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QUASILINEAR EVOLUTION OF THE TRANSVERSE INSTABILITY DRIVEN BY TEMPERATURE ANISOTROPY OF AN UNMAGNETIZED PLASMA

Norman Webster Albright
(Ph. D. Thesis)

January 28, 1969

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QUASILINEAR EVOLUTION OF THE TRANSVERSE INSTABILITY DRIVEN BY TEMPERATURE ANISOTROPY OF AN UNMAGNETIZED PLASMA

Norman Webster Albright

ABSTRACT

The subject of this thesis is the quasilinear evolution of the transverse instability driven by the temperature anisotropy of an unmagnetized plasma. This instability was first studied by E. S. Weibel (Phys. Rev. Letters 1959), A. E. Stefanovich (Soviet Physics-Technical Physics 1962), and F. D. Kahn (J. Fluid Mech. 1962). Starting with a temperature anisotropic plasma, a spectrum of purely growing magnetic waves develops in k-space. These waves cause particle diffusion in velocity space. This leads to temperature isotropy and thus to stabilization of the plasma. After a brief period of rapid diffusion, the anisotropy decreases with time like $t^{-2/3}$. Simultaneously the wave spectrum shifts to lower and lower wave numbers, and eventually decays.

I. INTRODUCTION

The Transverse Instability

The transverse instability is an instability in the low frequency transverse electromagnetic waves of a plasma, not in the high frequency transverse waves (modified light waves). It is driven by anisotropy either in temperature or in any of the even moments of the velocity distributions of either the electrons or the ions.[7] The contribution of a group of resonant particles to the current is an essential part of the instability. Let α denote the temperature anisotropy

$$\alpha = (T_{max} - T_{min})/T_{mean} \tag{I-1}$$

The instability occurs for very low wave numbers

$$k \lesssim \alpha^{1/2}\, \omega_e/c \sim (v_{th}/c)\ \alpha^{1/2}\, k_D \tag{I-2}$$

where ω_e is the electron plasma frequency, v_{th} is the mean electron thermal velocity, and k_D is the inverse of the electron Debye length. The instability has very slow growth rates

$$\gamma \lesssim \alpha\, k\, v_{th} \sim (v_{th}/c)\ \alpha^{3/2}\, \omega_e \tag{I-3}$$

For an unmagnetized plasma the instability is purely growing.

The circumstances under which a plasma will have a transverse instability are clarified by considering the 16 cases of Fig. I-1. In these cases we consider a plasma with different electron temperatures parallel and perpendicular to a symmetry axis, $T_{\parallel}$ and $T_{\perp}$. The 16 cases are for the absence or presence of a static magnetic field, B_o, along the symmetry axis; $T_{\parallel} > T_{\perp}$ or $T_{\parallel} < T_{\perp}$; the k-vector along or

perpendicular to the symmetry axis; and for the two transverse low frequency modes. Modes I and II refer to the modes I and II of the section on linear analysis. Numbers in parentheses are references. We note that the right circularly polarized wave corresponds to the magnetosonic-whistler-electron cyclotron branch of the dispersion relation, and the left circularly polarized wave corresponds to the Alfven wave-ion cyclotron branch. SMS and FMS refer to slow and fast magnetosonic waves.

For $B_o = 0$ we see that there are three unstable cases. In each case the region of unstable waves is

$$0 \leq k^2 \leq k^2_{crit} = \left|\frac{T_\perp - T_\parallel}{T_\parallel}\right| \omega_e^2 / c^2 \qquad \text{(I-4)}$$

These modes are purely growing.

The addition of a static magnetic field causes two new instabilities to appear, the "firehose" and the "mirror" instabilities. However these are different kinds of instabilities. These instabilities require the presence of the magnetic field. They require the anisotropy to exceed some critical value, whereas the other three unstable cases exist for small anisotropy. Furthermore these instabilities may be calculated with the double adiabatic equations; resonance is not essential. We do not include the "firehose" and "mirror" as transverse instabilities, and we will not consider them further.

The right wave instability involves a resonance between electrons and the electron cyclotron wave. This instability is very similar to Mode I and Mode II for $k_\parallel$. The region of instability is[10,19]

$$0 \leq k^2 \leq k^2_{crit} = \left(\frac{T_\perp - T_\parallel}{T_\parallel}\right) \omega_e^2/c^2 + \left(\frac{T_\perp - T_\parallel}{T_\perp}\right)^2 \Omega_e^2/c^2 \qquad \text{(I-5)}$$

where Ω_e is the electron gyrofrequency. The region of instability has been modified by the presence of the static magnetic field. Also the instability now has a real part to its frequency. The region of frequencies for unstable waves is[10,19]

$$\omega \leq \omega_{crit} = \left(\frac{T_\perp - T_\parallel}{T_\perp}\right)|\Omega_e| \tag{I-6}$$

The growth rate however has been reduced. It is exponentially small for small anisotropy.[3,4] The magnetic field has had a stabilizing effect.

For the left wave or ion cyclotron instability the situation is more complicated. For massive, isotropic ions this mode is stable. Finite mass, anisotropic ions are necessary for the instability which involves a resonance between ions and the ion cyclotron wave.[3,4,19] The region of frequencies for unstable waves is[19]

$$\omega \leq \omega_{crit} = \left(\frac{T_\perp - T_\parallel}{T_\perp}\right)_i \Omega_i \tag{I-7}$$

The critical frequency has the same formula here as in the electron cyclotron instability, except that ion parameters (temperatures and gyrofrequency) have been substituted for electron parameters. The other properties of this instability are similar to those of the electron cyclotron instability. However the formulas describing the region of instability in k-space and the growth rate are much more complicated and involve both ion and electron parameters. Generally if a transverse instability exists for electron anisotropy, then the presence or absence of ion anisotropy makes very small changes. However if ion anisotropy is essential, then the presence or absence of electron anisotropy has a strong effect on the instability.[19]

The SMS instability is similar to the Mode I, $k_\perp$ instability. The region of unstable waves is[12]

$$k^2 + \frac{T_\parallel}{T_\perp} e^{-\epsilon} I_0(\epsilon)\, \omega_e^2 / c^2 \leq \left(\frac{T_\parallel - T_\perp}{T_\perp}\right) \omega_e^2 / c^2 \qquad \text{(I-8)}$$

where I_0 is a modified Bessel function,

$$\epsilon = k^2 r_L^2 \qquad \text{(I-9)}$$

and r_L is the electron Larmor radius. The RHS of Eq. (I-8) is just k^2_{crit} for $B_0 = 0$. For instability both terms of the LHS must be less than the RHS. This requires $\epsilon \gg 1$ and thus

$$k^2 \gg 1 / r_L^2 \qquad \text{(I-10)}$$

The region of instability has been narrowed by gyroradius effects. It no longer extends to $k = 0$, and k_{crit} has been reduced. We see that in all three cases, right wave, left wave, and SMS, the magnetic field has a stabilizing effect.

Historical Survey

The first reference to the transverse instability was by E. S. Weibel.[1] In this paper the electrons were described by the Vlasov equation. The ions were taken to be a fixed background. A distribution function of the form

$$f_0 = f(v_\perp, v_z) \qquad \text{(I-11)}$$

was assumed. There was a static magnetic field, B_0, parallel to the preferred z-axis. A dispersion equation was obtained for the case k parallel to B_0 and E_k perpendicular to k. The dispersion equation was

solved for the special case of a two temperature Maxwellian distribution, for $B_o = 0$, and for very different thermal velocities, $v_{\perp th} \gg v_{zth}$. A purely growing mode was found with growth rate

$$\gamma \doteq \omega_e k v_{\perp th} / (\omega_e^2 + k^2 c^2)^{1/2} \doteq \omega_e v_{\perp th} / c \qquad \text{(I-12)}$$

He states that more detailed study shows that the perturbation is damped for large enough k.

B. D. Fried[2] commented on the above paper. He proposed a simple physical picture for transverse wave instability. He considered fixed ions, $B_o = 0$, and a distribution function

$$f_o = \delta(v_x)\ \delta(v_z)\ \delta(v_y^2 - a^2) \qquad \text{(I-13)}$$

A perturbation of the form

$$\delta B_z \sim e^{ikx} \qquad \text{(I-14)}$$

will perturb the two "current streams" of f_o. Curvature of electrons in δB causes a perturbation current which increases δB with growth rate

$$\gamma = a\omega_e / c \qquad \text{(I-15)}$$

R. Z. Sagdeev and V. D. Shafranov[3] calculated electron and ion cyclotron wave instability. A Vlasov calculation was made for the two cases of electrons plus fixed ions, and ions plus fixed electrons. A two temperature Maxwellian was used and the case k parallel to B_o was considered. For the ion wave assuming $kv_{\parallel} \ll \omega$, $p_{\parallel} \ll B_o^2/8\pi$, $\omega \ll \Omega_i$, and Re $\omega \gg$ Im ω they found that maximum growth occurred for

$$k = \frac{\Omega_i}{v_A} \left| \frac{T_{\parallel} - T_{\perp}}{T_{\parallel}} \right|_i \left\{ 1 - \frac{T_{\parallel i}}{m_i v_A^2} \left(\frac{T_{\parallel} - T_{\perp}}{T_{\parallel}} \right)_i^2 \right\} \qquad \text{(I-16)}$$

where v_A is the Alfven speed, and Ω_i is the ion gyrofrequency. The corresponding frequency and growth rate were

$$\omega = \Omega_i \left| \frac{T_\parallel - T_\perp}{T_\parallel} \right|_i \tag{I-17}$$

$$\gamma = \omega_i T_{\perp i} (2 T_{\parallel i} m_i c^2)^{-1/2} \exp\left\{ - \frac{B^2}{8\pi p_{\parallel i}} \left(\frac{T_\parallel}{T_\perp - T_\parallel} \right)_i^2 \right\} \tag{I-18}$$

where ω_i is the ion plasma frequency. The instability occurs for $T_\perp > T_\parallel$.

For the electron wave assuming $kv_\parallel \ll \omega$, $p_\parallel \ll B_o^2/8\pi$, $\omega < \Omega_e$, $\mathrm{Re}\,\omega \gg \mathrm{Im}\,\omega$, and $\Omega_e^2 \ll \omega_e^2$ they found that at maximum growth the frequency and growth rate were

$$\omega = |\Omega_e| \left| \frac{T_\perp - T_\parallel}{T_\perp} \right| \tag{I-19}$$

$$\gamma = \omega_e \frac{\pi^{1/2}}{2} \left(\frac{T_\perp - T_\parallel}{T_\perp} \right)^{3/2} T_\perp (2 T_\parallel m c^2)^{-1/2} \exp\left\{ - \frac{B^2}{8\pi p_\parallel} \left(\frac{T_\perp}{T_\perp - T_\parallel} \right) \right\} \tag{I-20}$$

In each case we have exponentially small growth for small anisotropy. It is stated that the cyclotron resonance condition is

$$\omega = n\Omega + k_\parallel v_\parallel \qquad n = 0, \pm 1, \pm 2, \ldots \tag{I-21}$$

However since no calculation was made for $\omega > \Omega$, we have no way of knowing if the wave is unstable in that case.

The cases of electron and ion cyclotron wave instability are considered in the excellent review article by A. A. Vedenov, E. P. Velikhov, and R. Z. Sagdeev.[4] This article reiterates the results of the preceeding paper.

A. B. Kitsenko and K. N. Stepanov[5] showed that an anisotropic ion distribution gives a cyclotron wave instability for

$$k \sim 1/r_{Li} \quad \text{and} \quad \omega \doteq n\Omega_i \tag{I-22}$$

where r_{Li} is the ion Larmor radius. The distribution function which they used was

$$f_o = \delta(v_\perp - v_o)\, f(v_\parallel) \tag{I-23}$$

where $f(v_\parallel)$ was a Maxwellian. The growth rates which they obtained were very different from those of Sagdeev et al.

The paper by A. E. Stefanovich[6] was the first paper to state the region of instability for $B_o = 0$, and is the only paper to date which has considered the case of a small static magnetic field. It is a very clear paper. He assumed a distribution function

$$f_o = f_o(v_\perp^2, v_\parallel^2) \tag{I-24}$$

and considered waves with k parallel to the preferred axis. For this general distribution function he found the region of instability to be

$$k^2 \leq \sum_{i,e} \frac{\omega_s^2}{c^2}\, 2\pi \int v_\perp^3\, dv_\perp\, dv_\parallel \left(\frac{\partial f_o}{\partial v_\perp^2} - \frac{\partial f_o}{\partial v_\parallel^2}\right)_s \tag{I-25}$$

where ω_s^2 is the plasma frequency for species s. For the case of a two temperature Maxwellian Eq. (I-25) became

$$k^2 \leq \sum_{i,e} \left(\frac{T_\perp - T_\parallel}{T_\parallel}\right)_s \omega_s^2 / c^2 \tag{I-26}$$

He then added a small static magnetic field parallel to the preferred axis. He again considered waves with k parallel to the preferred axis, and with $\omega, \Omega \ll kv_{\parallel th}$. He noted that ion motion has an appreciable effect on the dispersion relation only if

$$\left(\frac{T_\perp - T_\parallel}{T_\parallel}\right)_i \gtrsim \frac{m_i}{m_e} \left(\frac{T_\perp - T_\parallel}{T_\parallel}\right)_e \tag{I-27}$$

and chose to avoid this case. The frequency and growth rate were then found for a two temperature Maxwellian to be

$$\omega = \left(\frac{T_\perp - T_\parallel}{T_\perp}\right) \Omega_e \tag{I-28}$$

$$\gamma = \left(\frac{2}{\pi}\right)^{1/2} k v_{th} \frac{T_\parallel}{T_\perp} \left\{ \left(\frac{T_\perp - T_\parallel}{T_\parallel}\right) - k^2 c^2 / \omega_e^2 \right\} \tag{I-29}$$

where only electron parameters are involved. Maximum growth occurs at

$$k^2 = \frac{1}{3}\left(\frac{T_\perp - T_\parallel}{T_\parallel}\right) \omega_e^2 / c^2 \tag{I-30}$$

with growth rate

$$\gamma = \left(\frac{8}{27\pi}\right)^{1/2} \omega_e \frac{v_{th}}{c} \left(\frac{T_\perp - T_\parallel}{T_\parallel}\right)^{3/2} \tag{I-31}$$

He stated that these formulas are valid only for small anisotropy

$$\left(\frac{T_\perp - T_\parallel}{T_\parallel}\right) \ll 1 \tag{I-32}$$

and for weak static magnetic field

$$(p_\perp - p_\parallel) \gg B_0^2 / 8\pi \tag{I-33}$$

that is for

$$p_{mag} \ll \Delta p \ll p \tag{I-34}$$

The most general formulation of the stability condition for transverse waves in a plasma with no static magnetic field was given by F.D. Kahn.[7] Assuming a distribution function with central symmetry

$$f(v_x, v_y, v_z) = f(-v_x, -v_y, -v_z) \tag{I-35}$$

he found the necessary conditions for stability are that

$$\int_0^\infty f(v,\theta,\varphi)\, v^2 dv \tag{I-36}$$

and

$$\int_0^\infty f(v,\theta,\varphi)\, v\, dv \tag{I-37}$$

be independent of θ and φ. Anisotropy in any even velocity moment will cause instability (unless it is balanced by an intricate combination of anisotropies of all the other velocity moments).

In a second paper, F. D. Kahn[8] considered an unmagnetized plasma with a <u>three</u> temperature Maxwellian distribution function for the electrons and fixed ions. For small anisotropy he found that the most unstable wave has k parallel to the axis of temperature minimum and E_k parallel to the axis of temperature maximum. He found the region of instability is

$$k^2 \leq \left(\frac{T_{max} - T_{min}}{T_{min}}\right) \omega_e^2 / c^2 \tag{I-38}$$

which is similar to Stefanovich's result, Eq. (I-26). He found a formula for the maximum growth rate which is similar to Stefanovich's, Eq. (I-31).

$$\gamma = \frac{8}{3\pi}\left(\frac{2}{3}\right)^{1/2} \omega_e \frac{V}{c} \left(\frac{T_{max} - T_{min}}{T_{max}}\right)^{3/2} \tag{I-39}$$

where V is a moment of the distribution function and $V \sim v_{th}$.

He also considered the case of an isotropic electron distribution and an anisotropic ion distribution. In that case he found the region of instability was

$$k^2 \leq \left(\frac{T_{max} - T_{min}}{T_{min}}\right)_i \omega_i^2 / c^2 \tag{I-40}$$

which for the same degree of anisotropy is a reduction of the region of instability by a factor $(m_e/m_i)^{1/2}$. The corresponding maximum growth rate became

$$\gamma = \frac{8}{3\pi}\left(\frac{2}{3}\right)^{1/2} \omega_i \frac{V_e}{c} \left(\frac{m_e}{m_i}\right)\left(\frac{T_{max} - T_{min}}{T_{max}}\right)^{3/2} \qquad \text{(I-41)}$$

which for the same degree of anisotropy is a reduction in the growth rate by a factor $(m_e/m_i)^{3/2}$. He considered the application of these results to fluid flow. Ellipsoidal velocity distributions arise in any compressive, expansive, or shear flow. The resulting microinstability leads to enhanced diffusion. In particular viscosity is decreased and becomes dependent on the local velocity gradient.

An interesting application of the transverse wave instability was considered in the paper by S. S. Moiseev and R. Z. Sagdeev.[9] They considered collisionless shocks for the cases $p_e \gg p_i$ and $p_i \gtrsim p_e$ in an unmagnetized plasma. In the latter case the shock structure is determined by the transverse wave instability. They considered an anisotropic ion distribution and an isotropic electron distribution. They stated that this is unstable for

$$k^2 \leq k_o^2 = \left| 1 + \int \frac{v_x^2}{v_z} \frac{\partial f_i}{\partial v_z} d^3v \right| \omega_i^2/c^2 \sim \left| \frac{T_z - T_x}{T_z} \right|_i \omega_i^2/c^2 \qquad \text{(I-42)}$$

where x is the direction normal to the shock, and the wave has k parallel to z and δB parallel to y. They found the nonlinear stabilization by equating the anisotropy driven growth to the nonlinear transfer of energy along the spectrum to shorter waves which are damped. They found that stabilization occurred at a value of δB for which the electron Larmor radius was

$$r_{Le} \sim 1/k_o \tag{I-43}$$

They considered ion velocity space diffusion and estimated the time required to make ions isotropic

$$\tau \sim k_o \frac{c^2}{\omega_i^2} \frac{m_i}{m_e} \frac{1}{v_{ith}} \tag{I-44}$$

where v_{ith} is the ion thermal velocity. They considered hydrodynamics in the relaxation time approximation and thereby calculated the viscosity and the effect on shock structure.

The paper by P. D. Noerdlinger[10] is a comprehensive study of the region of instability for k parallel to a static magnetic field, B_o, and for B_o parallel to the preferred axis of a two temperature Maxwellian distribution. He applied a Nyquist type of analysis to the dispersion relation. He considered two cases: an anisotropic electron distribution with fixed ions, and an electron positron plasma with equal anisotropies. In the first case he found that for $T_\perp < T_\parallel$ there are unstable waves for large enough B_o, but that these require resonance with particles traveling faster than light. Therefore $T_\perp < T_\parallel$ is stable for fixed ions. For $T_\perp > T_\parallel$ there are unstable waves for

$$k^2 < \left(\frac{T_\perp - T_\parallel}{T_\parallel}\right) \omega_e^2/c^2 + \left(\frac{T_\perp - T_\parallel}{T_\perp}\right)^2 \Omega_e^2/c^2 \tag{I-45}$$

He estimated that the frequency of the maximum growing mode is

$$\omega \sim \left(\frac{T_\perp - T_\parallel}{T_\perp}\right) \Omega_e \tag{I-46}$$

For the electron positron plasma he found a new unstable mode, a purely growing hydromagnetic wave for $T_\parallel > T_\perp$ and

$$G(k)\,(p_\parallel - p_\perp) > B_o^2/4\pi \tag{I-47}$$

where the function G lies between 1.0 and 1.64. This is the "firehose" instability.

For $T_\perp > T_\parallel$ the boundary for unstable waves is only slightly changed. However the waves are now of two types, one which exists for $\Omega_e \ll \omega_e$ and the other which exists for $\omega_e \lesssim \Omega_e$. Graphs are given showing the separate regions.

In the second paper, P. D. Noerdlinger[11] extends the above results to the case of a hydrogen plasma. The unstable region was graphed as a function of kc, Ω, and $T_\perp/T_\parallel$. B_o was parallel to the preferred axis of a two temperature Maxwellian, and k was parallel to B_o. For the case $T_\perp > T_\parallel$ there were two unstable modes. One was similar to an electron plasma with fixed ions, and the other was similar to an ion plasma with fixed electrons.

The case $T_\parallel > T_\perp$ was much more complicated. It was found to be unstable for

$$p_\parallel - p_\perp > B_o^2/2\pi \tag{I-48}$$

and to have growth rates of the order of a few tenths of the ion gyrofrequency. The classical firehose result is instability for

$$p_\parallel - p_\perp > B_o^2/4\pi \tag{I-49}$$

He suggested that plasmas satisfying

$$B_o^2/2\pi > p_\parallel - p_\perp > B_o^2/4\pi \tag{I-50}$$

probably have instabilities with growth rates much less than the ion gyrofrequency.

H. P. Furth[12] interpreted the transverse instability in terms of a Bennett pinch, and calculated several cases. "Two identical interpenetrating particle streams (as in a thermally anisotropic plasma) give rise to no net current flow, and therefore to no self-focusing field. The zero-order counterstreaming of like particles gives rise to instability modes that correspond essentially to the formation of local first-order pinches. The perturbation magnetic field is transverse to the direction of streaming, and tends to concentrate particles that are moving in the same direction. The marginal stability criteria for all such modes can be interpreted in terms of Bennett's pinch condition."

In all the cases he considered, there is an anisotropic species, denoted by subscript α, with a two temperature Maxwellian distribution. He assumed there is a neutralizing background species, denoted by subscript n, which is isotropic. The first case he considered was $B_o = 0$, k perpendicular to $T_{\|}$, and δE parallel to $T_{\|}$. (This is Mode I, $k_{\perp}$.) Assuming $m_n = \infty$ he obtained for the region of instability

$$k^2 < \left(\frac{T_{\|} - T_{\perp}}{T_{\perp}}\right) \omega_{\alpha}^2 / c^2 \qquad \text{(I-51)}$$

For $T_{\perp} > T_{\|}$ the RHS is negative, so this case is always stable. Unstable waves occur only for $T_{\|} > T_{\perp}$. For finite mass neutralizer the instability condition becomes

$$k^2 < \left\{ \frac{T_{\|} - T_{\perp}}{T_{\perp}} - \frac{T_{\|}}{T_{\perp}} F(\Lambda_{\alpha}) - \frac{m_{\alpha}}{m_n} F(\Lambda_n) \right\} \omega_{\alpha}^2 / c^2 \qquad \text{(I-52)}$$

where

$$\Lambda = \gamma / 2^{1/2} k v_{\perp th} \qquad \text{(I-53)}$$

and $F(\Lambda)$ is related to the complement of the error function

$$F(\Lambda) = \pi^{1/2} \Lambda e^{\Lambda^2} \operatorname{erfc}(\Lambda) \tag{I-54}$$

At the boundary of the unstable region $\gamma = 0$ and $F(\Lambda_\alpha) = 0$. However for a cold neutralizer $F(\Lambda_n) = 1$. In this case the instability condition becomes

$$k^2 < \left\{ \frac{T_\parallel - T_\perp}{T_\perp} - \frac{m_\alpha}{m_n} \right\} \omega_\alpha^2 / c^2 \tag{I-55}$$

Thus for α = electron and n = ion there is negligible change from the previous condition. However for α = ion and n = electron the plasma is stable except for very large anisotropy. The neutralizer is considered cold if its thermal speed satisfies

$$v_{nth} < \gamma/k \tag{I-56}$$

for an unstable mode with growth rate γ.

The next case he considered had B_o parallel to $T_\parallel$, k perpendicular to B_o, and δE parallel to B_o. (This is SMS, $k_\perp$.) The instability condition was found to be

$$k^2 < \left\{ \frac{T_\parallel - T_\perp}{T_\perp} - \frac{T_\parallel}{T_\perp} e^{-\epsilon_\alpha} I_o(\epsilon_\alpha) - \frac{m_\alpha}{m_n} e^{-\epsilon_n} I_o(\epsilon_n) \right\} \omega_\alpha^2 / c^2 \tag{I-57}$$

where I_o is a modified Bessel function,

$$\epsilon = k^2 r_L^2 \tag{I-58}$$

and r_L is the particle Larmor radius. For $T_\parallel < T_\perp$ all waves are stable. Consider the case $T_\perp < T_\parallel$. For α = ion and n = electron the plasma is stable except for very large anisotropy. Let α = electron and n = ion. The m_α/m_n term may be dropped. The instability condition is then

$$k^2 + \frac{T_\parallel}{T_\perp} e^{-\epsilon} I_0(\epsilon)\, \omega_e^2/c^2 < \left(\frac{T_\parallel - T_\perp}{T_\perp}\right) \omega_e^2/c^2 \qquad \text{(I-59)}$$

For small or moderate anisotropy, $T_\parallel \sim T_\perp$, both terms of the LHS must be less than the RHS. This requires $\epsilon \gg 1$ and thus

$$1/r_{Le}^2 \ll k^2 \ll \omega_e^2/c^2 \qquad \text{(I-60)}$$

The value of k which minimizes the LHS is

$$k^3 = (8\pi)^{-1/2} \frac{\omega_e^2}{c^2 r_{Le}} \frac{T_\parallel}{T_\perp} \qquad \text{(I-61)}$$

Substituting this into Eq. (I-57) gives the instability requirement

$$\frac{\Omega_e^2}{\omega_e^2} \frac{c^2}{v_{\perp e}^2} < \frac{8\pi}{27} \left(\frac{T_\parallel - T_\perp}{T_\perp}\right) \qquad \text{(I-62)}$$

Thus for instability the magnetic field must not exceed a maximum value given by Eq. (I-62). These results were applied to the Astron E-layer.

The last case he considered was the "firehose" and "mirror" instability conditions for waves with k-vector at an arbitrary angle θ relative to the static magnetic field. For the "firehose" instability the most unstable waves have k parallel to B_o, and require $T_\parallel > T_\perp$ and

$$p_\parallel - p_\perp > B_o^2/4\pi \qquad \text{(I-63)}$$

For the "mirror" instability the most unstable waves have k perpendicular to B_o, δB parallel to B_o, and require $T_\perp > T_\parallel$ and

$$(p_\perp - p_\parallel)\frac{p_\perp}{p_\parallel} > B_o^2/8\pi \qquad \text{(I-64)}$$

The first paper by R. N. Sudan[13] considered anisotropic electrons with a two temperature Maxwellian distribution and a background of fixed ions. He considered waves with k parallel to a static magnetic

field, and found the electron cyclotron wave instability for $T_\perp > T_\parallel$. For the region of unstable waves he found Noerdlinger's condition, Eq. (I-45). He also calculates the growth rate

$$\gamma = \pi^{1/2} k v_{\parallel th} \left(\frac{T_\parallel - T_\perp}{T_\perp}\right)\left(\frac{k^2}{k_o^2} - 1\right) \exp\left(- \frac{T_\parallel^2}{T_\perp^2} \frac{\Omega^2}{k^2 v_{\parallel th}^2}\right) \qquad \text{(I-65)}$$

where k_o^2 is the RHS in Eq. (I-45), that is, it is the largest unstable wave number. Sudan's formula does not have the minus in the exponent, however compare Eq. (I-65) with Sagdeev's Eq. (I-20).

The second paper by R. N. Sudan[14] was a relativistic calculation of the above results. For a nonrelativistic plasma he found the region of instability to be given by Eq. (I-45) plus small corrections.

D. B. Chang[15] studied the generation and amplification of whistlers (electron cyclotron wave) in an isotropic electron plasma with a non-isotropic, relativistic component and with a background of fixed ions. The relativistic Vlasov equation was used to calculate the case of k parallel to B_o. The results were applied to Jupiter's sporadic decameter radiation by assuming that Jupiter's Van Allen belt contains relativistic electrons.

V. D. Shapiro and V. I. Shevchenko[16] made a quasilinear calculation for the changes in the longitudinal and transverse thermal energies and also the electromagnetic field energy of an instability due to an anisotropic ion distribution function. There was a static magnetic field. They considered two cases, the "firehose" mode with k parallel to B_o, and the "mirror" mode with $k_\perp \gg k_\parallel$. Both instabilities are purely growing. If we denote

$$Y = \left(\frac{p_\parallel - p_\perp}{p_\parallel} - \frac{B_o^2}{4\pi p_\parallel}\right) \qquad \text{(I-66)}$$

and

$$Y^* = \left(\frac{p_\perp - p_\parallel}{p_\parallel} - \frac{B_o^2}{8\pi p_\perp}\right) \tag{I-67}$$

Then the "firehose" and "mirror" instabilities occur for

$$Y > 0 \qquad \text{and} \qquad Y^* > 0 \tag{I-68}$$

respectively. The calculation was limited to modes satisfying

$$\gamma \ll k_\parallel v_{\parallel th} \tag{I-69}$$

which requires

$$Y \ll 1 \qquad \text{and} \qquad Y^* \ll 1 \tag{I-70}$$

respectively. It was also assumed in both cases that

$$k_\parallel v_{\parallel th} \ll \Omega_i \quad \text{and} \quad k_\perp v_{\perp th} \ll \Omega_i \tag{I-71}$$

In each case it was found that the wave spectrum did not disappear when the system stabilized. A spectrum of stationary magnetic waves remained. It was found that the energy in the final state wave spectrum was proportional to the initial values of Y and Y^*[4] respectively.

J. Rowlands, V. D. Shapiro, and V. I. Shevchenko[17] solved the quasilinear equations for the ion cyclotron instability. They considered waves with k parallel to B_o and with

$$\gamma \ll \omega \qquad k v_{th} \ll |\Omega - \omega| \quad \text{and} \quad k v_{th} \ll \omega \tag{I-72}$$

They found the shape of the final distribution function, and found that a spectrum of stationary waves remained when the system had stabilized. They applied the results to an ion beam passing through the plasma. They assert that the instability can occur either for $T_\perp > T_\parallel$ (normal

Doppler effect) or for $T_{\parallel} > T_{\perp}$ (anomalous Doppler effect).

C. S. Wu[18] examined the stability of transverse waves in an unmagnetized plasma. He found the stability condition in terms of the distribution function and evaluated this for a two temperature Maxwellian with k parallel to the preferred temperature axis. He found the instability condition to be

$$0 < \sum_s \left(\frac{T_\perp - T_\parallel}{T_\parallel}\right)_s \omega_s^2 \tag{I-73}$$

which is the same result as Stefanovich's Eq. (I-26).

A. W. Trivelpiece and J. E. Scharer[19] studied the electron and ion cyclotron instabilities for k parallel to B_o. For a two temperature Maxwellian, the dispersion relation becomes

$$\omega^2 = k^2c^2 - \sum_s \omega_s^2 \left\{ \frac{\omega}{k v_s} Z(\phi_s) + \left(\frac{T_\perp - T_\parallel}{T_\parallel}\right)_s \left(1 + \phi_s Z(\phi_s)\right) \right\} \tag{I-74}$$

where Z is the Plasma Dispersion Function, ω_s is the plasma frequency for species s, v_s is $2^{1/2}$ times the parallel thermal velocity of species s, ω is the complex frequency,

$$\phi_s = \frac{\omega \pm \Omega_s}{k v_s} \tag{I-75}$$

where plus and minus refer to the right and left waves respectively, and Ω_s is the cyclotron frequency for species s (positive for ions, and negative for electrons).

They found that for $T_\perp > T_\parallel$ unstable electron cyclotron waves exist for frequencies

$$\omega < \left(\frac{T_\perp - T_\parallel}{T_\perp}\right)_e |\Omega_e| = \omega_{crit} \tag{I-76}$$

and for wave numbers corresponding to Noerdlinger's Eq. (I-45). They

stated that maximum growth rate occurs for a wave with frequency slightly below ω_{crit}, which agrees with Sagdeev's Eq. (I-19). They noted that the electron cyclotron wave is never unstable above the electron cyclotron frequency. This wave is stable for $T_{\parallel} > T_{\perp}$ for all frequencies and wave numbers.

They found that for $T_{\perp} > T_{\parallel}$ unstable ion cyclotron waves exist for frequencies

$$\omega < \omega_{crit} = \left(\frac{T_{\perp} - T_{\parallel}}{T_{\perp}}\right)_i \Omega_i \tag{I-77}$$

For cold electrons the region of unstable wave numbers is

$$k^2 < \frac{(T_{\perp} - T_{\parallel})^2}{T_{\parallel} T_{\perp}} \omega_i^2 / c^2 + \left(\frac{T_{\perp} - T_{\parallel}}{T_{\perp}}\right)^2 \Omega_i^2 / c^2 \tag{I-78}$$

They found that cold electrons have a stabilizing influence on an unstable ion cyclotron wave, however thermally anisotropic electrons with $T_{\perp} > T_{\parallel}$ have a destabilizing effect.

G. Kalman, C. Montes, and D. Quemada[20] made a comprehensive study of the transverse wave instability in an unmagnetized plasma. They assumed a two temperature Maxwellian distribution with $T_{\perp} > T_{\parallel}$. They found that each of the three low frequency modes can be unstable. Let θ denote the angle between the k-vector and the $T_{\parallel}$-axis. They found that one mode is purely transverse (Mode II). It has E_k perpendicular to both k and $T_{\parallel}$. It is purely growing, and has maximum growth for k parallel to $T_{\parallel}$, $\theta = 0^{\circ}$.

