sum_{K=1}^{\infty} \frac{(-1)^K \Gamma(K + \tau)}{\Gamma(K)\,\Gamma(\tau)} \left[\frac{1}{\eta + \frac{1}{2}(1 - \tau) - K} + \frac{1}{-\eta + \frac{1}{2}(1 - \tau) - K} \right] , \tag{5.4}$$

$$F_0[w,y] \pm F_0[w,x] = \sum_{K=1}^{\infty} \frac{\Gamma(K + \tau + \frac{1}{2})}{\Gamma(K)\,\Gamma(\tau + \frac{1}{2})} \left[\frac{1}{y - K} \pm \frac{1}{x - K} \right] = \sum_{K=1}^{\infty} \frac{\Gamma(K + \tau + \frac{1}{2})}{\Gamma(K)\,\Gamma(\tau + \frac{1}{2})} \left[\frac{1}{-\eta + \frac{1}{2}(1 - \tau) - K} \pm \frac{1}{+\eta + \frac{1}{2}(1 - \tau) - K} \right] . \tag{5.5}$$

V.A. I = 2 Sum Rules

Referring to Fig. 3.2 we see that $F_0[x,y] \to 0$ faster than any power, so according to Sec. IV.B. A_2^t, in the model, superconverges.

Up to factors of π, the discontinuity arising from A_2^t can be read off from (5.4) and (5.1), and

$$D_2^t(\eta,\tau) = \sum_{K=1}^{\infty} (-1)^K B^{-1}(K+\tau) \left[\delta(\eta + \tfrac{1}{2}(1-\tau) - K) - \delta(-\eta + \tfrac{1}{2}(1-\tau) - K)\right] . \quad (5.6)$$

Defining $z_s = 1 + \dfrac{2\tau}{x - \frac{1}{2}}$, $z_u = 1 + \dfrac{2\tau}{y - \frac{1}{2}}$, we have

$$D_2^t(\eta,\tau) = \tfrac{1}{4}[P_0(z_s) - P_1(z_s)]\,\delta(\eta - \tfrac{1}{2}) + \tfrac{3}{8}[P_2(z_s) - P_1(z_s)]\delta(\eta - \tfrac{3}{2}) + \cdots + \{z_s \to z_u,\ \eta \to -\eta\}\cdots \quad (5.7)$$

in agreement with (4.39). Consulting (5.6) or Fig. 5.1 we see that, in the $\mu = 0$, $a = \frac{1}{2}$, $b = 1\ \mathrm{GeV}^{-2}$ case, A_2^t has zeros along $t = 0,-1,-2\cdots$ etc. The zero along $t = 0$ has been associated with the zero of Adler's consistency condition (4.3).[11] It forces the $P_0 - P_1$ combination in (5.7). Along $\tau = 0$, $D_2^t = 0$, and the superconvergence is accomplished by having each degenerate set of states at a particular mass cancel among themselves in the $I = 2$ sum rule. Let us check

and see that (5.6) reproduces Table I.1. Since there are no $I = 2$ poles, $D_2^{\,t}$ crosses into

$$D_2^{\,t} = \frac{1}{3} D_0^{\,s} - \frac{1}{2} D_1^{\,s}$$

$$= \frac{1}{3} \sum_{J=0}^{\infty} (2J + 1)\, b_0(J,\nu)\, P_J(z_s) \cdot \frac{1}{2}(1 + (-1)^J)$$

$$- \frac{1}{2} \sum_{J=1}^{\infty} (2J + 1)\, b_1(J,\nu)\, P_J(z_s)\, \frac{1}{2}(1 - (-1)^J) \quad , \tag{5.8}$$

where the u, left-hand discontinuity, has been suppressed. Assuming $D_2^{\,t}\big|_{t=0} = 0$, and that we have degenerate towers, as in the model, at the lowest mass tower,

$$\frac{1}{3}\, b_0(0,\nu_1) - \frac{3}{2}\, b_1(1,\, \nu_1) = 0 \quad , \tag{5.9}$$

while at the next mass, where spins 0,1,2 are possible

$$\frac{1}{3}\, b_0(0,\nu_2) + \frac{5}{3}\, b_0(2,\nu_2) - \frac{3}{2}\, b_1(1,\nu_2) = 0 \quad . \tag{5.10}$$

