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Dennis Sivers and Joel Yellin

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ABSTRACT

We discuss the dynamical behavior of narrow resonance models for $\pi\pi$ scattering in the framework of what we call atonous duality, by which we mean that the pole locations and residues in a particular channel completely specify the narrow resonance amplitude as a Mittag-Leffler expansion containing no entire functions.

We operate under the following assumptions:

(i) Crossing symmetry, isospin conservation, and Bose statistics hold.

(ii) The only singularities of the amplitude are simple poles at locations defined by $a + b\,\xi\ (\equiv \alpha(\xi)) = 0, 1, 2, \cdots$ where ξ is a Mandelstam variable.

(iii) There are no $I = 2$ poles.

(iv) The pole at $s = \xi$ has a residue which is a polynomial in t of order $\leq \alpha(\xi)$.

(v) Atonous duality holds.

(vi) All partial widths are positive.

Under these assumptions we find that the $I \neq 1$ amplitude $A_1^{\,t}(\nu,t)$ has Regge behavior for $\nu \to \infty$ at fixed t, while $A_2^{\,t}(\nu,t)$ has pure fixed pole behavior. There are no Regge cuts.

Furthermore, we find it impossible to formulate partial wave dispersion relations for these models.

As an example of such a model we discuss the one term Veneziano ansatz for $\pi\pi$ scattering.

I. INTRODUCTION

We discuss here the detailed dynamics of narrow resonance models for $\pi\pi$ scattering.

The model is defined as a choice of $F_0(x,y)$ in[1,2]

$$X^t = \begin{pmatrix} A_0^{\,t} \\ A_1^{\,t} \\ A_2^{\,t} \end{pmatrix} = g \begin{pmatrix} -\frac{1}{2}F_0[\alpha(s),\alpha(u)] + \frac{3}{2}F_0[\alpha(t),\alpha(s)] + \frac{3}{2}F_0[\alpha(t),\alpha(u)] \\ F_0[\alpha(t),\alpha(u)] - F_0[\alpha(t),\alpha(s)] \\ F_0[\alpha(s),\alpha(u)] \end{pmatrix} \tag{1.1}$$

where $\alpha(x) = a + bx$, and $A_I^{\,t}$ is the t-channel amplitude for isospin I. The assumptions under which we work are listed and discussed in Section VI. The form of (1.1) insures that Bose statistics, crossing symmetry, and isospin conservation hold. For clarity, in Sections II through V we develop the detailed properties of the one term Veneziano ansatz for $\pi\pi$ scattering[3]

$$F_0(x,y) = \frac{\Gamma(1-x)\,\Gamma(1-y)}{\Gamma(1-x-y)} . \tag{1.2}$$

We emphasize that our qualitative results also apply to a convergent sum of such terms.

For the amplitude choice (1.2), in Section II we give several useful expansions of X^t as partial fractions, and note that the model exhibits what we call atonous duality.[4] In Section III we discuss the

positions and residues of the model's Regge poles; in Section IV we discuss its fixed poles; in Section V we discuss the asymptotic behavior of the Regge residue functions $\beta(t)$ and the related fixed angle and fixed J bounds.

In Section VI we list a set of assumptions under which we discuss the general class of narrow resonance models.

Appendix A contains a proof of the relations of Section II, gleaned from Whittaker and Watson, while Appendix B contains a discussion of the behavior of resonance widths in the one-term Veneziano model, and a rough argument for their positivity.

II. EXPLICIT PARTIAL-FRACTION EXPANSIONS

To investigate the properties of the amplitude (1.2) in the J plane, we make use of two partial-franction expansions. For convenience we define the variables ν, τ, x, y, w, z, where[5]

$$\begin{aligned} x &= \alpha(s) = \frac{1}{2} + s, \\ y &= \alpha(t) = \frac{1}{2} + t\ , \\ w &= \alpha(u) = \frac{1}{2} + u, \\ \nu &= \frac{1}{2}(x - w)\ , \\ \tau &= 1 - x - w\ , \\ z &= 2\nu/\tau\ . \end{aligned} \tag{2.1}$$

In the limit $\mu = m_\pi = 0$, we have $z = \cos\theta_t$, $\tau = t$, and

$$x + y + w = 3/2\ . \tag{2.2}$$

The partial-fraction expansions are then,

$$F_0(x,y) = \sum_{K=1}^{\infty} \frac{\Gamma(K+y)}{\Gamma(K)\,\Gamma(y)} \frac{1}{x-K} \tag{2.3}$$

and

$$F_0(x,w) = \sum_{K=1}^{\infty} (-1)^K \frac{\Gamma(K+\tau)}{\Gamma(K)\Gamma(\tau)} \left\{ \frac{1}{\nu + \frac{1}{2}(1-\tau) - K} + \frac{1}{-\nu + \frac{1}{2}(1-\tau) - K} \right\}. \tag{2.4}$$

In Appendix A, we give a proof of (2.3) and (2.4) that demonstrates that these series converge absolutely for $\mathrm{Re}\, y < 0$ and $\mathrm{Re}\, \tau < 0$, respectively.

The expansion (2.3), expressing the function (1.2) entirely in terms of its poles in s(or x), illustrates explicitly the atonous duality property[4] of the model. The poles in t (or y) can be found by analytically continuing (2.3) to positive y, where the series diverges, using

$$\frac{1}{x-K} = -\frac{1}{K} \sum_{n=0}^{\infty} \left(\frac{x}{K}\right)^n \quad (\text{for } |x| < K) \tag{2.5}$$

and the asymptotic expansion for a ratio of two gamma functions,[6]

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} \underset{\substack{|z| \to \infty \\ |\arg z| < \pi - \epsilon}}{\sim} \sum_{n=0}^{\infty} c_n(\alpha-\beta, \beta) z^{\alpha-\beta-n}, \tag{2.6}$$

where

$$C_0 = 1,$$

$$C_1 = \frac{1}{2}(\alpha - \beta)(\alpha + \beta - 1),$$

$$C_2 = \frac{1}{12}\binom{\alpha - \beta}{2}\left[3(\alpha + \beta - 1)^2 - \alpha + \beta - 1\right],$$

$$\vdots$$

$$C_n(\alpha-\beta, \beta) = \frac{1}{n}\sum_{m=0}^{n-1}\left[\binom{\alpha-\beta-m}{n-m+1} - (-1)^{n+m}(\alpha-\beta)\beta^{n-m}\right]C_m(\alpha-\beta, \beta). \tag{2.7}$$

Investigating the divergence of (2.3), we have, from (2.5) and (2.6),

$$F_0(x,y) \sim -\frac{1}{\Gamma(y)}\sum_{K=1}^{\infty}\left[K^y + \frac{1}{2}y(y-1)K^{y-1} + O(K^{y-2})\right]\left[K^{-1} + x\,K^{-2} + O(K^{-3})\right]$$

$$\sim -\frac{1}{\Gamma(y)}\left\{\zeta(1 - y) + [\tfrac{1}{2}y(y-1) + x]\,\zeta(2 - y) + \cdots\right\}, \tag{2.8}$$

where $\zeta(z)$ is the Riemann zeta function, which is analytic except for a pole of unit residue at $z = 1$.[7] From (2.8) we see that the series for $F_0(x,y)$ diverges at $y = 1$ to produce a pole with residue x, as we know it must.

The series in (2.4) is interesting because the poles in s, $[\nu + \frac{1}{2}(1 - \tau)]$ and u, $[-\nu + \frac{1}{2}(1 - \tau)]$ are simultaneously exhibited, with polynomial residues in τ.[8] Although (2.4) converges absolutely only for $\mathrm{Re}\,\tau < 0$, the conditional convergence due to the factor $(-1)^K$

guarantees the absence of poles in τ in its analytic continuation to positive τ. This can be seen by following the argument leading up to (2.8) and using[7]

$$\sum_{K=1}^{\infty} (-1)^K K^{-z} = (2^{1-z} - 1)\, \zeta(z) . \tag{2.9}$$

It is interesting to note how the Regge asymptotic behavior arises. Consider first the exchange of a tower of direct-channel resonances. The divergent part of this series generates a Regge pole. We have, for $P < |x| < P + 1$ and large x,[9]

$$\sum_{K=P+1}^{\infty} \frac{\Gamma(K + y)}{\Gamma(y)\,\Gamma(K)} \frac{1}{x - K} \sim - \sum_{L=0}^{\infty} x^L \sum_{K=P+1}^{\infty} \frac{K^{y-L-1}}{\Gamma(y)}$$

$$\sim \frac{1}{\Gamma(y)} \sum_{L=0}^{\infty} \frac{x^L}{y - L} = \frac{1}{y\Gamma(y)} F(1, -y;\ 1-y;\ x)$$

$$\underset{x\to\infty}{\sim} \frac{\pi\, e^{-i\pi y}\, x^y}{\Gamma(y) \sin \pi y} . \tag{2.10}$$

