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J. H. Weis

February 27, 1969

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# UNIQUENESS OF THE PARAMETERIZATION OF REGGE RESIDUES AND TRAJECTORIES FOR SMALL $t$*

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## ABSTRACT

The small $t$ expansion of Regge residues $\beta(t)$ and trajectory functions $\alpha(t)$ for arbitrary external masses and spins is discussed. We assume the usual analyticity properties of helicity amplitudes, the factorization of Regge residues, and the requirement of consistency with coupling to unequal-mass channels. The leading $t$ behavior (i.e., the lowest permissible power of $t$ and its coefficient) of $\alpha(t)$ and $\beta(t)$ for the trajectories in a "family" is shown to be uniquely determined by the leading $t$ behavior of the highest-lying trajectory(-ies) in the family. It is also shown that certain nonleading terms in the expansions of $\alpha(t)$ and $\beta(t)$ are uniquely determined by higher-lying trajectories. The relationship of various recently proposed parameterizations of $\alpha(t)$ and $\beta(t)$ to our general results is discussed. In particular, it is shown that the parameterization given by Cosenza, Sciarrino, and Toller has the most general leading $t$ behavior permitted for residues.

# I. INTRODUCTION

The behavior of the Regge pole expansion of the scattering matrix for small values of the momentum-transfer variable $t$ has been studied intensively in the past few years. Toller[1] first studied the case of equal-mass (elastic) scattering at $t \equiv 0$ and found that a single Lorentz pole satisfies the analyticity properties of the scattering matrix and corresponds to an infinite descending sequence of integrally spaced Regge poles with factorizable residues. Toller's original group-theoretical approach has since been generalized by Cosenza, Sciarrino, and Toller[2] to treat other mass configurations [equal (EE), equal-unequal (EU), and unequal (UU)] and small nonzero $t$. The group-theoretical method elegantly provides a solution to the conspiracy problem for arbitrary external spins and masses but leaves open the question of the uniqueness of the solution.

Recently it has been realized that direct application of the analyticity properties of scattering amplitudes and the factorization of Regge residues gives a method of solution which is in a sense complementary to the group-theoretical method. The analyticity-factorization method readily answers questions of uniqueness but provides only a clumsy method for explicit calculation of residues and trajectory functions. This method has been used to explicitly show that the unequal-mass residues and analyticity and factorization in EU scattering uniquely determine the equal-mass residues to be those of a single Lorentz pole when the equal-mass channel consists of two spinless particles,[3,4,5] nucleon plus antinucleon,[5] and two rho

mesons.[6] This author has shown[7] that the equal-mass residues must be those of a single Lorentz pole for arbitrary equal-mass channels by also using analyticity and factorization in EU scattering. (If only EE processes are considered, then there is the possibility of only one solution in addition to a single Lorentz pole if channels of arbitrarily high spin are considered.[7]) In Ref. 3 through 7 only the dominant residue at $t = 0$ was considered (we henceforth call this "$t \equiv 0$ behavior"). In this paper we extend the method to $t \not\equiv 0$.

We should first like to make a few remarks about our terminology. Following Bitar and Tindle,[8] we call the family of Regge poles, integrally spaced at $t = 0$, which is required by analyticity and factorization, a "Lorentz family" and reserve the name Lorentz pole for the special case of EE scattering and $t \equiv 0$. Also, following Ref. 2, we call the members of a Lorentz family "conspirators" and the analyticity and factorization conditions "conspiracy conditions." By "minimal kinematic behavior" (MKB) we shall mean the lowest power of $t$ (and its coefficient) that is allowed by the conspiracy conditions. For example, if $\beta(t) = t^x \gamma[1 + O(t)]$ where $x \geqslant x_o$, the MKB is $t^{x_o}\gamma$. The MKB is by definition a unique quantity.

Our basic assumptions, in addition to (1) the usual analyticity properties of helicity amplitudes,[9] are (2) the conspiracy conditions are satisfied by a sum of Regge poles,[10] (3) there is only one trajectory of given quantum numbers and therefore its residue factorizes, and (4) the solution must be consistent with nonvanishing coupling to

unequal-mass channels at $t = 0$. For simplicity we do not discuss internal symmetries except to give the additional conditions that statistics and charge conjugation place on equal-mass residues. Furthermore, since trajectories of opposite signature $(\tau)$ give relatively imaginary contributions to the scattering amplitude, they must individually satisfy the conspiracy conditions. Thus by assumption (3) we consider at most a parity doublet for each $\alpha(t)$.

Using assumptions (1) to (4) we derive the MKB of the Regge residues and trajectories (and of the differences of these quantities for parity doublets). We prove that the MKB for all members of a Lorentz family is determined uniquely by the MKB of the leading member(s).[11] We also consider the general form of the power series expansion of residues and trajectories and show that certain derivatives are uniquely determined by derivatives of higher trajectories. This result is useful for testing the generality of proposed parameterizations; any parameterization which has only the specified derivatives constrained is the most general possible parameterization. We give our results in terms of particular linear combinations of the usual residues - these "crossed residues" are defined by Eq. (6). The usefulness of the crossed residues (particularly for equal-mass channels) will become clear in the course of the derivations. Throughout the paper particular attention is paid to the requirements of parity conservation.

At this point we should like to warn the reader that actual trajectories could have quite different behavior than that given here under certain "unphysical" circumstances. Firstly, if there is

"accidental" crossing of trajectories [a violation of assumption (3)], the small $t$ behavior of the crossing trajectories and lower members of the family can be entirely different. Secondly, if a trajectory couples only to a restricted type of channel [violating assumption (4)], the small $t$ behavior and family relationships may be entirely different. As a simple example, if a trajectory couples only to spinless equal-mass channels, then only one trajectory (or an arbitrary number of such trajectories) satisfies the (trivial) conspiracy conditions.[12] We believe that a sufficient condition for this not to occur is that not all unequal-mass residues be identically zero. The reader should notice that here we assume the stronger condition of nonvanishing coupling to unequal-mass channels at $t = 0$.

In Section II we introduce our notation and derive the MKB for leading trajectories. We treat all mass configurations in a unified manner. The leading trajectories have been discussed before by many authors,[13-18] but in no single reference have all the conditions on the residues and trajectories been given. This section will also serve to introduce the reader to the type of arguments made in the following sections. In Sections III, IV, and V we discuss respectively the unequal-mass (U), equal-mass (E), and zero-mass (Z) residues for all members of the family. We derive the U residues by considering a UU process and then derive the E residues from these by using factorization for a EU process. In Sections II through V we always impose the analyticity properties in terms of s-channel helicity amplitudes. The small $t$ behavior is most naturally expressed in terms of s-channel amplitudes, since $t = 0$ is near the s-channel physical boundary.