The next mode they considered was quasitransverse (Mode I). It has B_k perpendicular to both k and $T_{\parallel}$. It is transversely polarized for $\theta = 0^{\circ}$ and $\theta = 90^{\circ}$. It is purely growing for

$$0 < k < k_{crit}(\theta) \quad \text{and} \quad 0 < \theta < \theta_{crit} \tag{I-79}$$

where θ_{crit} is approximately 45°. It is purely damped otherwise. It has maximum growth for $\theta = 0^{\circ}$. (Note this mode is also unstable for $T_{\parallel} > T_{\perp}$, in which case maximum growth occurs for k perpendicular to $T_{\parallel}$, $\theta = 90^{\circ}$.)

The last mode they considered was the quasilongitudinal ion-acoustic mode. It is longitudinal for $\theta = 0^{\circ}$ and $\theta = 90^{\circ}$. It is driven unstable by its coupling with the quasitransverse mode. Its growth rate is small compared to the transverse mode. This mode was studied for the case of equal ion and electron anisotropies

$$\left(\frac{T_{\perp}}{T_{\parallel}}\right)_e = \left(\frac{T_{\perp}}{T_{\parallel}}\right)_i \quad \text{and} \quad T_e \gg T_i \tag{I-80}$$

This mode has a nonzero real part to its frequency for most values of k and θ. The region of instability for a given k value is

$$\theta_1 < \theta < \theta_4 \tag{I-81}$$

Interior to this is a region

$$\theta_2 < \theta < \theta_3 \tag{I-82}$$

in which the real part of the frequency is zero. The authors gave graphs showing growth rate, region of instability, and polarization.

Thesis Summary

An infinite plasma with a bi-Maxwellian initial velocity distribution is considered. The electrons have a temperature maximum along the z-axis and a temperature minimum in the plane perpendicular to the z-axis. The case of small electron temperature anisotropy, α, is considered

$$\alpha = (T_{\parallel} - T_{\perp})/T_{mean} \ll 1 \tag{I-83}$$

Ion anisotropy is assumed to be small enough so that its effects may be neglected. A set of ordered equations is found based on a multiple time scale expansion of the Vlasov and Maxwell equations in powers of α.

Linear analysis shows that the unstable waves are purely growing and have growth rate

$$\gamma \sim \alpha k v_{thermal} \tag{I-84}$$

The region of unstable waves is

$$0 < k^2 < (1 - 2\cos^2\theta)\ \alpha\ \omega_e^2/c^2 \tag{I-85}$$

where ω_e is the electron plasma frequency, and θ is the angle between the k-vector and the z-axis. Maximum growth occurs for k-vectors lying in the plane perpendicular to the z-axis, that is for $\theta = 90^o$. Generally the longitudinal electric field is the same order as the transverse electric field, however the electric field becomes transversely polarized for $\theta = 90^o$.

A study of energy flow shows that the magnetic field energy density of the waves is of order α^2 times the kinetic energy density of the plasma. The electric field energy density of the waves is in turn of order α^2 times the magnetic field energy density. This is because the real part of the frequency of the wave is zero, and the imaginary part, γ, is very small compared to kc.

The quasilinear approximation consists of a pair of coupled equations for the magnetic field, B_k, and the average particle distribution function, f_o. f_o is broken into two parts

$$f_o = f_o^{(o)} + f_o^{(2)} \tag{I-86}$$

$f_o^{(o)}$ is defined to be the isotropic part of f_o at $t = 0$, and $f_o^{(2)}$ is the anisotropic part of f_o at $t = 0$. It is shown that $f_o^{(o)}$ is constant. $f_o^{(2)}$ is shown to remain purely anisotropic for all time, that is the moments of v^2, v^4, etc. with respect to $f_o^{(2)}$ remain zero throughout its evolution. The coupled equations for B_k and $f_o^{(2)}$ have the form

$$\frac{\partial}{\partial t} B_k(t) = \gamma_k(t)\ B_k(t) \tag{I-87}$$

$$\frac{\partial}{\partial t} f_o^{(2)}(v,t) = \int d^3k\ |B_k(t)|^2\ B(k,v,\partial/\partial v)\ f_o^{(2)}(v,t)$$

$$+ \int d^3k\ \frac{\partial}{\partial t}|B_k(t)|^2\ C(k,v,\partial/\partial v)\ f_o^{(o)}(v) \tag{I-88}$$

The equation for B_k is actually nonlinear. Although it is linear in the field, the growth rate γ_k is a functional of f_o which itself changes on the shortest time scale. That is, when the amplitude of the wave spectrum, $|B_k|^2$, becomes large enough, f_o changes significantly over one growth period.

The $B(k,v,\partial/\partial v)$ term in the equation for $f_o^{(2)}$ is a diffusion term. The diffusion coefficient for this term is a symmetric tensor. As a result this term satisfies an H-theorem. From this one finds that the plasma evolves to temperature isotropy within the quasilinear approximation under the influence of the B term. The B term is responsible for isotropization. The diffusion coefficient for the $C(k,v,\partial/\partial v)$ term in the equation for $f_o^{(2)}$ is an antisymmetric tensor to lowest order in α. An antisymmetric diffusion coefficient corresponds to dissipationless flow in velocity space. The C term is a perfect time derivative. The contribution of this term to $f_o^{(2)}(t)$ depends only on B_k at time t, and

is independent of the state of the plasma at previous times. The C term therefore makes no contribution to the net change in $f_o^{(2)}$ from initial to final states, since the initial and final wave spectra are negligibly small. The equations for the evolution of α and B_k have been solved using a bi-Maxwellian model for f_o. The evolution of α may be broken up into three distinct phases: an initial phase, a rapid phase, and an asymptotic phase. During the initial phase the wave spectrum grows, whereas particle diffusion occurs very slowly at first, and hence little change occurs in the anisotropy. Eventually the amplitude of the wave spectrum becomes so large that rapid particle diffusion begins. This is due mainly to the C term in the equation for f_o which is larger than the B term at first. The temperature anisotropy then changes rapidly until about half of the wave spectrum with smaller wave numbers is growing, and the half with larger wave numbers is decaying. This reduces the C term and makes it negative. The C term then partially cancels the B term. This ends the rapid phase. Particle diffusion then proceeds ever more slowly as temperature isotropy is approached. Asymptotic analysis shows that the anisotropy decreases with time like $t^{-2/3}$. Simultaneously the wave spectrum shifts to lower and lower wave numbers, and eventually decays. The asymptotic phase is characterized by the wave spectrum shifting so as to always be about half growing and half decaying. Shifting in this manner the C term continues to partially cancel the B term. If the C term were to become less negative, then f_o and hence α would change more rapidly. The spectrum would then damp more quickly until the C term became more negative again.

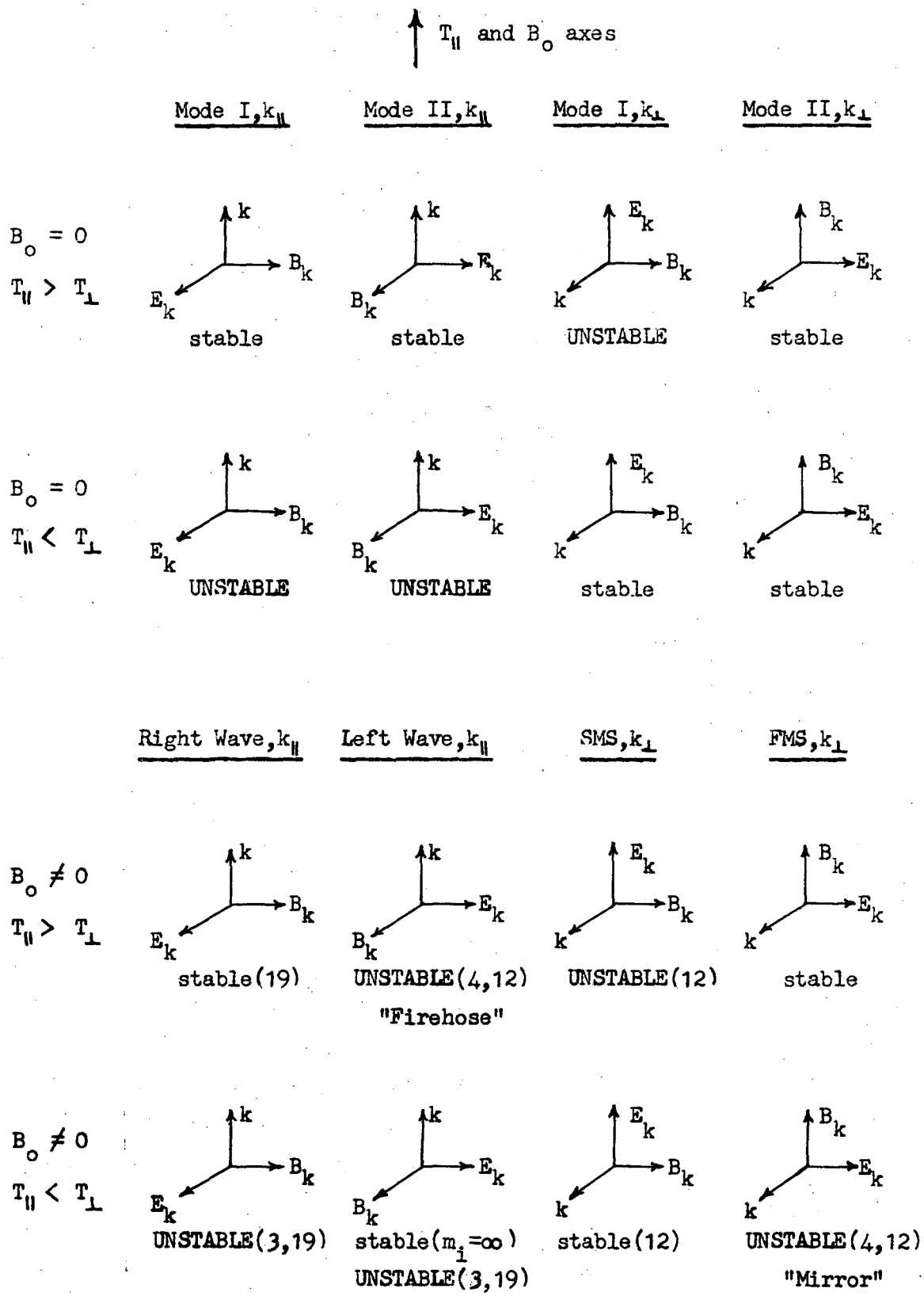
$T_{\parallel}$ and B_o axes
Mode I, $k_{\parallel}$
Mode II, $k_{\parallel}$
Mode I, $k_{\perp}$
Mode II, $k_{\perp}$
$B_o = 0$
$T_{\parallel} > T_{\perp}$
stable
stable
UNSTABLE
stable
$B_o = 0$
$T_{\parallel} < T_{\perp}$
UNSTABLE
UNSTABLE
stable
stable
Right Wave, $k_{\parallel}$
Left Wave, $k_{\parallel}$
SMS, $k_{\perp}$
FMS, $k_{\perp}$
$B_o \neq 0$
$T_{\parallel} > T_{\perp}$
stable(19)
UNSTABLE(4,12)
"Firehose"
UNSTABLE(12)
stable
$B_o \neq 0$
$T_{\parallel} < T_{\perp}$
UNSTABLE(3,19)
stable($m_i = \infty$)
UNSTABLE(3,19)
stable(12)
UNSTABLE(4,12)
"Mirror"

FIGURE I-1

II. BASIC EQUATIONS

Equivalence of Vlasov and Klimontovich Cumulant Hierarchies

Define the microdensity for species s by

$$N^s(r,v,t) = \sum_{i \in s} \delta(r - r_i(t))\ \delta(v - v_i(t)) \tag{II-1}$$

The microdensity satisfies the Klimontovich equation

$$\left\{ \frac{\partial}{\partial t} + v \cdot \frac{\partial}{\partial r} + \frac{e_s}{m_s}\left(E(r,t) + \frac{v}{c} \times B(r,t)\right) \cdot \frac{\partial}{\partial v} \right\} N^s(r,v,t) = 0 \tag{II-2}$$

where the electric and magnetic fields are calculated from Maxwell's equations using the current given by

$$j(r,t) = \sum_s e_s \int d^3v\ \ v\ \ N^s(r,v,t) \tag{II-3}$$

Let $F^s(r,v,t)$ be the distribution function for a Vlasov fluid. It satisfies the Vlasov equation

$$\left\{ \frac{\partial}{\partial t} + v \cdot \frac{\partial}{\partial r} + \frac{e_s}{m_s}\left(E(r,t) + \frac{v}{c} \times B(r,t)\right) \cdot \frac{\partial}{\partial v} \right\} F^s(r,v,t) = 0 \tag{II-4}$$

where the fields are calculated using the current given by

$$j(r,t) = \sum_s e_s \int d^3v\ \ v\ \ F^s(r,v,t) \tag{II-5}$$

Because of the similarity of the Klimontovich equation with the Vlasov equation, their cumulant hierarchies are similar. The cumulant hierarchy of an ensemble of Vlasov fluids lacks the particle-particle collision terms which are present in the cumulant hierarchy of an ensemble of point-particle plasmas. The relationship between the Vlasov and Klimontovich cumulant hierarchies has been studied in detail by R. C. Davidson.[21] We shall confine our attention to the Vlasov equation.

Basic Equations and Definitions

We introduce the notation

$$L^s(r,v,t) = \frac{e_s}{m_s}\left(E(r,t) + \frac{v}{c} \times B(r,t)\right) \cdot \frac{\partial}{\partial v} \tag{II-6}$$

Our basic equations are the Poisson equation, the Faraday equation, the Maxwell-Ampere equation, and the Vlasov equation

$$\frac{\partial}{\partial r} \cdot E(r,t) = \sum_s 4\pi e_s \int d^3v \; F^s(r,v,t) \tag{II-7}$$

$$\frac{\partial}{\partial t} B(r,t) = -c \frac{\partial}{\partial r} \times E(r,t) \tag{II-8}$$

$$\frac{\partial}{\partial t} E(r,t) - c \frac{\partial}{\partial r} \times B(r,t) + \sum_s 4\pi e_s \int d^3v \; v \, F^s(r,v,t) = 0 \tag{II-9}$$

$$\left\{ \frac{\partial}{\partial t} + v \cdot \frac{\partial}{\partial r} + L^s(r,v,t) \right\} F^s(r,v,t) = 0 \tag{II-10}$$

We define the space average of any quantity by

$$B_o(t) = \underset{V \to \infty}{\text{limit}} \; \frac{1}{V} \int_V d^3r \; B(r,t) \tag{II-11}$$

Henceforth in any equation in which V occurs, the limit $V \to \infty$ will be implied but not written. We define the deviation of any quantity by

$$\delta B(r,t) = B(r,t) - B_o(t) \tag{II-12}$$

Taking the space average of Eqs. (II-7) through (II-10) gives

$$\sum_s 4\pi e_s \int d^3v \; F_o^s(v,t) = 0 \tag{II-13}$$

$$\frac{\partial}{\partial t} B_o(t) = 0 \tag{II-14}$$

$$\frac{\partial}{\partial t} E_o(t) + \sum_s 4\pi e_s \int d^3v \; v \, F_o^s(v,t) = 0 \tag{II-15}$$

$$\left\{ \frac{\partial}{\partial t} + L_o^s(v,t) \right\} F_o^s(v,t) = -\frac{1}{V} \int_V d^3r \; \delta L^s(r,v,t) \; \delta F^s(r,v,t) \tag{II-16}$$

Subtracting Eqs. (II-13) through (II-16) from Eqs. (II-7) through (II-10) gives equations for the deviations

$$\frac{\partial}{\partial r} \cdot \delta E(r,t) = \sum_s 4\pi e_s \int d^3v \;\; \delta F^s(r,v,t) \tag{II-17}$$

$$\frac{\partial}{\partial t} \delta B(r,t) = - c \frac{\partial}{\partial r} \times \delta E(r,t) \tag{II-18}$$

$$\frac{\partial}{\partial t} \delta E(r,t) - c \frac{\partial}{\partial r} \times \delta B(r,t) + \sum_s 4\pi e_s \int d^3v \;\; v \; \delta F^s(r,v,t) = 0 \tag{II-19}$$

$$\left\{\frac{\partial}{\partial t} + v \cdot \frac{\partial}{\partial r} + L^s_o(v,t)\right\} \delta F^s(r,v,t) + \delta L^s(r,v,t) F^s_o(v,t)$$

$$= - \delta L^s(r,v,t) \; \delta F^s(r,v,t) + \frac{1}{V} \int_V d^3r \;\; \delta L^s(r,v,t) \; \delta F^s(r,v,t) \tag{II-20}$$

We define the Fourier transform of a deviation by

$$B_k(t) = \int d^3r \; e^{-ik\cdot r} \;\; \delta B(r,t) \tag{II-21}$$

with inversion formula

$$\delta B(r,t) = \int \frac{d^3k}{(2\pi)^3} \;\; e^{ik\cdot r} \;\; B_k(t) \tag{II-22}$$

In order to avoid any confusion, we shall use the subscripts $k = 0$ and 0 respectively to distinguish between $B_{k=0}$ (Fourier transform of a deviation) and B_o (space average). Our definitions require that

$$B_{k=0}(t) = E_{k=0}(t) = L^s_{k=0}(v,t) = F^s_{k=0}(v,t) = 0 \tag{II-23}$$

We note that the Fourier transform of $B(r,t)$ is related to $B_k(t)$ by

$$\mathcal{F}[B(r,t)] = B_k(t) + (2\pi)^3 \, \delta(k) \, B_o(t) \tag{II-24}$$

That is

$$\mathcal{F}[B(r,t)] = B_k(t) \qquad \text{for } k \neq 0 \tag{II-25}$$

and

$$\mathfrak{F}[B(r,t)] = V\, B_o(t) \qquad \text{for} \quad k = 0 \tag{II-26}$$

Evidently it is easier to deal separately with the deviations and the space averages.

Using the Fourier transform, Eqs. (II-16) through (II-20) may be written

$$\left\{\frac{\partial}{\partial t} + L_o^s(v,t)\right\} F_o^s(v,t) = -\frac{1}{V}\int \frac{d^3k}{(2\pi)^3}\, L_{-k}^s(v,t)\, F_k^s(v,t) \tag{II-27}$$

$$i\,k\cdot E_k(t) = \sum_s 4\pi e_s \int d^3v\; F_k^s(v,t) \tag{II-28}$$

$$\frac{\partial}{\partial t} B_k(t) = -ic\,k\times E_k(t) \tag{II-29}$$

$$\frac{\partial}{\partial t} E_k(t) - ic\,k\times B_k(t) + \sum_s 4\pi e_s \int d^3v\; v\; F_k^s(v,t) = 0 \tag{II-30}$$

$$\left\{\frac{\partial}{\partial t} + ik\cdot v + L_o^s(v,t)\right\} F_k^s(v,t) + L_k^s(v,t)\, F_o^s(v,t)$$
$$= -\int \frac{d^3\ell}{(2\pi)^3}\, L_{k-\ell}^s(v,t)\, F_\ell^s(v,t) + \delta_{k,0}\int \frac{d^3\ell}{(2\pi)^3}\, L_{-\ell}^s(v,t)\, F_\ell^s(v,t) \tag{II-31}$$

The last term in Eq. (II-31) cancels the other integral term for $k = 0$ and allows Eq. (II-31) to be consistent with Eq. (II-23). Henceforth we shall drop the last term in Eq. (II-31) and restrict our attention to values of $k \neq 0$.

In the problem considered in this thesis there are no uniform fields or currents, and average charge neutrality holds. Thus

$$B_o(t) = E_o(t) = \sum_s 4\pi e_s \int d^3v\; F_o^s(v,t) = \sum_s 4\pi e_s \int d^3v\; v\, F_o^s(v,t) = 0 \tag{II-32}$$

Therefore Eqs. (II-13), (II-14), and (II-15) are satisfied.

III. ORDERING SCHEME

Ordering the Equations

The parameter in which we shall order our equations is the electron temperature anisotropy

$$\alpha = (T_{\parallel} - T_{\perp})/T_{\text{isotropic}} = (b^2 - a^2)/d^2 \qquad \text{(III-1)}$$

where a and b are the perpendicular and parallel electron thermal velocities respectively. d is the thermal velocity which the electrons would have if their velocity distribution were isotropic

$$3d^2 = 2a^2(t=0) + b^2(t=0) \qquad \text{(III-2)}$$

We shall assume that effects due to the ion anisotropy may be neglected. The condition for the validity of this assumption will be given in the section on linear analysis.

We shall order Eqs. (II-16) through (II-20), and then make a multiple time scale expansion of the corresponding Fourier transformed Eqs. (II-27) through (II-31). To simplify writing our equations we define two quantities

$$\delta R^s(r,t) = \frac{e_s}{m_s c}\left(v \times \delta B(r,t)\right) \cdot \frac{\partial}{\partial v} \qquad \text{(III-3)}$$

$$\delta S^s(r,t) = \frac{e_s}{m_s}\, \delta E(r,t) \cdot \frac{\partial}{\partial v} \qquad \text{(III-4)}$$

Then Eqs. (II-16) through (II-20) become

$$\dot{F}_o = -\frac{1}{V}\int_V d^3r\, (\delta R + \delta S)\, \delta F \qquad \text{(III-5)}$$

$$\frac{\partial}{\partial r} \cdot \delta E = \sum_s 4\pi e_s \int d^3v\, \delta F^s \qquad \text{(III-6)}$$

$$\dot{\delta B} = -c\frac{\partial}{\partial r}\times\delta E \tag{III-7}$$

$$\dot{\delta E} - c\frac{\partial}{\partial r}\times\delta B + \sum_s 4\pi e_s\int d^3v\, v\,\delta F^s = 0 \tag{III-8}$$

$$\dot{\delta F} + v\cdot\frac{\partial}{\partial r}\delta F + (\delta R+\delta S)F_o + (\delta R+\delta S)\delta F - \frac{1}{V}\int_V d^3r\,(\delta R+\delta S)\delta F = 0 \tag{III-9}$$

where the dot denotes $\partial/\partial t$.

Linear analysis shows that the growing waves have growth rate

$$\gamma \sim \alpha\, k d \tag{III-10}$$

and have wave numbers k given by

$$k^2c^2 \sim \alpha\,\omega_e^2 \tag{III-11}$$

where ω_e is the electron plasma frequency. Linear analysis further shows that $E_k^{\text{longitudinal}} \sim E_k^{\text{transverse}}$. Using these we shall estimate the magnitudes of the various terms in our equations. For typical spatial gradients and time scales we take $\partial/\partial r \sim k$ and $\partial/\partial t \sim \gamma$. We begin by defining the dimensionless parameter ϵ

$$\frac{e}{mc}\,\delta B \sim \epsilon\, k d \tag{III-12}$$

To estimate δR we must take note of a property of $v \times \partial/\partial v$. $v \times \partial/\partial v$ operating on a spherically symmetric function of velocity gives zero. For example $v \times \partial/\partial v$ operating on a function of v^2 gives zero. Now F_o is equal to a spherically symmetric part plus a nonsymmetric part. The nonsymmetric part is $\sim \alpha F_o$. Therefore $v \times \partial/\partial v\, F_o \sim \alpha F_o$. Thus

$\delta R \sim \epsilon\, kd$ on something quite asymmetric, (III-13)

and $\delta R \sim \alpha\epsilon\, kd$ on F_o (III-14)

Since our waves have $E_k^{long.} \sim E_k^{trans.}$ we may assume the two terms in Faraday's Eq. (III-7) to be the same order

$$\begin{array}{ccc} \delta\dot{B} & = & -c\,\frac{\partial}{\partial r} \times \delta E \\ 1 & : & 1 \end{array} \qquad \text{(III-15)}$$

Using this we may estimate δE

$$\delta E \sim \frac{\gamma}{kc}\,\delta B \sim \alpha\left(\frac{d}{c}\right)\delta B \qquad \text{(III-16)}$$

and therefore

$$\delta S \sim \frac{e}{md}\,\delta E \sim \alpha\,\frac{e}{mc}\,\delta B \sim \alpha\epsilon k d \qquad \text{(III-17)}$$

from which it follows that

$$\delta S\, F_o \sim \delta R\, F_o \qquad \text{but} \qquad \text{—} \qquad \text{(III-18)}$$

$$\delta S\, \delta F \sim \alpha\, \delta R\, \delta F \qquad \text{(III-19)}$$

We estimate the relative magnitude of the two field terms in the Maxwell-Ampere Eq. (III-8)

$$\delta\dot{E} \sim \gamma\,\delta E \sim \frac{\gamma^2}{kc}\,\delta B \sim \left(\frac{\gamma}{kc}\right)^2 (kc\,\delta B) \sim \alpha^2\left(\frac{d}{c}\right)^2 \left(c\,\frac{\partial}{\partial r} \times \delta B\right) \qquad \text{(III-20)}$$

Therefore Eq. (III-8) may be ordered

$$\begin{array}{l} \delta\dot{E} - c\,\frac{\partial}{\partial r} \times \delta B + \sum_s 4\pi e_s \int d^3v\, v\, \delta F^s = 0 \\ \alpha^2\left(\frac{d}{c}\right)^2 \;:\; 1 \;:\; 1 \end{array} \qquad \text{(III-21)}$$

Next we find the magnitude of δF by using Eq. (III-21). Denote

$$n_o = \int d^3v\, F_o = \text{average density, and} \qquad \text{(III-22)}$$

$$\omega_e^2 = 4\pi n_o e^2/m = \text{square of plasma frequency} \qquad \text{(III-23)}$$

If we multiply Eq. (III-21) by ke/m we obtain

$$k^2c^2 \frac{e}{mc} \delta B \sim k \frac{e}{m} 4\pi e \int d^3v \, v \, \delta F \sim \frac{4\pi e^2}{m} \frac{m_o}{F_o} kd \, \delta F$$

$$\sim \omega_e^2 \, kd \, \delta F/F_o \qquad \text{(III-24)}$$

Hence

$$\delta F \sim \frac{\alpha}{kd} \frac{e}{mc} \delta B \, F_o \sim \alpha \epsilon F_o \qquad \text{(III-25)}$$

These estimates yield the following ordering for the Vlasov Eq. (III-9), after dividing out the common factor $\alpha\epsilon kdF_o$

$$\dot{\delta F} + v \cdot \frac{\partial}{\partial r} \delta F + \delta R \, F_o + \delta S \, F_o + \delta R \, \delta F + \delta S \, \delta F - \frac{1}{V} \int_V d^3r \, (\delta R + \delta S) \delta F = 0 \qquad \text{(III-26)}$$

$$\alpha : \quad 1 \quad : \quad 1 \quad : \quad 1 \quad : \quad \epsilon \quad : \alpha\epsilon \quad : \qquad (\epsilon : \alpha\epsilon)$$

The above estimates yield for the ordering of the Poisson Eq. (III-6)

$$\frac{\partial}{\partial r} \cdot \delta E = \sum_s 4\pi e_s \int d^3v \, \delta F^s \qquad \text{(III-27)}$$

$$\alpha \left(\frac{d}{c}\right)^2 : \qquad 1$$

This says that to lowest order in α the RHS is zero. In the section on linear analysis we will find that this gives us an equation for $E_k^{long.}$ in terms of B_k.

We consider finally the ordering of Eq. (III-5). On the basis of the above estimates we order this equation

$$\dot{F}_o = -\frac{1}{V} \int_V d^3r \, (\delta R + \delta S) \, \delta F \qquad \text{(III-28)}$$

$$1 \; : \qquad (\epsilon^2 : \alpha\epsilon^2)$$

This says that to lowest in ϵ F_o does not change on the γ^{-1} time scale. We will find that when F_o is expanded in powers of ϵ

$$F_o = F_o^{(0)} + F_o^{(1)} + F_o^{(2)} + \cdots \qquad \text{(III-29)}$$

that $F_o^{(2)}$ does change on the γ^{-1} time scale. The change is caused by the $\delta R \delta F$ term.

A precaution is to be observed in calculating velocity moments of Eq. (III-28)

$$\int d^3v \; g(v) \; \dot{F}_o \;=\; \int d^3v \; g(v) \cdot RHS \qquad \text{(III-30)}$$

If g(v) is spherically symmetric then integration by parts shows that the $\delta R \delta F$ term gives identically zero because of the factor $v \times \partial/\partial v$. The largest contribution to this moment is the $\delta S \delta F$ term, although in Eq. (III-28) it is higher order than the $\delta R \delta F$ term. The ordered expansion of a velocity moment of (III-28) is not equal to the moment of the ordered expansion of (III-28). g(v) must also be expanded.

Estimating How Large ϵ Becomes

We define the following energy densities

$$\mathcal{E}_{kin} \;=\; \tfrac{1}{2} m n_o \langle v_\perp^2 + v_\parallel^2 \rangle \qquad \text{(III-31)}$$

$$\mathcal{E}_{\perp kin} = \tfrac{1}{2} m n_o \langle v_\perp^2 \rangle \;=\; m n_o a^2 \qquad \text{(III-32)}$$

$$\mathcal{E}_{mag} \;=\; \frac{1}{V} \int_V d^3r \; |\delta B|^2 / 8\pi \qquad \text{(III-33)}$$

From the energy appendix we have

$$\dot{\mathcal{E}}_{mag} \sim \dot{\mathcal{E}}_{kin} \sim \alpha \, \dot{\mathcal{E}}_{\perp kin} \qquad \text{(III-34)}$$

The above relations hold nonlinearly. Thus we have

$$\frac{\partial}{\partial t} \frac{1}{V} \int_V d^3r \; |\delta B|^2 / 8\pi \;\sim\; \alpha m n_o \frac{\partial}{\partial t} a^2 \qquad \text{(III-35)}$$

We integrate and take $\mathcal{E}_{mag}(t=0) \doteq 0$, then

$$\frac{1}{V}\int_V d^3r\,|\delta B(t)|^2 \sim |\delta B(t)|^2 \sim 8\pi\alpha m n_0\left(a^2(t)-a^2(0)\right) \lesssim 8\pi\alpha m n_0\,\alpha d^2 \qquad \text{(III-36)}$$

where for Δa^2 we have substituted its maximum value. Therefore

$$\left|\frac{e}{mc}\,\delta B(\max)\right|^2 \sim \frac{8\pi n_0 e^2}{mc^2}\,\alpha^2 d^2 \sim \frac{\omega_e^2}{c^2}\,\alpha^2 d^2 \sim \alpha\, k^2 d^2 \qquad \text{(III-37)}$$

where we have used $k^2c^2 \sim \alpha\,\omega_e^2$. Comparing Eq. (III-37) with (III-12) we see that

$$\epsilon^2(\max) \sim \alpha \qquad \text{(III-38)}$$

Thus if we choose the ordering scheme $\epsilon = \alpha^{1/2}$ then we shall be able to follow the evolution of our waves through their maximum amplitude and to their subsequent decay.

Multiple Time Scale Expansion[22,23,24,25]

We now consider a multiple time scale expansion of the Fourier transform of Eqs. (III-5) through (III-9). The customary practice when we have a function having multiple time scales, $F(t,\alpha^{1/2}t,\alpha t,\ldots)$, is to introduce new variables, τ_0, τ_1, τ_2 ..., such that

$$F(t,\alpha^{1/2}t,\alpha t,\ldots) = F(\tau_0,\tau_1,\tau_2,\ldots) \qquad \text{(III-39)}$$

then

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau_0} + \alpha^{1/2}\frac{\partial}{\partial \tau_1} + \alpha\frac{\partial}{\partial \tau_2} + \cdots \qquad \text{(III-40)}$$

However this is inconvenient if α is itself a function of time. Therefore we choose

$$F(t,\alpha^{1/2}t,\alpha t,\ldots) = F(t_0,\alpha^{1/2}t_1,\alpha t_2,\ldots) \qquad \text{(III-41)}$$

thus

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t_0} + \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} + \cdots \tag{III-42}$$

At the end of a calculation, we equate the various time scales by

$$t = t_0 = t_1 = t_2 = \cdots \tag{III-43}$$

We expand each of our functions in powers of $\alpha^{1/2}$

$$F_0 = F_0^{(0)} + F_0^{(1)} + F_0^{(2)} + F_0^{(3)} + \cdots \tag{III-44}$$

$$F_k = F_k^{(0)} + F_k^{(1)} + F_k^{(2)} + \cdots \tag{III-45}$$

$$B_k = B_k^{(0)} + B_k^{(1)} + \cdots \tag{III-46}$$

$$E_k = E_k^{(0)} + E_k^{(1)} + \cdots \tag{III-47}$$

where superscript (o) refers to the largest nonzero term in the expansion. Thus from Eqs. (III-25, 12, 16, and 10) we have

$$F_k^{(0)} \sim \alpha^{3/2} F_0^{(0)} \tag{III-48}$$

$$B_k^{(0)} \sim \alpha^{1/2} kd \frac{mc}{e} \tag{III-49}$$

$$E_k^{(0)} \sim \alpha \frac{d}{c} B_k^{(0)} \tag{III-50}$$

$$\frac{\partial}{\partial t_0} \sim \alpha kd \tag{III-51}$$

Using

$$k^2c^2 \sim \alpha \omega_e^2 \tag{III-52}$$

from Eq. (III-11) we may eliminate k from the above expressions to obtain

$$F_k^{(o)} \sim \alpha^{3/2} F_o^{(o)} \qquad = \mathcal{O}(\alpha^{3/2}) \qquad \text{(III-53)}$$

$$B_k^{(o)} \sim \alpha \ (n_o m d^2)^{1/2} \qquad = \mathcal{O}(\alpha) \qquad \text{(III-54)}$$

$$E_k^{(o)} \sim \alpha^2 \frac{d}{c} (n_o m d^2)^{1/2} \qquad = \mathcal{O}(\alpha^2) \qquad \text{(III-55)}$$

$$\frac{\partial}{\partial t_o} \sim \alpha^{3/2} \frac{e}{mc} (n_o m d^2)^{1/2} \qquad = \mathcal{O}(\alpha^{3/2}) \qquad \text{(III-56)}$$

$$kd \sim \alpha^{1/2} \frac{e}{mc} (n_o m d^2)^{1/2} \qquad = \mathcal{O}(\alpha^{1/2}) \qquad \text{(III-57)}$$

For initial conditions we choose

$$F_o^{(o)}(o) = \text{the isotropic part of } F_o(o) \qquad \text{(III-58)}$$

$F_o^{(o)}(o)$ is defined to be an isotropic distribution, all of whose moments, $\langle u^2 \rangle$, $\langle u^4 \rangle$, etc. are the same as those of $F_o(o)$. We shall see in the section on qualitative considerations that this definition is unique.

$$F_o^{(2)}(o) = F_o(o) - F_o^{(o)}(o) = \text{anisotropic part of } F_o(o) \qquad \text{(III-59)}$$

Then the moments of u^2, u^4, etc. with respect to $F_o^{(2)}(o)$ are all zero.

$$F_o^{(1)}(o) = F_o^{(3)}(o) = F_o^{(4)}(o) = \ldots = 0 \qquad \text{(III-60)}$$

$$F_k^{(o)}(o) = F_k(o) \qquad F_k^{(1)}(o) = F_k^{(2)}(o) = \ldots = 0 \qquad \text{(III-61)}$$

$$B_k^{(o)}(o) = B_k(o) \qquad B_k^{(1)}(o) = B_k^{(2)}(o) = \ldots = 0 \qquad \text{(III-62)}$$

$$E_k^{(o)}(o) = E_k(o) \qquad E_k^{(1)}(o) = E_k^{(2)}(o) = \ldots = 0 \qquad \text{(III-63)}$$

We shall find that our plasma evolves in the direction of velocity isotropy. The anisotropic part of F_o is largest at $t = 0$ where it is of order α. There is thus no need for $F_o^{(1)}$ and we shall find that we may take $F_o^{(1)} = 0$ for all time. Likewise since $F_o^{(o)}(o)$ must be the

final state of F_o, there is no need for $F_o^{(o)}$ to change on any time scale. We shall find that we may take $F_o^{(o)}$ to be constant.