Now consider the exchange of a cross-channel tower. Again, with $P < |x| < P + 1$, we have

$$\lim_{\substack{x \to -\infty \\ \text{fixed } y}} \sum_{K=1}^{\infty} \frac{\Gamma(K + x)}{\Gamma(K)\,\Gamma(x)} \frac{1}{y - K}$$

$$= \lim_{P \to \infty} \left(\sum_{K=1}^{P} + \sum_{K=P+1}^{\infty} \right) \frac{\Gamma(K + x)}{\Gamma(K)\,\Gamma(x)} \frac{1}{y - K}$$

$$\approx \lim_{P \to \infty} \sum_{K=1}^{P} \frac{x^K}{\Gamma(K)} \frac{1}{y - K} + \lim_{P \to \infty} \sum_{K=P+1}^{\infty} \frac{K^x}{\Gamma(x)} \frac{1}{y - K}$$

$$\approx \lim_{P \to \infty} \sum_{K=1}^{P} \frac{x^K}{\Gamma(K)} \frac{1}{y - K} \,. \tag{2.11}$$

The incomplete Γ function satisfies[10]

$$\gamma(\alpha, x) = \int_0^x e^{-t}\, t^{\alpha-1}\, dt = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{\alpha+n}}{n!\,(\alpha + n)} \,. \tag{2.12}$$

If we set $-y = -1 + \alpha$, analytically continuing to $\operatorname{Re} \alpha < 0$, we have

$$e^{-i\pi y}\, x^y\, \gamma(1-y, -x) = \sum_{K=1}^{\infty} \frac{x^K}{\Gamma(K)} \frac{1}{y - K} \,, \tag{2.13}$$

and using $\lim_{x \to -\infty} \gamma(1-y, -x) = \Gamma(1 - y)$, we get the limit we want,

$$\lim_{\substack{x \to -\infty \\ \text{fixed } y}} \sum_{K=1}^{\infty} \frac{\Gamma(K + x)}{\Gamma(K)\,\Gamma(x)} \frac{1}{y - K} \sim e^{-i\pi y}\, x^y\, \Gamma(1 - y) \,, \tag{2.14}$$

which matches (2.10), as it should.

III. MOVING POLES IN ANGULAR MOMENTUM

Using (1.1) and the partial fraction expansions (2.3) and (2.4), we can write the t-channel amplitudes as

$$A_2^{\,t}(\nu,\tau) = g \sum_{K=1}^{\infty} \frac{(-1)^K\, T_K(\tau)}{\Gamma(K)} \left[\frac{1}{\nu + \frac{1}{2}(1-\tau) - K} + \frac{1}{-\nu + \frac{1}{2}(1-\tau) - K} \right], \tag{3.1}$$

$$A_1^{\,t}(\nu,\tau) = g \sum_{K=1}^{\infty} \frac{T_K(\tau + \frac{1}{2})}{\Gamma(K)} \left[\frac{1}{\nu + \frac{1}{2}(1-\tau) - K} - \frac{1}{-\nu + \frac{1}{2}(1-\tau) - K} \right], \tag{3.2}$$

$$a(\nu,\tau) = \frac{2}{3} A_0^{\,t} + \frac{1}{3} A_2^{\,t} = g \sum_{K=1}^{\infty} \frac{T_K(\tau + \frac{1}{2})}{\Gamma(K)} \times \left[\frac{1}{\nu + \frac{1}{2}(1-\tau) - K} + \frac{1}{-\nu + \frac{1}{2}(1-\tau) - K} \right], \tag{3.3}$$

where $T_K(x)$ is Pochhammer's polynomial,[11]

$$T_K(x) = \frac{\Gamma(K + x)}{\Gamma(x)} .$$

Extracting the s- and u-channel discontinuities of these expressions, we can then compute the Froissart-Gribov partial-wave amplitudes of definite signature

$$a^{\pm}(J,\tau) = \frac{1}{\pi} \int_0^{\infty} dz'\, Q_J(z')[D_u(\tau,z') \pm D_s(\tau,z')] . \tag{3.4}$$

Recalling

$$Q_J(z) = -\frac{1}{2}\int_{-1}^{+1} \frac{dz'}{z' - z} \; P_J(z') ,$$

we have, with $z = 2\nu/\tau$,

$$a_2^-(J,\tau) = 0 ,$$

$$a_2^+(J,\tau) = 2g \sum_{K=1}^{\infty} \frac{(-1)^K \, T_K(\tau)}{\Gamma(K)} \; Q_J\left(1 + \frac{2K - 1}{\tau}\right) , \qquad (3.5)$$

$$a_1^+(J,\tau) = 0 , \qquad (3.6)$$

$$a_1^-(J,\tau) = 2g \sum_{K=1}^{\infty} \frac{T_K(\tau + 1/2)}{\Gamma(K)} \; Q_J\left(1 + \frac{2K - 1}{\tau}\right) . \qquad (3.7)$$

Corresponding to $a(\nu,\tau)$ in (3.3), we find the partial-wave amplitudes $a_0^{\pm}(J,\tau)$,

$$a_0^-(J,\tau) = 0 , \qquad (3.8)$$

$$a_0^+(J,\tau) = 2g \sum_{K=1}^{\infty} \frac{T_K(\tau + 1/2)}{\Gamma(K)} \; Q_J\left(1 + \frac{2K - 1}{\tau}\right) = a_1^-(J,\tau). \qquad (3.9)$$

Regge poles appear as divergences of the sums in (3.5) and (3.7), which determine the partial-wave amplitudes. For example, the $I = 1$ Regge poles can be found by examining the behavior for large K of the summand in (3.7).

Using

$$Q_\nu(z) \underset{z \to \infty}{\sim} \frac{\pi^{1/2}\,\Gamma(\nu+1)}{(2z)^{\nu+1}\,\Gamma(\nu+3/2)} \left[1 + \frac{(\nu+1)(\nu+2)}{(4\nu+6)} \frac{1}{z^2} + O\left(\frac{1}{z^4}\right) \right] \tag{3.10}$$

and the asymptotic expansion (2.6), we have, setting $\alpha = \tau + 1/2$,

$$a_1^-(J,\tau) \sim \frac{1}{2(\pi)^{1/2}} \frac{1}{(4/\tau)^J} \frac{\Gamma(J+1)}{\Gamma(J+3/2)\Gamma(\alpha)} \sum_{K=1}^{\infty} \left\{ K^\alpha + \frac{1}{2}\alpha(\alpha-1)K^{\alpha-1} + \cdots \right\}$$

$$\times \left\{ K^{-J-1} - (J+1)\,\frac{1}{2}(\alpha - 3/2)K^{-J-2} + \cdots \right\}, \tag{3.11}$$

which can be summed to give a series of Riemann zeta functions, $\zeta(z)$,

$$a_1^-(J,\tau) \sim \frac{1}{2(\pi)^{1/2}} \frac{1}{(4/\tau)^J} \frac{\Gamma(J+1)}{\Gamma(J+3/2)\,\Gamma(\alpha)} \sum_{n=1}^{\infty} b_n(J,\alpha)\,\zeta(n+J-\alpha). \tag{3.12}$$

Since the only singularity of $\zeta(z)$ is a simple pole of unit residue at $z = 1$,[7] poles occur in (3.12) at $J = \alpha, \alpha-1, \alpha-2, \ldots$.

In principle we can compute any number of Regge residues by an iterative procedure evident from (3.10) and (2.6), but in practice combining the two expansions becomes cumbersome very quickly. The first two residues are

$$\beta_0(\tau) = \frac{g}{2(\pi)^{1/2}} \frac{\alpha(4/\tau)^{-\alpha}}{\Gamma(\alpha + 3/2)} \qquad (\text{for } J = \alpha), \tag{3.13}$$

$$\beta_1(\tau) = \frac{g}{2(\pi)^{1/2}} \frac{(4/\tau)^{-\alpha+1}}{\Gamma(\alpha + 1/2)} \left[\frac{1}{2}\alpha(\alpha - 1) - \frac{\alpha(\tau - 1)}{2}\right]$$

$$= \frac{g}{8(\pi)^{1/2}} \frac{\alpha(4/\tau)^{-\alpha+1}}{\Gamma(\alpha + 1/2)} \qquad (\text{for } J = \alpha - 1). \tag{3.14}$$

These residues are of the form predicted for linearly rising trajectories by Mandelstam.[12] In addition to the threshold factor $(4/\tau)^{-\alpha} = q^{2\alpha}$ in (3.13), note the zeros which appear at the negative half-integers, as they must because of Mandelstam symmetry and the absence of compensating trajectories.[13] It should be noted that since (2.6) is an asymptotic expansion rather than a convergent series, there is no guarantee that the infinite sum (3.12) converges to (3.7). In fact, this is not the case, and the asymptotic expansion (3.12) diverges for all finite τ and J. In addition, we will see in Section V that (3.12) is a poor approximation to the continuation of (3.7) as $\tau \to \infty$ for fixed J.