For the E and U channels we are content to show the uniqueness of the MKB, since, as pointed out in the first paragraph, the analyticity-factorization method most easily answers this question. Therefore only the subset of conspiracy conditions necessary to show uniqueness is considered. For a proof that the MKB given satisfies all the conditions we rely on the work of Cosenza, Sciarrino, and Toller.[2] They have shown that their solution has this property and we shall show in Section VI that it also has the most general MKB (even though it has parallel trajectories). In Section VI the relationship of other proposed parameterizations[8,19,20] to our results is also discussed. We should point out here that Klein[19] has previously given the MKB for the U residues and discussed a number of aspects of general parameterizations.

## II. THE LEADING TRAJECTORIES

We consider the s-channel process $1 + 2 \to 3 + 4$ and the corresponding t-channel process $\bar{4} + 2 \to 3 + \bar{1}$. The contribution of a Regge pole to the t-channel c.m. helicity amplitude is given by[21]

$$H^{t}_{\mu_3\mu_1;\mu_4\mu_2}(s,t) = \frac{(-1)^{\mu-\mu'}[2\alpha(t)+1]}{4} \frac{\{\tau + e^{-i\pi[\alpha(t)-v]}\}}{\cos\pi[\alpha(t)-v]}$$

$$\times \beta^{P\tau}_{\mu_3\mu_1}(t)\, \beta^{P\tau}_{\mu_4\mu_2}(t)\, e^{-\alpha(t)-1}_{\mu\mu'}(z_t) \quad , \qquad (1)$$

where $\mu = \mu_4 - \mu_2$, $\mu' = \mu_3 - \mu_1$; $v$ is $0$ and $1/2$ for boson and fermion trajectories respectively; $P$ and $\tau$ are the parity and signature respectively. The e function is given by[22]

$$e^{-\alpha-1}_{\mu\mu'}(z) = \frac{e^{\frac{i\pi}{2}(\mu-\mu')}}{2\Gamma(-2\alpha)} [\Gamma(\mu-\alpha)\,\Gamma(-\mu-\alpha)\,\Gamma(\mu'-\alpha)\,\Gamma(-\mu'-\alpha)]^{\frac{1}{2}}$$

$$\times \left(\frac{z-1}{2}\right)^{\alpha} \left(\frac{z+1}{z-1}\right)^{\frac{\mu+\mu'}{2}} F(\mu-\alpha, \mu'-\alpha; -2\alpha; -\frac{2}{z-1}) \quad . \qquad (2)$$

It is convenient to define the reduced residues (suppressing the t dependence)

$$\bar{\beta}_{\mu_i\mu_j}{}^{P\tau} \equiv (2p_{ij})^{-\alpha} \left\{\frac{e^{-i\pi(\mu+1)}\Gamma(\mu-\alpha)\,\Gamma(-\mu-\alpha)[\tau + e^{-i\pi(\alpha-v)}]}{8\,\Gamma(-2\alpha-1)\cos\pi(\alpha-v)}\right\}^{\frac{1}{2}}$$

$$\times \beta_{\mu_i\mu_j}{}^{P\tau} \quad , \qquad (3)$$

where $i,j$ is $\bar{4},2$ or $3,\bar{1}$. Combining (1), (2), and (3), we obtain

$$H^{t}_{\mu_3\mu_1;\mu_4\mu_2}(s,t) = e^{+i\pi\mu'}\,\bar{\beta}_{\mu_4\mu_2}\,\bar{\beta}_{\mu_3\mu_1}\left(\frac{4\,p_{42}p_{31}}{x}\right)^{\alpha}$$

$$\times\ (1+x)^{\frac{\mu+\mu'}{2}}\ F(\mu-\alpha,\ \mu'-\alpha;\ -2\alpha;\ -x)\ , \tag{4}$$

where $x = \dfrac{2}{z_t - 1}$ .

The Regge pole contribution to the s-channel c.m. helicity amplitude is obtained from (4) and the crossing relation[23]

$$H^{s}_{\lambda_3\lambda_4;\lambda_1\lambda_2}(s,t) = \sum_{\{\mu\}}\left\{\prod_{i=1}^{4} d_{\mu_i\lambda_i}^{\ s_i}[\chi(s,t)]\right\}H^{t}_{\mu_3\mu_1;\mu_4\mu_2}(s,t)\ . \tag{5}$$

The parameterization of the residues is most conveniently given in terms of the "crossed residues", which are defined as follows,

$$R_{\lambda_i\lambda_j}^{\ P\tau}(t) = \sum_{\mu_i\mu_j} d_{\mu_i\lambda_i}^{\ s_i}(\chi_i^{\,o})\ d_{\mu_j\lambda_j}^{\ s_j}(\chi_j^{\,o})\ \bar{\beta}_{\mu_i\mu_j}^{\ P\tau}(t)\ , \tag{6}$$

where

$$\begin{aligned} \chi^{o}_{2,\bar{4}} &\equiv \lim_{t\to 0}\ \lim_{s\to\infty}\ \chi_{2,\bar{4}}(s,t)\ , \\ \chi^{o}_{\bar{1},3} &\equiv \lim_{t\to 0}\ \lim_{s\to\infty}\ [\pi - \chi_{\bar{1},3}(s,t)]\ . \end{aligned} \tag{7}$$

For the various external mass configurations we obtain:[9]

Unequal masses (U), $m_i \neq m_j$:

$$\cos \chi_i^{\,o} \equiv \epsilon_i = \epsilon_j = \pm 1, \text{ for } m_i \lessgtr m_j \quad , \tag{8a}$$

where $i,j$ is $\bar{4},2$ or $3,\bar{1}$ as usual;

Equal masses (E), $m_i = m_j \neq 0$:

$$\chi_i^{\,o} = \pi/2 \quad ; \tag{8b}$$

Zero masses (Z), $m_i = m_j = 0$:

$$\cos \chi_i = -\cos \chi_j = -1 \quad . \tag{8c}$$

From the requirement of parity conservation

$$\beta_{\mu_i \mu_j}^{\;P\tau} = \eta\eta_i\eta_j(-1)^{s_i+s_j-v}\,\beta_{-\mu_i-\mu_j}^{\;P\tau} \quad ,$$

we obtain

$$\bar{\beta}_{\mu_i \mu_j}^{\;P\tau} = \eta\eta_i\eta_j\, e^{-i\pi v}(-1)^{s_i+s_j-\mu}\,\bar{\beta}_{-\mu_i-\mu_j}^{\;P\tau} \tag{9}$$

and

$$R_{\lambda_i \lambda_j}^{\;P\tau} = \eta\eta_i\eta_j\, e^{-i\pi v}(-1)^{s_i+s_j-\lambda}\,R_{-\lambda_i-\lambda_j}^{\;P\tau} \quad , \tag{10}$$

where $\lambda = \lambda_i - \lambda_j$ and $\eta_i \left(= (-1)^{2s_i}\,\eta_{\bar{i}}\right)$ is the intrinsic parity of $i$. The "parity sequence" $\eta$ is equal to $P\tau$.