Substituting our expansions into Eqs. (II-27) through (II-31), and collecting terms of the same order in $\alpha^{1/2}$, we obtain the following sets of equations

$$\frac{\partial}{\partial t_o} B_k^{(o)} = -ick \times E_k^{(o)} \tag{III-64}$$

$$-ick \times B_k^{(o)} + \sum_s 4\pi e_s \int d^3v \; v \; F_k^{s(o)} = 0 \tag{III-65}$$

$$ik \cdot v \; F_k^{(o)} + R_k^{(o)} F_o^{(2)} + S_k^{(o)} F_o^{(o)} = 0 \tag{III-66}$$

$$\sum_s 4\pi e_s \int d^3v \; F_k^{s(o)} = 0 \tag{III-67}$$

$$\frac{\partial}{\partial t_o} B_k^{(1)} + \frac{\partial}{\partial t_1} B_k^{(o)} = -ick \times E_k^{(1)} \tag{III-68}$$

$$-ick \times B_k^{(1)} + \sum_s 4\pi e_s \int d^3v \; v \; F_k^{s(1)} = 0 \tag{III-69}$$

$$ik \cdot v \; F_k^{(1)} + R_k^{(1)} F_o^{(2)} + R_k^{(o)} F_o^{(3)} + S_k^{(1)} F_o^{(o)} + S_k^{(o)} F_o^{(1)} + \int \frac{d^3\ell}{(2\pi)^3} R_{k-\ell}^{(o)} F_\ell^{(o)} = 0 \tag{III-70}$$

$$\sum_s 4\pi e_s \int d^3v \; F_k^{s(1)} = 0 \tag{III-71}$$

$$\frac{\partial}{\partial t_o} B_k^{(2)} + \frac{\partial}{\partial t_1} B_k^{(1)} + \frac{\partial}{\partial t_2} B_k^{(o)} = -ick \times E_k^{(2)} \tag{III-72}$$

$$-ick \times B_k^{(2)} + \sum_s 4\pi e_s \int d^3v \; v \; F_k^{s(2)} = 0 \tag{III-73}$$

$$\begin{aligned} &\frac{\partial}{\partial t_o} F_k^{(o)} + ik \cdot v \; F_k^{(2)} + R_k^{(2)} F_o^{(2)} + R_k^{(1)} F_o^{(3)} + R_k^{(o)} F_o^{(4)} + S_k^{(2)} F_o^{(o)} + S_k^{(1)} F_o^{(1)} \\ &\qquad + S_k^{(o)} F_o^{(2)} + \int \frac{d^3\ell}{(2\pi)^3} \left(R_{k-\ell}^{(o)} F_\ell^{(1)} + R_{k-\ell}^{(1)} F_\ell^{(o)} \right) = 0 \end{aligned} \tag{III-74}$$

$$ik \cdot E_k^{(o)} = \sum_s 4\pi e_s \int d^3v \; F_k^{s(2)} \tag{III-75}$$

$$\frac{\partial}{\partial t_0} F_0^{(0)} = 0 \tag{III-76}$$

$$\frac{\partial}{\partial t_0} F_0^{(1)} + \frac{\partial}{\partial t_1} F_0^{(0)} = 0 \tag{III-77}$$

$$\frac{\partial}{\partial t_0} F_0^{(2)} + \frac{\partial}{\partial t_1} F_0^{(1)} + \frac{\partial}{\partial t_2} F_0^{(0)} = -\frac{1}{V}\int \frac{d^3k}{(2\pi)^3} R_{-k}^{(0)} F_k^{(0)} \tag{III-78}$$

$$\frac{\partial}{\partial t_0} F_0^{(3)} + \cdots + \frac{\partial}{\partial t_3} F_0^{(0)} = -\frac{1}{V}\int \frac{d^3k}{(2\pi)^3} \left(R_{-k}^{(1)} F_k^{(0)} + R_{-k}^{(0)} F_k^{(1)} \right) \tag{III-79}$$

$$\frac{\partial}{\partial t_0} F_0^{(4)} + \cdots + \frac{\partial}{\partial t_4} F_0^{(0)} = -\frac{1}{V}\int \frac{d^3k}{(2\pi)^3} \left(R_{-k}^{(2)} F_k^{(0)} + R_{-k}^{(1)} F_k^{(1)} + R_{-k}^{(0)} F_k^{(2)} + S_{-k}^{(0)} F_k^{(0)} \right) \tag{III-80}$$

$$\frac{\partial}{\partial t_0} F_0^{(5)} + \cdots + \frac{\partial}{\partial t_5} F_0^{(0)} = -\frac{1}{V}\int \frac{d^3k}{(2\pi)^3} \left(R_{-k}^{(3)} F_k^{(0)} + \cdots + S_{-k}^{(1)} F_k^{(0)} + S_{-k}^{(0)} F_k^{(1)} \right) \tag{III-81}$$

$$\frac{\partial}{\partial t_0} F_0^{(6)} + \cdots + \frac{\partial}{\partial t_6} F_0^{(0)} = -\frac{1}{V}\int \frac{d^3k}{(2\pi)^3} \left(R_{-k}^{(4)} F_k^{(0)} + \cdots + S_{-k}^{(2)} F_k^{(0)} + \cdots \right) \tag{III-82}$$

IV. QUASILINEAR APPROXIMATION

The Equation for F_o

The quasilinear approximation consists of Eqs. (III-64) through (III-67) and (III-76) through (III-78). We consider now the evolution of F_o. Equation (III-76) shows that $F_o^{(o)}$ is not a function of t_o. Integrate Eq. (III-77) on the t_o time scale. $\partial F_o^{(o)}/\partial t_1$ is a constant on this time scale, hence $F_o^{(1)}$ will grow like

$$F_o^{(1)} = \text{constant} \cdot t_o \tag{IV-1}$$

Eventually $F_o^{(1)}$ would become comparable to $F_o^{(o)}$ contrary to the ordering assumptions. This unwanted secular behavior is suppressed by requiring

$$\frac{\partial}{\partial t_1} F_o^{(o)} = 0 \tag{IV-2}$$

from which it follows that

$$\frac{\partial}{\partial t_o} F_o^{(1)} = 0 \tag{IV-3}$$

Thus $F_o^{(o)}$ is not a function of t_1 and $F_o^{(1)}$ is not a function of t_o.

Let us integrate Eq. (III-78) on the t_o time scale. $R_{-k}^{(o)} F_k^{(o)}$ is a function of t_o, however $\partial F_o^{(o)}/\partial t_2$ and $\partial F_o^{(1)}/\partial t_1$ are constants on the t_o time scale. These would therefore lead to secular behavior in $F_o^{(2)}$. In order to suppress this secularity we must require

$$\frac{\partial}{\partial t_2} F_o^{(o)} + \frac{\partial}{\partial t_1} F_o^{(1)} = 0 \tag{IV-4}$$

from which follows

$$\frac{\partial}{\partial t_o} F_o^{(2)} = -\frac{1}{V} \int \frac{d^3k}{(2\pi)^3} R_{-k}^{(o)} F_k^{(o)} \tag{IV-5}$$

If we had a problem with oscillating waves instead of purely growing waves, the t_o dependence of F_k would have been like $\exp -i\Omega_k t_o$ where Ω_k is real. Since $R_{-k} = R_k^*$, the two exponentials in the product $R_{-k}F_k$ would have canceled leaving $R_{-k}F_k$ constant on the t_o time scale. The $R_{-k}F_k$ term would then have contributed to secular behavior in $F_o^{(2)}$. To suppress the secularity we would have required $F_o^{(2)}$ to not be a function of t_o. The $R_{-k}F_k$ term would then have determined either the t_1 dependence of $F_o^{(1)}$ or the t_2 dependence of $F_o^{(o)}$.

Continuing with the case at hand, we integrate Eq. (IV-4) on the t_1 time scale. $\partial F_o^{(o)}/\partial t_2$ is a constant on this time scale and would lead to secular behavior in $F_o^{(1)}$. To suppress the secularity we must require

$$\frac{\partial}{\partial t_2} F_o^{(o)} = 0 \qquad \text{(IV-6)}$$

from which follows

$$\frac{\partial}{\partial t_1} F_o^{(1)} = 0 \qquad \text{(IV-7)}$$

Thus $F_o^{(o)}$ is not a function of t_o, t_1, or t_2, and $F_o^{(1)}$ is not a function of t_o or t_1. Repetition of this analysis with higher order equations leads to the conclusion that $F_o^{(o)}$ and $F_o^{(1)}$ are constant on all time scales. From our initial condition Eqs. (III-58) and (III-60) it then follows that

$$F_o^{(o)} = F_o^{(o)}(o) \qquad \text{(IV-8)}$$

$$F_o^{(1)} = 0 \qquad \text{(IV-9)}$$

The Equation for $f_o^{(2)}$

Since $F_o^{(o)}$ is constant and $F_o^{(1)}$ is zero, the lowest order piece of the distribution function that undergoes change is $F_o^{(2)}$. We define distribution functions which have the average density divided out

$$F_o^{(n)} = n_o f_o^{(n)} \qquad \text{and} \qquad F_k^{(n)} = n_o f_k^{(n)} \tag{IV-10}$$

We solve Eq. (III-66) for $F_k^{(o)}$ and substitute this into Eq. (IV-5). We divide the latter by n_o to obtain

$$\begin{aligned} \frac{\partial}{\partial t_o} f_o^{(2)} &= -\frac{1}{V}\int \frac{d^3k}{(2\pi)^3} \; R_{-k}^{(o)} \; f_k^{(o)} \\ &= -\frac{1}{V}\int \frac{d^3k}{(2\pi)^3} \; R_{-k}^{(o)} \; \frac{i}{k\cdot v} \left\{ R_k^{(o)} f_o^{(2)} + S_k^{(o)} f_o^{(o)} \right\} \end{aligned} \tag{IV-11}$$

From Eq. (III-3,4) we have

$$R_k^{(o)} = \frac{e}{mc} \left(v \times B_k^{(o)} \right) \cdot \frac{\partial}{\partial v} \tag{IV-12}$$

$$S_k^{(o)} = \frac{e}{m} E_k^{(o)} \cdot \frac{\partial}{\partial v} = \frac{e}{m} \left(E_k^{(o)\,trans} + E_k^{(o)\,long} \right) \cdot \frac{\partial}{\partial v} \tag{IV-13}$$

Denote the amplitude of $E_k^{(o)long}$ by e_k^{ℓ} and denote a unit vector in the k-direction by $\hat{k}$, then

$$E_k^{(o)\,long} = e_k^{\ell} \; \hat{k} \tag{IV-14}$$

We may solve Eq. (III-64) for $E_k^{(o)trans}$

$$E_k^{(o)\,trans} = \frac{-i}{k^2 c} \; k \times \frac{\partial}{\partial t_o} B_k^{(o)} \tag{IV-15}$$

Substituting these relations into Eq. (IV-11) gives

$$\begin{aligned} \frac{\partial}{\partial t_o} f_o^{(2)} = -\frac{1}{V}\int \frac{d^3k}{(2\pi)^3} \; \frac{e}{mc} \left(v \times B_{-k}^{(o)} \right) \cdot \frac{\partial}{\partial v} \; \frac{i}{k\cdot v} \Big\{ & \frac{e}{mc} \left(v \times B_k^{(o)} \right) \cdot \frac{\partial}{\partial v} f_o^{(2)} \\ & - \frac{i}{k^2} \frac{e}{mc} \left(k \times \frac{\partial}{\partial t_o} B_k^{(o)} \right) \cdot \frac{\partial}{\partial v} f_o^{(o)} + \frac{e}{m} e_k^{\ell} \; \hat{k} \cdot \frac{\partial}{\partial v} f_o^{(o)} \Big\} \end{aligned} \tag{IV-16}$$

Consider the $E_k^{(o)long}$ contribution

$$(v\times B_{-k}^{(0)})\cdot\frac{\partial}{\partial v}\ \frac{1}{k\cdot v}\ \hat{k}\cdot\frac{\partial}{\partial v}\ f_o^{(0)} = (v\times B_{-k}^{(0)})\cdot\frac{\partial}{\partial v}\ \frac{1}{k\cdot v}\ \hat{k}\cdot v\left(\frac{-1}{d^2}\right) f_o^{(0)}$$

$$= B_{-k}^{(0)}\cdot\left(v\times\frac{\partial}{\partial v}\right)\ \frac{1}{kd^2}\ f_o^{(0)} = 0 \qquad \text{(IV-17)}$$

where we have used

$$f_o^{(0)} = d^{-3}(2\pi)^{-3/2}\ \exp -\frac{1}{2}\frac{v^2}{d^2} \qquad \text{(IV-18)}$$

Thus $E_k^{(o)long}$ does not affect the evolution of $f_o^{(2)}$. Equation (IV-16) is reduced to

$$\frac{\partial}{\partial t_o} f_o^{(2)} = -\frac{1}{V}\frac{e^2}{m^2c^2}\int\frac{d^3k}{(2\pi)^3}\left(v\times B_{-k}^{(0)}\right)\cdot\frac{\partial}{\partial v}\ \frac{1}{k\cdot v}\left\{i\left(v\times B_k^{(0)}\right)\cdot\frac{\partial}{\partial v}\ f_o^{(2)}\right.$$

$$\left. + \frac{1}{k^2}\left(k\times\frac{\partial}{\partial t_o}B_k^{(0)}\right)\cdot\frac{\partial}{\partial v}\ f_o^{(0)}\right\} \qquad \text{(IV-19)}$$

Note that the first term is odd in k and thus only the pole contributes to this term. The second term is even in k and thus only the principal value is nonzero.

In the section on linear analysis we find that there are two modes, one growing and one damped for $T_{\parallel} > T_{\perp}$. We ignore the damped mode and set

$$B_k^{(0)} = b_k\,\hat{B}_k \qquad \text{and} \qquad B_{-k}^{(0)} = B_k^{(0)*} = b_k^*\,\hat{B}_k \qquad \text{(IV-20)}$$

where $\hat{B}_k$ is the unit vector for the growing mode and b_k is the amplitude. Then Eq. (IV-19) may be written

$$\frac{\partial}{\partial t_o} f_o^{(2)} = -\frac{1}{V}\frac{e^2}{m^2c^2}\int\frac{d^3k}{(2\pi)^3}\left\{|B_k^{(0)}|^2\left(v\times\hat{B}_k\right)\cdot\frac{\partial}{\partial v}\ \frac{i}{k\cdot v}\left(v\times\hat{B}_k\right)\cdot\frac{\partial}{\partial v}\ f_o^{(2)}\right.$$

$$\left. + \frac{1}{k^2}\left(b_k^*\frac{\partial}{\partial t_o}b_k\right)\left(v\times\hat{B}_k\right)\cdot\frac{\partial}{\partial v}\ \frac{1}{k\cdot v}\left(k\times\hat{B}_k\right)\cdot\frac{\partial}{\partial v}\ f_o^{(0)}\right\} \qquad \text{(IV-21)}$$

In the second term we add the contributions from k and -k and divide by 2

$$\left(b_k^* \frac{\partial}{\partial t_0} b_k\right) \rightarrow \frac{1}{2}\left(b_k^* \frac{\partial}{\partial t_0} b_k + b_k \frac{\partial}{\partial t_0} b_k^*\right) = \frac{1}{2}\frac{\partial}{\partial t_0}|B_k^{(0)}|^2 \qquad \text{(IV-22)}$$

Thus Eq. (IV-21) becomes finally

$$\frac{\partial}{\partial t_0} f_0^{(2)} = -\frac{1}{V}\frac{e^2}{m^2c^2}\int\frac{d^3k}{(2\pi)^3}\left\{|B_k^{(0)}|^2 (v\times\hat{B}_k)\cdot\frac{\partial}{\partial v}\frac{i}{k\cdot v}(v\times\hat{B}_k)\cdot\frac{\partial}{\partial v} f_0^{(2)}\right.$$

$$\left. + \frac{1}{2k^2}\frac{\partial}{\partial t_0}|B_k^{(0)}|^2 (v\times\hat{B}_k)\cdot\frac{\partial}{\partial v}\frac{1}{k\cdot v}(k\times\hat{B}_k)\cdot\frac{\partial}{\partial v} f_0^{(0)}\right\} \qquad \text{(IV-23)}$$

Diffusion Equation

An equation of the diffusion form may be produced by adding terms of order α to Eq. (IV-19). First, since

$$\frac{\partial}{\partial t_0} f_0^{(0)} = 0 \qquad \text{(IV-24)}$$

we may add $\partial f_0^{(0)}/\partial t_0$ to the LHS. Likewise since

$$\left(v\times\frac{\partial}{\partial v}\right) f_0^{(0)} = 0 \qquad \text{(IV-25)}$$

we may add $f_0^{(0)}$ to $f_0^{(2)}$ in the first term on the RHS. Adding $f_0^{(2)}$ to $f_0^{(0)}$ in the second term on the RHS causes a change only of order α in this term. The result is

$$\frac{\partial}{\partial t_0}\left(f_0^{(0)} + f_0^{(2)}\right) = -\frac{1}{V}\frac{e^2}{m^2c^2}\int\frac{d^3k}{(2\pi)^3}\left\{\frac{\partial}{\partial v}\cdot(v\times B_{-k}^{(0)})\frac{i}{k\cdot v}(v\times B_k^{(0)})\cdot\frac{\partial}{\partial v}\left(f_0^{(0)} + f_0^{(2)}\right)\right.$$

$$\left. + \frac{\partial}{\partial v}\cdot(v\times B_{-k}^{(0)})\frac{1}{k^2}\frac{1}{k\cdot v}\left(k\times\frac{\partial}{\partial t_0}B_k^{(0)}\right)\cdot\frac{\partial}{\partial v}\left(f_0^{(0)} + f_0^{(2)}\right)\right\} \qquad \text{(IV-26)}$$

This is clearly of the diffusion form.

We shall add to Eq. (IV-26) another term of order α to simplify the symmetry. Equation (IV-11) from which we started contains the RR term

and the RS term of the equation for $\dot{F}_o$. As we have seen, the SR term of the equation for $\dot{F}_o$ is of order α times the RS term. The part of the SR term due to $E_k^{(o)trans}$ is

$$\frac{\partial}{\partial v}\cdot\left(-k\times\frac{\partial}{\partial t_0}B_{-k}^{(o)}\right)\frac{1}{k^2}\frac{1}{k\cdot v}\left(v\times B_k^{(o)}\right)\cdot\frac{\partial}{\partial v}f_o^{(2)} \tag{IV-27}$$

The above term will cause only a change of order α when added to Eq. (IV-26). Finally since

$$\left(v\times\frac{\partial}{\partial v}\right)f_o^{(o)} = 0 \tag{IV-28}$$

we may add $f_o^{(o)}$ to $f_o^{(2)}$ in (IV-27) with no change at all. The result of adding these terms to Eq. (IV-26) is

$$\frac{\partial}{\partial t_0}\left(f_o^{(o)}+f_o^{(2)}\right) = \frac{\partial}{\partial v}\cdot D\cdot\frac{\partial}{\partial v}\left(f_o^{(o)}+f_o^{(2)}\right) \tag{IV-29}$$

where

$$D = -\frac{1}{V}\frac{e^2}{m^2c^2}\int\frac{d^3k}{(2\pi)^3}\left\{\left(v\times B_{-k}^{(o)}\right)\frac{i}{k\cdot v}\left(v\times B_k^{(o)}\right)\right.$$
$$\left.+\frac{1}{k^2}\frac{1}{k\cdot v}\left[\left(v\times B_{-k}^{(o)}\right)\left(k\times\frac{\partial}{\partial t_0}B_k^{(o)}\right)-\left(k\times\frac{\partial}{\partial t_0}B_{-k}^{(o)}\right)\left(v\times B_k^{(o)}\right)\right]\right\} \tag{IV-30}$$

If we separate the amplitude and unit vector of $B_k^{(o)}$, then D may be written

$$D = -\frac{1}{V}\frac{e^2}{m^2c^2}\int\frac{d^3k}{(2\pi)^3}\left\{\left|B_k^{(o)}\right|^2\left(v\times\hat{B}_k\right)\frac{i}{k\cdot v}\left(v\times\hat{B}_k\right)\right.$$
$$\left.+\frac{1}{2k^2}\frac{1}{k\cdot v}\frac{\partial}{\partial t_0}\left|B_k^{(o)}\right|^2\left[\left(v\times\hat{B}_k\right)\left(k\times\hat{B}_k\right)-\left(k\times\hat{B}_k\right)\left(v\times\hat{B}_k\right)\right]\right\} \tag{IV-31}$$

We note that the $|B_k^{(o)}|^2$ term of D is a symmetric tensor, whereas the $\frac{\partial}{\partial t_o}|B_k^{(o)}|^2$ term is purely antisymmetric. Furthermore since the pole prescription is that the contour of integration go under the pole, an additional factor of i is produced in performing $\int d^3k$ in the symmetric

term. The consequence of this is that

$$\varphi \cdot D^{sym} \cdot \varphi = \text{positive definite form} \qquad \text{(IV-32)}$$

These results will be used in the H-theorem appendix.

The Equation for $B_k^{(o)}$

We solve Eq. (III-66) for $F_k^{(o)}$ and substitute this into Eq. (III-65)

$$-ick \times B_k^{(o)} + \sum_s 4\pi n_{os} e_s \int d^3v \; v \frac{i}{k\cdot v} \left\{ R_k^{s(o)} f_o^{s(2)} + S_k^{s(o)} f_o^{s(o)} \right\} = 0 \qquad \text{(IV-33)}$$

Substituting R and S from Eqs. (IV-12) through (IV-15) yields

$$-ick \times B_k^{(o)} + \sum_s \omega_s^2 \int d^3v \; v \frac{i}{k\cdot v} \left\{ \frac{1}{c}(v \times B_k^{(o)}) \cdot \frac{\partial}{\partial v} f_o^{s(2)} \right.$$

$$\left. - \frac{i}{k^2 c}\left(k \times \frac{\partial}{\partial t_o} B_k^{(o)}\right) \cdot \frac{\partial}{\partial v} f_o^{s(o)} + e_k^{\ell} \, \hat{k} \cdot \frac{\partial}{\partial v} f_o^{s(o)} \right\} = 0 \qquad \text{(IV-34)}$$

where the square of the plasma frequency is

$$\omega_s^2 = 4\pi n_{os} e_s^2 / m_s \qquad \text{(IV-35)}$$

Consider the $E_k^{(o)long}$ contribution

$$\int d^3v \; v \frac{1}{k\cdot v} \, k \cdot \frac{\partial}{\partial v} f_o^{(o)} = \int d^3v \; v \frac{1}{k\cdot v} \, k \cdot v \left(\frac{-1}{d^2}\right) f_o^{(o)}$$

$$= -\frac{1}{d^2} \int d^3v \; v \, f_o^{(o)} = 0 \qquad \text{(IV-36)}$$

where we have used Eq. (IV-18) for $f_o^{(o)}$. Thus $E_k^{(o)long}$ does not affect the evolution of $B_k^{(o)}$. The equation for $B_k^{(o)}$ is reduced to

$$ck \times B_k^{(o)} + \sum_s \frac{\omega_s^2}{c} H_{2s} \cdot B_k^{(o)} + \sum_s \frac{\omega_s^2}{k^2 c} H_{1s} \cdot \left(k \times \frac{\partial}{\partial t_o} B_k^{(o)}\right) = 0 \qquad \text{(IV-37)}$$

where

$$H_{1s} = i \int d^3v \, \frac{v}{k\cdot v} \frac{\partial}{\partial v} f_o^{s(o)} \tag{IV-38}$$

and

$$H_{2s} = \int d^3v \, \frac{v}{k\cdot v} \left(v \times \frac{\partial}{\partial v}\right) f_o^{s(2)} \tag{IV-39}$$

Notice that Eq. (IV-37) is nonlinear. Although it is linear in the fields, the term $H_{2s}B_k^{(o)}$ makes the equation nonlinear because $f_o^{s(2)}$ itself changes on the t_o time scale. Equations (IV-19) and (IV-37) form a coupled pair of equations for the evolution of $f_o^{(2)}$ and $B_k^{(o)}$, and must be integrated simultaneously.

We can form an equation for the evolution of the spectrum $|B_k^{(o)}|^2$ by taking the scalar product of Eq. (IV-37) with $(k \times B_{-k}^{(o)})$

$$c(k\times\hat{B}_k)\cdot(k\times\hat{B}_k)\,|B_k^{(o)}|^2 + \sum_s \frac{\omega_s^2}{c}\,(k\times\hat{B}_k)\cdot H_{2s}\cdot\hat{B}_k\,|B_k^{(o)}|^2$$

$$+ \sum_s \frac{\omega_s^2}{k^2 c}\,(k\times\hat{B}_k)\cdot H_{1s}\cdot(k\times\hat{B}_k)\left(b_{-k}\frac{\partial}{\partial t_o} b_k\right) = 0 \tag{IV-40}$$

where we have used b_k and $\hat{B}_k$ to denote the amplitude and unit vector of $B_k^{(o)}$ respectively. We shall find that H_{2s} is odd in k since the principal value is nonzero whereas the residue of the pole vanishes. H_{1s} on the other hand is even in k since only the residue is nonzero. Using this information we evaluate Eq. (IV-40) for -k, add the result to Eq. (IV-40), and divide by 2. The resulting equation is identical to Eq. (IV-40) except for

$$\left(b_{-k}\frac{\partial}{\partial t_o} b_k\right) \longrightarrow \frac{1}{2}\left(b_{-k}\frac{\partial}{\partial t_o} b_k + b_k \frac{\partial}{\partial t_o} b_{-k}\right) = \frac{1}{2}\frac{\partial}{\partial t_o}|B_k^{(o)}|^2 \tag{IV-41}$$

Finally since k is perpendicular to $\hat{B}_k$ we have

$$(k\times\hat{B}_k)\cdot(k\times\hat{B}_k) = k^2 \tag{IV-42}$$

Thus Eq. (IV-40) may be written

$$\left\{k^2c^2 + \sum_s \omega_s^2 G_{2s}\right\} |B_k^{(o)}|^2 + \sum_s \frac{\omega_s^2}{2k^2} G_{1s} \frac{\partial}{\partial t_o} |B_k^{(o)}|^2 = 0 \qquad \text{(IV-43)}$$

where

$$G_{1s} = (k \times \hat{B}_k) \cdot H_{1s} \cdot (k \times \hat{B}_k) \qquad \text{(IV-44)}$$

and

$$G_{2s} = (k \times \hat{B}_k) \cdot H_{2s} \cdot \hat{B}_k \qquad \text{(IV-45)}$$

The Equation for $E_k^{(o)\text{long}}$

We solve Eq. (III-66) for $F_k^{(o)}$ and substitute this into Eq. (III-67)

$$\sum_s 4\pi n_{os} e_s \int d^3v \, \frac{i}{k \cdot v} \left\{ R_k^{s(o)} f_o^{s(2)} + S_k^{s(o)} f_o^{s(o)} \right\} = 0 \qquad \text{(IV-46)}$$

Substituting R and S from Eqs. (IV-12) and (IV-13) yields

$$\sum_s \omega_s^2 \int d^3v \, \frac{1}{k \cdot v} \left\{ \frac{1}{c} (v \times B_k^{(o)}) \cdot \frac{\partial}{\partial v} f_o^{s(2)} + E_k^{(o)} \cdot \frac{\partial}{\partial v} f_o^{s(o)} \right\} = 0 \qquad \text{(IV-47)}$$

Consider the integral

$$I = \int d^3v \, \frac{1}{k \cdot v} \frac{\partial}{\partial v} f_o^{(o)} = -\frac{1}{d^2} \int d^3v \, \frac{v}{k \cdot v} f_o^{(o)} \qquad \text{(IV-48)}$$

$f_o^{(o)}$ is isotropic. Therefore I must be proportional to k since it is the only vector in I. Thus

$$I = (\hat{k} \cdot I)\hat{k} = -\frac{1}{kd^2} \int d^3v \, f_o^{(o)} \hat{k} = -\frac{1}{kd^2} \hat{k} \qquad \text{(IV-49)}$$

Consequently Eq. (IV-47) may be written

$$\sum_s \frac{\omega_s^2}{kd_s^2} e_k^{\ell} = -\sum_s \frac{\omega_s^2}{c} H_{3s} \cdot B_k^{(o)} \qquad \text{(IV-50)}$$

where

$$H_{3s} = \int d^3v \, \frac{1}{k\cdot v} \left(v \times \frac{\partial}{\partial v}\right) f_o^{s(2)} \tag{IV-51}$$

Thus $E_k^{(o)long}$ is given once $B_k^{(o)}$ is known. However as we have seen $E_k^{(o)long}$ is not needed to find the evolution of $f_o^{(2)}$ and $B_k^{(o)}$. We shall investigate the polarization of the electric field in the section on linear analysis, and thereafter we shall not concern ourselves with the longitudinal electric field.

V. LINEAR ANALYSIS

Expansion of a Bi-Maxwellian in Powers of α

We introduce the coordinate system shown in Fig. V-1. We let $\hat{1}$, $\hat{2}$, and $\hat{3}$ denote unit vectors along the principal axes. Throughout its evolution the plasma will have its temperature maximum along the 3-axis and its temperature minimum on the 1,2-plane. We consider a bi-Maxwellian distribution function which has the same maximum and minimum temperature as the plasma.

$$f_{bimax} = a^{-2} b^{-1} (2\pi)^{-3/2} \exp -\tfrac{1}{2}\left\{ v_\perp^2/a^2 + v_3^2/b^2 \right\} \tag{V-1}$$

where

$$v_\perp^2 = v_1^2 + v_2^2 \qquad\qquad v^2 = v_\perp^2 + v_3^2 \tag{V-2}$$

The exponent may be decomposed into an isotropic term of order one and an anisotropic term of order α.

$$\begin{aligned} f_{bimax} &= d^{-3}(2\pi)^{-3/2}\left(\frac{d^3}{a^2 b}\right) \exp -\tfrac{1}{2}\left\{ \frac{v^2}{d^2} + \frac{v_\perp^2}{d^2}\left(\frac{d^2-a^2}{a^2}\right) + \frac{v_3^2}{d^2}\left(\frac{d^2-b^2}{b^2}\right)\right\} \\ &= d^{-3}(2\pi)^{-3/2}\left(\exp -\tfrac{1}{2}\frac{v^2}{d^2}\right)\left(\frac{d^3}{a^2 b}\right)\left[1 - \tfrac{1}{2}\frac{v_\perp^2}{d^2}\left(\frac{d^2-a^2}{a^2}\right) - \tfrac{1}{2}\frac{v_3^2}{d^2}\left(\frac{d^2-b^2}{b^2}\right) + \cdots\right] \\ &= f_{max}\left(\frac{d^3}{a^2 b}\right)\left[1 - \tfrac{1}{2}\frac{v_\perp^2}{d^2}\left(\frac{d^2-a^2}{a^2}\right) - \tfrac{1}{2}\frac{v_3^2}{d^2}\left(\frac{d^2-b^2}{b^2}\right) + \cdots\right] \end{aligned} \tag{V-3}$$

where

$$f_{max} = d^{-3}(2\pi)^{-3/2} \exp -\tfrac{1}{2} v^2/d^2 \tag{V-4}$$

In the energy appendix it is shown that throughout the evolution of the plasma

$$a^2 = d^2\left[1 - \tfrac{1}{3}\alpha + \mathcal{O}(\alpha^2)\right] \tag{V-5}$$

$$b^2 = d^2\left[1 + \tfrac{2}{3}\alpha + \mathcal{O}(\alpha^2)\right] \tag{V-6}$$

from which it follows that

$$\left(\frac{d^2 - a^2}{a^2}\right) = \tfrac{1}{3}\alpha + \mathcal{O}(\alpha^2) \tag{V-7}$$

$$\left(\frac{d^2 - b^2}{b^2}\right) = -\tfrac{2}{3}\alpha + \mathcal{O}(\alpha^2) \tag{V-8}$$

$$a^2 b = d^3\left(1 + \mathcal{O}(\alpha^2)\right) \tag{V-9}$$

therefore

$$f_{bimax} = f_{max}\left[1 + \frac{\alpha}{6d^2}\left(2v_3^2 - v_\perp^2\right) + \mathcal{O}(\alpha^2)\right] \tag{V-10}$$

The initial state of the plasma is assumed to be bi-Maxwellian with negligible energy in waves. Now $f_o^{(o)}(t=0)$ is defined to be the isotropic part of $f_o(t=0)$ and $f_o^{(2)}(t=0)$ is the anisotropic part. Therefore

$$f_o^{(o)}(t=0) = f_{max} \tag{V-11}$$

and

$$f_o^{(2)}(t=0) = \frac{\alpha}{6d^2}\,(2v_3^2 - v_\perp^2) f_{max} \tag{V-12}$$

In the linear approximation $f_o^{(2)}$ is treated as a constant. This will be valid until sufficiently large amplitude waves have grown up. After that time $f_o^{(2)}$ will change on the t_o time scale along with the waves, and the quasilinear description will be needed.