If we apply the same methods to (3.5) we find that there are, as expected, no $I = 2$ Regge poles. Although (3.5) converges absolutely only for $\text{Re } J > \tau$, we can use (2.9) to verify the absence of moving poles. The conditional convergence of (3.5) caused by the factor $(-1)^K$ eliminates the kind of poles appearing in (3.12).

IV. FIXED POLES IN THE J PLANE[14]

The $Q_J(z)$ have poles at negative integral J. Let us see whether these appear in our Froissart-Gribov amplitudes.

The $I = 2$ fixed poles, from (3.5), have residues

$$\gamma_2^{(N)} = 2g \sum_{K=1}^{\infty} \frac{(-1)^K T_K(\tau)}{\Gamma(K)} P_{N-1}\left(1 + \frac{2K-1}{\tau}\right), \quad (4.1)$$

while the $I = 1$ fixed poles have residues, from (3.7),

$$\gamma_1^{(N)} = 2g \sum_{K=1}^{\infty} \frac{T_K(\tau + 1/2)}{\Gamma(K)} P_{N-1}\left(1 + \frac{2K-1}{\tau}\right). \quad (4.2)$$

In deriving (4.1) and (4.2) we have used the fact that

$$Q_J(z) \approx \frac{P_{N-1}(z)}{J+N} \qquad (\text{near} \quad J = -N).$$

In fact, $\gamma_1(N) = 0$, as expected. The proof goes as follows. Set $\alpha \equiv \tau + 1/2$, and choose $\alpha < -N$. Then (4.2) converges, and if it is zero for $\alpha < -N$, analytic continuation tell us it is zero everywhere. (There can never be a barrier of singularities because the zeta function has only a single pole.)

The sum in (4.2) is zero for $\alpha < -N$. Define (for $\alpha < -N$)

$$D(\alpha, N, P) = \sum_{K=1}^{P} \frac{K^{N-1} T_K(\alpha)}{\Gamma(K)}. \quad (4.3)$$

By induction it follows easily that

$$D(\alpha, 1, P) = \frac{T_{P+1}(\alpha)}{(\alpha + 1)\,\Gamma(P)} , \tag{4.4}$$

and we then have

$$\lim_{P \to \infty} D(\alpha, 1, P) = \frac{P^{\alpha+1}}{(\alpha + 1)\,\Gamma(\alpha)} \to 0 . \tag{4.5}$$

Now

$$D(\alpha, 2, P) = \sum_{K=1}^{P} \frac{K\, T_K(\alpha)}{\Gamma(K)} = \alpha[D(\alpha + 1, 1, P) - D(\alpha, 1, P)], \tag{4.6}$$

and therefore

$$\lim_{P \to \infty} D(\alpha, 2, P) \sim P^{\alpha+2} \to 0 \quad (\alpha + 2 < 0). \tag{4.7}$$

This procedure can be extended to arbitrary N, which gives us the required result.

Similarly, we define, to study (4.1),

$$G(x, N, P) = \sum_{K=1}^{P} \frac{(-1)^K\, T_K(x)\, K^{N-1}}{\Gamma(K)} . \tag{4.8}$$

Just as for the D's above, the study of $G(x, N, P)$ can be reduced by an iteration procedure to the study of $G(x, 1, P)$, but here $G(x, 1, \infty) \neq 0$. In fact we have[15]

$$G(x, 1, \infty) = \frac{x \sin \pi x}{\pi} \sum_{K=0}^{\infty} (-1)^K B(-x, K + 1 + x)$$

$$= \frac{x \sin \pi x}{\pi} \int_0^1 \sum_{K=0}^{\infty} (-1)^K t^{K+x} (1 - t)^{-x-1}$$

$$= \frac{x \sin \pi x}{\pi} \int_0^1 dt\, t^x (1 - t)^{-x-1} (1 + t)^{-1} . \tag{4.9}$$

Using[15]

$$\int_0^1 t^{a-1} (1 - t)^{b-1} (1 + t)^{-b-a} dt = \frac{\Gamma(a)\, \Gamma(b)}{\Gamma(a + b)} 2^{-a} , \tag{4.10}$$

we get

$$G(x, 1, \infty) = -x\, 2^{-(x+1)} . \tag{4.11}$$

Therefore the $I = 0$ and $I = 2$ amplitudes contain fixed poles, whose residues are functions of τ.[16] The poles manifest themselves in the amplitudes a_0^+ and a_2^+ at odd integer (wrong signature) values of J, and hence, from the symmetry properties of the $P_K(z)$, we can see they give no contribution to the asymptotic behavior of the physical amplitudes.[17]

V. ASYMPTOTIC BEHAVIOR OF $\beta(t)$

If we use the Stirling approximation for $\Gamma(z)$, (3.13) yields

$$\beta_0(\tau) \underset{\tau \to \infty}{\sim} \frac{1}{4\pi} e^{3/2} (4/e)^{-\alpha(\tau)} \propto e^{-\alpha \ln(4/e)} , \qquad (5.1)$$

and as τ goes to ∞ along a wedge near the negative real axis we get an exponential blowup. Since $\beta_0(\tau)$ is an analytic function of τ, and has an infinite string of zeros at $\alpha = -\frac{3}{2}, -\frac{5}{2} \cdots$, Carlson's theorem tells us this exponential blowup must occur, as pointed out by Jones and Teplitz.[18]

Jones and Teplitz further remark that in a theory with infinitely rising trajectories at least one of the following set of assumptions, considered in a related context by Khuri,[19] must fail:

(i) The amplitude $A(s,t)$ is analytic in the cut s plane and is bounded for fixed t by

$$f(s) = c \exp\left(|s|^{\frac{1}{2} - \epsilon}\right); \qquad (5.2)$$

(ii) $A(s,z)$ is bounded by $f(s)$ for fixed z;

(iii) The Sommerfeld-Watson transformation of the partial-wave amplitudes $a(J,s)$ exists, and $a(J,s)$ is bounded by $f(s)$ for fixed J;

(iv) $\alpha(s)$ and $\beta(s)$ are analytic with a single cut from $s = 4\mu^2$ to ∞, $\alpha(s)$ is polynomial bounded, and $\beta(s)$ is bounded by $f(s)$.

Since (i) and (iv) are satisfied by construction, we must check (ii) and (iii). Both conditions are violated. Checking the fixed z bound (ii) we have, with $x = \alpha(s)$, $w = \alpha(u)$, $\tau = 2\nu/z$,

$$F_0(x,w) = \frac{\Gamma(\frac{1}{2} + \nu + \frac{\nu}{z})\,\Gamma(\frac{1}{2} - \nu + \frac{\nu}{z})}{\Gamma(2\nu/z)} \tag{5.3}$$

$$\sim \frac{1}{e}\left(\frac{\pi z}{\nu}\right)^{1/2} \exp\left\{\nu[(1 + 1/z)\,\ell n(1 + 1/z) + (-1 + 1/z)\ell n(-1 + 1/z) - \frac{2}{z}\,\ell n(2/z)]\right\}. \tag{5.4}$$

The evaluation of the fixed J limit takes some care, since our partial-wave amplitudes are expressed as infinite sums.