One further definition is useful in order to partially diagonalize the crossing matrix:

$$R_{s\lambda}{}^{P}(t) \equiv \sum_{\lambda_i\lambda_j} C(s_i s_j s;\ \lambda_i - \lambda_j \lambda)(-1)^{s_j-\lambda_j} R_{\lambda_i\lambda_j}{}^{P}(t) \quad , \tag{11}$$

$$\begin{aligned} &= \eta\eta_i\eta_{\bar{j}}\, e^{-i\pi v}(-1)^{s-\lambda} R_{s-\lambda}{}^{P}(t) \quad , \\ &\equiv P\eta_\lambda R_{s-\lambda}{}^{P}(t) = P\eta_{-\lambda}{}^{-1} R_{s-\lambda}{}^{P}(t) \quad . \end{aligned} \tag{12}$$

From now on the signature label will be suppressed. The spin $s$ in (11) is the "channel spin" previously introduced by several authors.[1,19] Quantities $\bar{\beta}_{s\mu}{}^{P}$ can be defined by an expression similar to (11).

Near the physical region boundary $H^s$ must vanish at least as rapidly as $\left(\sin\frac{\theta_s}{2}\right)^{|\lambda-\lambda'|}$. It can be shown that in general[24]

$$\chi_i(s,t) - \lim_{\substack{s\to\infty \\ t\to 0}} \chi_i(s,t) = f_{ij}(s,t)\left(\sin\frac{\theta_s}{2}\right)^{\frac{1}{2}} \quad , \tag{13}$$

where $f_{ij}(s,t)$ is regular at $t = 0$. From the identity

$$d_{\mu_i\lambda_i}{}^{s_i}(\chi_i) = \sum_{\nu_i} d_{\mu_i\nu_i}{}^{s_i}\left[\lim_{\substack{s\to\infty \\ t\to 0}} \chi_i{}^{o}\right] d_{\nu_i\lambda_i}{}^{s_i}\left[f_{ij}(s,t)\left(\sin\frac{\theta_s}{2}\right)^{\frac{1}{2}}\right] ,$$

it then follows that we may replace $\chi_i$ by $\chi_i{}^{o}$ [or $(\pi - \chi_i{}^{o})$] in (5) without changing the behavior of the left-hand side. Hence our basic

analyticity requirement may be expressed as[24]

$$\frac{1}{\left(\sin\frac{1}{2}\theta_s\right)^{|\lambda-\lambda'|}} \sum_{\{\mu\}} d_{\mu_4\lambda_4}^{s_4}(\chi_4^{\,o})\, d_{\mu_2\lambda_2}^{s_2}(\chi_2^{\,o})\, d_{\mu_3\lambda_3}^{s_3}(\pi - \chi_3^{\,o})$$

$$\times\; d_{\mu_1\lambda_1}^{s_1}(\pi - \chi_1^{\,o})\, H^{t}_{\mu_3\mu_1;\mu_4\mu_2}(s,t) \;=\; \mathrm{NS}\;, \tag{14}$$

where NS means nonsingular at $t = 0$.

We now consider the leading trajectories in a Lorentz family. We substitute (4) in (14), allowing for the possibility of parity doublets, and take the leading term as $s \to \infty$,

$$t^{-\frac{1}{2}|\lambda-\lambda'|} \sum_{P=\pm} \sum_{\{\mu\}} d_{\mu_4\lambda_4}^{s_4}(\chi_4^{\,o})\, d_{\mu_2\lambda_2}^{s_2}(\chi_2^{\,o})\, d_{\mu_3\lambda_3}^{s_3}(\pi - \chi_3^{\,o})$$

$$\times\; d_{\mu_1\lambda_1}^{s_1}(\pi - \chi_1^{\,o})\, e^{i\pi\mu'}\, \bar{\beta}_{\mu_4\mu_2}^{\;P}(t)\, \bar{\beta}_{\mu_3\mu_1}^{\;P}(t)\, s^{\alpha^P}(t) \;=\; \mathrm{NS}\;. \tag{15}$$

We have used the fact that for all mass configurations, $\sin\frac{1}{2}\theta_s \xrightarrow[s\to\infty]{} (t/s)^{\frac{1}{2}}$. Using (11), (12), and the identity $d_{\mu\lambda}^{\,s}(\pi - \chi) = e^{-i\pi(s+\mu)}\, d_{\mu-\lambda}^{\,s}(\chi)$, we may rewrite (15) as

$$t^{-\frac{1}{2}|\lambda-\lambda'|} \sum_{P=\pm} P\; R_{s\lambda}^{\;P}(t)\, R_{s'\lambda'}^{\;P}(t)\, s^{\alpha^P}(t) \;=\; \mathrm{NS}\;. \tag{16}$$

This is our basic equation for leading trajectories.

If there is to be nonvanishing coupling at $t = 0$ for at least some channel as required by assumption (4), $R_{s\lambda}{}^{P}(0)\, R_{s'\lambda'}{}^{P}(0)$ must be nonzero for some $s$, $s'$, $\lambda$, and $P$. From (12), however, $R_{s\lambda}{}^{P}(0)\, R_{s'-\lambda}{}^{P}(0)$ is also nonzero and (16) would be violated by a factor $t^{-|\lambda|}$ unless there is some cancellation with a trajectory of opposite parity (for $\lambda \neq 0$). Therefore $\alpha^{-}(t) = \alpha^{+}(t) + O(t^{X})$ for some positive $X$ and parity doubling at $t = 0$ is necessary. For the moment we neglect the s dependence of (14) and return later to the determination of the power $X$.