Performing the First Integration for H_1, H_2, and H_3

We wish to calculate the fields in the linear approximation using Eqs. (IV-15), (IV-37), and (IV-50). To do this we shall have to calculate H_1, H_2, and H_3. It is convenient to introduce dimensionless velocities

$$u = v/d \tag{V-13}$$

Then

$$H_1 = \frac{i}{d}\int d^3u \, \frac{u}{k\cdot u} \, \frac{\partial}{\partial u} f_0^{(0)}(u) \tag{V-14}$$

$$H_2 = \int d^3u \, \frac{u}{k\cdot u} \left(u \times \frac{\partial}{\partial u}\right) f_0^{(2)}(u) \tag{V-15}$$

$$H_3 = \frac{1}{d}\int d^3u \, \frac{1}{k\cdot u} \left(u \times \frac{\partial}{\partial u}\right) f_0^{(2)}(u) \tag{V-16}$$

where

$$f_0^{(0)}(u) = (2\pi)^{-3/2} \exp -\tfrac{1}{2}u^2 \tag{V-17}$$

$$f_0^{(2)}(u) = \frac{\alpha}{6}\left(2u_3^2 - u_\perp^2\right) f_0^{(0)}(u) \tag{V-18}$$

Using

$$u \times \frac{\partial}{\partial u} = \left(u_2\frac{\partial}{\partial u_3} - u_3\frac{\partial}{\partial u_2}\right)\hat{1} + \left(u_3\frac{\partial}{\partial u_1} - u_1\frac{\partial}{\partial u_3}\right)\hat{2} + \left(u_1\frac{\partial}{\partial u_2} - u_2\frac{\partial}{\partial u_1}\right)\hat{3} \tag{V-19}$$

$$\left(u \times \frac{\partial}{\partial u}\right) f_0^{(2)}(u) = \alpha \, f_0^{(0)}(u) \left\{u_2u_3\hat{1} - u_1u_3\hat{2}\right\} \tag{V-20}$$

$$\frac{\partial}{\partial u} f_0^{(0)}(u) = -u \, f_0^{(0)}(u) \tag{V-21}$$

the integrals H_1, H_2, and H_3 become

$$H_1 = -\frac{i}{d}\int d^3u \, \frac{f_0^{(0)}(u)}{k\cdot u} \begin{pmatrix} u_1^2 & u_1u_2 & u_1u_3 \\ u_1u_2 & u_2^2 & u_2u_3 \\ u_1u_3 & u_2u_3 & u_3^2 \end{pmatrix} \tag{V-22}$$

$$H_2 = \alpha \int d^3u \; \frac{f_0^{(0)}(u)}{k\cdot u} \begin{pmatrix} u_1 u_2 u_3 & -u_1^2 u_3 & 0 \\ u_2^2 u_3 & -u_1 u_2 u_3 & 0 \\ u_2 u_3^2 & -u_1 u_3^2 & 0 \end{pmatrix} \tag{V-23}$$

$$H_3 = \frac{\alpha}{d} \int d^3u \; \frac{f_0^{(0)}(u)}{k\cdot u} \left(u_2 u_3 \; , \; -u_1 u_3 \; , \; 0 \right) \tag{V-24}$$

Without loss of generality we may take k to lie in the 1,3-plane. Then u_2 does not appear in the denominator and so we may integrate with respect to it. Performing the u_2 integration yields

$$H_1 = -\frac{i}{d} \int du_1 du_3 \; \frac{g}{k\cdot u} \begin{pmatrix} u_1^2 & 0 & u_1 u_3 \\ 0 & 1 & 0 \\ u_1 u_3 & 0 & u_3^2 \end{pmatrix} \tag{V-25}$$

$$H_2 = \alpha \int du_1 du_3 \; \frac{g}{k\cdot u} \begin{pmatrix} 0 & -u_1^2 u_3 & 0 \\ u_3 & 0 & 0 \\ 0 & -u_1 u_3^2 & 0 \end{pmatrix} \tag{V-26}$$

$$H_3 = -\frac{\alpha}{d} \int du_1 du_3 \; \frac{g}{k\cdot u} \; u_1 u_3 \; \hat{z} \tag{V-27}$$

where

$$g = (2\pi)^{-1} \exp - \tfrac{1}{2}(u_1^2 + u_3^2) \tag{V-28}$$

Performing the Second Integration

We shall make the change of variables shown in Fig. V-2, $u_1 u_3 \rightarrow xz$

$$x = u\cdot\hat{k} \qquad\qquad z = u\cdot\hat{n} \tag{V-29}$$

where

$$\hat{n} = \hat{k} \times \hat{2} \tag{V-30}$$

Then

$$\begin{pmatrix} u_1 \\ u_3 \end{pmatrix} = \begin{pmatrix} S & -C \\ C & S \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} \tag{V-31}$$

where

$$S = \sin\theta \qquad\qquad C = \cos\theta \tag{V-32}$$

and θ is the angle between the k-vector and the 3-axis. In terms of these variables

$$g = (2\pi)^{-1} \exp -\tfrac{1}{2}(x^2+z^2) \tag{V-33}$$

$$du_1\, du_3 = dx\, dz \tag{V-34}$$

$$k \cdot u = k\, x \tag{V-35}$$

and the various matrix elements become

$$u_1^2 = S^2x^2 - 2SCxz + C^2z^2 \tag{V-36}$$

$$u_3^2 = C^2x^2 + 2SCxz + S^2z^2 \tag{V-37}$$

$$u_1u_3 = SCx^2 + (S^2-C^2)xz - SCz^2 \tag{V-38}$$

$$u_3 = Cx + Sz \tag{V-39}$$

$$-u_1^2u_3 = -S^2Cx^3 + (2SC^2-S^3)x^2z + (2S^2C-C^3)xz^2 - SC^2z^3 \tag{V-40}$$

$$-u_1u_3^2 = -SC^2x^3 + (C^3-2S^2C)x^2z + (2SC^2-S^3)xz^2 + S^2Cz^3 \tag{V-41}$$

we may now perform the z integration

$$H_1 = -\frac{i}{kd}\int dx\ \frac{h}{x}\begin{pmatrix} S^2x^2+C^2 & 0 & SCx^2-SC \\ 0 & 1 & 0 \\ SCx^2-SC & 0 & C^2x^2+S^2 \end{pmatrix} \tag{V-42}$$

$$H_2 = \frac{\alpha}{k}\int dx\ \frac{h}{x}\begin{pmatrix} 0 & -S^2Cx^3+(2S^2C-C^3)x & 0 \\ Cx & 0 & 0 \\ 0 & -SC^2x^3+(2SC^2-S^3)x & 0 \end{pmatrix} \tag{V-43}$$

$$H_3 = -\frac{\alpha}{kd}\int dx\ \frac{h}{x}\ (SCx^2-SC)\ \hat{z} \tag{V-44}$$

where

$$h = (2\pi)^{-1/2}\ \exp -\tfrac{1}{2}x^2 \tag{V-45}$$

Performing the Third Integration

All the matrix elements for H_2 are well defined. All the matrix elements for H_1 and H_3 are odd in x and thus have zero principal value. Only the pole contributes to H_1 and H_3. If we had Fourier transformed in time our resonant denominator would have been $kx-\omega$. For a solution evolving forward in time the ω-contour of integration must be chosen to go above the pole. Therefore the contour for k or x integration must be chosen to go below the pole. With this pole prescription we have

$$\int dx\ \frac{h}{x} = \pi i\ h(x=0) = \pi i(2\pi)^{-1/2} = i\left(\frac{\pi}{2}\right)^{1/2} \tag{V-46}$$

The integrals H_1, H_2, and H_3 become

$$H_1 = \frac{1}{kd}\left(\frac{\pi}{2}\right)^{1/2}\begin{pmatrix} C^2 & 0 & -SC \\ 0 & 1 & 0 \\ -SC & 0 & S^2 \end{pmatrix} = \frac{1}{kd}\left(\frac{\pi}{2}\right)^{1/2}\hat{H}_1 \tag{V-47}$$

$$H_2 = \frac{\alpha}{k}\begin{pmatrix} 0 & -C(1-2S^2) & 0 \\ C & 0 & 0 \\ 0 & S(1-2S^2) & 0 \end{pmatrix} = \frac{\alpha}{k}\hat{H}_2 \tag{V-48}$$

$$H_3 = i\,\frac{\alpha}{kd}\,SC\left(\frac{\pi}{2}\right)^{1/2}\hat{z} \tag{V-49}$$

It is convenient to introduce a matrix representation of the operation $\hat{k}\times$

$$H_4 = \hat{k}\times = \begin{pmatrix} 0 & -C & 0 \\ C & 0 & -S \\ 0 & S & 0 \end{pmatrix} \tag{V-50}$$

We shall need one other relationship

$$\hat{H}_1 \cdot H_4 = H_4 \tag{V-51}$$

The Normal Mode Equation

We shall look for normal mode solutions. In anticipation that they will be purely growing we set

$$\frac{\partial}{\partial t_0} = \gamma_k \tag{V-52}$$

then Eq. (IV-37) becomes

$$\left\{k^2c^2 H_4 + \sum_s \alpha_s\omega_s^2\hat{H}_2 + \sum_s \frac{\omega_s^2}{kd_s}\left(\frac{\pi}{2}\right)^{1/2}\gamma_k\,\hat{H}_1\cdot H_4\right\}\cdot B_k^{(0)} = 0 \tag{V-53}$$

We may now state the condition for which the effect of ion anisotropy on the dispersion relation may be neglected. This is

$$\omega_e^2 \alpha_e \gg \omega_i^2 \alpha_i \tag{V-54}$$

This condition is equivalent to

$$\alpha_i \ll \frac{m_i}{m_e} \alpha_e \tag{V-55}$$

which is very easily satisfied. Of course if we wished to calculate the ion-acoustic wave, we would not be able to neglect the ion contribution to the mode equation. Kalman et al.[20] have shown that this mode is basically stable, but may be driven unstable by the unstable transverse mode. The growth rate of the transverse mode is $\gamma \sim \alpha kd$. The coupling between the ion-acoustic and the transverse mode is of order α. The growth rate of the ion-acoustic mode is thus considerably less than that of the transverse mode, and we shall not consider it further.

Continuing with the normal mode equation we disregard ion anisotropy and define

$$p = k^2c^2/\omega_e^2 + \alpha_e + \left(\frac{\pi}{2}\right)^{1/2} \gamma_k / kd^* \tag{V-56}$$

$$q = k^2c^2/\omega_e^2 + \alpha_e(1-2S^2) + \left(\frac{\pi}{2}\right)^{1/2} \gamma_k / kd^* \tag{V-57}$$

$$r = k^2c^2/\omega_e^2 + \left(\frac{\pi}{2}\right)^{1/2} \gamma_k / kd^* \tag{V-58}$$

$$p - r = \alpha_e \tag{V-59}$$

$$s = C^2 p + S^2 r = k^2c^2/\omega_e^2 + C^2\alpha_e + \left(\frac{\pi}{2}\right)^{1/2} \gamma_k / kd^* \tag{V-60}$$

where d^* is defined by

$$\frac{1}{d^*} = \frac{1}{d_e} + \frac{\omega_i^2}{\omega_e^2} \frac{1}{d_i} \tag{V-61}$$

We note that if the rather mild restriction

$$T_i \gg \frac{m_e}{m_i} T_e \tag{V-62}$$

is satisfied then d^* is equal to d_e. With these substitutions Eq. (V-53) becomes

$$\begin{pmatrix} 0 & -Cq & 0 \\ Cp & 0 & -Sr \\ 0 & Sq & 0 \end{pmatrix} \cdot B_k^{(0)} = 0 \tag{V-63}$$

This is an equation for the 1,2,3-components of B. It is convenient to transform to an equation for the k,2,n-components of B since we have the side condition that B_k must be perpendicular to k.

$$B(1,2,3) = R(\theta) \cdot B(k,2,n) = \begin{pmatrix} S & 0 & -C \\ 0 & 1 & 0 \\ C & 0 & S \end{pmatrix} \cdot B(k,2,n) \tag{V-64}$$

Substituting Eq. (V-64) into (V-63) and multiplying on the left by $R^{-1}(\theta)$ gives

$$\begin{pmatrix} 0 & 0 & 0 \\ SC\alpha & 0 & -s \\ 0 & q & 0 \end{pmatrix} \cdot B(k,2,n) = 0 \tag{V-65}$$

We evidently have two normal modes

$$\text{Mode (I):} \quad \hat{B}_k = \hat{2} \quad \text{and} \quad q = 0 \tag{V-66}$$

$$\text{Mode (II):} \quad \hat{B}_k = \hat{n} \quad \text{and} \quad s = 0 \tag{V-67}$$

Using Eqs. (IV-15) and (IV-50) we may determine the values of $E_k^{(o)\text{trans}}$ and $E_k^{(o)\text{long}}$ corresponding to a unit amplitude $B_k^{(o)}$ vector.

$$E_k^{(0)\text{trans}} = -\frac{i\gamma_k}{kc} \hat{k} \times B_k^{(0)} \tag{V-68}$$

$$\sum_s \frac{\omega_s^2}{k^2 d_s^2} e_k^{\ell} = -\sum_s \frac{\omega_s^2}{kc} \frac{\alpha_s}{k d_s} i SC \left(\frac{\pi}{2}\right)^{1/2} (\hat{2} \cdot B_k^{(0)}) \tag{V-69}$$

The condition for the neglect of ion anisotropy in Poisson's Eq. (V-69) is

$$\omega_e^2 \alpha_e / d_e \gg \omega_i^2 \alpha_i / d_i \tag{V-70}$$

which may be written

$$\alpha_i \ll \left(\frac{T_i}{T_e} \frac{m_i}{m_e}\right)^{1/2} \alpha_e \tag{V-71}$$

We assume this condition is satisfied and define

$$\beta = \sum_s \frac{\omega_s^2}{d_s^2} \cdot \frac{d_e^2}{\omega_e^2} = 1 + \frac{n_e T_e}{n_i T_i} \tag{V-72}$$

whereupon Eq. (V-69) becomes

$$e_k^{\ell} = -\frac{i}{\beta} \alpha_e \frac{d_e}{c} SC \left(\frac{\pi}{2}\right)^{1/2} (\hat{z} \cdot B_k^{(0)}) \tag{V-73}$$

Using the above relations we may describe our two normal modes. For Mode I

$$\hat{B}_k = \hat{z} \tag{V-74}$$

$$E_k^{(0)\,long} = -\frac{i}{\beta} \alpha_e \frac{d_e}{c} SC \left(\frac{\pi}{2}\right)^{1/2} \hat{k} \tag{V-75}$$

$$E_k^{(0)\,trans} = -i \frac{\gamma}{kc} \hat{n} \tag{V-76}$$

It follows from $q = 0$ that the dispersion relation is

$$\gamma_k = k d^* \left(\frac{2}{\pi}\right)^{1/2} \left\{\alpha_e (2S^2 - 1) - k^2 c^2 / \omega_e^2\right\} \tag{V-77}$$

For $\alpha > 0$ (that is for $T_{\parallel} > T_{\perp}$) we have growth for

$$45^\circ \leq \theta \leq 90^\circ \qquad \text{and} \qquad 0 \leq k \leq k(\text{crit}, \theta) \tag{V-78}$$

Maximum growth occurs for $\theta = 90^{\circ}$. Polarization varies from transverse at $\theta = 90^{\circ}$ (maximum growth) to longitudinal at $\theta = \theta(\text{crit})$ (zero growth). For Mode II we have

$$\hat{B}_k = \hat{n} \tag{V-79}$$

$$E_k^{(o)\,long} = 0 \tag{V-80}$$

$$E_k^{(o)\,trans} = i\,\frac{\gamma}{kc}\,\hat{2} \tag{V-81}$$

It follows from $s = 0$ that the dispersion relation is

$$\gamma_k = -kd^*\left(\frac{2}{\pi}\right)^{1/2}\left\{\alpha_e C^2 + k^2c^2/\omega_e^2\right\} \tag{V-82}$$

For $\alpha > 0$ this mode is damped for all k and θ. Electric polarization is purely transverse to $\hat{k}$.

Discussion of the Dispersion Relation

We shall henceforth assume that $d^* = d_e$ and shall omit the species subscript on d_e and α_e. The growth rate of our unstable Mode I is

$$\gamma = kd\left(\frac{2}{\pi}\right)^{1/2}\left\{\alpha\left(2S^2-1\right) - k^2c^2/\omega_e^2\right\} \tag{V-83}$$

We see that for a given value of θ, the wave is unstable for

$$0 \leqq k \leqq k(\text{crit}) \tag{V-84}$$

where k(crit) satisfies

$$k^2(\text{crit})c^2 = \alpha(2S^2 - 1)\omega_e^2 \tag{V-85}$$

In terms of k(crit) we may write the growth rate as

$$\gamma = kd\,\frac{c^2}{\omega_e^2}\left(\frac{2}{\pi}\right)^{1/2}\left\{k^2(\text{crit}) - k^2\right\} \tag{V-86}$$

This is graphed in Fig. V-3. For a fixed value of θ, γ reaches its maximum at k(γ max) which satisfies

$$k^2(\gamma_{max})\, c^2 = \frac{1}{3}\alpha\,(2S^2-1)\,\omega_e^2 = \frac{1}{3}k^2(\text{crit})\, c^2 \qquad \text{(V-87)}$$

Thus

$$k(\gamma_{max}) = 0.577 \;\; k(\text{crit}) \qquad \text{(V-88)}$$

For a given value of k the wave is unstable for

$$90^\circ \geqq \theta \geqq \theta(\text{crit}) \qquad \text{(V-89)}$$

where θ(crit) satisfies

$$\sin^2\theta(\text{crit}) = \frac{1}{2}\left(1 + k^2c^2/\alpha\omega_e^2\right) \qquad \text{(V-90)}$$

In terms of θ(crit) we may write the growth rate as

$$\gamma = 2\,\alpha\, kd\left(\frac{2}{\pi}\right)^{1/2}\left\{\sin^2\theta - \sin^2\theta(\text{crit})\right\} \qquad \text{(V-91)}$$

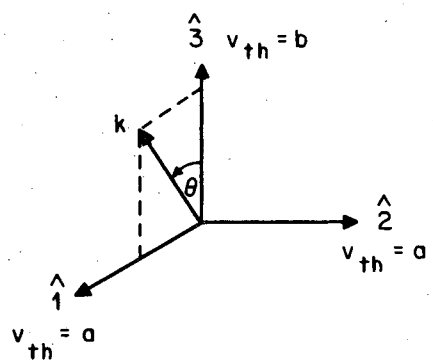

FIGURE V-1

COORDINATE SYSTEM

3 = axis of temperature maximum

1,2 = plane of temperature minimum

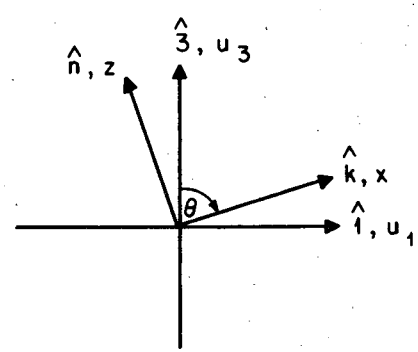

FIGURE V-2

CHANGE OF VARIABLES

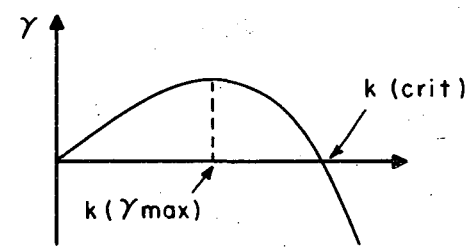

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FIGURE V-3

GROWTH RATE $\gamma(k)$

FOR FIXED θ

VI. QUALITATIVE CONSIDERATIONS

The quasilinear evolution of our plasma is described by Eqs. (IV-23) and (IV-43) for $f_o^{(2)}$ and $|B_k^{(o)}|^2$. Since our problem has cylindrical and reflectional symmetries, f_o will be a function of v_3^2 and $v_\perp^2$ only and $|B_k|^2$ will be a function of k_3^2 and $k_\perp^2$ only. However even with this simplifying knowledge, the coupled differential-integral equations for $f_o^{(2)}$ and $|B_k^{(o)}|^2$ are too difficult to solve directly. In order to study the evolution of our plasma, we must choose a simplified model to represent it. This choice will be guided by a consideration of the qualitative properties of Eqs. (IV-23) and (IV-43).

Dependence of the Growth Rate on the Distribution Function

In this section we shall use γ_k to denote

$$\gamma_k = |B_k^{(o)}|^{-2} \frac{\partial}{\partial t_o} |B_k^{(o)}|^2 \tag{VI-1}$$

which is of course twice the growth rate of $B_k^{(o)}$. γ_k is given by Eq. (IV-43).

$$\gamma_k = -\frac{2k^2}{G_1}\left\{G_2 + k_c^2 c^2/\omega_e^2\right\} \tag{VI-2}$$

where

$$G_1 = (k \times \hat{B}_k) \cdot H_1 \cdot (k \times \hat{B}_k) \tag{VI-3}$$

and

$$G_2 = (k \times \hat{B}_k) \cdot H_2 \cdot \hat{B}_k \tag{VI-4}$$

Using the coordinate system shown in Fig. V-1

$$\hat{k} = S\hat{1} + C\hat{3} \tag{VI-5}$$

$$\hat{B}_k = \hat{2} \tag{VI-6}$$

$$\hat{n} = \hat{k} \times \hat{B}_k = S\hat{3} - C\hat{1} \tag{VI-7}$$

where

$$S = \sin\theta \qquad \text{and} \qquad C = \cos\theta \tag{VI-8}$$

From Eq. (IV-38) it follows that H_1 is constant. Since it is a constant we shall evalute it for a Maxwellian $f_o^{(o)}$. However for the other considerations of this section we shall not assume that $f_o^{(o)}$ is Maxwellian or that f_o is bi-Maxwellian. We shall only assume that f_o is sufficiently close to a Maxwellian so that it may be expanded as a convergent sum of polynomials times a Maxwellian. For the above coordinate system, H_1 is given by Eq. (V-47). Carrying out the matrix multiplication we find that

$$\hat{n} \cdot \hat{H}_1 \cdot \hat{n} = 1 \tag{VI-9}$$

whence

$$G_1 = \frac{k}{d}\left(\frac{\pi}{2}\right)^{1/2} \tag{VI-10}$$

Thus γ_k is given by

$$\gamma_k = \left(\frac{8}{\pi}\right)^{1/2} kd \left\{ - G_2 - k^2c^2/\omega_e^2 \right\} \tag{VI-11}$$

Growing waves occur for G_2 sufficiently negative. γ_k changes in time because G_2 does. H_2 is given by Eq. (IV-39). For the above coordinate system, G_2 is given by

$$G_2 = \int d^3u \; \frac{\hat{n}\cdot u}{\hat{k}\cdot u} \left(u_3 \frac{\partial}{\partial u_1} - u_1 \frac{\partial}{\partial u_3}\right) f_0^{(2)}(u)$$

$$= \int d^3u \; \frac{\hat{n}\cdot u}{\hat{k}\cdot u} \left(u_3 \frac{\partial}{\partial u_1} - u_1 \frac{\partial}{\partial u_3}\right) \left(f_0^{(0)}(u) + f_0^{(2)}(u)\right)$$

$$= \int d^3u \; \frac{\hat{n}\cdot u}{\hat{k}\cdot u} \left(u_3 \frac{\partial}{\partial u_1} - u_1 \frac{\partial}{\partial u_3}\right) f_0(u) \qquad \text{(VI-12)}$$

because $f_0^{(0)}$ is isotopic and therefore contributes nothing. We have introduced the dimensionless velocity variable

$$u = v/d \qquad \text{(VI-13)}$$

and have denoted

$$f_0^{(2)}(u) = d^3 \; f_0^{(2)}(v) \qquad \text{(VI-14)}$$

<u>Expansion of the Distribution Function in Moment Excesses</u>

Since f_0 differs only slightly from f_{max}, we consider an expansion of the form

$$f_0 = \Big\{1 + a_1\left(\frac{\partial^2}{\partial u_1^2} + \frac{\partial^2}{\partial u_2^2}\right) + a_3 \frac{\partial^2}{\partial u_3^2} + a_{11}\left(\frac{\partial^2}{\partial u_1^2} + \frac{\partial^2}{\partial u_2^2}\right)^2$$

$$+ 2a_{13}\left(\frac{\partial^2}{\partial u_1^2} + \frac{\partial^2}{\partial u_2^2}\right)\frac{\partial^2}{\partial u_3^2} + a_{33}\frac{\partial^4}{\partial u_3^4} + \cdots\Big\} f_{max}(u) \qquad \text{(VI-15)}$$

where

$$f_{max}(u) = (2\pi)^{-3/2} \exp -\tfrac{1}{2}\left(u_1^2 + u_2^2 + u_3^2\right) \qquad \text{(VI-16)}$$

The expansion in Eq. (VI-15) is reflectionally and cylindrically symmetric. The terms in the expansion are Hermite polynomials. We consider the moments of f_0 and denote

$$\langle m(u)\rangle = \int d^3u \; m(u) \; f_o(u) \tag{VI-17}$$

Substituting f_o from Eq. (VI-15) and integrating by parts gives formulas for the various moments.

$$\langle 1\rangle = 1 \tag{VI-18}$$

$$\langle u_1^2\rangle = \langle u_2^2\rangle = 1 + 2a_1 \tag{VI-19}$$

$$\langle u_3^2\rangle = 1 + 2a_3 \tag{VI-20}$$

$$\langle u_1^4\rangle = \langle u_2^4\rangle = 3 + 12a_1 + 24a_{11} \tag{VI-21}$$

$$\langle u_3^4\rangle = 3 + 12a_3 + 24a_{33} \tag{VI-22}$$

$$\langle u_1^2 u_3^2\rangle = \langle u_2^2 u_3^2\rangle = 1 + 2a_1 + 2a_3 + 8a_{13} \tag{VI-23}$$

$$\langle u_1^2 u_2^2\rangle = 1 + 4a_1 + 8a_{11} \tag{VI-24}$$

The excesses in the moments of f_o over the moments of f_{max} is given by various combinations of the a_n.

In the section on the ordering scheme, $f_o^{(o)}(t=0)$ was defined to be an isotropic distribution all of whose moments, $\langle u^2\rangle^{(o)}$, $\langle u^4\rangle^{(o)}$, ..., are the same as those of $f_o(t=0)$. We have denoted

$$\langle m(u)\rangle^{(o)} = \int d^3u \; m(u) \; f_o^{(o)}(u) \tag{VI-25}$$

Let us see what relationships exist between the a_n of f_o and the $a_n^{(o)}$ of $f_o^{(o)}$. Combining Eqs. (VI-18) through (VI-24) we find

$$\langle u^2\rangle = 3 + 4a_1 + 2a_3 \tag{VI-26}$$

$$\langle u^4\rangle = 15 + 40a_1 + 20a_3 + 64a_{11} + 32a_{13} + 24a_{33} \tag{VI-27}$$

$$\langle u^2 \rangle^{(0)} = 3 + 4a_1^{(0)} + 2a_3^{(0)} \tag{VI-28}$$

$$\langle u^4 \rangle^{(0)} = 15 + 40a_1^{(0)} + 20a_3^{(0)} + 64a_{11}^{(0)} + 32a_{13}^{(0)} + 24a_{33}^{(0)} \tag{VI-29}$$

But for an isotropic distribution

$$a_1^{(0)} = a_3^{(0)} \tag{VI-30}$$

$$a_{11}^{(0)} = a_{33}^{(0)} = a_{13}^{(0)} \tag{VI-31}$$

therefore

$$a_1^{(0)} = \frac{1}{3}(2a_1 + a_3) \tag{VI-32}$$

$$a_{11}^{(0)} = \frac{1}{15}(8a_{11} + 4a_{13} + 3a_{33}) \tag{VI-33}$$

which uniquely specifies the distribution $f_o^{(o)}(t=0)$ in terms of $f_o(t=0)$.

We shall now calculate G_2 in terms of the a_n by using Eq. (VI-15) for f_o. Carrying out the differentiation we obtain

$$\begin{aligned} f_o = f_{max} \{ & 1 + a_1(u_1^2+u_2^2-2) + a_3(u_3^2-1) \\ & + a_{11}\left[u_1^4 - 6u_1^2 + 3 + u_2^4 - 6u_2^2 + 3 + 2(u_1^2-1)(u_2^2-1)\right] \\ & + 2a_{13}(u_1^2+u_2^2-2)(u_3^2-1) + a_{33}(u_3^4 - 6u_3^2 + 3) + \cdots \} \end{aligned} \tag{VI-34}$$

whereupon

$$\begin{aligned} & \left(u_3 \frac{\partial}{\partial u_1} - u_1 \frac{\partial}{\partial u_3}\right) f_o(u) = \\ & f_{max}(u) \{ u_3[2a_1u_1 + 4a_{11}u_1^3 - 12a_{11}u_1 + 4a_{11}u_1(u_2^2-1) + 4a_{13}u_1(u_3^2-1)] \\ & - u_1[2a_3u_3 + 4a_{13}u_3(u_1^2+u_2^2-2) + 4a_{33}u_3^3 - 12a_{13}u_3] + \cdots \} \end{aligned} \tag{VI-35}$$

The u_2-integration just replaces $u_2^2 \to 1$. G_2 then becomes

$$G_2 = \int du_1 du_3 \frac{\hat{n}\cdot u}{\hat{k}\cdot u}\, g \left\{ \left[2(a_1-a_3) + 12(a_{33}-a_{11})\right] u_1 u_3 + 4(a_{11}-a_{13})\, u_1^3 u_3 + 4(a_{13}-a_{33})\, u_3^3 u_1 + \cdots \right\} \qquad \text{(VI-36)}$$

where

$$g = (2\pi)^{-1} \exp -\tfrac{1}{2}(u_1^2 + u_3^2) \qquad \text{(VI-37)}$$

We note that G_2 is a function only of the anisotropy, that is a function of $(a_1 - a_3)$, $(a_{33} - a_{13})$, etc. Thus an isotropic change in f_o does not affect G_2 and hence will not affect the growth of the waves. This is an obvious consequence of the factor $(u \times \partial/\partial u)$ which appears in the expression for G_2.

To continue we make the change of variables given by Eqs. (V-29), etc.

$$x = \hat{k}\cdot u \qquad z = \hat{n}\cdot u \qquad \text{etc.} \qquad \text{(VI-38)}$$

Then our integrands become

$$u_1 u_3 = SC x^2 + (S^2 - C^2) xz - SC z^2 \qquad \text{(VI-39)}$$

$$u_1^3 u_3 = S^3 C x^4 + (S^4 - 3S^2C^2) x^3 z + 3SC(C^2 - S^2) x^2 z^2 + (3S^2C^2 - C^4) x z^3 - SC^3 z^4 \qquad \text{(VI-40)}$$

$$u_1 u_3^3 = SC^3 x^4 + (3S^2C^2 - C^4) x^3 z + 3SC(S^2 - C^2) x^2 z^2 + (S^4 - 3S^2C^2) x z^3 - S^3 C z^4 \qquad \text{(VI-41)}$$

and G_2 may be written

$$G_2 = \int dx\, dz\, \frac{z}{x}\, g \left\{ \alpha_1 u_1 u_3 + \alpha_2 u_1^3 u_3 + \alpha_3 u_1 u_3^3 + \cdots \right\} \qquad \text{(VI-42)}$$

where the α_n are the combinations of the a_n given in Eq. (VI-36). We perform the z-integration and obtain

$$G_2 = \int dx \; \frac{h}{x} \left\{ \alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3 + \cdots \right\} \tag{VI-43}$$

where

$$A_1 = (S^2 - C^2)\, x \tag{VI-44}$$

$$A_2 = (S^4 - 3S^2C^2)\, x^3 + 3(3S^2C^2 - C^4)\, x \tag{VI-45}$$

$$A_3 = (3S^2C^2 - C^4)\, x^3 + 3(S^4 - 3S^2C^2)\, x \tag{VI-46}$$

$$h = (2\pi)^{-1/2} \exp - \tfrac{1}{2} x^2 \tag{VI-47}$$

The x-integration yields finally

$$G_2 = \alpha_1 (S^2 - C^2) + \alpha_2 \left[(S^4 - 3S^2C^2) + 3(3S^2C^2 - C^4) \right] + \alpha_3 \left[(3S^2C^2 - C^4) + 3(S^4 - 3S^2C^2) \right] + \cdots$$

$$= \alpha_1 (2S^2 - 1) + \alpha_2 (-8S^4 + 12S^2 - 3) + \alpha_3 (8S^4 - 4S^2 - 1) + \cdots \tag{VI-48}$$

which may be written in the form

$$-G_2 = 2(a_3 - a_1)(2S^2 - 1) + 8(a_{11} - a_{13})(4S^4 - 3S^2) + 8(a_{13} - a_{33})(4S^4 - 5S^2 + 1) \tag{VI-49}$$

The symmetry of the 2nd and 3rd term is clarified by noting that

$$4S^4 - 5S^2 + 1 = 4C^4 - 3C^2 \quad \text{and} \quad C(\theta) = S(90^\circ - \theta) \tag{VI-50}$$

The functions $2S^2-1$, $4S^4-3S^2$, and $4C^4-3C^2$ are shown in Fig. VI-1.