Equation (5.9) gives the 9/2 ratio for $\Gamma_{\epsilon\pi\pi}/\Gamma_{\rho\pi\pi}$. At $x = 2$, the extra zero at $\tau = -1$, implies $\Gamma_{\epsilon'\pi\pi} = b_0(0,\nu_2) = 0$, so that $b_0(2,\nu_2)/b_1(1,\nu_2) = 10/9 = \Gamma_{f\pi\pi}/\Gamma_{\rho'\pi\pi}$ as in Table I.1. The coefficients of the various Legendre polynomials up to $x = N = 4$, and for the $\mu = 0$, $a = 1/2$, $b = 1\ \mathrm{GeV}^{-2}$ case, are shown in Table 5.1.

Table 5.1. Coefficients of the Legendre polynomials contained in the model amplitude, normalized to $L = N = 1$. For $N = 4$, $c = 4^{-4}/\Gamma(4)$.

L \ N	0	1	2	3	4
4					$c \cdot 7^3 \cdot 8/5$
3				$\frac{25}{16}$	$c \cdot 7^3 \cdot 8/5$
2			$\frac{3}{2}$	$\frac{25}{16}$	$c \cdot 112/3$
1		$\underline{1}$	$\frac{3}{2}$	$\frac{5}{16}$	$c \cdot 208/5$
0	0	$\underline{1}$	0	$\frac{5}{16}$	$c \cdot 64/15$

$I_t = 2$ Sum Rules for $\tau < 0$

As τ gets negative, poles in $F_0[x,y]$ begin to move out into the unphysical, double spectral region, as shown in Fig. 5.1, and the s and u poles cross.

We can check that each sum in (5.6) still separately superconverges. For simplicity, consider $\tau = -N$, $(N = 1,2,\cdots)$ and take any odd moment. Then we should have

$$\int d\eta \sum_{J=1}^{\infty} \eta^{2P+1} (-1)^J B^{-1}(-N,J)\, \delta(\eta + \tfrac{1}{2} - J - \tfrac{N}{2}) = 0 \quad . \tag{5.11}$$

Because we have chosen $\tau = -N$, the sum truncates at N, and changing variables, (5.11) becomes

$$\sum_{Q=-\frac{1}{2}(N-1)}^{\frac{1}{2}(N-1)} \frac{Q^{2P+1}\, \Gamma(N+1)}{\Gamma(\frac{1}{2}(N+1) - Q)\, \Gamma(\frac{1}{2}(N+1) + Q)} = 0 \quad , \tag{5.12}$$

showing the cancellation explicitly.

$I_t = 2$ Sum Rules for $\tau > 0$. FESR.[8]

We now consider the lowest moment finite energy sum rule on the right hand discontinuity in (5.6),

$$\frac{1}{2} \int_{-U}^{+U} \eta d\eta \sum_{K=1}^{\infty} (-1)^K B^{-1}(K,\tau) \cdot \delta(\eta + \tfrac{1}{2} - K - \tfrac{\tau}{2}) = 0 \quad . \tag{5.13}$$

We choose U so that the highest mass pole included has $K = N$. $(-\frac{1}{2} + N + \frac{\tau}{2} \leqslant U \leqslant -\frac{1}{2} + N + 1 + \frac{\tau}{2}$.) The left-hand side of (5.13) then becomes

$$\frac{1}{2} \sum_{K=1}^{N} (-1)^K B^{-1}(\tau,K)(2K - 1 + \tau)$$

$$= (-1)^N B^{-1}(\tau,N) \frac{1}{2}(N + \tau)$$

$$= \frac{1}{2}(-1)^N T_{N+1}(\tau)/\Gamma(N) \quad . \tag{5.14}$$

Equation (5.14) can be easily proved by induction. The sum changes sign and grows in absolute value as each new resonance is included, so that there are violent cancellations. If we give a finite width to the resonances, we can always find a point intermediate between any pair of neighboring resonances such that the sum vanishes. This remains true for all the moments.