We now write an expression like (16) except with $-\lambda'$ and use (12) to obtain

$$t^{-\frac{1}{2}|\lambda+\lambda'|}\left\{ R_{s\lambda}{}^{+}(t)\, R_{s'\lambda'}{}^{+}(t) + R_{s\lambda}{}^{-}(t)\, R_{s'\lambda'}{}^{-}(t)\right\} = \mathrm{NS} \quad . \tag{17}$$

We may add and subtract (16) and (17) after multiplying by an appropriate power of $t$ and obtain

$$t^{-\frac{1}{2}m_{\lambda\lambda'}}\, R^{\pm}_{s\lambda}(t)\, R^{\pm}_{s'\lambda'}(t) = \mathrm{NS} \quad , \tag{18}$$

where $m_{\lambda\lambda'} = \min(|\lambda - \lambda'|, |\lambda + \lambda'|)$. We now choose $s'$, $\lambda'$ such that $R^{+}_{s'\lambda'}(0) \neq 0$. Let $M \geqslant 0$ denote this value of $\lambda'$. From (18) it follows that

$$t^{-\frac{1}{2}m_{\lambda M}}\, R^{+}_{s\lambda}(t) = \mathrm{NS} \quad .$$

We may therefore write the MKB as

$$R^{+}_{s\lambda}(t) = t^{\frac{1}{2}|\lambda-M|}\, \gamma_{s\lambda}{}^{+,M}(t) + \eta_{\lambda}\, t^{\frac{1}{2}|\lambda+M|}\, \gamma_{s\lambda}{}^{+,-M}(t) \quad ,$$

where $\gamma_{s-\lambda}{}^{+,-M}(t) = \gamma_{s\lambda}{}^{+,M}(t)$ by (12). Similar expressions may be obtained for $R^{+}_{s'\lambda'}$ and $R^{-}$; they must both have the same $M$ to be consistent with (16). Therefore we have the MKB:

$$R_{s\lambda}{}^{P}(t) = t^{\frac{1}{2}|\lambda-M|}\, \gamma_{s\lambda}{}^{PM}(t) + P\eta_{\lambda}\, t^{\frac{1}{2}|\lambda+M|}\, \gamma_{s\lambda}{}^{P-M}(t) \quad , \tag{19}$$

where $\gamma_{s-\lambda}{}^{P-M}(t) = \gamma_{s\lambda}{}^{PM}(t)$ .

Returning to (16), we now obtain

$$\begin{aligned} t^{-\frac{1}{2}|\lambda-\lambda'|} \Big\{ & t^{\frac{1}{2}(|\lambda-M|+|\lambda'-M|)}\Big[\gamma_{s\lambda}{}^{+,M}(t)\, \gamma_{s'\lambda'}{}^{+,M}(t) - \gamma^{-,M}_{s\lambda}(t)\, \gamma^{-,M}_{s'\lambda'}(t)\Big] \\ & + \eta_{\lambda}\eta_{\lambda'}\, t^{\frac{1}{2}(|\lambda+M|+|\lambda'+M|)}\Big[\gamma^{+,-M}_{s\lambda}(t)\, \gamma^{+,-M}_{s'\lambda'}(t) - \gamma^{-,-M}_{s\lambda}(t)\, \gamma^{-,-M}_{s'\lambda'}(t)\Big] \\ & + \eta_{\lambda'}\, t^{\frac{1}{2}(|\lambda-M|+|\lambda'+M|)}\Big[\gamma^{+,M}_{s\lambda}(t)\, \gamma^{+,-M}_{s'\lambda'}(t) + \gamma^{-,M}_{s\lambda}(t)\, \gamma^{-,-M}_{s'\lambda'}(t)\Big] \\ & + \eta_{\lambda}\, t^{\frac{1}{2}(|\lambda+M|+|\lambda'-M|)}\Big[\gamma^{+,-M}_{s\lambda}(t)\, \gamma^{+,M}_{s'\lambda}(t) + \gamma^{-,-M}_{s\lambda}(t)\, \gamma^{-,M}_{s'\lambda'}(t)\Big] \Big\} \\ & = NS \quad . \end{aligned} \tag{20}$$

The first two terms are nonsingular but the second two are not for $M \neq 0$. Therefore the last two quantities in brackets must vanish

sufficiently rapidly. We thus write $\gamma_{s\lambda}^{-,\pm M}(t) = i\,\gamma_{s\lambda}^{+,\pm M}(t) + O[t^{Y(s,\lambda)}]$, or alternatively,

$$\gamma_{s\lambda}^{P\pm M}(t) = e^{i\frac{\pi}{4}(1-P)}\left[\gamma_{s\lambda}^{\pm M}(t) + P g_{s\lambda}^{\pm M}(t)\, t^{Y(s,\lambda)}\right] . \tag{21}$$

The minimum power $Y$ is independent of $s$, since the power of $t$ required depends only on $\lambda$. It is easy to show that the power $Y$ is given by[25]

$$Y(\lambda) = \begin{cases} |\lambda|, & 0 \leqslant |\lambda| \leqslant M \\ M, & M \leqslant |\lambda| \end{cases} . \tag{22}$$

We notice that the two terms in (19) differ precisely by the power $Y(\lambda)$ and therefore $g^{\pm M}$ may be absorbed into $\gamma^{\mp M}$ and (18), with the additional condition

$$\gamma_{s\lambda}^{P\pm M}(t) = e^{i\frac{\pi}{4}(1-P)}\,\gamma_{s\lambda}^{\pm M}(t) ,$$

has the general MKB. With this condition the last two terms of (20) vanish identically for $\alpha^{+}(t) = \alpha^{-}(t)$.

Finally we must examine the relationship between $\alpha^{+}(t)$ and $\alpha^{-}(t)$. For the required cancellation of the last two terms in (20) to take place, the s dependence cannot be too different for the two trajectories. It is clear that the power $X$ defined above cannot be smaller than $Y$. Since trajectory functions are helicity independent, $X$ must equal $M$.

We may now collect the above results for the MKB[26] (the definition of the $\gamma$'s is slightly modified):

$$H^{s}_{s\lambda;s'\lambda'}(s,t) \sim \left[\eta_{-\lambda'}\, t^{\frac{1}{2}(|\lambda-M|+|\lambda'-M|)} + \eta_{\lambda}\, t^{\frac{1}{2}(|\lambda+M|+|\lambda'+M|)}\right]$$

$$C_{s\lambda;s'\lambda'}(t)\, s^{\alpha(0)} ,$$

$$C_{s-\lambda;s'-\lambda'}(t) = \overline{C}_{s\lambda;s'\lambda'}(t) \quad ; \tag{23}$$

$$R_{s\lambda}{}^{P}(t) = e^{i\frac{\pi}{4}(1-P)}\left[t^{\frac{1}{2}|\lambda-M|}\gamma_{s\lambda}{}^{M}(t) + Pn_{\lambda}\, t^{\frac{1}{2}|\lambda+M|}\, \gamma_{s\lambda}{}^{-M}(t)\right] ,$$

$$\gamma_{s-\lambda}{}^{-M}(t) = \gamma_{s\lambda}{}^{M}(t) \quad ; \tag{24}$$

$$\alpha^{P}(t) = \alpha(t) + Pa(t)\, t^{M} . \tag{25}$$

The functions $C$, $\gamma$, $\alpha$, and $a$ are analytic in $t$ (or $t^{\frac{1}{2}}$ for fermion trajectories) at $t = 0$ with the possible exception of a factor $t^{\frac{1}{2}}$ common to all $\gamma$. The parameter $M$ which gives the absolute value of the helicity flip in leading order may be identified with the $O(3,1)$ quantum number[1] for EE processes.