Near $z = 1$ [20] we have

$$Q_\nu(z) \approx -\frac{1}{2}\log(\frac{1}{2}z - \frac{1}{2}) - \gamma - \psi(\nu + 1) \tag{5.5}$$

(for $\nu \neq -1, -2, \cdots$) ,

where

$$\psi(z) = \frac{d}{dz}\log\Gamma(z) \tag{5.6}$$

and γ is the Euler-Mascheroni constant

$$\gamma = -\psi(1) = \lim_{z \to 1}\left\{\zeta(z) - \frac{1}{z - 1}\right\} . \tag{5.7}$$

We rewrite (3.5) for $P \leq |\tau| \leq P + 1$:

$$a_2^{+}(J,\tau) = 2g\left(\sum_{K=1}^{P} + \sum_{K=P+1}^{\infty}\right)\frac{(-1)^K\,T_K(\tau)}{\Gamma(K)}\,Q_J\left(1 + \frac{2K - 1}{\tau}\right)$$

(Eq. 5.8 Cont.)

$$\approx 2g \sum_{K=1}^{P} \frac{(-1)^K \tau^K}{\Gamma(K)} \left\{ -\frac{1}{2} \log\left(\frac{2K-1}{2\tau}\right) - \gamma - \psi(J+1) \right\}$$

$$+ 2g \sum_{K=P+1}^{\infty} \frac{(-1)^K K^\tau}{\Gamma(\tau)} \frac{(\pi)^{1/2} \Gamma(1+J)}{\Gamma(J+3/2)} \left(\frac{4K}{\tau}\right)^{-J-1} . \tag{5.8}$$

The second term on the RHS of (5.8) gives no trouble for τ large and negative. For the first term, however, we need to consider the asymptotic behavior of

$$g(\tau) = \sum_{K=1}^{\infty} \frac{(-\tau)^K}{\Gamma(K)} = -\tau\, e^{-\tau} , \tag{5.9}$$

which violates (iii) for $\tau \to -\infty$. Looking at the singularity structure for $a_2^+(J,\tau)$ diagrammed in Fig. 1, we see that this asymptotic behavior is what we expect, if we approximate the amplitude for fixed J by the pole at $J = -1$, with residue $G(\tau,1,\infty)$, using (4.1) and (4.11), so that

$$a_2^+(J,\tau) \approx \frac{-\tau\, e^{-(\tau+1)\ln 2}}{J+1} . \tag{5.10}$$

Now consider (3.7) for $P < \tau + 1/2 < P + 1$:

$$a_1^-(J,\tau) = 2g \left(\sum_{K=1}^{P} + \sum_{K=P+1}^{\infty} \right) \frac{\Gamma(K+\tau+1/2)}{\Gamma(K)\,\Gamma(\tau+1/2)} Q_J\left(1 + \frac{2K-1}{\tau}\right),$$

$$a_1^-(J,\tau) \sim 2g \sum_{K=1}^{P} \frac{\tau^K}{\Gamma(K)} \left\{ -\frac{1}{2}\log\left(\frac{2K-1}{2\tau}\right) - \gamma - \psi(J+1) \right\}$$

$$+ 2g \sum_{K=P+1}^{\infty} \frac{\{K^{\tau+1/2} + \cdots\}}{\Gamma(\tau+1/2)} \frac{(\pi)^{1/2}\,\Gamma(1+J)}{\Gamma(J+3/2)}$$

$$\times \left\{ \left(\frac{4K}{\tau}\right)^{-J-1} + \cdots \right\} . \qquad (5.11)$$

As τ goes to $+\infty$, we leave the region of convergence of the Froissart-Gribov projection (see Fig. 2) and, for fixed J, must consider more and more pole terms coming from the divergences of the second sum in (5.11). We see here, however, that the first sum in (5.11) becomes exponentially more important than the second as we go to higher τ. We must therefore interpret the expression for $a_1^-(J,\tau)$ in terms of an infinite number of Regge poles, (3.12), as an asymptotic expansion which is good for $\tau \to -\infty$ at fixed J, but fails for $\tau \to +\infty$. The first sum on the right of (5.11) corresponds roughly to the contribution of the background integral, and the second is the sum of Regge poles. Our statement above about the asymptotic behavior of $a_1^-(J,\tau)$ can alternatively be expressed as the fact that as one moves the background contour to the left in the J plane, in this model the background integral grows exponentially for large τ and dominates the Regge series.

Since assumption (iii) fails both for $a_1^-(J,\tau)$ and for $a_2^+(J,\tau)$, we cannot write fixed-J dispersion relations for the partial-wave amplitudes. Partial-wave dispersion relations have served as a useful dynamical tool in traditional Regge theory,[21] and their failure here is an important consequence of the Veneziano model.

VI. POLE MODELS, REGGE BEHAVIOR, AND ATONOUS DUALITY

In this section we discuss the general class of models for $\pi\pi$ scattering containing only simple poles in place of the physical cut.

We choose to restrict our considerations to that class of narrow resonance models whose defining characteristic is that the amplitude is completely specified by the residues and locations of the poles. This assumption eliminates from consideration, for example, models of the type recently proposed by Jengo, which are forms of the interference model.[22]

We will use here the following assumptions:

(a) Crossing symmetry, isospin conservation, and Bose statistics are satisfied.

(b) The only singularities of the amplitude are simple poles at $s = s_K$, where $a + bs_K$ $(\equiv \alpha(s_K)) = 0, 1, 2, \cdots$, and similarly for the t and u channels.

(c) There are no $I = 2$ poles.[23]

(d) The pole at $s = s_K$ has a residue which is a polynomial in t (or u) of order $\leq \alpha(s_K)$.

(e) Given the locations and residues of the poles in s, the amplitude is completely determined.

(f) All partial widths are positive.

Assumption (a) is of course also satisfied by the physical $\pi\pi$ amplitude. Assumptions (b) and (d) specify the pole locations and their possible angular momentum content. Assumption (c) is made purely for reasons of convenience. Assumption (d) is probably not independent of assumptions (a), (b), (c), and (e). Assumption (e) is

the defining characteristic we refer to above, which we call "atonous" duality in order to differentiate it from Dolen-Horn-Schmid duality.[4] We do not discuss assumption (f) further in this section, except to mention that of all possible individual Veneziano terms, only (1.2) yields all positive widths. We touch on the positivity condition further in Appendix B.

Assumptions (a), (b), and (c) are implemented by taking the t channel amplitudes to be

$$\begin{pmatrix} A_0^{\,t} \\ A_1^{\,t} \\ A_2^{\,t} \end{pmatrix} = g \begin{pmatrix} \frac{3}{2}\{H[\alpha(t),\alpha(s)] + H[\alpha(t),\alpha(u)]\} - \frac{1}{2}H[\alpha(s),\alpha(u)] \\ H[\alpha(t),\alpha(u)] - H[\alpha(t),\alpha(s)] \\ H[\alpha(s),\alpha(u)] \end{pmatrix} \tag{6.1}$$

where (a) is satisfied by taking $H(x,y) = H(y,x)$, (b) is to be implemented by hand, by choosing the pole locations as above, and (c) is satisfied by insisting that $H[\alpha(s),\alpha(u)]$ have no singularities for $4\mu^2 > s + u$.

For fixed y, the most general $H(x,y)$ having given pole locations and residues, by the Mittag-Leffler theorem,[24] is

$$H(x,y) = \sum_{K=0}^{\infty} \left\{ \frac{R_K(y)}{x - K} + E_K(x,y) \right\} \tag{6.2}$$

where $E_K(x,y)$ is entire in x.

By assumption (e), atonous duality, we mean that, at least in the physical region in the channel where x is the energy,

$$H(x,y) = \sum_{K=0}^{\infty} \frac{R_K(y)}{x - K} \tag{6.3}$$

so that $E_K(x,y) = 0$. Now (6.3), by construction, will converge for all x, and for $\mathrm{Re}\, y < 0$.

The symmetry of H then tells us that

$$\sum_{K=0}^{\infty} \frac{R_K(x)}{y - K} = \sum_{K=0}^{\infty} \frac{R_K(y)}{x - K} \tag{6.4}$$

provided $\mathrm{Re}\, x$ <u>and</u> $\mathrm{Re}\, y$ are negative. In regions where x (and/or y) is positive, the divergence of the series of poles in x generates the poles in y, and vice versa.

Assumption (e) is essentially the statement that for some fixed t we can write an unsubtracted dispersion relation in s. We call this "atonous" duality, in order to distinguish it from Dolen-Horn-Schmid duality.[4]

If we express the pole residues of $H(x,y)$ in terms of $\tau = 1 - x - y$, and let $\nu = \frac{1}{2}(x - y)$, we see that both sets of poles can be simultaneously exhibited in the form

$$H(x,y) = \sum_{K=0}^{\infty} R_K(1 - K - \tau) \left[\frac{1}{\nu + \frac{1}{2}(1-\tau) - K} + \frac{1}{-\nu + \frac{1}{2}(1-\tau) - K} \right], \tag{6.5}$$

again with no additional entire function.

Below, we will argue semi-rigorously that in the general case:

(A) $H(x,y)$ has pure power behavior of the form

$$H(x,y) \underset{\substack{x \to \infty \\ \text{fixed } y}}{\sim} \sum_{L=0}^{\infty} x^{y-L} c_L(y) . \tag{6.6}$$

(B) For $\nu \to \infty$ in (6.5) for fixed τ, $H[x(\nu,\tau), y(\nu,\tau)]$ has fixed power behavior of the form

$$H \underset{\substack{\nu \to \infty \\ \text{fixed } \tau}}{\sim} \sum_{L=0}^{\infty} \nu^{-2L-1} e_L(\tau) . \tag{6.7}$$

Property (A) is Regge behavior, and property (B) is the Froissart-Gribov phenomenon.[25] From (6.6) and (6.7) the $I = 0$ amplitude has fixed and moving poles in the J plane; the $I = 1$ amplitude has moving poles only; and the $I = 2$ amplitude contains only fixed poles.