V.B. The $I_t = 1,0$ Sum Rules

From (5.1) and (5.5), the $I_t = 1$ model amplitude is

$$\frac{\Gamma(\frac{1}{2} - \eta + \frac{\tau}{2})\,\Gamma(\frac{1}{2} - \tau)}{\Gamma(-\eta - \frac{\tau}{2})} - \frac{\Gamma(\frac{1}{2} + \eta + \frac{\tau}{2})\,\Gamma(\frac{1}{2} - \tau)}{\Gamma(\eta - \frac{\tau}{2})}$$

$$= \sum_{K=1}^{\infty} \frac{\Gamma(K + \tau + \frac{1}{2})}{\Gamma(\tau + \frac{1}{2})\,\Gamma(K)} \left\{ \frac{1}{-\eta + \frac{1}{2}(1 - \tau) - K} - \frac{1}{\eta + \frac{1}{2}(1 - \tau) - K} \right\} . \tag{5.15}$$

For $\tau = 0$, (5.15) gives

$$\frac{\Gamma(\frac{1}{2} - \eta)\,\Gamma(\frac{1}{2})}{\Gamma(-\eta)} - \frac{\Gamma(\frac{1}{2} + \eta)}{\Gamma(\eta)}\,\Gamma(\tfrac{1}{2})$$

$$= \Gamma(\tfrac{1}{2}) \cdot \eta \left[- \frac{\Gamma(\frac{1}{2} - \eta)}{\Gamma(1 - \eta)} - \frac{\Gamma(\frac{1}{2} + \eta)}{\Gamma(1 + \eta)} \right]$$

$$= \sum_{K=1}^{\infty} \frac{\Gamma(K + \frac{1}{2})}{\Gamma(K)\,\Gamma(\frac{1}{2})} \left\{ \frac{-2\eta}{(K - \frac{1}{2})^2 - \eta^2} \right\} . \tag{5.16}$$

At $\eta = 0$, this gives the Adler $\pi\pi$ sum rule for zero mass pions

$$\pi = \sum_{K=1}^{\infty} \frac{\Gamma(K - \frac{1}{2})}{\Gamma(K)\,\Gamma(\frac{1}{2})(K - \frac{1}{2})} = 2 + \frac{1}{3} + \frac{3}{20} + \frac{5}{56} + \cdots . \tag{5.17}$$

From (5.17) we see that in this model the sum rule is saturated 64% by (ρ,ϵ); 11% by (f, ρ', ϵ'); 5% by the g family, etc.

Let us find out how sensitive L is to changes in the pion mass, μ, and the intercept a. We define $\delta = a - \frac{1}{2}$, $\lambda = 4\mu^2 b$, and expand (5.15) about $\lambda = \delta = 0$, at x_T. We have

$$\frac{L}{2\mu b g_\pi} = G(\lambda,\delta) = -\frac{\Gamma(\frac{1}{2} - \delta)}{\pi} \left\{ \frac{\Gamma(\frac{1}{2} - \delta - \lambda)}{\Gamma(-\delta - \lambda)} - \frac{\Gamma(\frac{1}{2} - \lambda - \delta)}{\Gamma(-\delta)} \right\}$$

$$= \frac{\Gamma(\frac{1}{2} - \delta)}{\pi} \left\{ \frac{\sin \pi\lambda}{\pi\lambda} \cos \pi\delta\, \Gamma(1 + \lambda + \delta)\, \Gamma(\tfrac{1}{2} - \lambda - \delta) \right.$$

$$\left. + \frac{\sin \pi\delta}{\pi\lambda} \left[\cos \pi\lambda\, \Gamma(1 + \lambda + \delta)\, \Gamma(\tfrac{1}{2} - \lambda - \delta) - \Gamma(\tfrac{1}{2} - \delta)\, \Gamma(1 + \delta) \right] \right\} . \tag{5.18}$$

The polynomial approximation for $\Gamma(1 + z)$ gives

$$\Gamma(1 + z) \cong 1 - a_1 z , \tag{5.19}$$

with $a_1 \cong +0.57$.