For $M = 0$ it is evident that (20) is satisfied with only one leading trajectory and (24) and (25) show that the residues and trajectories of opposite parity are unrelated. In the absence of any dynamical requirement of a parity doublet, we henceforth assume that for $M = 0$ there is just one trajectory. For $M = 0$, in (24) $\gamma_{s\lambda}{}^{-M}$

should be replaced by $\gamma_{s-\lambda}{}^{0}$ and the requirement $\gamma_{s-\lambda}{}^{-M} = \gamma_{s\lambda}{}^{M}$ omitted. Similar modifications are to be understood throughout the paper.

At no time in the above derivation of the MKB have the external masses been mentioned, hence (22), (23), and (24) hold for all processes. The only dependence on the masses arises when Eq. (6) is now inverted to solve for the ordinary reduced residues $\overline{\beta}$. We find, changing the notation slightly:

Unequal masses:

$$U_{\overline{\beta}_{\mu_i\mu_j}}{}^{P}(t) = e^{i\frac{\pi}{4}(1-P)} \Big[ t^{\frac{1}{2}|\epsilon_i\mu-M|} \gamma_{\mu_i\mu_j}{}^{M}(t)$$

$$+ \eta\eta_i\eta_j e^{-i\pi v\epsilon_i} (-1)^{s_i+s_j-\mu} t^{\frac{1}{2}|\epsilon_i\mu+M|} \gamma_{\mu_i\mu_j}{}^{-M}(t)\Big] , \qquad (26)$$

where $\gamma^{-M}_{-\mu_i-\mu_j}(t) = \gamma_{\mu_i\mu_j}{}^{M}(t)$ . The factor $\epsilon_i(\epsilon_i = \pm 1$ for $m_i \lesseqgtr m_j)$ takes into account the eccentric behavior (8a) of the crossing angles.

Equal masses:

$$E_{\overline{\beta}_{\mu_i\mu_j}}{}^{P}(t) = e^{i\frac{\pi}{4}(1-P)} \sum_{m_i m_j} d_{\mu_i m_i}{}^{s_i}(\tfrac{\pi}{2})\, d_{\mu_j m_j}{}^{s_j}(\tfrac{\pi}{2})$$

$$\times \Big[ t^{\frac{1}{2}|m-M|} \gamma_{m_i m_j}{}^{M}(t) + \eta\eta_i\eta_j e^{-i\pi v}(-1)^{s_i+s_j-m} t^{\frac{1}{2}|m+M|} \gamma^{-M}_{m_i m_j}(t)\Big] , \qquad (27)$$

where $\gamma^{-M}_{-m_i-m_j}(t) = \gamma_{m_i m_j}{}^{M}(t)$ and $m = m_i - m_j$.

Zero masses:

$$Z_{\bar{\beta}_{\mu_i\mu_j}}{}^{P}(t) = e^{i\frac{\pi}{4}(1-P)} \left[ t^{\frac{1}{2}|\nu+M|} \gamma_{\mu_i\mu_j}{}^{M}(t) + \eta\eta_i\eta_j \, e^{-i\pi\nu}(-1)^{s_i+s_j+\nu} \right.$$

$$\left. \times \; t^{\frac{1}{2}|\nu-M|} \gamma_{\mu_i\mu_j}{}^{-M}(t) \right] , \qquad (28)$$

where $\gamma^{-M}_{-\mu_i-\mu_j}(t) = \gamma_{\mu_i\mu_j}{}^{M}(t)$ and $\nu = \mu_i + \mu_j$. For minimal behavior the various $\gamma$ are finite at $t = 0$. In general they are analytic at $t = 0$ with the possible exception of an additional factor $t^{\frac{1}{2}}$ common to all residues of a family. For fermion trajectories $\alpha$ and $\beta$ are analytic functions of $t^{\frac{1}{2}}$ rather than $t$, as is allowed on general principles.

## III. UNEQUAL-MASS RESIDUES

In order to derive the MKB for U residues we apply the basic equation (14) to a UU process. For small t and large s we find

$$z_s = \frac{2st + s^2 - (m_1^2 + m_2^2 + m_3^2 + m_4^2)s + (m_1^2 - m_2^2)(m_3^2 - m_4^2)}{\mathcal{S}_{12}\mathcal{S}_{34}}$$

$$= \left\{1 + \frac{2t}{s}\left[1 + \frac{(m_4^2 - m_2^2)(m_3^2 - m_1^2)}{st}\right]\right\}\left[1 + O\left(\frac{1}{s}\right)\right] ,$$

and

$$\sin\frac{1}{2}\theta_s = \left(\frac{z_s - 1}{2}\right)^{\frac{1}{2}} \approx \left(\frac{t}{s}\right)^{\frac{1}{2}}\left(1 + \frac{(m_4^2 - m_2^2)(m_3^2 - m_1^2)}{st}\right) .$$

Furthermore, choosing $m_4 < m_2$, $m_3 < m_1$ for definiteness, we find

$$x = \frac{2}{z_t - 1} = \frac{(m_4^2 - m_2^2)(m_3^2 - m_1^2)}{st}\left[1 + O\left(t, \frac{1}{s}\right)\right] ,$$

and therefore

$$\frac{4\, p_{42}p_{31}}{x} \approx s + O(1) ,$$

$$\sin\frac{1}{2}\theta_s = \left(\frac{t}{s}\right)^{\frac{1}{2}}(1 + x)^{\frac{1}{2}}\left[1 + O\left(t, \frac{1}{s}\right)\right] .$$