The initial state of our plasma has a bi-Maxwellian f_o. Comparing Eq. (VI-34) with Eq. (V-10) we see that

$$a_1(t=0) = -\frac{1}{6}\alpha(t=0) \tag{VI-51}$$

$$a_3(t=0) = \frac{1}{3}\alpha(t=0) \tag{VI-52}$$

$$a_{11}(t=0)\,,\; a_{13}(t=0)\,,\; \text{and} \;\; a_{33}(t=0) = \mathcal{O}(\alpha^2) \tag{VI-53}$$

The growth rate for waves is proportional to $-G_2$. Therefore since the initial state of the plasma has

$$a_3(t=0) - a_1(t=0) = \frac{1}{2}\alpha \tag{VI-54}$$

$$a_{11}(t=0) - a_{13}(t=0) = \mathcal{O}(\alpha^2) \tag{VI-55}$$

$$a_{13}(t=0) - a_{33}(t=0) = \mathcal{O}(\alpha^2) \tag{VI-56}$$

a spectrum of waves grows up which is peaked about $\theta = 90^\circ$ where $2S^2-1$ is most positive and thus where G_2 is most negative. We are interested in the subsequent growth and decay of these waves. We therefore note that for $\theta = 90^\circ$

$$-G_2 = 2(a_3 - a_1) + 8(a_{11} - a_{13}) + \cdots \tag{VI-57}$$

Thus any changes in the anisotropy, whether in the temperature, the fourth moment, etc., can contribute to the growth and decay of waves near $\theta = 90^\circ$, whereas isotropic changes in the distribution function have no effect on the waves.

Factors Affecting the Distribution Function

If we introduce the dimensionless velocity $u = v/d$, then Eq. (IV-23) for $f_0^{(2)}$ may be written

$$\frac{\partial}{\partial t_0} f_0^{(2)}(u) = -\frac{1}{V}\frac{e^2}{m^2c^2}\int\frac{d^3k}{(2\pi)^3}\left\{\frac{1}{kd}\,|B_k^{(0)}|^2\,(u\times\hat{B}_k)\cdot\frac{\partial}{\partial u}\,\frac{i}{\hat{k}\cdot u}\,(u\times\hat{B}_k)\cdot\frac{\partial}{\partial u}\right\} f_0^{(2)}(u)$$

$$+\frac{1}{V}\frac{e^2}{m^2c^2}\int\frac{d^3k}{(2\pi)^3}\left\{\frac{1}{2k^2d^2}\,\frac{\partial}{\partial t_0}\,|B_k^{(0)}|^2\,(u\times\hat{B}_k)\cdot\frac{\partial}{\partial u}\,\frac{1}{\hat{k}\cdot u}\,(\hat{k}\times\hat{B}_k)\cdot u\right\} f_0^{(0)}(u) \tag{VI-58}$$

The first term on the RHS is a diffusion term. The diffusion coefficient for this term is a symmetric tensor: D^{sym}. The quantity $\varphi\cdot D^{sym}\cdot\varphi$ is a

positive definite form. As a result, this term satisfies an H-theorem, and its effect is discussed in the H-theorem appendix. Any anisotropic part of $f_o^{(2)}$ will eventually diffuse away. However because of the factor $(u \times \partial/\partial u)$, any isotropic part of $f_o^{(2)}$ is unaffected by this term. The term $(\partial/\partial u_i)^n f_o^{(o)}$ in the moment-excess expansion of $f_o^{(2)}$ will be an nth degree polynomial in velocity times $f_o^{(o)}$. This term in the moment excess expansion will diffuse at a rate proportional to n^2 because of the double velocity derivative in the diffusion term. Thus higher moments diffuse away more rapidly than lower moments. The diffusion term therefore has a smoothing effect on $f_o^{(2)}$.

We have seen that the diffusion coefficient for the second term on the RHS of Eq. (VI-58) is an antisymmetric tensor to lowest order in α. An antisymmetric diffusion coefficient corresponds to dissipationless flow in velocity space. We shall therefore refer to this term as a flow term. It is a perfect time derivative. The contribution of this term to $f_o^{(2)}(t)$ depends only on $B_k^{(o)}$ at time t, and is independent of the state of the plasma at previous times. This term therefore makes no contribution to the <u>net</u> change in $f_o^{(2)}$ from initial to final states, since the initial and final wave spectra are negligibly small. (This term is large and positive for a short time, and small and negative for a long time in just the right proportions so that the integral of this term from $t = 0$ to $t = \infty$ is zero.)

The flow term may be written

$$(\text{flow term}) = (u \times \frac{\partial}{\partial u}) \cdot (\text{some vector}) \qquad \text{(VI-59)}$$

It therefore follows that the moments $\langle u^n \rangle$ of the flow term

$$\langle u^n \rangle = \int d^3u\, u^n \cdot \text{(flow term)} \tag{VI-60}$$

are all zero. Thus no isotropic contribution to $f_o^{(2)}$ arises from the flow term. Since $f_o^{(2)}$ is initially purely anisotropic, <u>it remains so forever</u>. Thus the moments of u^2, u^4, etc. with respect to $f_o^{(2)}$ remain zero forever. The final state of the plasma has no anisotropy because of the diffusion term. It follows that

$$f_o^{(2)}(t=\infty) = 0 \tag{VI-61}$$

and thus the final state of the plasma has

$$f_o(t=\infty) = f_o^{(o)}(t=\infty) = f_o^{(o)}(t=0) \tag{VI-62}$$

since $f_o^{(o)}$ is constant.

<u>Expansion of the Flow Term in Moment Excesses</u>

In order to examine in more detail the effect of the flow term on $f_o^{(2)}$ we shall expand the flow term in moment excesses

$$\text{(flow term)} = \left\{ b_1 \left(\frac{\partial^2}{\partial u_1^2} + \frac{\partial^2}{\partial u_2^2} \right) + b_3 \frac{\partial^2}{\partial u_3^2} + \cdots \right\} f_{max}(u) \tag{VI-63}$$

If we consider moments of the flow term and denote

$$\langle m(u) \rangle = \int d^3u\, m(u)\ \text{(flow term)} \tag{VI-64}$$

Substituting the (flow term) from Eq. (VI-63) and integrating by parts gives formulas for the various moments

$$\langle u_\perp^2 \rangle = 4 b_1 \tag{VI-65}$$

$$\langle u_3^2 \rangle = 2 b_3 \tag{VI-66}$$

$$\langle u_\perp^4 \rangle = 32 b_1 + 64 b_{11} \tag{VI-67}$$

$$\langle u_3^4 \rangle = 12 b_3 + 24 b_{33} \tag{VI-68}$$

$$\langle u_\perp^2 u_3^2 \rangle = 4 b_1 + 4 b_3 + 16 b_{13} \tag{VI-69}$$

We have used $u_\perp$ in the above formulas because it is convenient not to choose coordinate axes for u_1 and u_2 until after we interchange the order of the u and k integrations. If we now substitute the (flow term) from Eq. (VI-58) into Eq. (VI-64) we obtain

$$\langle m(u) \rangle = \frac{1}{V} \frac{e^2}{m^2 c^2} \int \frac{d^3k}{(2\pi)^3} \frac{1}{2k^2 d^2} \frac{\partial}{\partial t_0} |B_k^{(0)}|^2 \; c_k(m) \tag{VI-70}$$

where

$$c_k(m) = \int d^3u \; m(u) \; (u \times \hat{B}_k) \cdot \frac{\partial}{\partial u} \frac{1}{\hat{k} \cdot u} (\hat{k} \cdot \hat{B}_k) \cdot u \; f_0^{(0)}(u) \tag{VI-71}$$

Equations (VI-70) and (VI-71) have the u and k integrations interchanged. We may therefore choose the u-coordinate axes to depend on k. We choose the coordinate system shown in Fig. V-1 which has k in the u_1, u_3-plane. We then make the same change of variables as before

$$x = \hat{k} \cdot u \qquad y = \hat{B}_k \cdot u \qquad z = \hat{n} \cdot u \qquad \text{etc.} \tag{VI-72}$$

whereupon

$$(u \times \hat{B}_k) \cdot \frac{\partial}{\partial u} = -\hat{B}_k \cdot \left(u \times \frac{\partial}{\partial u}\right) = \left(-z \frac{\partial}{\partial x} + x \frac{\partial}{\partial z}\right) \tag{VI-73}$$

$$u_3^2 = C^2 x^2 + 2SCxz + S^2 z^2 \tag{VI-74}$$

$$u_\perp^2 = u_1^2 + u_2^2 = y^2 + S^2 x^2 - 2SCxz + C^2 z^2 \tag{VI-75}$$

Integrating by parts in Eq. (VI-60) gives for $c_k(m)$

$$c_k(m) = \int dx\, dy\, dz \quad \left\{ m(u) \left(z \frac{\overleftarrow{\partial}}{\partial x} - x \frac{\overleftarrow{\partial}}{\partial z} \right) \frac{z}{x} \right\} f_o^{(o)}(u) \tag{VI-76}$$

For the various moments, the expression in the curly brackets in Eq. (VI-76) becomes

$$\langle u_\perp^2 \rangle \rightarrow \left\{ \frac{z^2}{x}(2xS^2 - 2SCz) - z(-2SCx + 2zC^2) \right\} \tag{VI-77}$$

$$\langle u_3^2 \rangle \rightarrow \left\{ \frac{z^2}{x}(2xC^2 + 2SCz) - z(2SCx + 2zS^2) \right\} \tag{VI-78}$$

$$\langle u_\perp^4 \rangle \rightarrow \left\{ 2\left[y^2 + S^2x^2 - 2SCxz + C^2z^2\right]\left[\frac{z^2}{x}(2xS^2 - 2SCz) - z(-2SCx + 2zC^2)\right] \right\} \tag{VI-79}$$

$$\langle u_3^4 \rangle \rightarrow \left\{ 2\left[C^2x^2 + 2SCxz + S^2z^2\right]\left[\frac{z^2}{x}(2xC^2 + 2SCz) - z(2SCx + 2zS^2)\right] \right\} \tag{VI-80}$$

$$\begin{aligned} \langle u_\perp^2 u_3^2 \rangle \rightarrow \Big\{ & \left[y^2 + S^2x^2 - 2SCxz + C^2z^2\right]\left[\frac{z^2}{x}(2xC^2 + 2SCz) - z(2SCx + 2zS^2)\right] \\ & + \left[C^2x^2 + 2SCxz + S^2z^2\right]\left[\frac{z^2}{x}(2xS^2 - 2SCz) - z(-2SCx + 2zC^2)\right] \Big\} \end{aligned} \tag{VI-81}$$

We take $f_o^{(o)}$ to be Maxwellian and perform the y and z integrations to obtain the expressions

$$\langle u_\perp^2 \rangle \rightarrow \left\{ 2S^2 - 2C^2 \right\} \tag{VI-82}$$

$$\langle u_3^2 \rangle \rightarrow \left\{ 2C^2 - 2S^2 \right\} \tag{VI-83}$$

$$\langle u_\perp^4 \rangle \rightarrow \left\{ 2(1 + S^2x^2)[2S^2 - 2C^2] - 4SCx\left[-\frac{6}{x}SC + 2SCx\right] + 2C^2[6S^2 - 6C^2] \right\} \tag{VI-84}$$

$$\langle u_3^4 \rangle \rightarrow \left\{ 2C^2x^2[2C^2 - 2S^2] + 4SCx\left[\frac{6}{x}SC - 2SCx\right] + 2S^2[6C^2 - 6S^2] \right\} \tag{VI-85}$$

$$\begin{aligned} \langle u_\perp^2 u_3^2 \rangle \rightarrow \Big\{ & (1 + S^2x^2)[2C^2 - 2S^2] - 2SCx\left[\frac{6}{x}SC - 2SCx\right] + C^2[6C^2 - 6S^2] \\ & + C^2x^2[2S^2 - 2C^2] + 2SCx\left[-\frac{6}{x}SC + 2SCx\right] + S^2[6S^2 - 6C^2] \Big\} \end{aligned} \tag{VI-86}$$

We perform the x integration and obtain

$$c_k(u_\perp^2) = \{2S^2 - 2C^2\} \tag{VI-87}$$

$$c_k(u_3^2) = \{2C^2 - 2S^2\} \tag{VI-88}$$

$$c_k(u_\perp^4) = \{2(1+S^2)(2S^2-2C^2) + 16\,S^2C^2 + 2C^2(6S^2-6C^2)\} \tag{VI-89}$$

$$c_k(u_3^4) = \{2C^2(2C^2-2S^2) + 16\,S^2C^2 + 2S^2(6C^2-6S^2)\} \tag{VI-90}$$

$$\begin{aligned} c_k(u_\perp^2 u_3^2) = \{&(1+S^2)(2C^2-2S^2) - 8\,S^2C^2 + C^2(6C^2-6S^2) \\ &+ C^2(2S^2-2C^2) - 8\,S^2C^2 + S^2(6S^2-6C^2)\} \end{aligned} \tag{VI-91}$$

We define functions $d_n(k)$ to have the same relationship to the $c_k(m)$ that the b_n have to the $\langle m \rangle$, namely

$$c_k(u_\perp^2) = 4\,d_1(k) \tag{VI-92}$$

$$c_k(u_3^2) = 2\,d_3(k) \qquad \text{etc.} \tag{VI-93}$$

then

$$b_n = \frac{1}{V}\,\frac{e^2}{m^2c^2}\int\frac{d^3k}{(2\pi)^3}\,\frac{1}{2k^2d^2}\,\frac{\partial}{\partial t_0}\left|B_k^{(0)}\right|^2\,d_n(k) \tag{VI-94}$$

and Eqs. (VI-87) through (VI-91) may be solved for the d_n.

$$2d_1 = (2S^2-1) \tag{VI-95}$$

$$d_3 = (1-2S^2) \tag{VI-96}$$

$$8d_{11} = (-4S^4+3S^2) \tag{VI-97}$$

$$3d_{33} = (-4S^4+5S^2-1) \tag{VI-98}$$

$$4 d_{13} = (8S^4 - 8S^2 + 1) \tag{VI-99}$$

The spectrum of waves grows up peaked about $\theta = 90^\circ$. We shall see that for typical initial conditions the spectrum is sharply peaked about $\theta = 90^\circ$. Therefore we may estimate b_n roughly as

$$b_n \doteq d_n(S=1) \frac{1}{V} \frac{e^2}{m^2c^2} \int \frac{d^3k}{(2\pi)^3} \frac{1}{2k^2d^2} \frac{\partial}{\partial t_d} |B_k^{(0)}|^2 = d_n(S=1) \cdot \beta \tag{VI-100}$$

We note that β is positive when the wave spectrum is growing. From Eq. (VI-100) it follows that

$$b_1 \doteq \frac{1}{2}\beta \tag{VI-101}$$

$$b_3 \doteq -\beta \tag{VI-102}$$

$$b_{11} \doteq -\frac{1}{8}\beta \tag{VI-103}$$

$$b_{33} \doteq 0 \tag{VI-104}$$

$$b_{13} \doteq \frac{1}{4}\beta \tag{VI-105}$$

whence

$$(b_3 - b_1) \doteq -\frac{3}{2}\beta \tag{VI-106}$$

$$(b_{11} - b_{13}) \doteq -\frac{3}{8}\beta \tag{VI-107}$$

$$(b_{13} - b_{33}) \doteq \frac{1}{4}\beta \tag{VI-108}$$

Comparison of Eqs. (VI-106) through (VI-108) with Eq. (VI-49) and Fig. VI-1 show that all the contributions of the flow term to $f_o^{(2)}$ have a

damping effect on the waves near $\theta = 90^{\circ}$. The moment-excesses generated by the flow term all have effects with the same sign on the waves near $\theta = 90^{\circ}$.

Resonant Particles

Waves grow up corresponding to every k-vector lying near the perpendicular plane. Consequently every particle resonates with some of the waves, that is, has its velocity vector perpendicular to the k-vector of the wave. Let the angular width of the spectrum about the perpendicular plane be Δ. Particles whose velocity vector lies within an angle Δ of the parallel axis will interact with more waves than particles whose velocity vector lies outside this cone. This causes particles with large $v_{\parallel}/v_{\perp}$ to diffuse more rapidly than particles with small $v_{\parallel}/v_{\perp}$. This reduces $T_{\parallel}$ and increases $T_{\perp}$ thereby causing stabilization.

Note that the region of rapid diffusion, Δ, is large. Δ is not of order α, rather $\Delta \sim 1$. Thus we do not expect f_o to develop a structure which varies over distances δv small compared to the thermal velocity. That is we expect that the distribution function should be describable by a smooth function.

For a particular k-value, there is a resonant contribution to the current, j_k, and a nonresonant contribution to the current. The resonant contribution is essential for the correct calculation of the growth rate, for it is the residue of the pole which is the coefficient of γ in the Eq. (IV-43) for γ. The residue corresponds to the density of particles at $v = 0$, which does not change to lowest order. Thus the residue is constant to lowest order and the growth rate changes only as the anisotropy changes. The growth rate is not affected by the evolution of the resonant

particles. If one neglected the resonant contribution to the current, but kept the $\partial E/\partial t$ term in the Maxwell-Ampere equation, one would still obtain an equation for γ. However the result would be completely wrong. This is precisely what happens in a fluid calculation of this instability.

Rationale for a Bi-Maxwellian Model

The distribution function should be describable by a smooth function. The smoother f_o is the fewer terms are required in the moment-excess expansion. Anisotropy in any of the moments will eventually diffuse away, higher moments diffusing faster. The growth rate of the waves is determined by the moments of the bulk of the distribution function and not by a special group of particles. Anisotropy in any of the moments will contribute to the growth and decay of the waves near $\theta = 90^{\circ}$. The moment-excesses generated by the flow term all have effects with the same sign on the waves near $\theta = 90^{\circ}$.

Anisotropies of all even moments have similar effects on the waves near $\theta = 90^{\circ}$, and similarly all anisotropies diffuse away. We may think of the temperature anisotropy of a bi-Maxwellian as representing the combined anisotropies of the actual distribution. The bi-Maxwellian model is then a model of the "lumped parameter" type. Since the initial state of the plasma has a bi-Maxwellian distribution function, a bi-Maxwellian model gives a rigorously accurate description of the period during which the wave spectrum grows up. The bi-Maxwellian model is not a rigorous description of the subsequent period during which the wave spectrum damps, however it possesses all of the essential qualitative features.

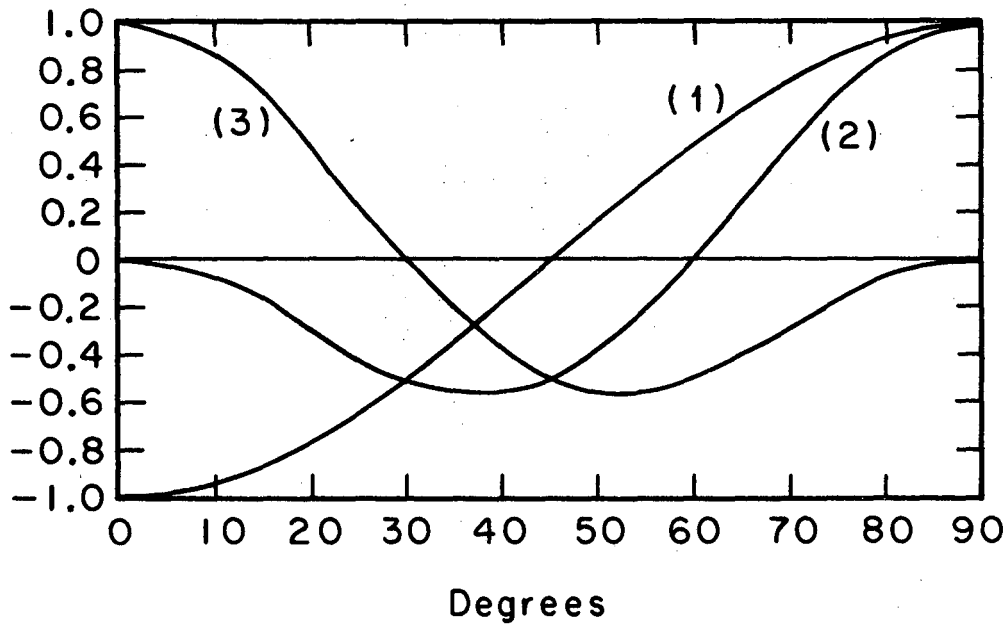

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FIGURE VI-1

(1) = 2 $SIN^2\theta$ - 1

(2) = 4 $SIN^4\theta$ - 3 $SIN^2\theta$

(3) = 4 $SIN^4\theta$ - 5 $SIN^2\theta$ + 1

VII. QUASILINEAR EVOLUTION OF A BI-MAXWELLIAN MODEL

Quasilinear Equations for a Bi-Maxwellian

In the energy appendix it is shown that for the exact evolution of the plasma the two temperatures are not independent to first order in α. Rather they are related by

$$a^2/d^2 = \langle u_1^2 \rangle = 1 - \frac{1}{3}\alpha + \cdots \tag{VII-1}$$

$$b^2/d^2 = \langle u_3^2 \rangle = 1 + \frac{2}{3}\alpha + \cdots \tag{VII-2}$$

Therefore in a bi-Maxwellian model the two parameters $a^2(t)$ and $b^2(t)$ may be reduced to one parameter $\alpha(t)$.

For a bi-Maxwellian model the spectrum evolution is given by Eqs. (VI-11) and (VI-49) as

$$\frac{\partial}{\partial t_0}|B_k^{(0)}|^2 = \left(\frac{8}{\pi}\right)^{1/2} kd \left\{\alpha(2S^2-1) - k^2c^2/\omega_e^2\right\} \tag{VII-3}$$

where we have used Eqs. (VI-19) and (VI-20), and the definition of α

$$\langle u_1^2 \rangle = 1 + 2a_1 \tag{VII-4}$$

$$\langle u_3^2 \rangle = 1 + 2a_3 \tag{VII-5}$$

$$\alpha = (b^2 - a^2)/d^2 = 2(a_3 - a_1) \tag{VII-6}$$

Equation (VII-3) and an equation for $\alpha(t)$ form a closed set of equations. To obtain an equation for α it suffices to take the moment of Eq. (VI-58) with u_3^2. Then from Eq. (VI-20), (VI-94), and (VI-96) we have

$$RHS = \frac{\partial}{\partial t_0}(2a_3) = \frac{\partial}{\partial t_0}\left(\frac{2}{3}\alpha\right) \tag{VII-7}$$

$$\text{Flow term} = 2b_3 = \frac{1}{V}\frac{e^2}{m^2c^2}\int\frac{d^3k}{(2\pi)^3}\,\frac{1}{2k^2d^2}\,\frac{\partial}{\partial t_0}|B_k^{(0)}|^2\,2(1-2S^2) \qquad \text{(VII-8)}$$

It only remains to calculate the moment of u_3^2 with the diffusion term. This is

$$-\int d^3u\; u_3^2\;\frac{1}{V}\frac{e^2}{m^2c^2}\int\frac{d^3k}{(2\pi)^3}\left\{\frac{1}{kd}|B_k^{(0)}|^2(u\times\hat{B}_k)\cdot\frac{\partial}{\partial u}\frac{i}{\hat{k}\cdot u}(u\times\hat{B}_k)\cdot\frac{\partial}{\partial u}\right\}f_0^{(2)}(u) \qquad \text{(VII-9)}$$

To first order in α, $f_0^{(2)}$ is given by Eq. (V-12)

$$f_0^{(2)}(u) = \frac{1}{6}\alpha\left(2u_3^2 - u_\perp^2\right)f_0^{(0)}(u)$$

$$= \frac{1}{6}\alpha\left(3u_3^2 - u^2\right)f_0^{(0)}(u) \qquad \text{(VII-10)}$$

We take advantage of the fact that the $u^2 f_0^{(0)}$ term gives zero because of the factor $(u \times \partial/\partial u)$ in Eq. (VII-9). Interchanging the u and k-integrations, we choose the coordinate system of Fig. V-1. We then make the same change of variables as before

$$x = \hat{k}\cdot u \qquad y = \hat{B}_k\cdot u \qquad z = \hat{n}\cdot u \qquad \text{etc.} \qquad \text{(VII-11)}$$

whereupon

$$(u\times\hat{B}_k)\cdot\frac{\partial}{\partial u} = \left(-z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z}\right) \qquad \text{(VII-12)}$$

$$u_3^2 = (Cx + Sz)^2 \qquad \text{(VII-13)}$$

Integrating by parts once in Eq. (VII-9) gives for the diffusion term

$$\text{Diffusion term} = \frac{1}{V}\frac{e^2}{m^2c^2}\int\frac{d^3k}{(2\pi)^3}\,\frac{1}{kd}\,|B_k^{(0)}|^2\,p_k \qquad \text{(VII-14)}$$

where

$$p_k = \int d^3u\,(Cx+Sz)^2\left(-z\frac{\overleftarrow{\partial}}{\partial x} + x\frac{\overleftarrow{\partial}}{\partial z}\right)\frac{i}{x}\left(-z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z}\right)\frac{\alpha}{2}(Cx+Sz)^2 f_0^{(0)}(u)$$

$$= 2\alpha i\int d^3u\,\frac{1}{x}\left(C^2x^2 + 2CSxz + S^2z^2\right)\left(S^2x^2 - 2SCxz + C^2z^2\right)f_0^{(0)}(u) \qquad \text{(VII-15)}$$

Performing the y and z integrations yields

$$p_k = 2\alpha i \int dx \; \frac{h}{x} \left\{ C^2x^2(S^2x^2+C^2) + 2SCx(-2SCx) + S^2(S^2x^2+3C^2) \right\} \quad \text{(VII-16)}$$

where

$$h = (2\pi)^{-1/2} \exp -\tfrac{1}{2}x^2 \quad \text{(VII-17)}$$

Since the integrand is odd in x, only the pole contributes. Hence

$$p_k = 2\alpha i \cdot i\left(\tfrac{\pi}{2}\right)^{1/2} \cdot 3S^2C^2 = -6\alpha S^2C^2\left(\tfrac{\pi}{2}\right)^{1/2} \quad \text{(VII-18)}$$

Consequently the equation for $\alpha(t)$ is

$$\frac{2}{3}\frac{\partial}{\partial t_0}\alpha(t_0) = -\frac{1}{V}\frac{e^2}{m^2c^2}\int\frac{d^3k}{(2\pi)^3}\frac{1}{kd}|B_k^{(0)}|^2 S^2C^2 \; 6\left(\tfrac{\pi}{2}\right)^{1/2}\alpha(t_0)$$

$$+\frac{1}{V}\frac{e^2}{m^2c^2}\int\frac{d^3k}{(2\pi)^3}\frac{1}{k^2d^2}\frac{\partial}{\partial t_0}|B_k^{(0)}|\;(1-2S^2) \quad \text{(VII-19)}$$

Dimensionless Scaling

In order to analyze Eqs. (VII-3) and (VII-19) it is convenient to introduce dimensionless variables. We define dimensionless variables: K, A, T, and R(K) by

$$k = k_0 K \quad \text{(VII-20)}$$

$$\alpha(t_0) = A\,\alpha(0) \quad \text{(VII-21)}$$

$$\frac{\partial}{\partial t_0} = \gamma_0 \frac{\partial}{\partial T} \quad \text{(VII-22)}$$

$$\frac{e^2}{m^2c^2}|B_k^{(0)}|^2 = q\,R(K) \quad \text{(VII-23)}$$

Also we let

$$\mu = \cos\theta \quad \text{(VII-24)}$$

Then Eq. (VII-3) becomes

$$\gamma_0 \frac{\partial}{\partial T} R = \left(\frac{8}{\pi}\right)^{1/2} k_0 d\, K\, \alpha(0) \left\{ A(1-2\mu^2) - K^2 k_0^2 c^2 / \alpha(0)\, \omega_e^2 \right\} R \qquad \text{(VII-25)}$$

The most convenient choice of scaling constants k_0 and γ_0 is evidently

$$k_0^2 c^2 / \alpha(0)\, \omega_e^2 = 1 \qquad \text{(VII-26)}$$

$$\gamma_0 = \left(\frac{8}{\pi}\right)^{1/2} k_0 d\, \alpha(0) \qquad \text{(VII-27)}$$

whereupon Eq. (VII-25) becomes

$$\frac{\partial}{\partial T} R = \left\{ (1-2\mu^2) AK - K^3 \right\} R \qquad \text{(VII-28)}$$

Substituting our dimensionless variables into Eq. (VII-19) we obtain

$$\frac{2}{3} \gamma_0 \alpha(0) \frac{\partial}{\partial T} A = -\frac{1}{V} \frac{k_0^3}{(2\pi)^3} \int d^3K \frac{1}{K} \frac{1}{k_0 d} q R \mu^2 (1-\mu^2)\, 6 \left(\frac{\pi}{2}\right)^{1/2} \alpha(0) A$$

$$- \frac{1}{V} \frac{k_0^3}{(2\pi)^3} \int d^3K \frac{1}{K^2} \frac{1}{k_0^2 d^2} \gamma_0 \frac{\partial}{\partial T} q R (1-2\mu^2) \qquad \text{(VII-29)}$$

Obviously the most convenient choice for the constant q is

$$\frac{2}{3} \gamma_0 \alpha(0) = \frac{1}{V} \frac{k_0^3}{(2\pi)^3} \frac{1}{k_0^2 d^2} \gamma_0 q \qquad \text{(VII-30)}$$

then

$$\left(\frac{1}{V} \frac{k_0^3}{(2\pi)^3} q \right) = \frac{2}{3} \alpha(0) k_0^2 d^2 \qquad \text{(VII-31)}$$

and

$$\frac{1}{\frac{2}{3} \gamma_0 \alpha(0)} \left(\frac{1}{V} \frac{k_0^3}{(2\pi)^3} q \right) \frac{1}{k_0 d} \alpha(0) = \frac{1}{\gamma_0} \alpha(0) k_0 d = \left(\frac{\pi}{8}\right)^{1/2} \qquad \text{(VII-32)}$$

whereupon Eq. (VII-29) becomes

$$\frac{\partial}{\partial T} A = -\frac{3\pi}{2} \int d^3K \frac{1}{K} \mu^2 (1-\mu^2) R(K) A - \int d^3K \frac{1}{K^2} (1-2\mu^2) \frac{\partial}{\partial T} R(K) \qquad \text{(VII-33)}$$

The physical significance of our scaling constants may be understood by noticing that

$$k_0 = k(\text{crit}, \theta = 0, t_0 = 0) \tag{VII-34}$$

$$\gamma = \gamma_0 \left\{ (1-2\mu^2) AK - K^3 \right\} \sim \gamma_0 \tag{VII-35}$$

and that R is proportional to the magnetic field energy density: $\mathcal{E}_{mag}(K)$. From Eqs. (VII-28) or (VII-35) it follows that the region of growing waves is

$$0 < K^2 < K^2(\text{crit}) = (1-2\mu^2)A \tag{VII-36}$$

Maximum growth rate occurs for

$$\mu = 0 \qquad \text{and} \qquad K^2(\gamma_{max}) = \tfrac{1}{3}A \tag{VII-37}$$

Initial Energy Conditions

The magnetic field energy density is a quantity of direct physical interest. We shall find an expression for it in terms of our dimensionless variables.

$$\mathcal{E}_{mag} = \frac{1}{V} \int \frac{d^3k}{(2\pi)^3} \frac{|B_k^{(0)}|^2}{8\pi}$$

$$= \frac{k_0^3}{V(2\pi)^3} \frac{q}{8\pi} \frac{m^2c^2}{e^2} \int d^3K \; R(K) \tag{VII-38}$$

Now

$$\left(\frac{k_0^3 q}{V(2\pi)^3}\right) \frac{m^2c^2}{8\pi e^2} = \left(\frac{2}{3}\alpha(0) k_0^2 d^2\right) \frac{n_0 m c^2}{2} \frac{m}{4\pi n_0 e^2}$$

$$= \frac{1}{3}\alpha(0) \; n_0 m d^2 \; \frac{k_0^2 c^2}{\omega_e^2}$$

$$= \frac{1}{3}\alpha^2(0) \; n_0 m d^2 \tag{VII-39}$$

And the kinetic energy density is

$$\mathcal{E}_{kin} = \frac{3}{2} n_o m d^2 \tag{VII-40}$$

We define the dimensionless variable M to be the ratio of the magnetic field energy density to the kinetic energy density, divided by $\alpha^2(o)$

$$M = \mathcal{E}_{mag} / (\mathcal{E}_{kin}\, \alpha^2(o)) = \frac{2}{9} \int d^3K \;\; R(K) \tag{VII-41}$$

We have mentioned in the section on the ordering scheme that $M \sim 1$ when the wave spectrum has reached maximum amplitude. Let us see how large M might be initially. We first consider the electromagnetic field energy in thermal equilibrium. For a box of length L the number of degrees of freedom for waves with wave number less than k_1 is

$$\text{degrees of freedom} = (k_1 L)^3 \tag{VII-42}$$

Let Θ denote the temperature then the total field energy in the box for waves with wave number less than k_1 is

$$\text{total field energy in box} = \Theta\, (k_1 L)^3 \tag{VII-43}$$

Let $\lambda_D = d/\omega_e$ denote the Debye length and $\Lambda = n\lambda_D^3$ denote the plasma parameter, then the field energy density for waves with wave number less than k_1 in thermal equilibrium is

$$\begin{aligned}
\mathcal{E}_{field}(\text{thermal}, k < k_1) &= \Theta\, k_1^3 = \frac{n\Theta}{n\lambda_D^3}\, k_1^3 \lambda_D^3 \\
&= \frac{2}{3}\mathcal{E}_{kin}\, \Lambda^{-1} k_1^3 \frac{d^3}{\omega_e^3} = \frac{2}{3}\mathcal{E}_{kin}\, \Lambda^{-1} \frac{d^3}{c^3}\, \alpha^{3/2}(0)\; k_1^3 \frac{c^3}{\alpha^{3/2}(0)\, \omega_e^3} \\
&= \frac{2}{3}\mathcal{E}_{kin}\, \Lambda^{-1} \left(\frac{d}{c}\right)^3 \alpha^{3/2}(0) \left(\frac{k_1}{k_o}\right)^3
\end{aligned} \tag{VII-44}$$

If we assume for initial conditions that the magnetic field energy density at $T = 0$ is the same order of magnitude as the above thermal equilibrium field energy density.

$$\mathcal{E}_{mag}(T=0, k<k_1) \sim \mathcal{E}_{field}(\text{thermal}, k<k_1) \tag{VII-45}$$

then

$$M(T=0, K<K_1) \sim \Lambda^{-1} \left(\frac{d}{c}\right)^3 \alpha^{-1/2}(0) \; K_1^3 \tag{VII-46}$$

but from Eq. (VII-41)

$$M(T=0, K<K_1) = \frac{2}{9} \int^{K_1} d^3K \; R(0,K) = \frac{2}{9} \frac{4\pi}{3} Q K_1^3 \tag{VII-47}$$

where we have taken our initial wave spectrum to be

$$R(0,K) = Q = \text{constant} \tag{VII-48}$$

Comparing Eqs. (VII-46) and (VII-47) we see that

$$Q \sim \Lambda^{-1} \left(\frac{d}{c}\right)^3 \alpha^{-1/2}(0) \tag{VII-49}$$

Thus, for example, the values

$$\Lambda = 1000 \qquad \frac{d}{c} = 0.1 \qquad \alpha(o) = 0.01 \tag{VII-50}$$

give

$$Q \sim 10^{-5} \tag{VII-51}$$

The unstable wave have $K < 1$ and for them

$$M(T=0, K<1) \sim 10^{-5} \tag{VII-52}$$

Performing the μ Integration

When the initial wave spectrum has a very small amplitude compared to a later wave spectrum, then the initial spectrum shape is a less important determinant of the later spectrum shape than is the growth rate. For the growth rate has an exponential effect on the later spectrum.

$$\text{(later spectrum)} = \text{(initial spectrum)}\cdot\exp\left(\int dT\cdot\text{growth rate}\right) \qquad \text{(VII-53)}$$

Our initial wave spectrum is assumed to be quite small compared to our maximum wave spectrum. Therefore for simplicity we shall choose our initial spectrum to be

$$R(T=0,K) = \begin{cases} Q = \text{constant} & \text{for } 0 \leq K \leq 1 \\ 0 & \text{for } 1 < K \end{cases} \qquad \text{(VII-54)}$$

The damped modes will never contribute significantly to the spectrum moments which occur in Eq. (VII-33). Consequently we have ignored them from the beginning.