Using the duplication formula

$$\Gamma(2z) = \frac{2^{2z-1}}{(\pi)^{\frac{1}{2}}} \Gamma(z)\, \Gamma(z + \tfrac{1}{2}) , \tag{5.20}$$

we get

$$\frac{\Gamma(\frac{1}{2} - z)}{(\pi)^{\frac{1}{2}}} \cong 1 + z(\ln 4 - a_1) . \tag{5.21}$$

Finally

$$G(\lambda,\delta) = 1 + \delta(3\ \ell n\ 4 + a_1) + \lambda\ \ell n\ 4 + \cdots$$

$$\cong 1 + 4.72\delta + 1.39\lambda + \text{quadratic terms} \cdots \quad . \tag{5.22}$$

To first order in δ and λ

$$\mu L \cong \frac{\pi\lambda g}{2}\,[1 + 4.72\delta + 1.39\lambda] \quad . \tag{5.23}$$

For $b = 1/50\mu^2 \cong 1\ \text{GeV}^{-2}$ and $g \cong 1$, $\mu L \cong \pi/25 \cong 0.125$.

<u>$I_t = 1$ Sum Rules For $\tau > 0$. FESR.</u>

If we go to positive τ, the sums in (5.15) diverge because of the t channel poles at $\tau = 1/2, 3/2, \cdots$. The discontinuity in η is, from (5.15)

$$D_1{}^t = \sum_{J=1}^{\infty} \frac{\Gamma(\frac{1}{2} + \tau + J)}{\Gamma(J)\ \Gamma(\frac{1}{2} + \tau)} \left[\delta(\eta + \tfrac{1}{2} - \tfrac{\tau}{2} - J) + \delta(-\eta + \tfrac{1}{2} - \tfrac{\tau}{2} - J)\right] \quad . \tag{5.24}$$

Just as for the $I = 2$ case, we take U such that $N + \frac{\tau}{2} - \frac{1}{2} < U < N + 1 + \frac{\tau}{2} - \frac{1}{2}$, and for the zeroth moment finite energy sum rule we have

$$\frac{1}{2}\int_{-U}^{+U} d\eta\ D_1{}^t(\eta,\tau) = \sum_{J=1}^{N} \frac{\Gamma(\frac{1}{2} + \tau + J)}{\Gamma(J)\ \Gamma(\frac{1}{2} + \tau)} = \frac{T_{N+1}(\tau + \frac{1}{2})}{\Gamma(N)(\tau + \frac{3}{2})} \quad , \tag{5.25}$$

which one easily can prove by induction. If we expand in powers of N, using (3.15), we have

$$\frac{1}{2}\int_{-U}^{+U} d\eta\, D_1^{\,t}(\eta,\tau) = \frac{N^{\frac{3}{2}+\tau}}{(\tau+\frac{3}{2})\,\Gamma(\tau+\frac{1}{2})}\left[1+\frac{(\frac{3}{2}+\tau)(\frac{1}{2}+\tau)}{2N}+O(N^{-2})\right], \tag{5.26}$$

or, inserting $\alpha(t) = \tau + \frac{1}{2}$, the right-hand side takes the familar FESR form[8]

$$\frac{N^{\alpha(t)+1}}{[\alpha(t)+1]\,\Gamma[\alpha(t)]}\left[1+\frac{[\alpha(t)+1]\,\alpha(t)}{2N}+\cdots\right] . \tag{5.27}$$

At $\alpha(t) = 1$ (i.e., at $t = m_\rho^{\,2}$) this becomes

$$\frac{N^2}{2}\left[1+\frac{1}{N}+\cdots\right] , \tag{5.28}$$

so that we commit a 50% error if we choose to keep the leading trajectory only on the right-hand side of the FESR, and take $N = 2$. (Meaning we keep the ρ,f families on the left.) Let us see what happens on the left if we keep only ρ and f. Rewriting the first term in (5.24) as Legendre polynomials in z_s we have