We now insert the above expressions into (14) and obtain

$$\frac{1}{t^{\frac{1}{2}|\lambda-\lambda'|}(1 + x)^{\frac{1}{2}|\lambda-\lambda'|}} \sum_{P=\pm 1} P\, {}^{U}R^{P}_{s\lambda}\, {}^{U}R^{P}_{s'\lambda'} \left(\frac{4p_{42}p_{31}}{x}\right)^{\alpha} (1 + x)^{\frac{\lambda-\lambda'}{2}}$$

$$\times\, F(\lambda - \alpha, -\lambda' - \alpha; -2\alpha; -x) = NS .$$

Since $x \approx (ts)^{-1}$, it is evident this equation is untrue as it stands and conspirator trajectories at $\alpha_n(0) = \alpha - n$ must be introduced. Therefore the basic equation for UU scattering is

$$\frac{(1+x)^{\frac{\lambda-\lambda'}{2}-\frac{|\lambda-\lambda'|}{2}}}{t^{\frac{1}{2}|\lambda-\lambda'|}} \sum_{p=\pm} \sum_{n=0}^{\infty} P(-1)^n \; U_{R_{s\lambda}}{}^{P(n)}(t) \; U_{R_{s'\lambda'}}{}^{P(n)}(t)$$

$$\times \left(\frac{4p_{42}p_{31}}{x}\right)^{\alpha_n{}^P(t)} F(\lambda - \alpha_n{}^P(t), -\lambda' - \alpha_n{}^P(t); -2\alpha_n{}^P(t); -x) = NS \; . \tag{29}$$

The factor $(-1)^n$ comes from (12) and the alternation of signature, $\tau_n = (-1)^n \tau$, necessary for obtaining the same phase of the amplitude.

As in Section II, we first determine the $\gamma^{\pm M(n)}(t)$. For this calculation we may neglect the parity dependence of $\alpha_n{}^P(t)$ [and hence set $\alpha_n{}^P(t) = \alpha - n$]. If one imagines solving for the $U_R{}^{P(n)}$ by iteration starting with (24), one easily sees that $\gamma^{\pm M(n)}(0)$ is completely determined by $\gamma^{\pm M}(0)$. To determine the precise relationship we write the general form,

$$U_{R_{s\lambda}}{}^{P(n)}(t) = e^{i\frac{\pi}{4}(1-P)} \Big[ B_\lambda^{M\alpha(n)} \; t^{\frac{1}{2}|\lambda-M|} \; \gamma_{s\lambda}{}^{M}(0)$$

$$+ \; P\eta_\lambda \; B_\lambda^{-M\alpha(n)} \; t^{\frac{1}{2}|\lambda+M|} \; \gamma_{s\lambda}{}^{-M}(0) \Big] \; (2p_{ij})^n \; . \tag{30}$$

The B's are functions to be determined. They depend only on the helicity difference $\lambda$, as will soom become clear, and by (12) satisfy

$$B_{\lambda}^{M\alpha(n)} = B_{-\lambda}^{-M\alpha(n)} \quad . \tag{31}$$

We now insert (30) in (29) and consider first $\lambda \geqslant M$ and $0 \leqslant \lambda' \leqslant M$. The requirement that the coefficient of the singularity $(-x)^N$ vanish becomes

$$\sum_{n=0}^{N} B_{\lambda}^{M\alpha(n)} B_{\lambda'}^{M\alpha(n)} \frac{}{(N-n)!} \frac{\Gamma(\lambda - \alpha + N)}{\Gamma(\lambda - \alpha + n)} \frac{\Gamma(-\lambda' - \alpha + N)}{\Gamma(-\lambda' + \alpha + n)}$$

$$\times \frac{\Gamma(-2\alpha + 2n)}{\Gamma(-2\alpha + N + n)} = 0 \quad . \tag{32}$$

The solution of this set of equations for all $N$ is readily found to be[7]

$$B_{\lambda}^{M\alpha(n)} B_{\lambda'}^{M\alpha(n)} = \left[\frac{(-1)^n \Gamma(n - 2\alpha - 1)}{n!\, \Gamma(2n - 2\alpha - 1)}\right] \left[\frac{\Gamma(\lambda - \alpha + n)\, \Gamma(-\lambda' - \alpha + n)}{\Gamma(\lambda - \alpha)\, \Gamma(-\lambda' - \alpha)}\right] . \tag{33}$$

Taking $\lambda = \lambda' = M$ yields

$$B_{M}^{M\alpha(n)} = \left[\frac{(-1)^n \Gamma(n - 2\alpha - 1)\, \Gamma(M - \alpha + n)\, \Gamma(-M - \alpha + n)}{n!\, \Gamma(2n - 2\alpha - 1)\, \Gamma(M - \alpha)\, \Gamma(-M - \alpha)}\right]^{\frac{1}{2}} \quad .$$

Then from (33) it follows that

$$B_\lambda^{M\alpha(n)} = \frac{\Gamma(\lambda - \alpha + n)}{\Gamma(\lambda - \alpha)}$$

$$\left[\frac{(-1)^n\,\Gamma(n - 2\alpha - 1)}{n!\,\Gamma(2n - 2\alpha - 1)}\,\frac{\Gamma(M - \alpha)}{\Gamma(M - \alpha + n)}\,\frac{\Gamma(-M - \alpha + n)}{\Gamma(-M - \alpha)}\right]^{\frac{1}{2}} \quad \lambda \geqslant M \quad ,$$

$$B_\lambda^{M\alpha(n)} = \frac{\Gamma(-\lambda - \alpha + n)}{\Gamma(-\lambda - \alpha)} \tag{34}$$

$$\left[\frac{(-1)^n\,\Gamma(n - 2\alpha - 1)}{n!\,\Gamma(2n - 2\alpha - 1)}\,\frac{\Gamma(M - \alpha + n)}{\Gamma(M - \alpha)}\,\frac{\Gamma(-M - \alpha)}{\Gamma(-M - \alpha + n)}\right]^{\frac{1}{2}} \quad 0 \leqslant \lambda \leqslant M \quad .$$

Finally we consider $\lambda \geqslant M$ and $\lambda' \leqslant -M$, for which the condition (29) becomes

$$t^{-\frac{1}{2}|\lambda-\lambda'|} \sum_{n=0}^{\infty} \Big[(-1)^n\, B_\lambda^{M\alpha(n)}\, B_{\lambda'}^{M\alpha(n)}\, \gamma_{s\lambda}^{M}(0)\, \gamma_{s'\lambda'}^{M}(0)$$

$$+ \; (-1)^n\, B_\lambda^{-M\alpha(n)}\, B_{\lambda'}^{-M\alpha(n)}\, \gamma_{s\lambda}^{-M}(0)\, \gamma_{s'\lambda'}^{-M}(0)$$

$$+ \; B_\lambda^{M\alpha(n)}\, B_{\lambda'}^{-M\alpha(n)}\, \gamma_{s\lambda}^{M}(0)\, \gamma_{s'\lambda'}^{-M}(0)\, a_n(0)\, \log s\Big]$$

$$\times \quad x^n\, F(\lambda - \alpha + n,\, -\lambda' - \alpha + \eta;\, -2\alpha + 2n;\, -x) \;=\; NS \quad .$$