Property (A) follows from the behavior of the residues

$$\lim_{K \to \infty} R_K(y) = \tilde{q}_0(y)K^y + \tilde{q}_1(y)K^{y-1} + \cdots . \tag{6.8}$$

Property (B) follows from (6.8) and from the fact that $\beta[\alpha(t)]/q^{2\alpha(t)} \geq 0$ for $\alpha(t) = 1, 2, 3, \cdots$.

Furthermore, except for a violently pathological case, the fixed-J bound on $a_2^+(J,\tau)$ is violated, as in the specific example discussed in Section V.

Taken together, assumptions (b), (d), and (e) are almost equivalent to assuming Regge behavior. Let us see how the asymptotic behavior works out. By assumption (c)

$$R_K(x) = \left(\sum_{m=0}^{K} a_{Km} x^m \right) g(K) \tag{6.9}$$

where we have taken $a_{KK} = 1$, for convenience. Factoring out any multiplicative x dependence we define $\tilde{g}(x)$, $\sigma(K,x)$, and $\tilde{\sigma}(K,x)$ by

$$R_K(x) = \tilde{g}(x)\, \sigma(K,x) \tag{6.10}$$

and

$$\lim_{K \to \infty} R_K(x) = \tilde{g}(x)\, \tilde{\sigma}(K,x) . \tag{6.11}$$

In the example above $g(x) = \tilde{g}(x) = \tilde{q}_0(x)$, and $\tilde{\sigma}(K,x) = K^x$. By using (6.4) we will argue below that these relations are true in general.

Using the same procedure as in Section II, for $P < |x| < P + 1$

$$\lim_{x \to -\infty} \sum_{K=0}^{\infty} \frac{R_K(x)}{y - K} \sim \lim_{P \to \infty} \sum_{K=0}^{P} \frac{x^K g(K)}{y - K} + \lim_{P \to \infty} \sum_{K=P+1}^{\infty} \frac{\tilde{g}(x)\sigma(K,x)}{y-K} . \tag{6.12}$$

Similarly, for the direct channel exchange

$$\lim_{P \to \infty} \sum_{K=P+1}^{\infty} \frac{R_K(y)}{x - K} \sim - \sum_{L=0}^{\infty} x^L \sum_{K=P+1}^{\infty} K^{-L-1}\, \tilde{g}(y)\tilde{\sigma}(K,y) . \tag{6.13}$$

Physically, the second sum on the right in (6.12) should give no contribution to the asymptotic behavior. Matching the remaining terms, we get, in the limit of large $|x|$

$$\lim_{P \to \infty} - \sum_{L=0}^{\infty} x^L \sum_{K=P+1}^{\infty} K^{-L-1} \tilde{g}(y)\tilde{\sigma}(K,y) = \sum_{L=0}^{\infty} \frac{x^L g(L)}{y - L}$$

$$= \sum_{L=0}^{\infty} x^L g(L) \lim_{P \to \infty} \sum_{K=P+1}^{\infty} K^{-L+y+1} \tag{6.14}$$

using the property of the Riemann zeta function[7]

$$\frac{1}{z - 1} = \zeta(z) - E_\zeta(z) = \sum_{K=1}^{\infty} K^{-z} - E_\zeta(z)$$

$$= \sum_{K=M}^{\infty} K^{-z} + \left(\sum_{K=1}^{M-1} K^{-z} - E_\zeta(z) \right) \tag{6.15}$$

where $E_\zeta(z)$ is entire.

From this last relation we would of course like to conclude that $\tilde{g}(y) = g(L)$, and $\tilde{\sigma}(K,y) = K^y$. We do not know how to prove that this is true rigorously, but we conjecture that it is, and we note that if these relations do not hold, evidently we need more information to determine the system, and assumption (e) is therefore violated. If we assume $\tilde{\sigma}(K,y)$ is indeed K^y, (6.10) then yields, using (2.10),

$$\tilde{g}(y) \sum_{L=0}^{\infty} \frac{x^L}{y - L} = \frac{\tilde{g}(y)}{y} F(1, -y; 1-y; x)$$

$$\underset{x \to \infty}{\sim} \frac{\tilde{g}(y)\, \pi\, e^{-i\pi y}\, x^y}{\sin \pi y} \tag{6.16}$$

so that we have Regge behavior and are left with an unspecified entire function $\tilde{g}(y)$.[9]

Let us try to verify the conclusions above by using (6.4) in a somewhat different way. As $y \to 0-$, we know that the sum on the left in (6.4) must diverge and we have, using (6.9)

$$\lim_{y \to 0-} \sum_{K=0}^{\infty} \frac{R_K(y)}{x - K} \sim \frac{g(0)}{y} . \tag{6.17}$$

Since we are looking for the divergent piece of the series we can throw away the first M terms, taking $M \gg x$, and obtaining, from (6.15) and (6.11), for small y and arbitrary, large, M

$$\tilde{\sigma}(K,y) = K^{f(y)}, \qquad \tilde{g}(0) = g(0) = \tilde{q}(0) \tag{6.18}$$

$$f(0) = 0, \qquad f'(0) = 1. \tag{6.19}$$

If we subtract off the first divergence and work at successive integers we get

$$\tilde{g}(N) = g(N); \qquad f(N) = N; \qquad f'(N) = 1. \tag{6.20}$$

If we differentiate (6.4) and go through the same procedure we can in fact conclude that $f(y) = y$, since we have implicitly assumed

$\tilde{\sigma}(z,y)$ is holomorphic. The differentiation and subtraction procedure also enables us to conclude that the general power behavior (6.8) holds.

We therefore reach, semirigorously, the unsurprising conclusion that our assumptions imply that all the poles lie on Regge trajectories, and that there are no extra trajectories unassociated with poles.

Either from (6.16) or from the Froissart-Gribov continuation

$$a_1^-(J,\tau) = \sum_{K=0}^{\infty} R_K(y)\, Q_J\left(1 + \frac{2K-1}{\tau}\right), \tag{6.21}$$

we have the leading Regge residue

$$\beta_0(\tau) = \frac{\tilde{g}(y)\,\Gamma(y+1)}{\Gamma(y+3/2)}\left(4/\tau\right)^{-y}. \tag{6.22}$$

In order to prevent $\beta_0(t)$ from having poles at $y = -1, -2, \cdots$ we must insist

$$\tilde{g}(y) = \Gamma^{-1}(y+1)\, h(y) \tag{6.23}$$

where $h(y)$ is entire and arbitrary, except for the requirement that $h(N) \geq 0$. From (6.5) we can see that $a_2^+(J,\tau)$ has fixed poles in J at wrong signature negative integers. As in the example above, the residues of the fixed poles are nonzero if

$$\gamma_0(\tau) = \sum_{K=0}^{\infty} \frac{(-1)^K \tau^K}{\Gamma(K+1)}\, h(K) \tag{6.24}$$

is nonzero for large, negative τ. The sum in (6.24) is bounded below by $\epsilon^2 e^{-\tau}$, provided all $h(K) > \epsilon^2$ for $K >$ (some) M, and in fact $\gamma_0(\tau)$ is nonzero if at least one $h(K)$ is nonzero.

The model is now specified up to the entire functions $h_i(K)$, which enter each Regge residue. Except for a violently pathological choice of h_i, the $\beta_i(t)$ will then blow up exponentially as in the example above, as will the residues of the fixed poles, and the fixed J bound will be violated.[18,26]

VII. CONCLUSION AND SUMMARY

We have argued above that under a certain set of assumptions, including most importantly what we call atonous duality,[4] narrow resonance models for $\pi\pi$ scattering necessarily contain fixed poles in J in the $I = 0$ and 2 amplitudes, and moving poles in the $I = 1$ and 0 amplitudes.

The Regge residues can be determined up to entire functions. For any reasonable choice of these functions, the fixed J bounds for the Froissart-Gribov amplitudes are violated and the Regge residues will show the exponential behavior discussed by Jones and Teplitz.[18] The formulation of partial wave dispersion relations is therefore precluded.

We believe the pathology outlined above can only be cured by a frontal attack on the narrow resonance approximation itself.