$$\sum_{J=1}^{\infty} \frac{\Gamma(\frac{1}{2} + \tau + J)}{\Gamma(J)\ \Gamma(\frac{1}{2} + \tau)}\ \delta(\eta + \frac{1}{2} - \frac{\tau}{2} - J)$$

$$= (\tfrac{1}{2} + \tau)\ \delta(\eta - \tfrac{1}{2} - \tfrac{\tau}{2}) + (\tfrac{1}{2} + \tau)(\tfrac{3}{2} + \tau)\ \delta(\eta - \tfrac{3}{2} - \tfrac{\tau}{2}) + \cdots$$

$$= \tfrac{1}{4}[P_o(z) + P_1(z)]\ \delta(\eta - \tfrac{1}{2} - \tfrac{\tau}{2})$$

$$+ \tfrac{3}{8}[P_2(z) + P_1(z)]\ \delta(\eta - \tfrac{3}{2} - \tfrac{\tau}{2}) + \cdots \quad , \qquad (5.29)$$

so that the resonances cancel in the backward direction, as they should. At $\tau = \frac{1}{2}$, the ρ and f contributions to the left-hand side of the FESR are, from (5.29), using $z = 1 + \dfrac{2\tau}{x - \frac{1}{2}}$,

$$\frac{1}{4} \cdot 3 + \frac{3}{8} \cdot \frac{11}{3} = \frac{17}{8} \quad , \qquad (5.30)$$

while the ϵ and ρ' contribute

$$\frac{1}{4} \cdot 1 + \frac{3}{8} \cdot \frac{5}{3} = \frac{7}{8} \quad , \qquad (5.31)$$

making a total of 3, which checks with (5.36).

Therefore, while the exact relation reads $3 = 3$, the FESR at $t = m_\rho^{\ 2}$ reads $\frac{17}{8} \cong 2$, since compensating errors have been made.

The $I_t = 0$ sum rule, which is suspect in any case because we have neglected the Pomeranchon, contains the oscillating object already

associated with the $I = 2$ sum rule. The same calculation as was performed here for the $I = 1$ case can be done for $I = 0$, and is left as an exercise for the enterprising reader.

VI. SOME REMARKS ON THE UNIQUENESS OF THE VENEZIANO AMPLITUDE

The general requirements one would like to make on the function $F(x,y)$ do not specify it uniquely. If we ask that: (i) $F(x,y) = F(y,x)$; (ii) $F(x,y)$ has simple poles only at x and/or $y = 0,1,2,\cdots$; (iii) $F(x,y)$ has Regge asymptotic behavior in the average sense discussed above; we get an infinite class of functions. One such, though not necessarily the most general one, is[9]

$$F(x,y) = \sum_{m,n,p=0}^{\infty} C_p^{mn} \frac{\Gamma(m+p-x)\,\Gamma(n+p-y)}{\Gamma(m+n+p-x-y)} , \quad (6.1)$$

where the constants C_p^{mn} satisfy $C_p^{mn} = C_p^{nm}$.

Now we can make various general requirements on (6.1). If we want no ghosts, for example, we kill the residue at $x = 0$ by insisting

$$\gamma(0,y) = \sum_{m=0}^{\infty} C_0^{0n} = 0 . \quad (6.2)$$

In (6.1) we can choose any finite number of couplings arbitrarily.

Mandelstam[10] has pointed out that the alternate odd trajectories in (6.1) may be killed off if one likes, by a proper choice of the C_p^{mn}. However none of the even trajectories - 2nd, 4th, etc. below the leading one - may be eliminated.

Mandelstam's solution is

$$C_p^{mn} = \frac{\delta_{p0}\,\delta^{mn}\,T_m[-\frac{1}{2}(\lambda + 3a + 1)]}{\Gamma(m + 1)} , \tag{6.3}$$

and the new function in closed form is related to the hypergeometric function ${}_3F_2$, by

$$F(x,y) = B(-x, -y)\ {}_3F_2[-x, -y, -\tfrac{1}{2}(\lambda + 3a + 1), + \tfrac{1}{2}(1 - x - y); \tfrac{1}{4}] . \tag{6.4}$$

We can see fairly easily that trajectory number two cannot be eliminated from (6.1), by taking the asymptotic limit in $\cos\Theta = z$ and observing that the y dependence cannot be eliminated.