Equations (VII-28) and (VII-33) govern the evolution of our plasma. We define

$$G = \int_0^T A \, dT \qquad \text{(VII-55)}$$

and integrate Eq. (VII-28) in time to obtain

$$R = Q \exp\left\{(1-2\mu^2)GK - K^3T\right\} \qquad \text{(VII-56)}$$

Setting

$$d^3K = d\varphi \, d\mu \, K^2 dK \qquad \text{(VII-57)}$$

we integrate Eq. (VII-33) in φ and in time obtaining

$$A(T) = A(0) - 3\pi^2 \int_0^T dT' \int d\mu\, dK\, \mu^2(1-\mu^2)\, K\, R(T',K)\; A(T')$$

$$- 2\pi \int d\mu\, dK\, (1-2\mu^2) \left\{ R(T,K) - R(0,K) \right\} \qquad \text{(VII-58)}$$

If we let B denote the diffusion term and C denote the flow term and note that by definition

$$A(0) = 1.0 \qquad \text{(VII-60)}$$

then

$$A(T) = 1.0 - B(T) - C(T) \qquad \text{(VII-61)}$$

where

$$B(T) = 3\pi^2 \int_0^T dT' \int d\mu\, dK\; \mu^2(1-\mu^2)\, K\, R(T',K)\; A(T') \qquad \text{(VII-62)}$$

and

$$C(T) = 2\pi \int d\mu\, dK\; (1-2\mu^2)\; R(T,K) \qquad \text{(VII-63)}$$

The term R(0,K) makes negligible contribution to A, so we have dropped it from Eq. (VII-63).

We now define

$$P = Q \exp(GK - K^3 T) \qquad \text{(VII-64)}$$

$$R = P \exp(-2GK\mu^2) \qquad \text{(VII-65)}$$

and consider the μ-integration in Eqs. (VII-62) and (VII-63). In particular we consider the approximation

$$2\pi \int_0^1 dK\, P(K) \int_{-1}^{1} d\mu\, \exp(-2GK\mu^2) \doteq 2\pi \int_0^1 dK\, P(K) \int_{-\infty}^{\infty} d\mu\, \exp(-2GK\mu^2) \quad \text{(VII-66)}$$

The LHS is the major contribution to C. In order that the error in calculating C give a negligible error to A through Eq. (VII-61), the error must be small compared to 1.0. The error is

$$\text{Error} = 2\pi \int_0^1 dK\, P(K)\; 2\int_1^{\infty} d\mu\, \exp(-2GK\mu^2) \quad \text{(VII-67)}$$

The μ-integral in the error is worse for small K values, however P(K) is peaked about middle K values. As T increases, G increases and the error becomes exponentially less. We may place an upper bound on our error by using the inequality

$$(\mu-1)^2 = \mu^2 - 1 - 2(\mu-1) \leq \mu^2 - 1 \quad \text{for } \mu \geq 1 \quad \text{(VII-68)}$$

From this it follows that

$$2\int_1^{\infty} d\mu\, \exp(-2GK\mu^2) = \exp(-2GK)\cdot 2\int_1^{\infty} d\mu\, \exp\{-2GK(\mu^2-1)\}$$

$$\leq \exp(-2GK)\cdot 2\int_1^{\infty} d\mu\, \exp\{-2GK(\mu-1)^2\} = \exp(-2GK)(2\pi)^{1/2}(4GK)^{-1/2} \quad \text{(VII-69)}$$

Thus

$$\text{Error} \leq 2\pi \int_0^1 dK\, P(K)\, \exp(-2GK)\, (2\pi)^{1/2}\, (4GK)^{-1/2}$$

$$= (2\pi)^{3/2}\, (4G)^{-1/2} Q \int_0^1 dK\, K^{-1/2} \exp(-GK - K^3 T) \quad \text{(VII-70)}$$

But

$$\int_0^1 dK\, K^{-1/2} \exp(-GK - K^3T) \leq \int_0^1 dK\, K^{-1/2} \exp(-GK) \leq \int_0^{\infty} dK\, K^{-1/2} \exp(-GK) \quad \text{(VII-71)}$$

We make the change of variable

$$x^2 = K \quad \text{(VII-71a)}$$

then

$$\int_0^\infty dK\, K^{-1/2} \exp(-GK) = 2\int_0^\infty dx \exp(-Gx^2) = (2\pi)^{1/2}(2G)^{-1/2} \qquad \text{(VII-72)}$$

Therefore

$$\text{Error} \leq (2\pi)^{3/2}(4G)^{-1/2} Q\, (2\pi)^{1/2}(2G)^{-1/2} = 2^{1/2}\pi^2 Q/G \doteq 14.0\, Q/G \qquad \text{(VII-73)}$$

For $G \geqq 10$

$$\text{Error} \leq 1.4\, Q \sim 10^{-5} \qquad \text{(VII-74)}$$

At $T = 10$, B and C are still quite small so $A \doteq 1.0$ and $G \doteq 10$. Thus the error in calculating B and C which results from extending the μ-integration from 1 to ∞ is already negligible when B and C are still quite small compared to 1. For sufficiently small values of T the above approximation would not be valid. However for such times we have no need to calculate accurate values for B and C in order to find A. For we know them to be so small that $A = 1.0$.

Using the above approximation, the μ integrals for B and C are

$$C \rightarrow \int_{-\infty}^{\infty} d\mu\, (1-2\mu^2) \exp(-2GK\mu^2) = (2\pi)^{1/2}(4GK)^{-1/2}\{1-2(4GK)^{-1}\} \qquad \text{(VII-75)}$$

$$B \rightarrow \int_{-\infty}^{\infty} d\mu\, (\mu^2-\mu^4) \exp(-2GK\mu^2) = (2\pi)^{1/2}(4GK)^{-3/2}\{1-3(4GK)^{-1}\} \qquad \text{(VII-76)}$$

whereupon B and C become

$$C = \pi(2\pi)^{1/2}\, G^{-1/2} \int_0^1 dK\, K^{-1/2} \left\{1 - \frac{1}{2GK}\right\} P(K) \qquad \text{(VII-77)}$$

$$B = \frac{3}{8}\pi^2(2\pi)^{1/2} \int_0^T dT\, A\, G^{-3/2} \int_0^1 dK\, K^{-1/2} \left\{1 - \frac{3}{4GK}\right\} P(K) \qquad \text{(VII-78)}$$

Performing the μ integration for M yields

$$M = \frac{2}{9}(2\pi)\int_0^1 dK\ K^2 P(K) \int_{-\infty}^{\infty} d\mu\ \exp(-2GK\mu^2)$$

$$= \frac{2}{9}(2\pi)\int_0^1 dK\ K^2 P(K)\ (2\pi)^{1/2}(4GK)^{-1/2}$$

$$= \frac{1}{9}(2\pi)^{3/2}\ G^{-1/2}\int_0^1 dK\ K^{3/2}\ P(K) \qquad \text{(VII-79)}$$

Equations of Evolution

In summary, our equations for the evolution of our plasma have been reduced to the following set.

$$G = \int_0^T A\ dT \qquad \text{(VII-80)}$$

$$P = Q\ \exp(GK - K^3 T) \qquad \text{(VII-81)}$$

$$A = 1.0 - B - C \qquad \text{(VII-82)}$$

$$D = C_0\ G^{-1/2}\int_0^1 dK\ K^{-1/2}\ P(K) \qquad \text{(VII-83)}$$

$$L = C_1\ G^{-3/2}\int_0^1 dK\ K^{-3/2}\ P(K) \qquad \text{(VII-84)}$$

$$C = D - L \qquad \text{(VII-85)}$$

$$B = B_0 \int_0^T dT\ G^{-1} A\ (D - \tfrac{3}{2}L) \qquad \text{(VII-86)}$$

$$M = M_0\ G^{-1/2}\int_0^1 dK\ K^{3/2}\ P(K) \qquad \text{(VII-87)}$$

where

$$C_0 = \pi(2\pi)^{1/2} = 7.874\ 807 \qquad \text{(VII-88)}$$

$$C_1 = \tfrac{1}{2}C_0 = 3.937\ 404 \qquad \text{(VII-89)}$$

$$B_0 = \tfrac{3}{8}\pi = 1.178\ 097 \qquad \text{(VII-90)}$$

$$M_0 = \tfrac{1}{9}(2\pi)^{3/2} = 1.749\ 957 \qquad \text{(VII-91)}$$

These equations were integrated simultaneously on a computer and the results are shown in Figs. VII-1 through VII-4.

As we have mentioned, the flow term, C, makes no contribution to the net change in A from initial to final states. Our initial state is

$$A(0) = 1.0 \tag{VII-92}$$

$$B(0) = C(0) = 0 \tag{VII-93}$$

Our final state, when the anisotropy has disappeared, must be

$$A \rightarrow 0 \quad \text{as} \quad T \rightarrow \infty \tag{VII-94}$$

$$C \rightarrow 0 \quad \text{as} \quad T \rightarrow \infty \tag{VII-95}$$

It therefore follows that

$$B \rightarrow 1.0 \quad \text{as} \quad T \rightarrow \infty \tag{VII-96}$$

Whence

$$B_0 \int_0^\infty dT \; G^{-1} A \left(D - \tfrac{3}{2} L\right) = 1.0 \tag{VII-97}$$

It is convenient to define the complement of B

$$\overline{B} = B_0 \int_T^\infty dT \; G^{-1} A \left(D - \tfrac{3}{2} L\right) \tag{VII-98}$$

then

$$\overline{B} = 1.0 - B \tag{VII-99}$$

and

$$A = \overline{B} - C \tag{VII-100}$$

Description of the Evolution

Let us consider the system as described by the differentiated versions of Eqs. (VII-81) and (VII-82).

$$\dot{P} = (AK - K^3)P = \gamma P \tag{VII-101}$$

$$\dot{A} = -\dot{B} - \dot{C} \tag{VII-102}$$

where dot means d/dT. Equation (VII-101) gives the spectrum growth rate as a function of K. Equation (VII-102) says that the change in the anisotropy is due to two terms, each of which are functionals of the spectrum. $\dot{B}$ is the diffusion term and $\dot{C}$ is the dissipationless flow term. The $\dot{B}$ term is responsible for isotropization. The $\dot{C}$ term causes some flow in velocity space, but the net effect of this flow on the distribution function as $T \to \infty$ is zero. The initial state, $f_o(t=0)$, is given. The final state, $f_o(t=\infty)$, is determined by the $\dot{B}$ term. The $\dot{C}$ term has a dramatic effect on the path through function space which f_o follows in going from $f_o(t=0)$ to $f_o(t=\infty)$. However $\dot{C}$ has no effect on what the final state will be. Equation (VII-33) shows that $\dot{C}$ is proportional to the growth rate, $\gamma(K,\mu)$, averaged over the spectrum, while $\dot{B}$ is proportional to the angular width of the spectrum about the perpendicular plane.

The evolution of the anisotropy, A, may be broken up into three distinct phases: an intial phase, a rapid phase, and an asymptotic phase. This is shown in Fig. VII-1, where A(T) is graphed on log-log paper. On this graph each of the three phases may be approximated with a straight line. These three straight lines have the equations

$$A(\text{initial}) = \text{constant } T^{0} = \text{constant} \tag{VII-103}$$

$$A(\text{rapid}) = \text{constant } T^{-3.0} \tag{VII-104}$$

$$A(\text{asymptotic}) = \text{constant } T^{-2/3} \tag{VII-105}$$

The exact time at which one phase ends and another begins is somewhat arbitrary. A reasonable definition is: that time for which the slope of A(T) lies halfway between the slopes of the straight lines of the two adjacent phases. With this definition the initial phase runs from $T = 0$ to $T = 25$. The rapid phase is from $T = 25$ to $T = 42$, and the asymptotic phase is for $T > 42$.

During the initial phase the amplitude of the wave spectrum is small. Consequently $\dot{B}$ and $\dot{C}$ are small and therefore so is $\dot{A}$. It follows that A changes very little while P grows. That is, the initial phase is a period of exponential wave growth with almost constant exponent. The wave spectrum grows up peaked about

$$K(\gamma_{max}) = \left\{ \tfrac{1}{3} A(\text{initial}) \right\}^{1/2} = 0.577 \tag{VII-106}$$

The spectrum grows for

$$0 < K < K(\text{crit}) = A^{1/2} \tag{VII-107}$$

and damps for K larger than K(crit).

Eventually the amplitude of the wave spectrum becomes so large that the temperature anisotropy begins to change rapidly. This is due mainly to the $\dot{C}$ term which is larger than the $\dot{B}$ term at first. A decreases rapidly until about half of the wave spectrum with smaller wave numbers is growing and the half with larger wave numbers is damping. Thus

although the spectrum remains large, $\dot{C}$ is drastically reduced. This brings the rapid phase to an end. A comparison of the evolution of A, B, and C with time is shown in Fig. VII-2. The wave spectrum and growth rate at $T = 42$ are shown in Fig. VII-3. The spectrum peak has shifted slightly to

$$K = (G/3T)^{1/2} = 0.521 \qquad \text{(VII-108)}$$

The spectrum continued to grow throughout the rapid phase, however the growth was no longer exponential. The magnetic wave energy, M, reached a peak at $T = 40$. $M(T)$ is graphed in Fig. VII-4.

After a small additional decrease in A, slightly less than half of the wave spectrum is growing and slightly more than half is damping. This makes $\dot{C}$ negative which partially cancels the $\dot{B}$ term. Changes in A then proceed over more slowly as temperature isotropy is approached. Simultaneously the wave spectrum shifts to lower and lower wave numbers, and eventually decays. The asymptotic phase is characterized by the wave spectrum shifting so as to always be slightly more than half damping. Shifting in this manner the $\dot{C}$ term continues to partially cancel the $\dot{B}$ term. If $\dot{C}$ were to become less negative, then A would change more rapidly. The spectrum would then damp more quickly until $\dot{C}$ became more negative again. As time increases $\overline{B}$ is asymptotic to C since A approaches zero more rapidly than either $\overline{B}$ or C. Figure VII-4 shows that throughout the asymptotic phase M decreases somewhat faster than A. From the fact that A decreases like $T^{-2/3}$ we conclude that G grows approximately like $T^{1/3}$. Therefore the quantity Z

$$Z = G/(3T)^{1/3} \qquad \text{(VII-109)}$$

should be approximately constant throughout the asymptotic phase. This is shown in Fig. VII-4.. The degree of constancy may be seen from the following brief table

T	Z	T	Z
50	6.841	200	6.924
70	6.870	300	6.937
100	6.893	400	6.944
150	6.913	500	6.948

We shall exploit this property in seeking an asymptotic solution to our equations of evolution.

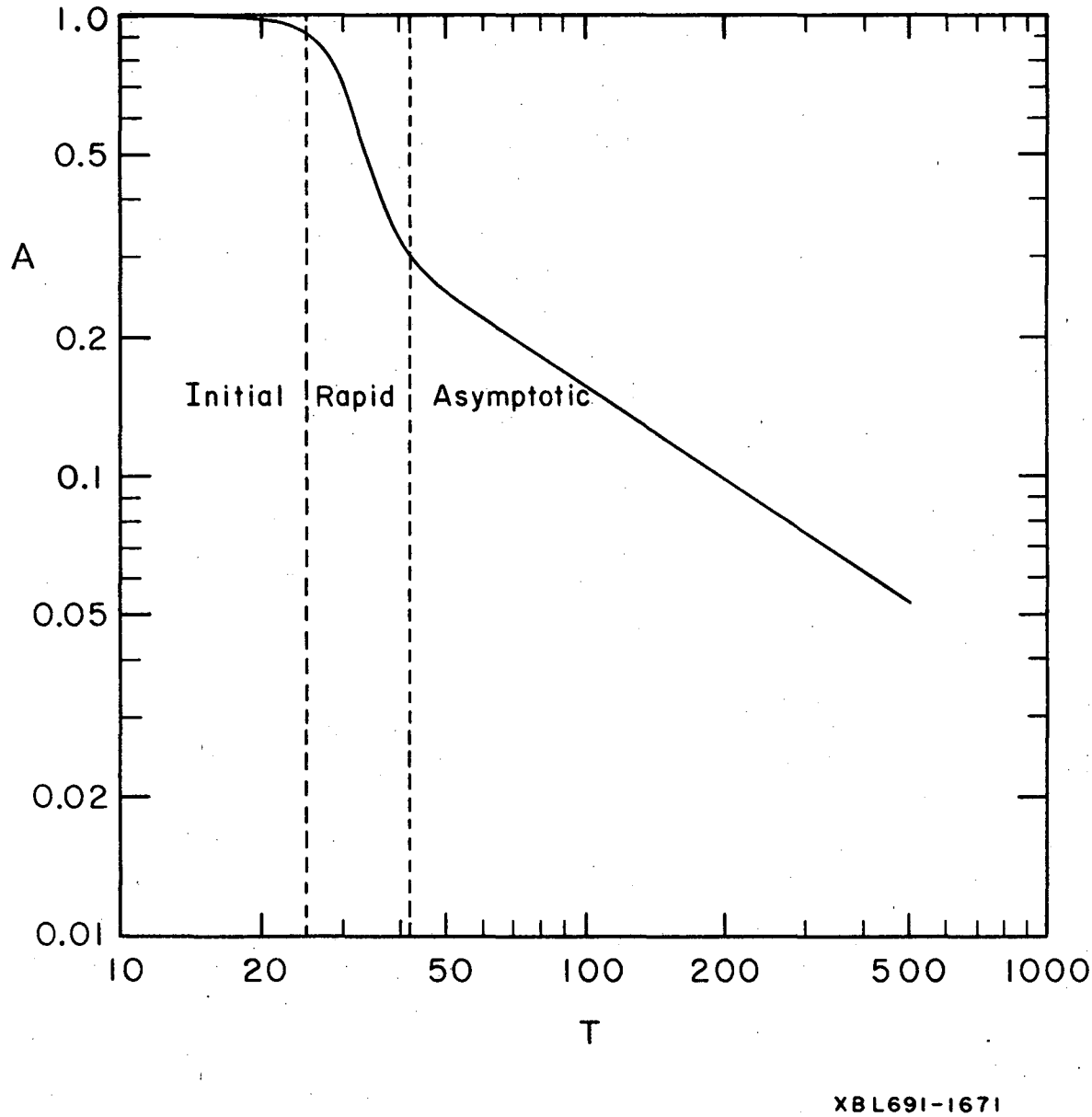

FIGURE VII-1
ANISOTROPY A

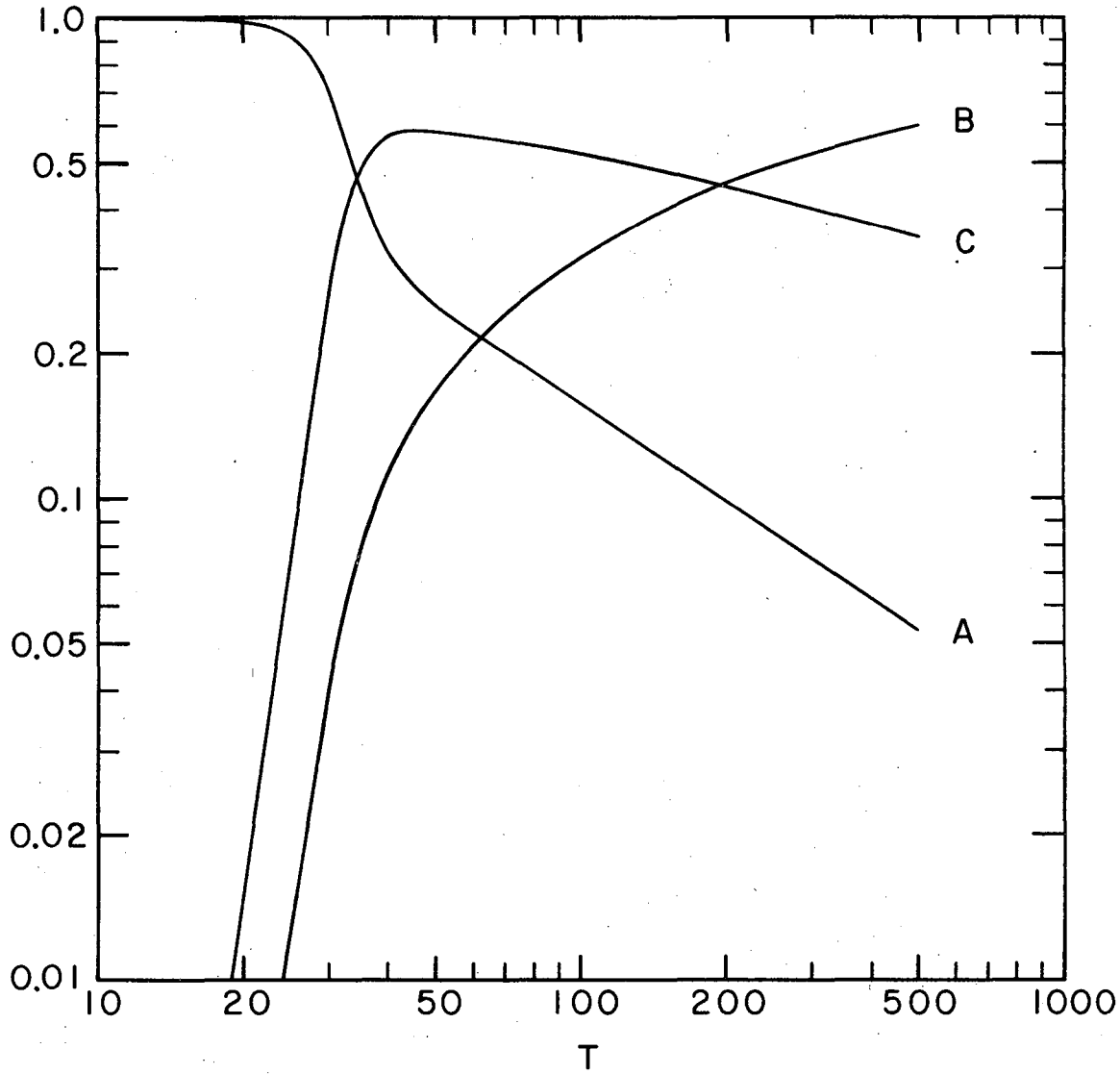

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FIGURE VII-2

ANISOTROPY A

DIFFUSION TERM B

DISSIPATIONLESS FLOW TERM C

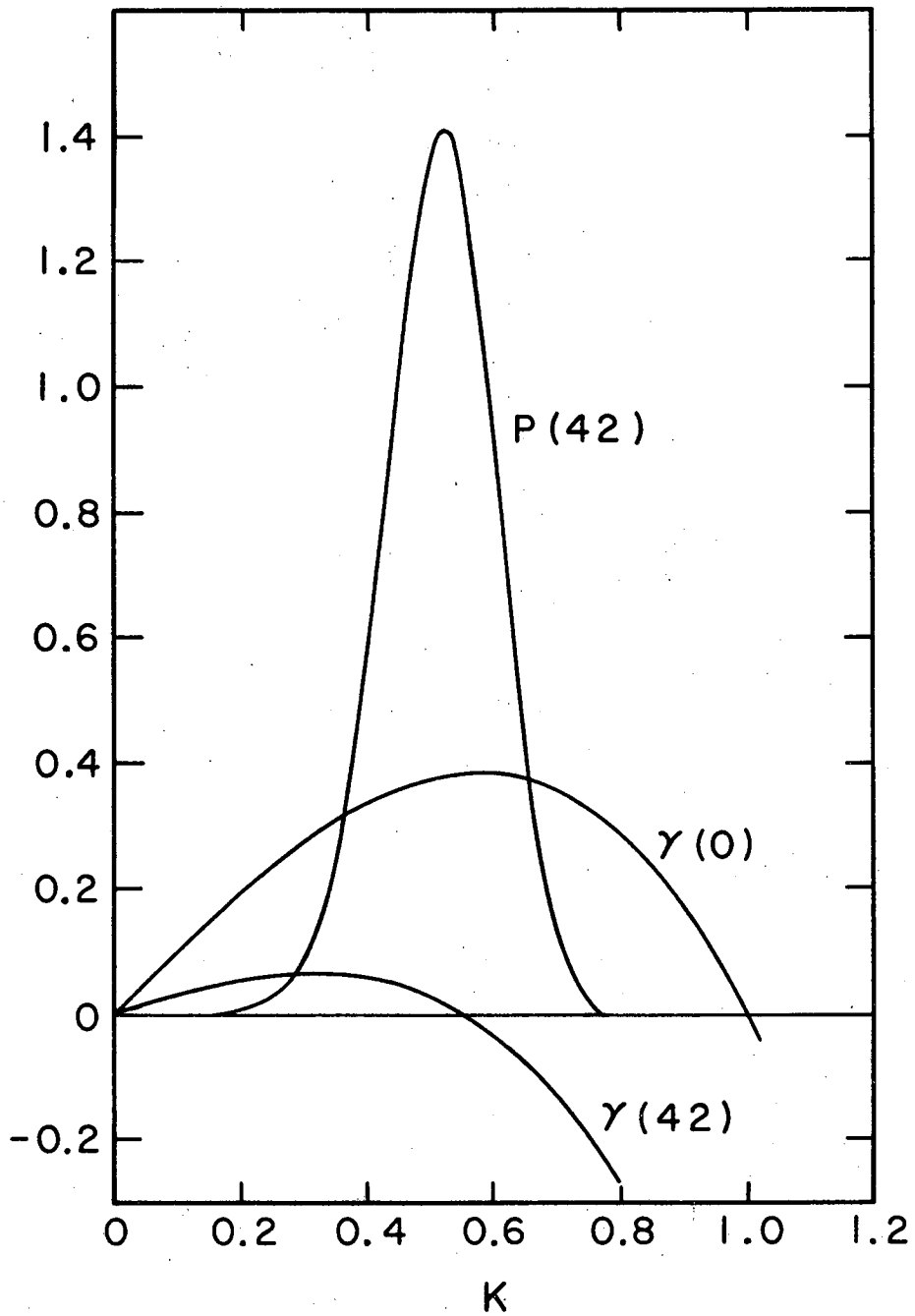

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FIGURE VII-3

WAVE SPECTRUM P(T=42)

GROWTH RATES γ(T=42) AND γ(T=0)

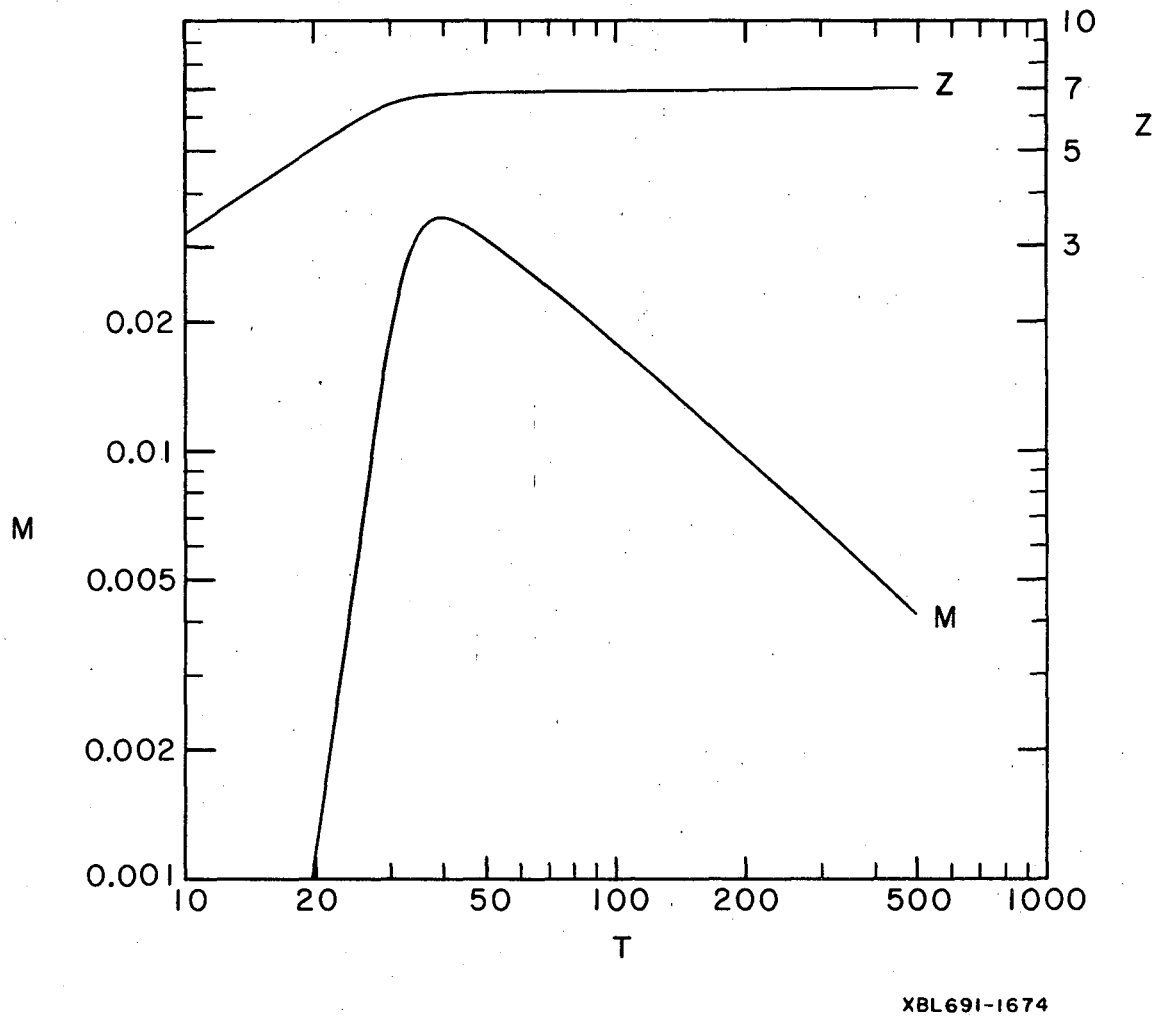

FIGURE VII-4
ASYMPTOTIC CONSTANT Z
MAGNETIC ENERGY M

VIII. ASYMPTOTIC ANALYSIS

Asymptotic Expansion of Spectral Integrals

We shall find asymptotic expansions for the integrals D, L, and M based on the sharpness of the spectrum. From Eq. (VII-83)

$$D = C_0 Q G^{-1/2} \int_0^1 dK \; K^{-1/2} \exp(GK - K^3 T) \tag{VIII-1}$$

Let K_p denote the K value for which the exponential has its peak

$$K_p = (G/3T)^{1/2} \tag{VIII-2}$$

We define

$$\lambda = Z^{3/2} = GK_p = K_p^3 \, 3T \tag{VIII-3}$$

whence

$$G = \lambda^{2/3} (3T)^{1/3} \qquad \text{and} \qquad K_p = \lambda^{1/3} (3T)^{-1/3} \tag{VIII-4}$$

We make a change of variable from K to x, where

$$K = K_p (1 + x) \tag{VIII-5}$$

then

$$\begin{aligned} GK - K^3 T &= GK_p + GK_p x - K_p^3 T (1 + 3x + 3x^2 + x^3) \\ &= \lambda + \lambda x - \tfrac{1}{3}\lambda (1 + 3x + 3x^2 + x^3) \\ &= \tfrac{2}{3}\lambda - \lambda (x^2 + \tfrac{1}{3} x^3) \end{aligned} \tag{VIII-6}$$

Thus

$$\begin{aligned} D &= C_0 Q G^{-1/2} K_p^{1/2} \exp(\tfrac{2}{3}\lambda) \; I(D) \\ &= C_0 Q \lambda^{-1/6} (3T)^{-1/3} \exp(\tfrac{2}{3}\lambda) \; I(D) \end{aligned} \tag{VIII-7}$$

where

$$I(D) = \int dx \; (1+x)^{-1/2} \exp -\lambda\left(x^2 + \tfrac{1}{3}x^3\right) \tag{VIII-8}$$

We make the transformation

$$y^2 = x^2 + \tfrac{1}{3}x^3 \tag{VIII-9}$$

then

$$I(D) = \int dy \; \frac{dx}{dy} \; (1+x)^{-1/2} \; \exp\left(-\lambda y^2\right) \tag{VIII-10}$$

Equation (VIII-9) may be solved for x

$$x = y - \tfrac{1}{6}y^2 + \tfrac{5}{72}y^3 + \cdots \tag{VIII-11}$$

whence

$$(1+x)^{-1/2} = 1 - \tfrac{1}{2}y + \tfrac{11}{24}y^2 + \cdots \tag{VIII-12}$$

and

$$\frac{dx}{dy} = 1 - \tfrac{1}{3}y + \tfrac{5}{24}y^2 + \cdots \tag{VIII-13}$$

thus

$$\frac{dx}{dy} \cdot (1+x)^{-1/2} = 1 - \tfrac{5}{6}y + \tfrac{5}{6}y^2 + \cdots \tag{VIII-14}$$

whence

$$I(D) = \int dy \; \left\{1 - \tfrac{5}{6}y + \tfrac{5}{6}y^2 + \cdots\right\} \exp\left(-\lambda y^2\right) \tag{VIII-15}$$

It is obvious from Fig. VII-3 that the spectrum is essentially zero outside the interval $K = (0,1)$. This means that we may take the limits of the y integration to be $y = (-\infty, +\infty)$. Then

$$\int dy \ \exp(-\lambda y^2) = \pi^{1/2} \lambda^{-1/2} \tag{VIII-16}$$

$$\int dy \ y \exp(-\lambda y^2) = 0 \tag{VIII-17}$$

$$\int dy \ y^2 \exp(-\lambda y^2) = \tfrac{1}{2} \pi^{1/2} \lambda^{-3/2} \tag{VIII-18}$$

Thus

$$I(D) = \pi^{1/2} \lambda^{-1/2} \left\{ 1 + \tfrac{5}{12} \lambda^{-1} + \cdots \right\} \tag{VIII-19}$$

and therefore

$$D = \pi^{1/2} C_0 Q \ \lambda^{-2/3} (3T)^{-1/3} \exp\left(\tfrac{2}{3}\lambda\right) \left\{ 1 + \tfrac{5}{12} \lambda^{-1} + \cdots \right\} \tag{VIII-20}$$

We note that in the asymptotic region $Z \doteq 7.0$ and therefore $\lambda \doteq 18.5$. Thus one term in our expansion should give better than 5% accuracy.