We have suppressed some irrelevant phase factors. The solution is easily obtained by examining the three terms separately. Using (32) and (34), we find that $B^{-M\alpha(n)}$ is given by an expression just like (34) with $M$ replaced by $-M$ and that

$$a_n(0) = \chi^{M\alpha(n)}\, a(0) \quad ,$$

where

$$\chi^{M\alpha(n)} = (-1)^n \frac{\Gamma(M - \alpha + n)}{\Gamma(M - \alpha)} \frac{\Gamma(-M - \alpha)}{\Gamma(-M - \alpha + n)} \quad . \tag{35}$$

We now collect the above results and give the MKB for the crossed residues:[27]

$$U_{R_{s\lambda}}{}^{P(n)}(t) = e^{i\frac{\pi}{4}(1-P)} \Big[ t^{\frac{1}{2}|\lambda-M|}\, B_\lambda{}^{M\alpha(n)}\, \gamma_{s\lambda}{}^{M(n)}(t)$$

$$+ \; Pn_\lambda\, t^{\frac{1}{2}|\lambda+M|}\, B_\lambda{}^{-M\alpha(n)}\, \gamma_{s\lambda}{}^{-M(n)}(t) \Big] \; (2p_{ij})^n \quad , \tag{36}$$

where

$$\gamma_{s\lambda}{}^{M(n)}(t) = \gamma_{s-\lambda}{}^{-M(n)}(t)$$

and

$$\gamma_{s\lambda}{}^{\pm M(n)}(0) = \gamma_{s\lambda}{}^{\pm M}(0) \quad . \tag{37}$$

The ordinary residues are simply related to these through (11) and (6). For trajectories the relationship is

$$\alpha_n{}^P(t) = \alpha_n(t) + P\, a_n(t)\, t^M \quad , \tag{38}$$

where

$$\alpha_n(0) = \alpha - N, \qquad a_n(0) = \chi^{M\alpha(n)}\, a(0) \quad . \tag{39}$$

The signature is $\tau_n = (-1)^n \tau$ and $\eta = P\tau_n$. Equations (37) and (39) show that the leading behavior at $t = 0$ is completely determined by the leading trajectory, as asserted in Section I.

For a discussion of the derivatives of the $\gamma^{\pm M(n)}$ it is sufficient to consider $\lambda' = \pm M$. For example, if we take $\lambda' = M$ and $\lambda \geqslant 0$, we find the nth trajectory gives in order $(-x)^N$ a contribution to (29) of the form

$$\gamma_{s\lambda}{}^{M(n)}(t) \quad \gamma_{s'M}{}^{M(n)}(t) \quad s^{\alpha_n(t)+n} \quad (-x)^N \quad ,$$

as well as terms involving the $\gamma^{-M(n)}$ and $a_n$ multiplied by extra factors of $t$. By factorization we may consider these various terms separately, since they depend on different independent parameters $\gamma$. We may also treat the various combinations of derivatives separately as they involve different parameters. Considering terms proportional to $\gamma_{s'M}{}^{M(n)}(0) = \gamma_{s'M}{}^{M}(0)$, which by assumption (4) may be taken to be nonzero, we see that the first $N-1$ derivatives of $\gamma_{s\lambda}{}^{M(N)}(t)$ are determined. By considering terms proportional to logs we reach a similar conclusion for $\alpha_N$. Finally, by considering other values of $\lambda$ and $\lambda' = -M$, we conclude that the first $N-1$ derivatives of $\gamma^{\pm M(N)}(t)$, $\alpha_N(t)$, and $a_N(t)$ are uniquely determined. They depend on the first $N-1$ derivatives of residues and trajectory functions for $n \leqslant N - 1$.

## IV. EQUAL-MASS RESIDUES

We derive the E residues from the U residues (36) and analyticity and factorization for EU scattering. We consider the s-channel process $m_1 + m \to m_3 + m$ and assume $m_3 < m_1$ for definiteness. It is convenient to define the singular quantity,

$$y \equiv \frac{1}{z_t} = \frac{\mathcal{J}_{31}(t - 4m^2)^{\frac{1}{2}}}{2t^{\frac{1}{2}}\left[s + \frac{1}{2}\left(t - 2m^2 - m_1^{2'} - m_3^2\right)\right]} \approx \frac{i\, m(m_1^2 - m_3^2)}{t^{\frac{1}{2}}\, s} , \tag{40}$$

where $\mathcal{J}_{31}$ is the usual threshold-pseudothreshold factor. In terms of $y$ we have

$$x = \frac{2y}{1 - y} ,$$

$$\sin\frac{\theta_s}{2} = \left(\frac{t}{s}\right)^{\frac{1}{2}} (1 - y^2)^{\frac{1}{2}} \left[1 + O(t, \tfrac{1}{s})\right] ,$$

$$\frac{4p_{42}p_{31}}{x} \approx s(1 - y) .$$

The basic formula (14) must be modified slightly (but significantly) for EU scattering. The crossing angles for the E particles have the singular behavior

$$\sin(\chi_i^{\circ} - \chi_i) = \cos\chi_i \equiv \sin(\phi_i + \psi_i) , \tag{41}$$

where

$$\phi_i = f_{ij}(s,t)\left(\sin\frac{\theta_s}{2}\right)^{\frac{1}{2}} ,$$

and

$$\sin \psi_i = -y[1 + O(t, \tfrac{1}{s})].$$

We may now combine (41) and the other expressions above with (14) to obtain the basic equation

$$\frac{1}{t^{\frac{1}{2}|\lambda-\lambda'|}\,(1 - y^2)^{\frac{1}{2}|\lambda-\lambda'|}} \sum_{P=\pm} \sum_{n=0}^{\infty} \sum_{\mu\nu} P(-1)^n\, d_{\lambda\nu}{}^{s}(-\sin^{-1} y)$$

$$\times\; d_{\nu\mu}{}^{s}(-\tfrac{\pi}{2})\; E_{\bar{\beta}_{s\mu}}{}^{P(n)}(t)\; U_{R_{s'\lambda'}}{}^{P(n)}(t)\left(\frac{4p_{42}p_{31}}{2y}\right)^{\alpha_n{}^{P}(t)} (1 - y)^{\alpha_n{}^{P}(t)}$$

$$\times \left(\frac{1 + y}{1 - y}\right)^{\frac{\mu-\lambda'}{2}} F\left(\mu - \alpha_n{}^{P}(t),\; -\lambda' - \alpha_n{}^{P}(t);\; -2\alpha_n{}^{P}(t);\; -\frac{2y}{1 - y}\right)$$

$$= NS \quad . \tag{42}$$

In (42) we have neglected terms of order $t$ and $\frac{1}{s}$ times the most singular contribution in the argument of the first rotation matrix. For our purposes it will be sufficient to consider $\lambda' = \pm M$ so that $\gamma_{s'M}{}^{\pm M}(0)$ may be taken nonzero by assumption (4).