In the s channel, define, following Mandelstam,[10]

$$w = x + \frac{y}{2} - \frac{1}{2}(3a + \lambda) , \tag{6.5}$$

where $x = \alpha(s)$, $y = \alpha(t)$, $w = \frac{1}{2} z_t(t - 4\mu^2)b$. Taking $w \to \infty$ in (6.1) and using (3.15, 3.16) we have, defining $\xi = \frac{1}{2}(\lambda + 3a + 2)$

$$\lim_{w\to\infty} F(x,y) = \sum_{m,n,p=0}^{\infty} \sum_{k=0}^{\infty} C_p^{mn}\, \Gamma(n+p-y)\, (-w)^{y-n-k}$$

$$\cdot\; c_k\left(y-n,\; m+n+p-\frac{y}{2}+\xi\right)$$

$$= \sum_{k'=0}^{\infty} \sum_{m,p=0}^{\infty} \sum_{n=0}^{k'} (-w)^{y-k'}\, C_p^{mn}\, \Gamma(n+p-y)$$

$$c_{k'-n}\left(y-n,\; m+n+p-\frac{y}{2}+\xi\right) \quad . \qquad (6.6)$$

In (6.6) we have interchanged summation orders in order to isolate powers of w. For k' even, the y dependence can never be eliminated so that no cancellations are possible, while for k' odd the y dependence in the c_k's cancels out that of the Γ functions, and Mandelstam's solution[10] (6.4) can then be chosen.

FOOTNOTES AND REFERENCES

* This work was supported in part by the U.S. Atomic Energy Commission.

1. We follow here the notation of N. N. Khuri, Phys. Rev. 153, 1477 (1967).

2. S. L. Adler, Phys. Rev. 137, 1022 (1968).

3. S. Weinberg, Phys. Rev. Letters 17, 616 (1966). See also Ref. 1.

4. This approach to the sum rule parallels that of M. A. B. Beg, Phys. Rev. Letters 17, 333 (1966) who considered low energy theorems for Compton scattering.

5. F. J. Gilman and H. Harari, Phys. Rev. 165, 1803 (1968). This paper gives a comprehensive discussion of sum rules for pion-hadron scattering.

6. Equation (4.35) is sometimes called the Kawarabayashi-Suzuki relation. See M. Suzuki and K. Kawarabayashi, Phys. Rev. Letters 16, 255, 384(E) (1966); F. Gilman and H. Schnitzer, Phys. Rev. 150, 1362 (1966); J. J. Sakurai, Phys. Rev. Letters 17, 552 (1966).

7. In general, just as for functions of one complex variable, additional terms which do not affect the discontinuity may appear, accompanying such an expression. These terms can be shown to be absent here.

8. See C. Schmid and J. Yellin, Finite Energy Sum Rules and the Process $0^- + 0^- \rightarrow 0^- + 0^-$, Lawrence Radiation Laboratory Report UCRL-18625 (1968) (to be published), for a detailed discussion of the FESR in connection with 0^-0^- scattering. Some additional points are discussed by C. Schmid, Phys. Rev. Letters 20, 628 (1968).

9. J. Mandula, Caltech preprint CALT-68-178, unpublished.

10. S. Mandelstam, Phys. Rev. Letters 21, 1724 (1968).

11. C. Lovelace, Phys. Letters 28B, 265 (1968); M. Ademollo, G. Veneziano, and S. Weinberg, Massachusetts Institute of Technology preprint (1968), unpublished. We wish to thank Dr. Veneziano for sending his work to us, and for a helpful discussion in connection with the PCAC zero.

FIGURE CAPTION

Fig. 5.1. Poles and zeros of $F[\alpha(s), \alpha(u)]$, the $I_t = 2$ amplitude, for $a = \frac{1}{2}$, $b = 1\ \mathrm{GeV}^{-2}$, $m_\pi = 0$.