We now expand L using the above procedure. From Eq. (VII-84)

$$L = \tfrac{1}{2} C_0 Q \ G^{-3/2} \int dK \ K^{-3/2} \ \exp(GK - K^3 T) \tag{VIII-21}$$

Thus

$$\begin{aligned} L &= \tfrac{1}{2} C_0 Q \ G^{-3/2} K_p^{-1/2} \exp\left(\tfrac{2}{3}\lambda\right) \ I(L) \\ &= \tfrac{1}{2} C_0 \ Q \ \lambda^{-7/6} \ (3T)^{-1/3} \exp\left(\tfrac{2}{3}\lambda\right) \ I(L) \end{aligned} \tag{VIII-22}$$

where

$$I(L) = \int dx \ (1+x)^{-3/2} \ \exp -\lambda\left(x^2 + \tfrac{1}{3} x^3\right) \tag{VIII-23}$$

We transform to the variable y then

$$I(L) = \int dy \ \frac{dx}{dy} \ (1+x)^{-3/2} \ \exp(-\lambda y^2) \tag{VIII-24}$$

From Eq. (VIII-11) we have

$$(1+x)^{-3/2} = 1 - \frac{3}{2}y + \frac{17}{8}y^2 + \cdots \tag{VIII-25}$$

thus

$$\frac{dx}{dy} \cdot (1+x)^{-3/2} = 1 - \frac{11}{6}y + \frac{17}{6}y^2 + \cdots \tag{VIII-26}$$

whence

$$I(L) = \int dy \left\{1 - \frac{11}{6}y + \frac{17}{6}y^2 + \cdots\right\} \exp(-\lambda y^2)$$

$$= \pi^{1/2} \lambda^{-1/2} \left\{1 + \frac{17}{12}\lambda^{-1} + \cdots\right\} \tag{VIII-27}$$

and therefore

$$L = \frac{1}{2}\pi^{1/2} C_0 Q \lambda^{-5/3} (3T)^{-1/3} \exp\left(\frac{2}{3}\lambda\right) \left\{1 + \frac{17}{12}\lambda^{-1} + \cdots\right\} \tag{VIII-28}$$

We now expand M the same way. From Eq. (VII-87)

$$M = M_0 Q G^{-1/2} \int dK \; K^{3/2} \exp(GK - K^3 T) \tag{VIII-29}$$

Thus

$$M = M_0 Q G^{-1/2} K_p^{3/2} \exp\left(\frac{2}{3}\lambda\right) I(M)$$

$$= M_0 Q \lambda^{1/2} (3T)^{-1} \exp\left(\frac{2}{3}\lambda\right) I(M) \tag{VIII-30}$$

where

$$I(M) = \int dx \; (1+x)^{3/2} \exp -\lambda\left(x^2 + \frac{1}{3}x^3\right) \tag{VIII-31}$$

We transform to the variable y then

$$I(M) = \int dy \; \frac{dx}{dy} (1+x)^{3/2} \exp(-\lambda y^2) \tag{VIII-32}$$

From Eq. (VIII-11) we have

$$(1+x)^{3/2} = 1 + \frac{3}{2}y + \frac{1}{8}y^2 + \cdots \tag{VIII-33}$$

thus

$$\frac{dx}{dy} \cdot (1+x)^{3/2} = 1 + \frac{7}{6}y - \frac{1}{6}y^2 + \cdots \tag{VIII-34}$$

whence

$$\begin{aligned} I(M) &= \int dy \left\{1 + \frac{7}{6}y - \frac{1}{6}y^2 + \cdots\right\} \exp(-\lambda y^2) \\ &= \pi^{1/2} \lambda^{-1/2} \left\{1 - \frac{1}{12}\lambda^{-1} + \cdots\right\} \end{aligned} \tag{VIII-35}$$

and therefore

$$M = \pi^{1/2} M_0 Q \, (3T)^{-1} \exp\left(\frac{2}{3}\lambda\right) \left\{1 - \frac{1}{12}\lambda^{-1} + \cdots\right\} \tag{VIII-36}$$

Combining Eqs. (VIII-20) and (VIII-28) gives

$$C = D - L = C_2 (3T)^{-1/3} \lambda^{-2/3} \exp\left(\frac{2}{3}\lambda\right) \left\{1 - \frac{1}{12}\lambda^{-1} + \cdots\right\} \tag{VIII-37}$$

$$\left(D - \frac{3}{2}L\right) = C_2 (3T)^{-1/3} \lambda^{-2/3} \exp\left(\frac{2}{3}\lambda\right) \left\{1 - \frac{1}{3}\lambda^{-1} + \cdots\right\} \tag{VIII-38}$$

where

$$C_2 = \pi^{1/2} C_0 Q \tag{VIII-39}$$

A Theorem on Differentiation

We note that the defining integrals (VIII-10) and (VIII-23) for I(D) and I(L) may be differentiated with respect to λ. It is further obvious that each of the $dI/d\lambda$ possess asymptotic power series expansions, which may be calculated in the same manner as the I's were calculated.

We therefore appeal to the following theorem from Erdélyi.[26]

Theorem: Let

$$f(z) \sim p(z) \quad \text{to N terms as} \quad z \to \infty \tag{VIII-40}$$

where p(z) is an asymptotic power series

$$p(z) = a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots \tag{VIII-41}$$

(1) If f(z) is differentiable, and

(2) if f'(z) possesses an asymptotic power series expansion, then

$$f'(z) \sim p'(z) \quad \text{to N-1 terms as} \quad z \to \infty \tag{VIII-42}$$

The theorem allows us to differentiate our expansions for D and L term by term.

The Differential Equation for λ

We differentiate Eqs. (VII-80), (VII-98), and (VII-100) with respect to T

$$\frac{d}{dT} G = A \tag{VIII-43}$$

$$\frac{d}{dT} A = \frac{d}{dT} \overline{B} - \frac{d}{dT} C \tag{VIII-44}$$

$$\frac{d}{dT} \overline{B} = - B_0 G^{-1} A \left(D - \tfrac{3}{2} L\right) \tag{VIII-45}$$

Combining these gives

$$\frac{d^2}{dT^2} G + \frac{d}{dT} C + B_0 G^{-1} \left(\frac{d}{dT} G\right) \left(D - \tfrac{3}{2} L\right) = 0 \tag{VIII-46}$$

We introduce the variable

$$s = (3T)^{1/3} \tag{VIII-47}$$

and substitute

$$G = s\,\lambda^{2/3} \tag{VIII-48}$$

and

$$\frac{d}{dT} = s^{-2}\frac{d}{ds} \tag{VIII-49}$$

then multiplying through by s^3 yields

$$s\frac{d}{ds}s^{-2}\frac{d}{ds}s\lambda^{2/3} + s\frac{d}{ds}C + s^3 B_0 s^{-1}\lambda^{-2/3}\left(s^{-2}\frac{d}{ds}s\lambda^{2/3}\right)\left(D-\tfrac{3}{2}L\right) = 0 \tag{VIII-50}$$

We perform the differentiation and let prime, ', denote d/ds

$$(\lambda^{2/3})'' - 2s^{-2}\lambda^{2/3} + sC' + B_0\left(1 + s\lambda^{-1}\tfrac{2}{3}\lambda'\right)\left(D-\tfrac{3}{2}L\right) = 0 \tag{VIII-51}$$

We may now substitute our asymptotic expansions for C and $(D - \frac{3}{2}L)$. If we denote

$$C_2 = \pi^{1/2} C_0 Q \tag{VIII-52}$$

then these become

$$C = C_2\, s^{-1}\lambda^{-2/3}\exp\left(\tfrac{2}{3}\lambda\right)\left\{1 - \tfrac{1}{12}\lambda^{-1} + \cdots\right\} \tag{VIII-53}$$

$$\left(D - \tfrac{3}{2}L\right) = C_2\, s^{-1}\lambda^{-2/3}\exp\left(\tfrac{2}{3}\lambda\right)\left\{1 - \tfrac{1}{3}\lambda^{-1} + \cdots\right\} \tag{VIII-54}$$

We substitute these expressions into Eq. (VIII-51) and multiply through by $\lambda^{2/3}$

$$\lambda^{2/3}(\lambda^{2/3})'' - 2s^{-2}\lambda^{4/3} + \tfrac{2}{3}\lambda' C_2\exp\left(\tfrac{2}{3}\lambda\right)\left\{1 + \left(B_0 - \tfrac{13}{12}\right)\lambda^{-1} + \cdots\right\}$$
$$+ s^{-1}C_2\exp\left(\tfrac{2}{3}\lambda\right)\left\{(B_0 - 1) - \left(\tfrac{1}{3}B_0 - \tfrac{1}{12}\right)\lambda^{-1} + \cdots\right\} = 0 \tag{VIII-55}$$

Ordered Solution

Our asymptotic expansions introduce a natural ordering into our differential equation. Since λ is approximately a constant over a long time interval, we shall seek an ordered solution of the form

$$\lambda = \lambda_0 + \lambda_1 + \lambda_2 + \cdots \tag{VIII-55a}$$

where

$$\lambda_0 = \text{constant} = \mathcal{O}(18) \tag{VIII-56}$$

$$\lambda_1 = \mathcal{O}(1) \tag{VIII-57}$$

$$\lambda_2 = \mathcal{O}(\lambda_0^{-1}) \tag{VIII-58}$$

With this ordering we may make the following expansions

$$\begin{aligned} \exp\left(\tfrac{2}{3}\lambda\right) &= \exp\left[\tfrac{2}{3}(\lambda_0+\lambda_1)\right] \exp\left(\tfrac{2}{3}\lambda_2 + \cdots\right) \\ &= \exp\left[\tfrac{2}{3}(\lambda_0+\lambda_1)\right] \left\{1 + \tfrac{2}{3}\lambda_2 + \cdots\right\} \end{aligned} \tag{VIII-59}$$

$$\lambda^{4/3} = \lambda_0^{4/3} \left\{1 + \tfrac{4}{3}\lambda_1\lambda_0^{-1} + \cdots\right\} \tag{VIII-60}$$

$$\begin{aligned} \lambda^{2/3}(\lambda^{2/3})'' &= \lambda_0^{4/3}\left\{1 + \tfrac{2}{3}\lambda_1\lambda_0^{-1} + \cdots\right\}\left\{1 + \tfrac{2}{3}\lambda_1\lambda_0^{-1} + \cdots\right\}'' \\ &= \tfrac{2}{3}\lambda_0^{1/3}\lambda_1'' + \cdots \end{aligned} \tag{VIII-61}$$

We denote

$$E = C_2 \exp\left[\tfrac{2}{3}(\lambda_0+\lambda_1)\right] \tag{VIII-61a}$$

and substitute these expressions into Eq. (VIII-55)

$$\frac{2}{3}\lambda_0^{1/3}\lambda_1'' + \cdots + 2s^{-2}\lambda_0^{4/3}\left\{1 + \frac{4}{3}\lambda_1\lambda_0^{-1} + \cdots\right\}$$
$$+ \frac{2}{3}\left\{\lambda_1' + \lambda_2' + \cdots\right\} E \left\{1 + \frac{2}{3}\lambda_2 + \cdots\right\}\left\{1 + (B_0 - \tfrac{13}{12})\lambda_0^{-1} + \cdots\right\}$$
$$+ s^{-1} E \left\{1 + \frac{2}{3}\lambda_2 + \cdots\right\}\left\{(B_0 - 1) - (\tfrac{1}{3}B_0 - \tfrac{1}{12})\lambda_0^{-1} + \cdots\right\} = 0 \qquad \text{(VIII-62)}$$

We must now separate this equation into different orders. For large s Eq. (VIII-62) will be dominated by the two terms $\frac{2}{3}\lambda_1' E$ and $(B_0-1)s^{-1}E$. Therefore we take these terms to be of the same order. A numerical extimate shows that $2s^{-2}\lambda_0^{4/3}$ is the same order as these two terms for values of s corresponding to the beginning of the asymptotic region. However we shall assume that $\frac{2}{3}\lambda_0^{1/3}\lambda_1''$ is of order λ_0^{-1} compared to the preceding three terms. After we find a solution, we shall show that this assumption is valid. Our ordered equations are thus

$$\lambda_0 = \text{constant} \qquad \text{(VIII-63)}$$

$$-2s^{-2}\lambda_0^{4/3} + \frac{2}{3}\lambda_1' E + (B_0 - 1)s^{-1}E = 0 \qquad \text{(VIII-64)}$$

$$\frac{2}{3}\lambda_0^{1/3}\lambda_1'' + \frac{8}{3}\lambda_0^{1/3}s^{-2}\lambda_1 + \frac{2}{3}\lambda_2' E + \frac{2}{3}\lambda_1' E\left\{\frac{2}{3}\lambda_2 + (B_0 - \tfrac{13}{12})\lambda_0^{-1}\right\}$$
$$+ s^{-1}E\left\{\frac{2}{3}\lambda_2(B_0 - 1) - (\tfrac{1}{3}B_0 - \tfrac{1}{12})\lambda_0^{-1}\right\} = 0 \qquad \text{(VIII-65)}$$

The solution of Eq. (VIII-64) is straightforward. We note that

$$\frac{2}{3}\lambda_1' E = E' \qquad \text{(VIII-66)}$$

then

$$E = e_0 s^{-\epsilon} - e_1 s^{-1} \qquad \text{(VIII-67)}$$

where

$$\epsilon = B_0 - 1 \tag{VIII-68}$$

and

$$e_1 = 2\lambda_0^{4/3} / (1-\epsilon) \tag{VIII-69}$$

e_0 is determined by the initial conditions at $s = s_0$. We choose our initial conditions to be

$$\lambda(s_0) = \lambda_0 \tag{VIII-70}$$

$$\lambda_1(s_0) = \lambda_2(s_0) = \cdots = 0 \tag{VIII-71}$$

Then

$$E(s_0) = C_2 \exp\left(\tfrac{2}{3}\lambda(s_0)\right) \tag{VIII-72}$$

and

$$e_0 s_0^{-\epsilon} = E(s_0) + e_1 s_0^{-1} \tag{VIII-73}$$

Having solved for E, we obtain λ, G, and A through first order, but C, $\bar{B}$, and M only through zeroth order.

$$\lambda \doteq \lambda_0 + \lambda_1 = \tfrac{3}{2}\log(E/C_2) \tag{VIII-74}$$

$$G = s\,\lambda^{2/3} \tag{VIII-75}$$

$$A = s^{-2}\frac{d}{ds}G = s^{-2}\lambda^{2/3} + s^{-1}\lambda^{-1/3}\tfrac{2}{3}\lambda'$$

$$\doteq s^{-2}\lambda^{2/3} + s^{-1}\lambda^{-1/3}E'E^{-1} \tag{VIII-76}$$

Now

$$E' = -\epsilon e_0 s^{-1-\epsilon} + e_1 s^{-2} \tag{VIII-77}$$

therefore

$$E' E^{-1} \leq \mathcal{O}(s^{-1}) \tag{VIII-78}$$

whence

$$\left(s^{-1} \lambda^{-1/3} E' E^{-1}\right) / \left(s^{-2} \lambda^{2/3}\right) \leq \mathcal{O}(\lambda^{-1}) \tag{VIII-79}$$

thus

$$A = s^{-2} \lambda_0^{2/3} \left\{1 + \mathcal{O}(\lambda^{-1})\right\} \tag{VIII-80}$$

which shows that the dominant behavior of A is a $T^{-2/3}$ decay. The exact expressions for C, $\bar{B}$, and M may be written

$$C = s^{-1} \lambda^{-2/3} E \left\{1 + \tfrac{2}{3}\lambda_2 + \cdots\right\} \left\{1 - \tfrac{1}{12} \lambda^{-1} + \cdots\right\} \tag{VIII-81}$$

$$\bar{B} = A + C \tag{VIII-82}$$

$$M = \frac{M_0}{C_0} s^{-3} E \left\{1 + \tfrac{2}{3}\lambda_2 + \cdots\right\} \left\{1 - \tfrac{1}{12} \lambda^{-1} + \cdots\right\} \tag{VIII-83}$$

Since we have not calculated λ_2, we shall neglect other terms of order λ^{-1}. Thus to zeroth order

$$C \doteq \lambda_0^{-2/3} s^{-1} E \tag{VIII-84}$$

$$\bar{B} = A + C \tag{VIII-85}$$

$$M \doteq \frac{M_0}{C_0} s^{-3} E \tag{VIII-86}$$

C thus has a two component decay, $T^{-1/3 - \epsilon/3}$ and $T^{-2/3}$. The decay of $\bar{B}$

is similar to that of C. $\bar{B}$ has the same $T^{-1/3-\epsilon/3}$ term as C but has a different $T^{-2/3}$ term. Thus as the faster $T^{-2/3}$ term decays away, $\bar{B}$ is asymptotic to C. $\bar{B}$ and C decay more slowly than A. M decays faster than A. M has a two component decay, $T^{-1-\epsilon/3}$ and $T^{-4/3}$.

We shall now show that $\frac{2}{3}\lambda_o^{1/3}\lambda_1''$ is of order λ_o^{-1} compared to $\frac{2}{3}\lambda_1' E$.

$$\frac{2}{3}\lambda_o^{1/3}\lambda_1'' = \lambda_o^{1/3}\frac{d}{ds}\left(\frac{2}{3}\lambda_1'\right) = \lambda_o^{1/3}\frac{d}{ds}\left(E'E^{-1}\right)$$

$$\leq \lambda_o^{1/3}\frac{d}{ds}\mathcal{O}(s^{-1}) = \lambda_o^{1/3}\mathcal{O}(s^{-2}) \qquad \text{(VIII-87)}$$

whereas

$$\frac{2}{3}\lambda_1' E = E' = -\epsilon e_o s^{-1-\epsilon} + e_1 s^{-2} \qquad \text{(VIII-88)}$$

We compare $\frac{2}{3}\lambda_o^{1/3}\lambda_1''$ to the fastest decaying part of $\frac{2}{3}\lambda_1' E$.

$$\lambda_o^{1/3}\mathcal{O}(s^{-2})/e_1 s^{-2} = \mathcal{O}(\lambda_o^{1/3}/e_1) = \mathcal{O}\left((1-\epsilon)\lambda_o^{1/3}/2\lambda_o^{4/3}\right)$$

$$= \mathcal{O}(\lambda_o^{-1}) \qquad \text{(VIII-89)}$$

Thus the λ'' term has the assumed order.

A Second Order Correction

We notice that $s^{-1}E(\frac{1}{3}B_o - \frac{1}{12})\lambda_o^{-1}$ in the second order equation could be absorbed with $(B_o - 1)s^{-1}E$ in the first order equation, and that $\frac{2}{3}\lambda_1' E(B_o - \frac{13}{12})\lambda_o^{-1}$ in the second order equation could be absorbed with $\frac{2}{3}\lambda_1' E$ in the first order equation. Each of these second order terms are driving terms in the equation for λ_2. Absorbing them in the equation for λ_1 thereby simplifies the equation for λ_2. However these terms are only part of the second order terms. We therefore would not

expect including them in the equation for λ_1 to make λ_1 more accurate, if it were not for the following accident. Namely that $(B_o - 1)$ is small compared to one. Consequently $(\frac{1}{3} B_o - \frac{1}{12})$ makes a 10% change in the coefficient of $s^{-1}E$, which in turn determines the long time decay rate of E. These terms in the second order equation cause λ_2 to compensate for the inaccurate decay rate of the first order equation. Such an effect would eventually cause λ_2 to become the same order as λ_1. We therefore transplant these terms into the first order equation. The effect on E is quite simple. If we define

$$r_o = \left\{ (B_o - 1) - \left(\tfrac{1}{3} B_o - \tfrac{1}{12}\right) \lambda_o^{-1} \right\} \tag{VIII-90}$$

and

$$r_1 = \left\{ 1 + \left(B_o - \tfrac{13}{12}\right) \lambda_o^{-1} \right\} \tag{VIII-91}$$

then

$$E = e_o s^{-\epsilon} - e_1 s^{-1} \tag{VIII-92}$$

where

$$\epsilon = r_o / r_1 \tag{VIII-93}$$

and

$$e_1 = 2\lambda_o^{4/3} / (r_1 - r_o) \tag{VIII-94}$$

Discussion of the Asymptotic Phase of Evolution

Numerical calculations were made of E, λ, G, A, C, $\overline{B}$, and M using the asymptotic representations of Eq. (VIII-67), (VIII-74 to 76), (VIII-84

to 86), and using the second order corrected expressions for ϵ and e_1. We chose $T_o = 50$ for our initial time and required

$$\lambda_o = \lambda(T=50) \tag{VIII-95}$$

for our initial condition. Table VIII-1 shows a brief comparison of the asymptotically calculated values with the exact values of λ, G, A, C, $\overline{B}$, and M. We see that C, $\overline{B}$, and M are all well within the 5% accuracy expected of terms of order λ^{-1}. λ, G, and A are all within the 0.3% accuracy expected of terms of order λ^{-2}, except for A(T=50). The asymptotic value for A(50) differs from the exact value by 1.1%. The point $T = 50$ is really too close to the beginning of the asymptotic phase to be described with $O(\lambda^{-2})$ accuracy. We see that by $T = 60$ the inaccuracy in A has indeed been reduced to the expected 0.3%.

The graphs in Figs. VIII-1 and 2 were made from calculations using the asymptotic expressions. In Fig. VIII-2 A, C, $\overline{B}$, and 10M are graphed as functions of s/s_o, that is, as functions of $(T/T_o)^{1/3}$. As we have already mentioned, M decays fastest like $T^{-1-\epsilon/3}$ and $T^{-4/3}$. A decays like $T^{-2/3}$. C and $\overline{B}$ each decay like $T^{-1/3-\epsilon/3}$ and $T^{-2/3}$. A is reduced to 0.01, that is, 1% of its initial value by $s = 5s_o = 26.5$ or $T = 125T_o = 6250$. M is also reduced to 0.00035 which is 1% of its maximum value at $s = 5s_o$. C and $\overline{B}$ are reduced to 0.01 at $s = 50s_o = 265$ or $T = 6.25 \times 10^6$. In Fig. VIII-1 E and λ are graphed as functions of s/s_o. E relaxes like s^{-1} to its maximum at

$$s^{1-\epsilon} = e_1 / \epsilon e_o \tag{VIII-96}$$

The maximum is at $s = 3.94s_o$. Thereafter E decays slowly like $s^{-\epsilon}$. Correspondingly λ starts with a value of 17.9, reaches a maximum of

18.35 and has declined to 17.35 by $s = 1000s_o$. λ_1 has thus risen from 0 to +0.45 and then declined to -0.55 by $s = 1000s_o$. Our basic assumptions for the asymptotic expansion of the spectral integrals and for the ordered solution were that $\lambda_o = O(18) \gg 1$ and $\lambda_1 = O(1)$ etc. We see that these assumptions are indeed valid throughout the range $s = s_o$ to $s = 1000s_o$ and beyond. By the time these assumptions break down A, C, $\bar{B}$, and M will be negligibly small. We may estimate this time as follows. E decays like $s^{-\epsilon}$ and will therefore eventually be reduced to $O(C_2)$. When E is $O(C_2)$ λ will be $O(1)$ and our assumptions will have broken down.

Equating

$$e_o s^{-\epsilon} = O(C_2) \tag{VIII-97}$$

gives

$$s/s_o = O\left[(e_o s_o^{-\epsilon}/C_2)^{1/\epsilon}\right] = O(10^{34.2}) \tag{VIII-98}$$

This in turn is equivalent to $T/T_o = O(10^{103})$ which is very remote indeed.

T	λ	λ(asymp)	$\Delta\lambda$	G	G(asymp)	ΔG
50	17.893	17.893	--	36.35	36.35	0.00
100	18.096	18.078	- 0.018	46.14	46.11	- 0.03
200	18.220	18.196	- 0.024	58.40	58.35	- 0.05
500	18.315	18.287	- 0.028	79.54	79.45	- 0.09

T	A	A(asymp)	ΔA	C	C(asymp)	ΔC
50	0.2541	0.2514	- 0.0027	0.5793	0.5820	+ 0.0027
100	0.1577	0.1573	- 0.0004	0.5224	0.5226	+ 0.0002
200	0.0989	0.0987	- 0.0002	0.4484	0.4486	+ 0.0002
500	0.0534	0.0534	0.0000	0.3508	0.3509	+ 0.0001

T	$\overline{B}$	$\overline{B}$(asymp)	$\Delta\overline{B}$	M	M(asymp)	ΔM
50	0.8334	0.8334	0.0000	0.03120	0.03134	+ 0.00014
100	0.6801	0.6799	- 0.0002	0.01786	0.01773	- 0.00013
200	0.5472	0.5472	0.0000	0.00970	0.00959	- 0.00011
500	0.4042	0.4043	+ 0.0001	0.00413	0.00407	- 0.00006

T	A	A(asymp)	ΔA
60	0.2230	0.2221	- 0.0009

TABLE VIII-1

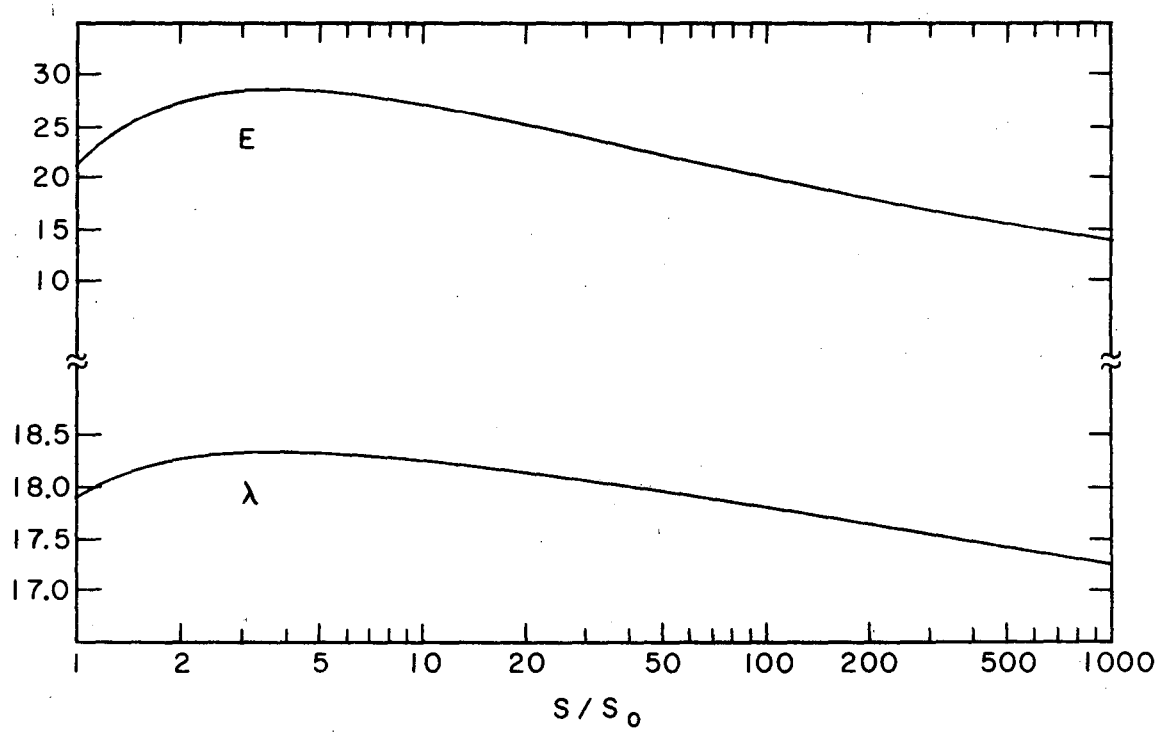

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FIGURE VIII-1

EXPONENTIAL E

EXPANSION PARAMETER λ

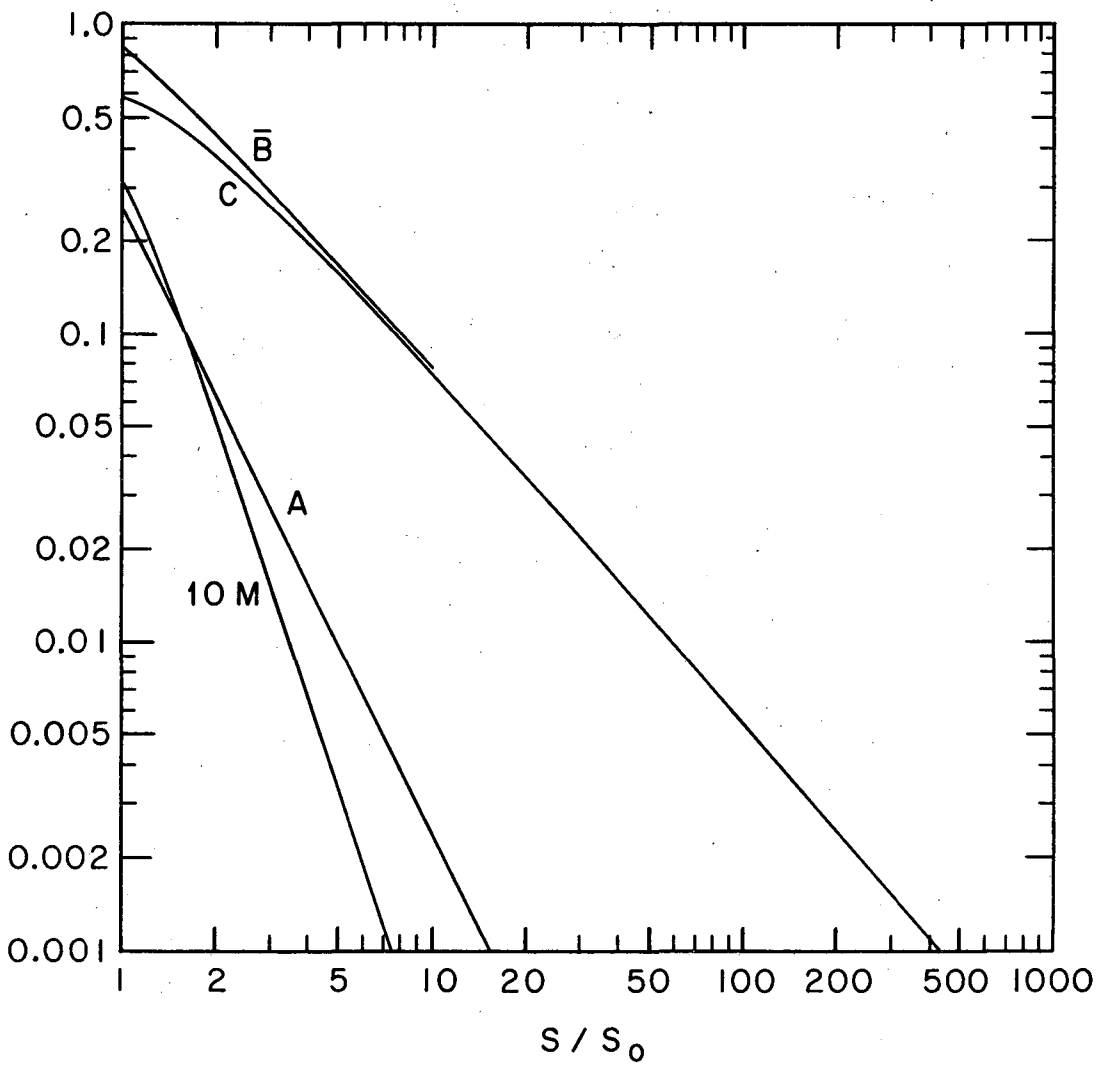

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FIGURE VII-2

ANISOTROPY A

COMPLEMENT OF DIFFUSION TERM $\bar{B}$

DISSIPATIONLESS FLOW TERM C

MAGNETIC ENERGY M

APPENDIX A - ENERGY CONSERVATION

We define the kinetic energy density for species s by

$$\mathcal{E}^s_{kin} = \int d^3v \; \tfrac{1}{2} m_s v^2 \; F^s_o \tag{A-1}$$

To find the time rate of change of $\mathcal{E}^s_{kin}$ we use the Fourier transform of the Vlasov equation for F_o (III-5), and integrate by parts

$$\begin{aligned}
\dot{\mathcal{E}}^s_{kin} &= \int d^3v \; \tfrac{1}{2} m_s v^2 \; \dot{F}^s_o = -\int d^3v \; \tfrac{1}{2} m_s v^2 \; \frac{1}{V}\int \frac{d^3k}{(2\pi)^3} \left(R^s_{-k} + S^s_{-k}\right) F^s_k \\
&= -\frac{1}{V}\int \frac{d^3k}{(2\pi)^3} \int d^3v \; \tfrac{1}{2} m_s v^2 \; \frac{e_s}{m_s}\left(E_{-k} + \frac{v}{c}\times B_{-k}\right)\cdot \frac{\partial}{\partial v} F^s_k \\
&= \frac{1}{V}\int \frac{d^3k}{(2\pi)^3} \int d^3v \; \tfrac{1}{2} e_s v^2 \; \overleftarrow{\frac{\partial}{\partial v}}\cdot\left(E_{-k} + \frac{v}{c}\times B_{-k}\right) F^s_k \\
&= \frac{1}{V}\int \frac{d^3k}{(2\pi)^3} \int d^3v \; e_s \, v\cdot\left(E_{-k} + \frac{v}{c}\times B_{-k}\right) F^s_k \\
&= \frac{1}{V}\int \frac{d^3k}{(2\pi)^3} \; E_{-k}\cdot \int d^3v \; e_s v \; F^s_k = \frac{1}{V}\int \frac{d^3k}{(2\pi)^3} \; E^*_k \cdot j^s_k
\end{aligned} \tag{A-2}$$

As we have previously noted the $R_{-k}F_k$ term is zero because v^2 is spherically symmetric. We define the kinetic, magnetic, and electric energy densities by

$$\mathcal{E}_{kin} = \sum_s \mathcal{E}^s_{kin} \tag{A-3}$$

$$\mathcal{E}_{mag} = \frac{1}{V}\int d^3r \; |\delta B|^2/8\pi = \frac{1}{V}\int \frac{d^3k}{(2\pi)^3} \; |B_k|^2/8\pi \tag{A-4}$$

$$\mathcal{E}_{elec} = \frac{1}{V}\int d^3r \; |\delta E|^2/8\pi = \frac{1}{V}\int \frac{d^3k}{(2\pi)^3} \; |E_k|^2/8\pi \tag{A-5}$$

To continue our calculation, we eliminate the current by using the Maxwell-Ampere Eq. (III-8), and we eliminate $k \times E_{-k}$ by using Faradays Eq. (III-7)

$$\dot{\mathcal{E}}_{kin} = \frac{1}{V}\int\frac{d^3k}{(2\pi)^3}\,\frac{1}{4\pi}\,E_{-k}\cdot\sum_s 4\pi e_s\int d^3v\; v\; F_k^s$$

$$= \frac{1}{V}\int\frac{d^3k}{(2\pi)^3}\,\frac{1}{4\pi}\,E_{-k}\cdot\left\{ick\times B_k - \dot{E}_k\right\}$$

$$= -\frac{1}{V}\int\frac{d^3k}{(2\pi)^3}\,\frac{1}{4\pi}\left\{(ick\times E_{-k})\cdot B_k + E_{-k}\cdot\dot{E}_k\right\}$$

$$= -\frac{1}{V}\int\frac{d^3k}{(2\pi)^3}\,\frac{1}{4\pi}\left\{\dot{B}_{-k}\cdot B_k + E_{-k}\cdot\dot{E}_k\right\}$$

$$= -\frac{d}{dt}\frac{1}{V}\int\frac{d^3k}{(2\pi)^3}\left\{|B_k|^2+|E_k|^2\right\}/8\pi = -\dot{\mathcal{E}}_{mag} - \dot{\mathcal{E}}_{elec} \qquad \text{(A-6)}$$

For the case of the transverse instability we note that

$$E_k \sim \frac{1}{kc}\dot{B}_k \sim \alpha\,\frac{v_{th}}{c}\,B_k \qquad \text{(A-7)}$$

Thus

$$\mathcal{E}_{elec} \sim \alpha^2\left(\frac{v_{th}}{c}\right)^2\mathcal{E}_{mag} \qquad \text{(A-8)}$$

The energy in the electric field is exceedingly small compared to that in the magnetic field.