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APPENDIX A. DERIVATION OF THE PARTIAL-FRACTION EXPANSIONS

To prove (2.3) we use Gauss' theorem for the hypergeometric function[27]

$$F(a,b;\ c;\ 1) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}, \tag{A.1}$$

which holds for $\mathrm{Re}\,[c - a - b] > 0$. We remind the reader that F satisfies the hypergeometric equation

$$z(1-z)u'' + [c - (a+b+1)z]u' - abu = 0, \tag{A.2}$$

and is defined by the hypergeometric series,

$$F(a,b;\ c;\ z) = \sum_{K=0}^{\infty} \frac{T_K(a)\,T_K(b)}{T_K(c)\,\Gamma(K+1)}\, z^K . \tag{A.3}$$

The series converges for $|z| < 1$. For $|z| = 1$ the series is[27]

(a) divergent for $\mathrm{Re}(a + b - c) \geq 1$,

(b) absolutely convergent for $\mathrm{Re}(a + b - c) < 0$,

(c) conditionally convergent for $0 \leq \mathrm{Re}(a + b - c) < 1$, the point $z = 1$ being excluded.

To verify (2.3) we note that

$$\sum_{K=1}^{\infty} \frac{\Gamma(K+y)}{\Gamma(K)\,\Gamma(y)}\,\frac{1}{x-K} = \sum_{m=0}^{\infty} \frac{\Gamma(m+1+y)}{\Gamma(y)\,\Gamma(m+1)}\,(-1)\,\frac{\Gamma(m+1-x)}{\Gamma(m+2-x)},$$

$$\text{(by A.3)} \qquad = -\frac{\Gamma(1+y)\Gamma(1-x)}{\Gamma(2-x)\,\Gamma(y)}\,F(1+y,\ 1-x;\ 2-x;\ 1) \tag{A.4}$$

$$\text{(by A.1)} \qquad = (-y)\,\frac{\Gamma(1-x)\,\Gamma(-y)}{\Gamma(1-x-y)}$$

$$= F_0[x,\ y],$$

whenever $\mathrm{Re}\, y < 0$.

Verification of (2.4) is more involved, since both sets of poles have been simultaneously exhibited.

In the notation of (2.4) we write

$$F_0[x,w] = \tau\, B[-\nu + \tfrac{1}{2}(1+\tau),\ +\nu + \tfrac{1}{2}(1+\tau)]. \tag{A.5}$$

Making use of the integral representation of the beta function,

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt$$

$$= \int_0^1 (1+t)^{-x-y} \left\{ t^{x-1} + t^{y-1} \right\} dt\,, \tag{A.6}$$

and of the hypergeometric function,[27]

$$F(a,b;\ c;\ z) = \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \int_0^1 dt\, t^{b-1}(1-t)^{c-b-1}\,(1-tz)^{-a}, \tag{A.7}$$

we have

$$B[-\nu + \tfrac{1}{2}(1+\tau),\ \nu + \tfrac{1}{2}(1+\tau)]$$

$$= \frac{\Gamma[-\nu + \frac{1}{2}(1+\tau)]}{\Gamma[-\nu + \frac{1}{2}(1+\tau) + 1]}\, F[\tau+1,\ -\nu + \tfrac{1}{2}(1+\tau);\ -\nu + \tfrac{1}{2}(1+\tau) + 1;\ -1]$$

$$+ \frac{\Gamma[\nu + \frac{1}{2}(1+\tau)]}{\Gamma[\nu + \frac{1}{2}(1+\tau) + 1]}\, F[\tau+1,\ \nu + \tfrac{1}{2}(1+\tau);\ \nu + \tfrac{1}{2}(1+\tau) + 1;\ -1]$$

$$= \frac{1}{\Gamma(\tau+1)} \sum_{n=0}^{\infty} \frac{\Gamma(\tau+1+n)}{\Gamma(n+1)}\,(-1)^n \left\{ \frac{1}{-\nu + \frac{1}{2}(1+\tau) + n} + \frac{1}{\nu + \frac{1}{2}(1+\tau) + n} \right\}, \tag{A.8}$$

and if we let $K = n + 1$, (2.4) follows. From the properties of F at $z = -1$ listed above, we see the series in (2.4) is absolutely convergent for $\mathrm{Re}\ \tau < 0$, and conditionally convergent for $0 \leq \mathrm{Re}\ \tau < 1$.

APPENDIX B. BEHAVIOR OF RESONANCE WIDTHS

We define $W(N,L)$ as the partial width of the $\pi\pi$ resonance having spin L and $(\text{mass})^2 \cdot (N - a)/b$. The behavior of $W(N,L)$ as a function of its arguments is contained in

$$W(N,L) = \int_{-1}^{+1} dz\ P_L(z)\ \frac{\Gamma(N + y)}{\Gamma(N)\ \Gamma(y)}, \tag{B.1}$$

where

$$y = z\left[\frac{1}{2}(N - D) + a\right] - \frac{1}{2}(N - D) \tag{B.2}$$

and

$$D = \alpha(s) + \alpha(t) + \alpha(u) = 3a + 4\mu^2 b\ . \tag{B.3}$$

The coefficients of the Legendre polynomials appropriate for (1.2) in the case $D = 3/2$, $a = 1/2$, $\mu^2 = 0$, $b = 1\ \mathrm{BeV}^{-2}$ are listed in Table B1.

For $N \gg D, a$ and $4\mu^2 b$, the polynomials $T_N(y) = \Gamma(N+y)/\Gamma(y)$ essentially consist of a huge forward peak between $0 \leq y \leq a$, or $1 - 2a/N \leq z \leq 1$.

We can therefore approximate $W(N,L)$ by

$$W(N,L) \cong \int_{1-2a/N}^{1} dz\ \frac{P_L(z)\ \Gamma(N + y)}{\Gamma(N)\ \Gamma(y)}\ . \tag{B.4}$$

For large L and small $1 - \eta$,[28]

$$P_L(z) = J_0(\eta) + \left(\frac{1-z}{2}\right)\left[\frac{J_1(\eta)}{2\eta} - J_2(\eta) + \frac{\eta}{6} J_3(\eta)\right] + O[(1-z)^2], \tag{B.5}$$

where $\eta = (2L+1)\left[\frac{1-z}{2}\right]^{1/2}$.

The first zero of $J_0(x)$ occurs at $x \approx 2.41$. Therefore the first zero of $P_L(z)$, from (B.5), occurs near the edge of the T_N peak for

$$L = L_0 \approx 1.2\ (N/a)^{1/2} . \tag{B.6}$$

For $L \ll L_0$ and N fixed, $W(N,L)$ will thus be roughly constant, since the entire forward peak in $T_N(y)$ is contributing to the integral.

As L gets larger than L_0 , $W(N,L)$ will begin to drop sharply. Taking $N \gg a$ in (B.4) and expanding $\Gamma^{-1}(y)$ in a power series about $y = a$, we obtain

$$W(N,L) \approx 2N^a \int_0^\infty d\xi\ e^{-N(\log N)\xi}\ J_0(2L\sqrt{\xi}) \sum_{j=0}^{\infty} C_j^{(N)}\ \xi^j$$

$$\approx 2N^a \sum_{j=0}^{\infty} C_j^{(N)}\ \Gamma(j+1)(N \log N)^{-j-1}\ e^{-L^2/N \log N}\ L_j(L^2/N \log N), \tag{B.7}$$

where $L_j(x)$ is a Laguerre polynomial.[29] In the interval of interest, $0 \le \xi \le 2/N$, we can bound $\Gamma^{-1}(y)$ by

$$\Gamma(1-a) \sin \pi a \left(1 - \frac{N\xi}{2}\right) = \Gamma(1-a)\left\{\sin \pi a \sum_{j=0}^{\infty} \frac{\phi^{2j}}{\Gamma(2j+1)} - \cos \pi a \sum_{j=0}^{\infty} \frac{\phi^{2j+1}}{\Gamma(2j+2)}(-1)^j\right\}, \tag{B.8}$$

where $\phi = \frac{1}{2} N \xi$. This expression roughly tells us what $C_j^{(N)}$ must be like as a function of j , and we can now check and see whether the approximated series converges.

To proceed, we need to know how L_j behaves for large j. Fejer's formula tells us that for fixed z [22]

$$\lim_{j \to \infty} L_j(z) = (\pi)^{1/2} e^{\frac{1}{2} z} z^{-1/4} \cos [2(jz)^{1/2} - \pi/4] + O(j^{-1/4}). \tag{B.9}$$

Therefore the approximated series converges for fixed and large (L,N).

For $L \ll N$ the argument of L_j becomes small and (B.9) is no longer applicable. Instead we have[30]

$$e^{-\frac{1}{2}z} L_j(z) \sim J_0 \left([(4j + 2)z]^{1/2} \right) + O(j^{-3/4}) , \tag{B.10}$$

and the approximated series still converged. For $L \ll N^{1/2}$, $W(N,L) \sim \exp(-L^2/N \log N)$, and is relatively insensitive to L, as we have pointed out above.