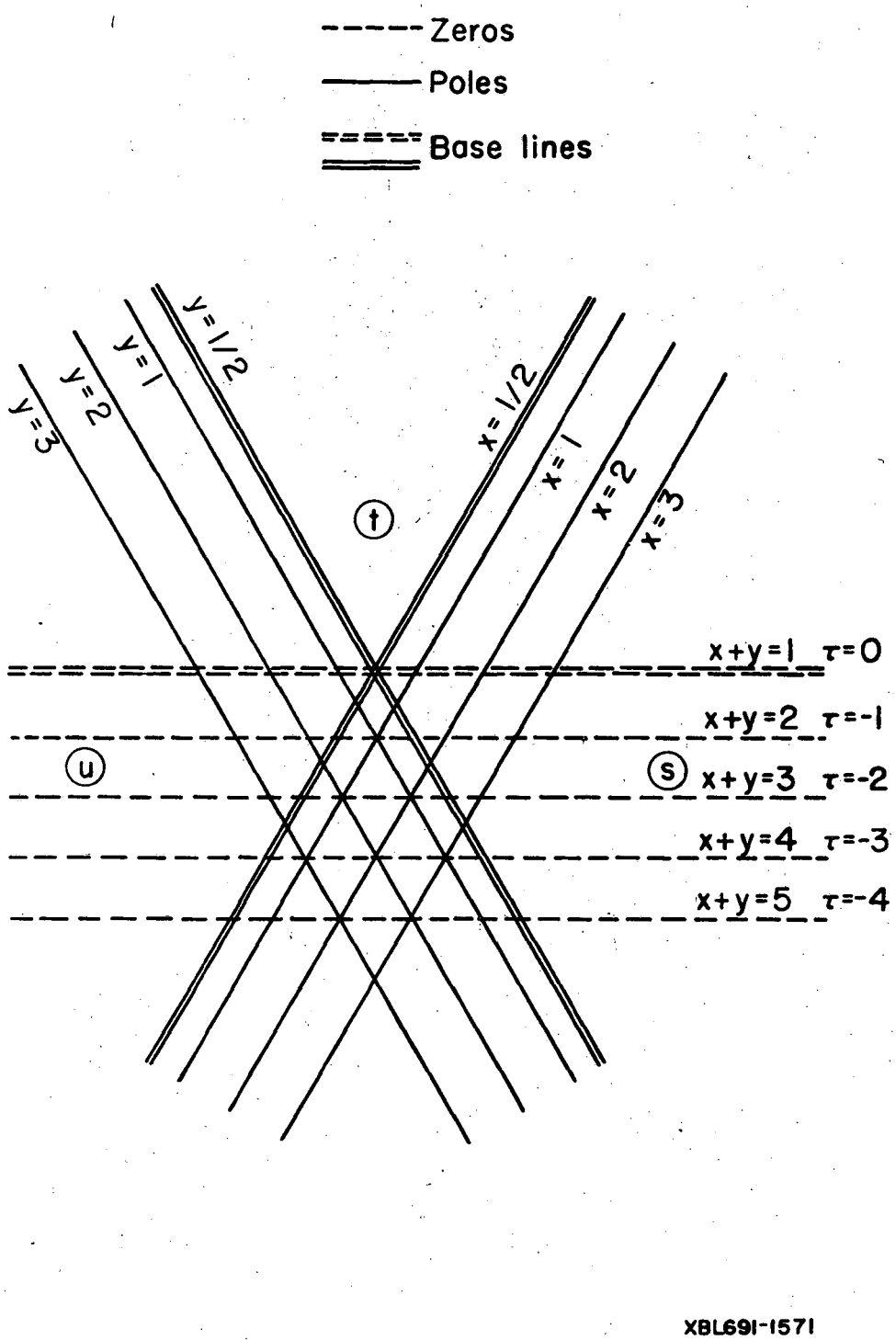
Zeros
Poles
Base lines
y=1/2
y=1
y=2
y=3
x=1/2
x=1
x=2
x=3
t
u
s
x+y=1 τ=0
x+y=2 τ=-1
x+y=3 τ=-2
x+y=4 τ=-3
x+y=5 τ=-4
XBL691-1571

Fig. 5.1

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