We consider separately the kinetic energy of motion perpendicular to and parallel to the axis of temperature maximum. We let $\hat{3}$ denote the axis of temperature maximum. From the Fourier transform of the Vlasov equation (III-28) we have to lowest order in α that

$$\dot{\mathcal{E}}_{\perp kin} = \sum_s\int d^3v\,\tfrac{1}{2}m_s v_\perp^2\,\dot{F}_0^s = -\sum_s\int d^3v\,\tfrac{1}{2}m_s v_\perp^2\,\frac{1}{V}\int\frac{d^3k}{(2\pi)^3}R_{-k}^s F_k^s \neq 0 \qquad \text{(A-9)}$$

and

$$\dot{\mathcal{E}}_{\parallel kin} = \sum_s\int d^3v\,\tfrac{1}{2}m_s v_3^2\,\dot{F}_0^s = -\sum_s\int d^3v\,\tfrac{1}{2}m_s v_3^2\,\frac{1}{V}\int\frac{d^3k}{(2\pi)^3}R_{-k}^s F_k^s \neq 0 \qquad \text{(A-10)}$$

since neither $v_\perp^2$ nor v_3^2 is spherically symmetric. But $v_\perp^2 + v_3^2 = v^2$ is spherically symmetric so

$$\int d^3v \; \frac{1}{2} m_s v^2 \frac{1}{V} \int \frac{d^3k}{(2\pi)^3} \; R^s_{-k} F^s_k = 0 \tag{A-11}$$

That is to lowest order in α

$$\dot{\mathcal{E}}_{\perp \text{ kin}} = - \dot{\mathcal{E}}_{\parallel \text{ kin}} \tag{A-12}$$

and as we have seen, to lowest nontrivial order in α, $\dot{\mathcal{E}}_{kin}$ is given by

$$\dot{\mathcal{E}}_{kin} = - \sum_s \int d^3v \; \frac{1}{2} m_s v^2 \frac{1}{V} \int \frac{d^3k}{(2\pi)^3} \; S^s_{-k} F^s_k \tag{A-13}$$

Thus since $S_k \sim \alpha R_k$

$$\dot{\mathcal{E}}_{kin} \sim \alpha \dot{\mathcal{E}}_{\parallel \text{ kin}} \qquad \text{and} \qquad \dot{\mathcal{E}}_{kin} \sim \alpha \dot{\mathcal{E}}_{\perp \text{ kin}} \tag{A-14}$$

Therefore since $\dot{\mathcal{E}}_{mag} = - \dot{\mathcal{E}}_{kin}$ to lowest order in α

$$\dot{\mathcal{E}}_{mag} \sim \alpha \dot{\mathcal{E}}_{\perp \text{ kin}} \tag{A-15}$$

Thus an amount of energy, $\Delta \mathcal{E}_{\parallel} \sim \alpha \mathcal{E}_{kin}$, flows from parallel motion into perpendicular motion. However only $\alpha \Delta \mathcal{E}_{\parallel} \sim \alpha^2 \mathcal{E}_{kin}$ of it flows into the magnetic field.

Assume now that the ions do not participate in the energy flow, and consider the electron temperatures. We defined d^2 to be the mean thermal velocity at $t = 0$. a^2 and b^2 are the thermal velocities perpendicular to and parallel to the axis of temperature maximum respectively. Then

$$a^2 = d^2 + \delta a^2 \tag{A-16}$$

$$b^2 = d^2 + \delta b^2 \tag{A-17}$$

where δa^2 and δb^2 are of order αa^2. Now

$$\frac{\partial}{\partial t} a^2 = \frac{\partial}{\partial t} \delta a^2 \neq 0 \tag{A-18}$$

but

$$\frac{\partial}{\partial t} (2a^2 + b^2) = \frac{\partial}{\partial t} (2\,\delta a^2 + \delta b^2) \sim \alpha \frac{\partial}{\partial t} \delta a^2 \tag{A-19}$$

therefore

$$(2\,\delta a^2(t) + \delta b^2(t)) = (2\,\delta a^2(0) + \delta b^2(0)) + \mathcal{O}(\alpha\,\delta a^2) \tag{A-20}$$

but

$$\mathcal{O}(\alpha\,\delta a^2) = \mathcal{O}(\alpha^2 a^2) \tag{A-21}$$

Let us define $\bar{\alpha}$ so that

$$\delta a^2 = -\frac{1}{3}\bar{\alpha}\, d^2 \tag{A-22}$$

then

$$\delta b^2 = \frac{2}{3}\bar{\alpha}\, d^2 + \mathcal{O}(\alpha^2 a^2) \tag{A-23}$$

but

$$\alpha\, d^2 = b^2 - a^2 = \delta b^2 - \delta a^2 = \bar{\alpha}\, d^2 + \mathcal{O}(\alpha^2 a^2) \tag{A-24}$$

Thus to lowest order

$$\bar{\alpha} = \alpha \tag{A-25}$$

whence

$$a^2 = d^2 \left\{ 1 - \frac{1}{3}\alpha + \mathcal{O}(\alpha^2) \right\} \tag{A-26a}$$

$$b^2 = d^2 \left\{ 1 + \frac{2}{3}\alpha + \mathcal{O}(\alpha^2) \right\} \tag{A-26b}$$

We now consider the energy relations for our ordered Eqs. (III-64) through (III-82). Recall that the order superscript refers to an expansion in powers of $\alpha^{1/2}$, and that $X^{(o)}$ refers to the lowest <u>nonzero</u> order term in the expansion of X. We define

$$\mathcal{E}_{kin}^{(n)} = \sum_s \int d^3v \; \tfrac{1}{2} m_s v^2 \; F_o^{s(n)} \tag{A-27}$$

and

$$\mathcal{E}_{mag}^{(o)} = \frac{1}{V} \int \frac{d^3k}{(2\pi)^3} \; |B_k^{(o)}|^2 / 8\pi \tag{A-28}$$

$$\mathcal{E}_{mag}^{(1)} = \frac{1}{V} \int \frac{d^3k}{(2\pi)^3} \; \{ B_k^{(1)*} \cdot B_k^{(o)} + B_k^{(o)*} \cdot B_k^{(1)} \} / 8\pi \tag{A-29}$$

$$\mathcal{E}_{mag}^{(2)} = \frac{1}{V} \int \frac{d^3k}{(2\pi)^3} \; \{ B_k^{(2)*} \cdot B_k^{(o)} + B_k^{(1)*} \cdot B_k^{(1)} + B_k^{(o)*} \cdot B_k^{(2)} \} / 8\pi \tag{A-30}$$

$$\cdots$$

and

$$\mathcal{E}_{elec}^{(o)} = \frac{1}{V} \int \frac{d^3k}{(2\pi)^3} \; |E_k^{(o)}|^2 / 8\pi \tag{A-31}$$

$$\cdots$$

Then since $F^{(o)}$ is constant, and $F^{(1)} = 0$ we have that

$$\mathcal{E}_{kin}^{(o)} = \text{constant} = \mathcal{E}_{total}(t=0) \tag{A-32}$$

$$\mathcal{E}_{kin}^{(1)} = 0 \tag{A-33}$$

As we have noted all terms in the equation for $\dot{F}_o^{(n)}$ which begin with R_{-k} give zero contribution to the energy flow. Thus from Eqs. (III-78) and (III-79)

$$\frac{\partial}{\partial \tau_o} \mathcal{E}_{kin}^{(2)} = 0 \tag{A-34}$$

$$\frac{\partial}{\partial \tau_1} \mathcal{E}_{kin}^{(2)} + \frac{\partial}{\partial \tau_o} \mathcal{E}_{kin}^{(3)} = 0 \tag{A-35}$$

Using the ordered Vlasov equations for F_o (III-80 to 82), using the ordered Maxwell-Ampere and Faraday equations, and following the same steps as in the above derivation for $\dot{\mathcal{E}}_{kin}$ we find

$$\frac{\partial}{\partial t_2}\mathcal{E}^{(2)}_{kin} + \frac{\partial}{\partial t_1}\mathcal{E}^{(3)}_{kin} + \frac{\partial}{\partial t_0}\mathcal{E}^{(4)}_{kin} = -\frac{\partial}{\partial t_0}\mathcal{E}^{(0)}_{mag} \tag{A-36}$$

$$\frac{\partial}{\partial t_3}\mathcal{E}^{(2)}_{kin} + \cdots + \frac{\partial}{\partial t_0}\mathcal{E}^{(5)}_{kin} = -\frac{\partial}{\partial t_1}\mathcal{E}^{(0)}_{mag} - \frac{\partial}{\partial t_0}\mathcal{E}^{(1)}_{mag} \tag{A-37}$$

$$\frac{\partial}{\partial t_4}\mathcal{E}^{(2)}_{kin} + \cdots + \frac{\partial}{\partial t_0}\mathcal{E}^{(6)}_{kin} = -\frac{\partial}{\partial t_2}\mathcal{E}^{(0)}_{mag} - \frac{\partial}{\partial t_1}\mathcal{E}^{(1)}_{mag} - \frac{\partial}{\partial t_0}\mathcal{E}^{(2)}_{mag} \tag{A-38}$$

$$\cdots$$

Integrate Eq. (A-36) on the t_o time scale. $\mathcal{E}^{(4)}_{kin}$ and $\mathcal{E}^{(0)}_{mag}$ are functions of t_o, however $\frac{\partial}{\partial t_2}\mathcal{E}^{(2)}_{kin}$ and $\frac{\partial}{\partial t_1}\mathcal{E}^{(3)}_{kin}$ are constants on the t_o time scale. These would therefore lead to secular behavior. In order to suppress this secularity we must require

$$\frac{\partial}{\partial t_2}\mathcal{E}^{(2)}_{kin} + \frac{\partial}{\partial t_1}\mathcal{E}^{(3)}_{kin} = 0 \tag{A-39}$$

Integrate Eq. (A-39) on the t_1 time scale, etc. From extension and repetition of this argument we conclude

$$\frac{\partial}{\partial t_0}\mathcal{E}^{(2)}_{kin} = \frac{\partial}{\partial t_1}\mathcal{E}^{(2)}_{kin} = \cdots = \frac{\partial}{\partial t_4}\mathcal{E}^{(2)}_{kin} = 0 \tag{A-40}$$

$$\frac{\partial}{\partial t_0}\mathcal{E}^{(3)}_{kin} = \cdots = \frac{\partial}{\partial t_3}\mathcal{E}^{(3)}_{kin} = 0 \tag{A-41}$$

$$\frac{\partial}{\partial t_0}\left\{\mathcal{E}^{(4)}_{kin} + \mathcal{E}^{(0)}_{mag}\right\} = 0 \qquad \text{also } \frac{\partial}{\partial t_1} = 0 \text{ and } \frac{\partial}{\partial t_2} = 0 \tag{A-42}$$

$$\frac{\partial}{\partial t_0}\left\{\mathcal{E}^{(5)}_{kin} + \mathcal{E}^{(1)}_{mag}\right\} = 0 \qquad \text{also } \frac{\partial}{\partial t_1} = 0 \tag{A-43}$$

$$\frac{\partial}{\partial t_0}\left\{\mathcal{E}^{(6)}_{kin} + \mathcal{E}^{(2)}_{mag}\right\} = 0 \tag{A-44}$$

$$\cdots$$

The generalization to higher orders is obvious and we conclude that on <u>all</u> time scales

$$\mathcal{E}^{(2)}_{kin} = 0 \tag{A-45}$$

$$\mathcal{E}^{(3)}_{kin} = 0 \tag{A-46}$$

$$\mathcal{E}^{(4)}_{kin} + \mathcal{E}^{(0)}_{mag} = 0 \tag{A-47}$$

$$\mathcal{E}^{(5)}_{kin} + \mathcal{E}^{(1)}_{mag} = 0 \tag{A-48}$$

$$\mathcal{E}^{(6)}_{kin} + \mathcal{E}^{(2)}_{mag} = 0 \tag{A-49}$$

$$\cdots$$

$$\mathcal{E}^{(8)}_{kin} + \mathcal{E}^{(4)}_{mag} + \mathcal{E}^{(0)}_{elec} = 0 \tag{A-50}$$

$$\cdots$$

Thus we see again that $\mathcal{E}_{elec} \sim \alpha^2 \mathcal{E}_{mag}$ and $\mathcal{E}_{mag} \sim \alpha^2 \mathcal{E}_{kin}$.

APPENDIX B - H-THEOREM

The diffusion equation

$$\frac{\partial}{\partial t} f_0 = \frac{\partial}{\partial v} \cdot D \cdot \frac{\partial}{\partial v} f_0 \tag{B-1}$$

satisfies an H-theorem with H given by

$$H = \frac{1}{2} \int d^3v \, (f_0)^2 \tag{B-2}$$

Thus

$$\dot{H} = \int d^3v \, f_0 \dot{f}_0 = \int d^3v \, f_0 \, \frac{\partial}{\partial v} \cdot D \cdot \frac{\partial}{\partial v} f_0$$

$$= - \int d^3v \left(\frac{\partial f_0}{\partial v}\right) \cdot D \cdot \left(\frac{\partial f_0}{\partial v}\right)$$

$$= - \int d^3v \left(\frac{\partial f_0}{\partial v}\right) \cdot D^{sym} \cdot \left(\frac{\partial f_0}{\partial v}\right) \tag{B-3}$$

where D^{sym} is the symmetric part of the diffusion tensor D, that is

$$D^{sym} = \frac{1}{2} (D + D^{transpose}) \tag{B-4}$$

Now if $\varphi \cdot D^{sym} \cdot \varphi$ is a positive definite form then $\dot{H}$ is negative definite. But H itself is positive definite. It follows that $\dot{H}$ must be zero in the final state to which the system evolves. Therefore the final state is characterized by

$$\left(\frac{\partial f_0}{\partial v}\right) \cdot D^{sym} \cdot \left(\frac{\partial f_0}{\partial v}\right) = 0 \tag{B-5}$$

As we have already noted, the diffusion tensor given by Eq. (IV-31) does indeed yield a positive definite form. It therefore follows that the final state of the transverse instability must satisfy Eq. (B-5). Let us write

$$D^{sym} = - v \times M \times v \tag{B-6}$$

and denote

$$\psi = \left(v \times \frac{\partial}{\partial v}\right) f_o \tag{B-7}$$

then

$$\left(\frac{\partial f_o}{\partial v}\right) \cdot D^{sym} \cdot \left(\frac{\partial f_o}{\partial v}\right) = \psi \cdot M \cdot \psi \tag{B-8}$$

From Eq. (IV-31) we have for M

$$\begin{aligned} M &= - \frac{1}{V} \frac{e^2}{m^2c^2} \int \frac{d^3k}{(2\pi)^3} \left|B_k^{(o)}\right|^2 \frac{i}{k \cdot v} \hat{B}_k \hat{B}_k \\ &= \left\{\frac{1}{V} \frac{e^2}{8\pi^2 m^2 c^2} \frac{1}{|v|}\right\} \int_{(k \cdot v = 0)} d^2k \left|B_k^{(o)}\right|^2 \hat{B}_k \hat{B}_k \end{aligned} \tag{B-9}$$

Now because f_o is axially symmetric, ψ lies in the plane perpendicular to the symmetry axis. But $\hat{B}_k$ also lies in this plane. Therefore

$$\hat{\psi} \cdot M \cdot \hat{\psi} = \{\text{constant}\} \int_{(k \cdot v = 0)} d^2k \left|B_k^{(o)}\right|^2 (\hat{\psi} \cdot \hat{B}_k)^2 \tag{B-10}$$

is positive definite and is zero only if $|B_k^{(o)}|^2$ is identically zero. Thus $\psi \cdot M \cdot \psi$ will be zero if and only if either

$$(v \times \frac{\partial}{\partial v}) f_o = 0 \qquad \text{or} \qquad |B_k^{(o)}|^2 = 0 \tag{B-11}$$

However we know that if $(v \times \partial/\partial v) f_o$ is not zero, that is, if the anisotropy is not zero then the spectrum will grow for some k-values. Conversely if the anisotropy is zero all waves are damped so the spectrum eventually becomes identically zero. Thus we conclude that the final state of our system is characterized by both of the relations in Eq. (B-11).

APPENDIX C - RELATIVISTIC LINEAR ANALYSIS

If we define the tensor

$$I_v = (1 - v^2/c^2)^{1/2} \, (I - vv/c^2) \tag{C-1}$$

then the relativistic Vlasov equation may be written[27]

$$\left\{ \frac{\partial}{\partial t} + v \cdot \frac{\partial}{\partial r} + \frac{e}{m} \frac{\partial}{\partial v} \cdot I_v \cdot \left(E + \frac{v}{c} \times B\right) \right\} f = 0 \tag{C-2}$$

We use the relation

$$\frac{\partial}{\partial v} \cdot I_v = I_v \cdot \frac{\partial}{\partial v} - \frac{5}{c^2} v \, (1 - v^2/c^2)^{1/2} \tag{C-3}$$

then

$$\begin{aligned} \frac{\partial}{\partial v} \cdot I_v \cdot (v \times B) &= \left(I_v \cdot \frac{\partial}{\partial v}\right) \cdot (v \times B) = \left[\left(I_v \cdot \frac{\partial}{\partial v}\right) \times v\right] \cdot B \\ &= \left[(1 - v^2/c^2)^{1/2} \left\{ \frac{\partial}{\partial v} - \frac{v}{c^2}\left(v \cdot \frac{\partial}{\partial v}\right) \right\} \times v \right] \cdot B \\ &= \left[(1 - v^2/c^2)^{1/2} \left(\frac{\partial}{\partial v} \times v\right)\right] \cdot B \end{aligned} \tag{C-4}$$

whereupon the relativistic Vlasov equation may be written

$$\left\{ \frac{\partial}{\partial t} + v \cdot \frac{\partial}{\partial r} + \frac{e}{m} \frac{\partial}{\partial v} \cdot I_v \cdot E - \frac{e}{mc} (1 - v^2/c^2)^{1/2} B \cdot \left(v \times \frac{\partial}{\partial v}\right) \right\} f = 0 \tag{C-5}$$

We order the Fourier transform of Eq. (C-5) in α but not in v/c. Thus

$$i k \cdot v \, f_k^{(0)} + \frac{e}{m} \frac{\partial}{\partial v} \cdot I_v \cdot E_k^{(0)} f_0^{(0)} - \frac{e}{mc} (1 - v^2/c^2)^{1/2} B_k^{(0)} \cdot \left(v \times \frac{\partial}{\partial v}\right) f_0^{(2)} = 0 \tag{C-6}$$

We see that the $B_k^{(0)} f_0^{(0)}$ term vanished identically as it did in the nonrelativistic calculation, because of the factor $(v \times \partial/\partial v)$ and the property that $f_0^{(0)}$ is spherically symmetric.

We substitute Eq. (C-6) into the Maxwell-Ampere Eq. (III-65).

$$-ick\times B_k^{(o)} + \frac{4\pi n_o e^2}{m}\int d^3v\; v\,\frac{i}{k\cdot v}\left\{\frac{\partial}{\partial v}\cdot I_v\cdot E_k^{(o)} f_o^{(o)} - \frac{1}{c}(1-v^2/c^2)^{1/2}\, B_k^{(o)}\cdot\left(v\times\frac{\partial}{\partial v}\right) f_o^{(2)}\right\} = 0 \qquad \text{(C-7)}$$

We consider the $E_k^{(o)\text{long}}$ contribution

$$\int d^3v\; v\,\frac{1}{k\cdot v}\left\{\frac{\partial}{\partial v}\cdot I_v\right\}\cdot k\, f_o^{(o)} =$$

$$\int d^3v\; v\,\frac{1}{k\cdot v}(1-v^2/c^2)^{1/2}\left\{-\frac{k\cdot v}{d^2} + \frac{k\cdot v}{c^2}\,\frac{v^2}{d^2} - \frac{5}{c^2}\,k\cdot v\right\} f_o^{(o)} \qquad \text{(C-8)}$$

Each term has a k•v in the numerator to cancel the k•v in the denominator, thus this integral is of the form

$$\int d^3v \quad v\cdot \text{function of } v^2 = 0 \qquad \text{(C-9)}$$

Thus as in the nonrelativistic calculation, $E_k^{(o)\text{long}}$ does not contribute to the equation for the evolution of $B_k^{(o)}$.

We find $E_k^{(o)\text{trans}}$ from the Faraday Eq. (III-64), and substitute it into Eq. (C-7) to obtain

$$ck\times B_k^{(o)} + \frac{\omega_e^2}{c^2}\, H_2^*\cdot B_k^{(o)} + \frac{\omega_e^2}{k^2c}\, H_1^*\cdot k\times \dot{B}_k^{(o)} = 0 \qquad \text{(C-10)}$$

where

$$H_1^* = i\int d^3v\;\frac{v}{k\cdot v}\,\frac{\partial}{\partial v}\cdot I_v\, f_o^{(o)} \qquad \text{(C-11)}$$

and

$$H_2^* = \int d^3v\;\frac{v}{k\cdot v}\,(1-v^2/c^2)^{1/2}\left(v\times\frac{\partial}{\partial v}\right) f_o^{(2)} \qquad \text{(C-12)}$$

where the asterisk is used to distinguish quantities in this appendix from similar quantities which occurred in the nonrelativistic calculation.

The asterisk does not denote complex conjugation. Equations (C-10 to 12) are very similar to Eqs. (IV-37 to 39).

The normal mode Eq. (C-10) may be written in the $\hat{k}$, $\hat{2}$, $\hat{n}$ coordinate system of Fig. V-2 more easily if we first multiply it by k ×.

$$\left\{-(k^2c^2/\omega_e^2)\, I_k + J_2 + (\gamma/kd)\, J_1\right\} \cdot \hat{B}_k = 0 \tag{C-13}$$

where

$$I_k = I - \hat{k}\hat{k} \tag{C-14}$$

$$J_1 = kd \;\; \hat{k} \times H_1^* \times \hat{k} \tag{C-15}$$

$$J_2 = k \times H_2^* \tag{C-16}$$

We introduce the dimensionless velocity

$$u = v/d \tag{C-17}$$

and define

$$\epsilon = d^2/c^2 \tag{C-18}$$

then J_1 may be written

$$\begin{aligned} J_1 &= i \int d^3u \; \frac{\hat{k}\times u}{\hat{k}\cdot u} \left\{ I_v \cdot \frac{\partial}{\partial u} - 5\epsilon u (1-\epsilon u^2)^{1/2} \right\} \times \hat{k} \; f_o^{(0)}(u) \\ &= i \int d^3u \; \frac{\hat{k}\times u \;\; \hat{k}\times u}{\hat{k}\cdot u} \left\{ (1-\epsilon u^2)^{3/2} + 5\epsilon (1-\epsilon u^2)^{1/2} \right\} f_o^{(0)}(u) \end{aligned} \tag{C-19}$$

We write $f_o^{(2)}(u)$ in the form

$$f_o^{(2)}(u) = \left(\frac{\alpha}{2} u_3^2 - \frac{\alpha}{6} u^2 \right) f_o^{(0)}(u) \tag{C-20}$$

then only the u_3^2 part contributes to J_2

$$J_2 = \int d^3u \; \frac{\hat{k}\times u}{\hat{k}\cdot u} \left(1-\epsilon u^2\right)^{1/2} \left(u \times \frac{\partial}{\partial u}\right) \frac{\alpha}{2} u_3^2 \; f_0^{(0)}(u)$$

$$= -\alpha \int d^3u \; \frac{\hat{k}\times u \;\; \hat{3}\times u \;\; \hat{3}\cdot u}{\hat{k}\cdot u} \left(1-\epsilon u^2\right)^{1/2} f_0^{(0)}(u) \tag{C-21}$$

We introduce the $\hat{k}$, $\hat{2}$, $\hat{n}$ coordinates

$$x = u\cdot\hat{k} \qquad y = u\cdot\hat{2} \qquad \text{and} \qquad z = u\cdot\hat{n} \tag{C-22}$$

then

$$u^2 = x^2 + y^2 + z^2 \tag{C-23}$$

$$\hat{k} \times u = y\hat{n} - z\hat{2} \tag{C-24}$$

$$\hat{3} = C\hat{k} + S\hat{n} \tag{C-25}$$

where

$$C = \cos\theta \qquad \text{and} \qquad S = \sin\theta \tag{C-26}$$

whence

$$\hat{3}\cdot u = Cx + Sz \tag{C-27}$$

$$\hat{3} \times u = -\,Sy\hat{k} + (Sx - Cz)\hat{2} + Cy\hat{n} \tag{C-28}$$

Using the above relations J_2 may be written

$$J_2 = -\alpha \int d^3u \; \frac{1}{x}\left(y\hat{n} - z\hat{2}\right)\left[-Sy\hat{k} + (Sx - Cz)\hat{2} + Cy\hat{n}\right]$$

$$\cdot \left(Cx + Sz\right)\left(1-\epsilon u^2\right)^{1/2} f_0^{(0)}(u) \tag{C-29}$$

Eliminating those terms which have an odd power of x, y, or z in the numerator gives

$$J_2 = -\alpha \int d^3u \left\{ y^2(-SC\hat{m}\hat{k} + C^2\hat{n}\hat{n}) + z^2(C^2-S^2)\hat{z}\hat{z} \right\} (1-\epsilon u^2)^{1/2} f_0^{(0)}(u) \qquad \text{(C-30)}$$

Likewise the tensor J_1 may be written

$$J_1 = i \int d^3u \; \frac{1}{x}(y\hat{n} - z\hat{z})(y\hat{n} - z\hat{z}) \left\{ (1-\epsilon u^2)^{3/2} + 5\epsilon(1-\epsilon u^2)^{1/2} \right\} f_0^{(0)}(u) \qquad \text{(C-31)}$$

We eliminate the odd powers of y and z, and perform the x integration. We denote

$$w^2 = y^2 + z^2 \qquad \text{(C-32)}$$

and

$$j(w) = (2\pi)^{-1} \exp -\tfrac{1}{2} w^2 \qquad \text{(C-33)}$$

then J_1 becomes

$$J_1 = -\left(\frac{\pi}{2}\right)^{1/2} \int dy\,dz \; (y^2\hat{n}\hat{n} + z^2\hat{z}\hat{z}) \left\{ (1-\epsilon w^2)^{3/2} + 5\epsilon(1-\epsilon w^2)^{1/2} \right\} j(w) \qquad \text{(C-34)}$$

The integrals in Eqs. (C-30) and (C-34) may be done by introducing the expansions

$$(1-\epsilon u^2)^{1/2} = \left(1 - \tfrac{1}{2}\epsilon u^2 + \cdots\right) \qquad \text{(C-35)}$$

$$\left\{ (1-\epsilon w^2)^{3/2} + 5\epsilon(1-\epsilon w^2)^{1/2} \right\} = \left\{ 1 + 5\epsilon - \tfrac{3}{2}\epsilon w^2 + \cdots \right\} \qquad \text{(C-36)}$$

Then

$$J_2 = -\alpha \left[-SC\hat{n}\hat{k} + C^2\hat{n}\hat{n} + (C^2-S^2)\hat{z}\hat{z} \right] \left\{ 1 - \tfrac{5}{2}\epsilon + \cdots \right\} \qquad \text{(C-37)}$$

and

$$J_1 = -\left(\frac{\pi}{2}\right)^{1/2} (\hat{n}\hat{n} + \hat{z}\hat{z}) \left\{ 1 - \epsilon + \cdots \right\} \qquad \text{(C-38)}$$

If we define

$$q^* = k^2c^2/\omega_e^2 + \alpha(1-2S^2)\{1-\tfrac{5}{2}\epsilon+\cdots\} + \left(\tfrac{\pi}{2}\right)^{1/2}(\gamma/kd)\{1-\epsilon+\cdots\} \tag{C-39}$$

$$s^* = k^2c^2/\omega_e^2 + \alpha C^2\{1-\tfrac{5}{2}\epsilon+\cdots\} + \left(\tfrac{\pi}{2}\right)^{1/2}(\gamma/kd)\{1-\epsilon+\cdots\} \tag{C-40}$$

$$t^* = -\alpha SC\{1-\tfrac{5}{2}\epsilon+\cdots\} \tag{C-41}$$

then the normal mode equation in the k, $\hat{2}$, n coordinate system becomes

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & q^* & 0 \\ t^* & 0 & s^* \end{pmatrix} \cdot \hat{B}_k(k,2,n) = 0 \tag{C-42}$$

We evidently have the same two modes as before

$$\text{Mode I:} \quad \hat{B}_k = \hat{2} \quad \text{and} \quad q^* = 0 \tag{C-43}$$

$$\text{Mode II:} \quad \hat{B}_k = \hat{n} \quad \text{and} \quad s^* = 0 \tag{C-44}$$

Equations (C-42 to 44) are similar to Eqs. (V-65 to 67). The relativistic calculation shows that the normal mode vectors are unchanged, and that the dispersion relation has corrections of order d^2/c^2. That is the growth rate is changed by

$$\gamma_k(\text{relativistic}) = \gamma_k(\text{nonrelativistic})\{1 + O(d^2/c^2)\} \tag{C-45}$$

For both Modes I and II the relativistic terms decrease the growth rate for a given k-value and reduce the region of unstable waves. Relativistic terms have a stabilizing effect on the transverse instability.

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