For $L \gg \sqrt{N}$, a much more complicated argument leads once more to a convergent series, falling off exponentially.

The behavior of $W(N,L)$ sketched above is equivalent to an impact parameter description in which the radius of interaction grows as $s^{1/2}$.[31]

For $L \ll N$, the first term in the series gives $N^{a-1}/\log N$ for the behavior in N , showing the power behavior one expects, along with the logarithmic shrinkage.

These general remarks lead to a rough argument as to the positivity of $W(N,L)$. As has been shown by Shapiro and Yellin,[3] if one chooses a such that $W(2,0) \geq 0$, then $W(N,0) \geq 0$.[32] Using the same arguments, it is probably possible to show a similar result for $W(N,1)$, the p-wave widths, though this has not been done. Furthermore, $W(N,N)$ and $W(N,N-1) \geq 0$ because the coefficient of z^L in $P_L(z)$ is positive.

Above, we have argued that for fixed large N, $W(N,L)$ falls monotonically with increasing L. Combining this property with positivity of the s- and p-wave widths on one hand, and that of the leading two trajectories on the other, we have a rough argument for the positivity of all $W(N,L)$.

FOOTNOTES AND REFERENCES

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† National Science Foundation Predoctoral Fellow.

1. G. Veneziano, Nuovo Cimento 57, 190 (1968).

2. J. Shapiro, Phys. Rev. (to be published) 1969. See also J. Shapiro and J. Yellin, Lawrence Radiation Laboratory Report UCRL-18500, September 26, 1968 (unpublished). C. Lovelace, Phys. Letters 28B, 265 (1968).

3. More calculational details of the work presented here are contained in Notes on $\pi\pi$ Scattering III: J-Plane Phenomena, D. Sivers and J. Yellin, Lawrence Radiation Laboratory Report UCRL-18665, Jan. 17, 1969 (unpublished).

4. R. Dolen, D. Horn, and C. Schmid, Phys. Rev. 166, 1768 (1968). Dolen-Horn-Schmid duality was originally formulated as a connection between low energy resonances and the leading Regge trajectory in the cross channel. We make no distinction between leading and nonleading objects in defining our "atonous" duality. This particular nomenclature was suggested to one of us (J.Y.) by Prof. Y. Ne'eman.

5. We have here made the choice of slope and intercept $b = 1\ \mathrm{GeV}^{-2}$, $a = \frac{1}{2}$ for definiteness. However, all the analysis can be carried out abstractly in terms of ν, z, and τ.

6. F. C. Tricomi and A. Erdelyi, Pacific Journal of Math 1, 133 (1951).

7. W. Magnus, F. Oberhettinger, and R. P. Soni, Formulas and Theorems for the Special Functions of Mathematical Physics (Springer-Verlag,

New York, 1966), Section 1.3. The entire function referred to in (6.15) is given by $E_\zeta(z) = -\frac{\sin \pi z}{\pi(z-1)} \int_0^1 [\psi'(1+t) - \frac{1}{1+t}]t^{1-z}\, dt$.

8. The sums in (2.4) do not exhibit the duality property. Separately, they do not give Regge asymptotic behavior, but blow up exponentially for large ν.

9. Compare these expressions with those of L. Durand III, Phys. Rev. 161, 1610 (1967). The ingredient which is absent in older treatments is the atonous duality statement that everything is determined in the narrow resonance limit if one knows the location of poles and their residues. This implies that the divergence of the series of cross-channel poles generates direct-channel poles, and vice versa.

10. Ref. 7, Section 9.1.

11. L. Pochhammer, Math. Ann. 36, 84 (1890).

12. S. Mandelstam, Phys. Rev. 166, 1539 (1968).

13. S. Mandelstam, Ann. Phys. (N.Y.), 19, 254 (1959).

14. M. A. Virasoro (University of Wisconsin) (private communication to S. Mandelstam) and G. Veneziano (Massachusetts Institute of Technology) (private communication) have also independently discovered the fixed poles we discuss here. It is easy to see that $F_0[\alpha(s), \alpha(t)]$ and $F_0[\alpha(u), \alpha(t)]$ contain no fixed poles, since these would show up as fixed powers of $\alpha(t)$ in the related asymptotic expansion.

15. Ref. 7, Section 1.1.

16. Since the residues are functions of τ, the fixed poles are of the Gribov-Pomeranchuk type instead of the type occurring in potential scattering which start to move when unitarity is implemented. (S. Mandelstam, private communication.)

17. The physical amplitude is given in terms of the definite signature amplitudes by $A(\tau,z) = \frac{1}{2}[A^+(\tau,z) + A^+(\tau,-z) + A^-(\tau,z) - A^-(\tau,-z)]$.

18. C. E. Jones and V. L. Teplitz, Phys. Rev. Letters 19, 135 (1967).

19. N. N. Khuri, Phys. Rev. Letters 18, 1094 (1967). One of us (J.Y.) would like to thank Professor Khuri for several helpful discussions.

20. A. Erdelyi et al., Higher Transcendental Functions (McGraw-Hill, N.Y., 1954), I, 3.9.2(7).

21. See for example, P. D. B. Collins and E. J. Squires, Regge Poles in Particle Physics (Springer, New York, 1968), p. 46. S. Matsuda and F. Drago, California Institute of Technology preprint CALT-68-181, to be published, have also considered the J-plane properties of the one term Veneziano $\pi\pi$ amplitude, and also discuss the failure of partial wave dispersion relations in that special case. D. I. Fivel and P. K. Mitter, University of Maryland preprint, March 1969 (unpublished) have also discussed the J-plane behavior of the one term Veneziano $\pi\pi$ amplitude. However, they do not discuss the fixed poles and the failure of partial wave dispersion relations.

22. R. Jengo, Phys. Letters 28B, 262 (1968).

23. Therefore by assumption we exclude multiplicative fixed poles in J, force the system to be exchange degenerate, and eliminate the possibility of including the Pomeranchon as an ordinary Regge trajectory. Assumption (c) allows us to ignore Virasoro's representation (M. Virasoro,

University of Wisconsin preprint, unpublished) and the generalization proposed by Mandelstam which encompasses both the Virasoro and Veneziano representations as special cases. (S. Mandelstam, Berkeley preprint, to be published).

24. L. Hormander, An Introduction to Complex Analysis in Several Variables (D. Van Nostrand Inc., Princeton, 1966), p. 11.

25. V. N. Gribov and I. Ya. Pomeranchuk, Phys. Rev. Letters 9, 238 (1962).

26. The reader may wonder why we have not mentioned sums of Veneziano terms in Section VI. The reason is that, while given a set of poles and residues, we can always construct a Veneziano series with at least those same residues and poles, we have no way of telling if, where, and how such a series will converge. We thank R. Brower for an interesting discussion regarding this question. K. Bardakci (private communication) has pointed out to us that the most general convergent series of Veneziano terms can be expressed as $P(x,y) = \int_0^1 du\, [f(u)]^{-x-1}\, [1 - f(u)]^{-y-1}\, h(u)$ where $f(0) = 0$, $f(1) = 1$, and $f(u)$ and $h(u)$ are analytic for u in $[0,1]$. One can easily constrain $f(u)$ and $h(u)$ so that $P(x,y)$ is symmetric. Bardakci, Brower, and the authors all conjecture that the symmetric $P(x,y)$ is the most general function consistent with our assumptions. We have been unable to find a proof however, and leave this as a challenge to the mathematically agile reader.

27. E. T. Whittaker and G. N. Watson, Modern Analysis (Cambridge University Press, 1929), 4th Edition, Section 14.11.

28. I. S. Gradshteyn and I. M. Ryzhik, Tables of Integrals, Series, and Products (Academic Press, New York, 1965), 8.722(1). (Read μ for ν in the exponent of the bracket on the left.)

29. A. Erdelyi et al., Tables of Integral Transforms (McGraw Hill, New York, 1964), 4.14(33), 4.14(35).

30. Ref. 20, Section 10.15.

31. Parallel arguments have been made by M. Kugler, Phys. Rev. Letters 21, 570 (1968). See also F. Wagner, CERN preprint Th 978, Dec. 1968, unpublished. B. Margolis, McGill University (private communication) has also investigated the impact parameter behavior of the model.

32. As noted by Veneziano in Ref. 1, it is possible to consider an infinite sum of terms of the form $\frac{\Gamma(M - x)\,\Gamma(N - x)}{\Gamma(M + N - K - x - y)} + (M \longleftrightarrow N)$. As we have pointed out earlier (see J. Yellin, Phys. Rev. to be published), Eq. (1.2), the term with $M = N = K = 1$, is the only term of the series which we can expect to yield all positive widths, as can easily be seen by drawing a picture of the general pole residue vs the physical region in $\cos\Theta$. It is an interesting unanswered question whether (1.2) is the only convergent series of Veneziano terms with all positive widths.

Table B.1. Coefficients of the Legendre polynomials contained in the model amplitude, normalized to $L = N = 1$. For $N = 4$, $c = 4^{-4}/\Gamma(4)$.

L \ N	0	1	2	3	4
4					$c \cdot 7^3 \cdot 8/5$
3				$\frac{25}{16}$	$c \cdot 7^3 \cdot 8/5$
2			$\frac{3}{2}$	$\frac{25}{16}$	$c \cdot 112/3$
1		$\underline{1}$	$\frac{3}{2}$	$\frac{5}{16}$	$c \cdot 208/5$
0	0	$\underline{1}$	0	$\frac{5}{16}$	$c \cdot 64/15$

FIGURE CAPTIONS

Fig. 1. Singularity structure of the partial wave amplitude of even signature $a_2^+(J,\tau)$ in the Re J, Re τ plane. Note that the series defining $a_2^+(J,\tau)$, (3.5), converges only conditionally below the double line.

Fig. 2. Singularity structure of the partial wave amplitude of odd signature $a_1^-(J,\tau)$ in the Re J, Re τ plane. Note that the series defining $a_1^-(J,\tau)$, (3.7), diverges beyond the double line marking the leading Regge trajectory.

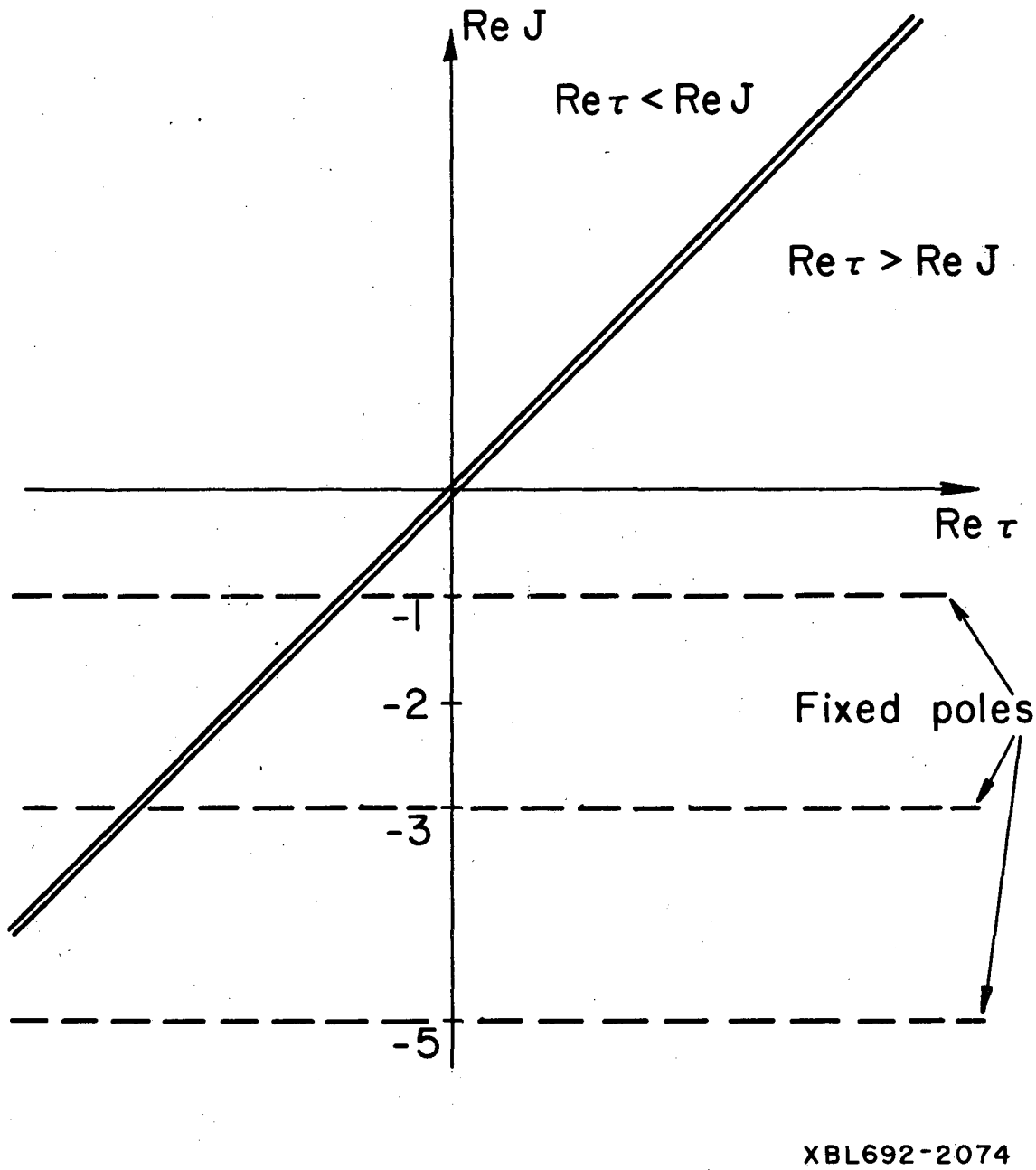

Fig. 1

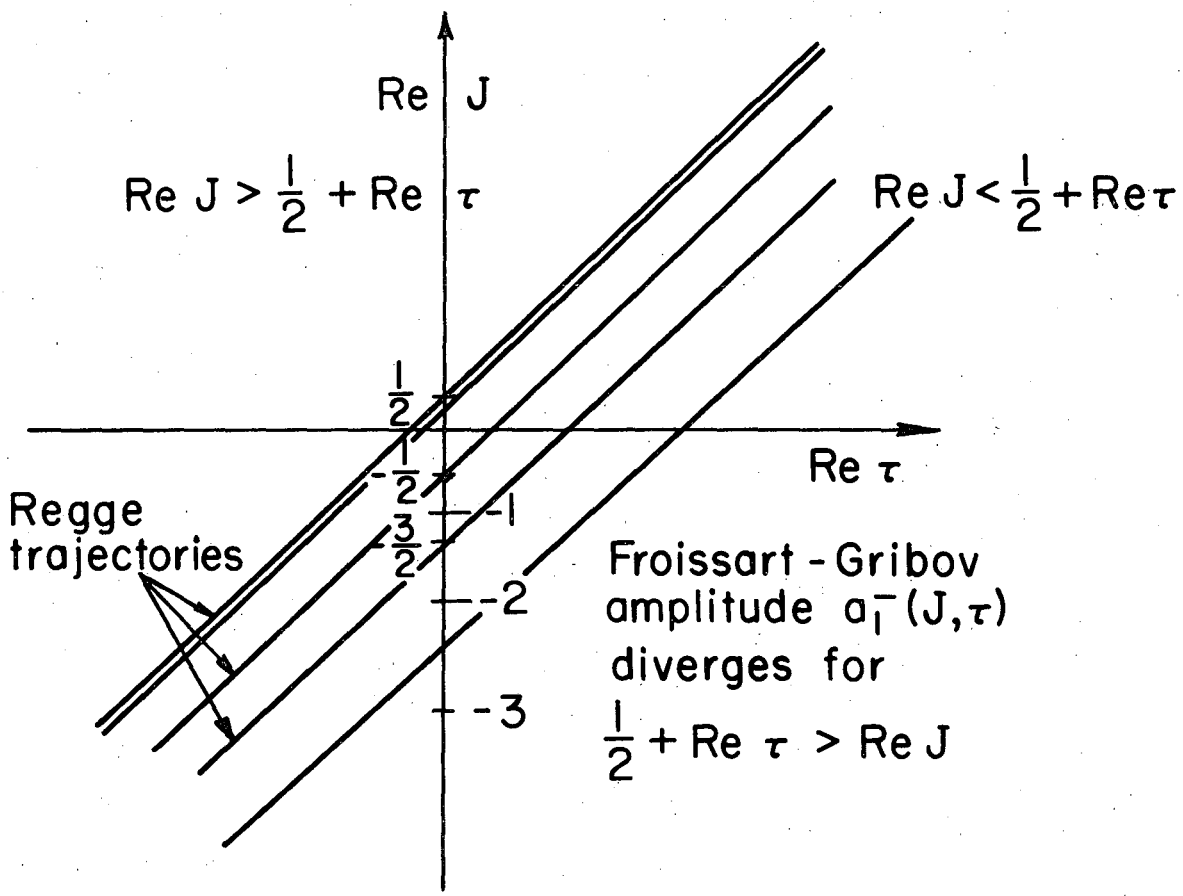

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Fig. 2

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