Before proceeding further we establish some additional notation and develop some of the properties of (42) by considering the solution for $t \equiv 0$. From (27) we obtain

$$E_{\bar{\beta}_{s\mu}}{}^{P(0)}(0) = e^{i\frac{\pi}{4}(1-P)} \left[d_{\mu M}{}^{s}(\tfrac{\pi}{2}) + P\eta_{-M}\, d_{\mu -M}{}^{s}(\tfrac{\pi}{2})\right] \gamma_{sM}{}^{M}(0) \quad . \tag{43}$$

Since $y \approx t^{-\frac{1}{2}} s^{-1}$, we must require that the coefficient of each power $y^N$ $(N > 0)$ vanish. From (42) we see the contribution of order $y^N$ from $n = 0$ can be written as at most $N$ powers of $J_y$ or $J_x$ between $\langle s\lambda|$ and $|s\pm M\rangle$ by using the identity

$$\sum_{\mu} d_{\nu\mu}{}^{s}(\tfrac{\pi}{2})\ (\mu)^k\ d_{\mu\pm M}{}^{s}(\tfrac{\pi}{2}) = \langle s\nu|(-J_x)^k|s \pm M\rangle \quad .$$

If (42) is solved iteratively for the crossed residues it is thus clear that $E_{R_{s\lambda}}{}^{P(n)}$ vanishes unless $\lambda = \pm M + k$, $-n \leqslant k \leqslant n$. Furthermore from the identity

$$(1 - y)^{\alpha} \left(\frac{1 + y}{1 - y}\right)^{\frac{\mu-\lambda'}{2}} F\left(\mu - \alpha,\ -\lambda' - \alpha;\ -2\alpha;\ -\frac{2y}{1 - y}\right)$$

$$= (1 + y)^{\alpha} \left(\frac{1 - y}{1 + y}\right)^{\frac{-\mu-\lambda'}{2}} F\left(-\mu - \alpha,\ -\lambda' - \alpha;\ -2\alpha;\ \frac{2y}{1 + y}\right) \quad ,$$

we see that the power of $\mu(J_x)$ is even (odd) when the power $N$ of $y$ is even (odd). The power of $J_y$ is also even (odd) when $N$ is even (odd) since $\sin^{-1} y$ is an odd function of $y$. Therefore $E_{R_{s\lambda}}{}^{P(n)}$ vanishes unless $n-k$ is even. After consideration of the parity properties (12) and use of (6) we find the unique solution

$$E_{\bar{\beta}_{s\mu}}{}^{P(n)}(0) = e^{i\frac{\pi}{4}(1-P)}\left[1 + P\eta_{\mu}(-1)^{M-s+n}\right] K_{s\mu}{}^{M\alpha(n)}\ (2p_{ij})^n$$

$$\times\ \gamma_{sM}{}^{M}(0) \quad , \tag{44}$$

where $K$ is given by[28]

$$K_{s\mu}^{M\alpha(n)} = \langle s\mu | R_y(\tfrac{\pi}{2}) \left\{ b_{sn}^{M\alpha(n)} \left(\frac{J_+}{2}\right)^n + b_{sn-2}^{M\alpha(n)} \left(\frac{J_+}{2}\right)^{n-2} + \cdots \right.$$

$$\left. + b_{s-n+2}^{M\alpha(n)} \left(\frac{J_-}{2}\right)^{n-2} + b_{s-n}^{M\alpha(n)} \left(\frac{J_-}{2}\right)^n \right\} | sM \rangle$$

$$= (-1)^{s+\mu+n} \; K_{s\mu}^{-M\alpha(n)} \quad . \tag{45}$$

The $b_{sk}^{M\alpha(n)}$ are given explicitly in Table I of Ref. 7 for $n \leqslant 3$; for general $n$ they can be obtained from the results of Ref. 1. In (44) the residues are those of a single Lorentz pole, since we have just shown[29] they are unique and Refs. 2, 9, and 19 prove that a single Lorentz pole satisfies our conditions.

We now derive the MKB for E residues from (42). To derive the residues we may set $\alpha_n^P(t) = \alpha - n$ and, as mentioned before, consider only $\lambda' = \pm M$. We solve the equation iteratively. Consider the coefficient of $y^N$. The contribution from $n = N$ is, for $\lambda' = \pm M$ respectively,

$$\left(\frac{2y}{2p_{42}}\right)^N \frac{1}{t^{\frac{1}{2}|\lambda - M|}} \left( E_{R_{s\lambda}}^{+(N)} \; B_M^{M\alpha(N)} + E_{R_{s\lambda}}^{-(N)} \; B_M^{M\alpha(N)} \right) \gamma_M^M(0) \quad ,$$

$$\eta_M \left(\frac{2y}{2p_{42}}\right)^N \frac{1}{t^{\frac{1}{2}|\lambda + M|}} \left( E_{R_{s\lambda}}^{+(N)} \; B_M^{-M\alpha(N)} - E_{R_{s\lambda}}^{-(N)} \; B_M^{-M\alpha(N)} \right) \gamma_M^M(0) \quad .$$

These will be equal to a function of the already determined residues for $n < N$. We can solve for $E_R{}^{\pm(N)}$ by multiplying one equation by a quantity proportional to $t$ raised to the power $||\lambda - M| - |\lambda + M||$ and then adding and subtracting. Using (34) we therefore obtain a unique result for $E_R{}^{P(N)}$.

We shall not burden the reader with the details of the calculation of the specific forms for the $E_{R_{s\lambda}}{}^{P(\eta)}$. It is not difficult to show that they are given by the following expressions,

$$E_{R_{s\lambda}}{}^{P(n)}(t) = e^{i\frac{\pi}{4}(1-P)} \left[ R_{s\lambda}{}^{M(n)}(t) + P\eta_{-M}(-1)^k R_{s\lambda}{}^{-M(n)}(t) \right] (2p_{ij})^n$$

$$= e^{i\frac{\pi}{4}(1-P)} \left[ R_{s\lambda}{}^{M(n)}(t) + P\eta_{\lambda} R_{s-\lambda}{}^{M(n)}(t) \right] (2p_{ij})^n \quad , \tag{46}$$

where $R^{M(n)}$ are given by

(i) $\lambda = M \pm n, \; M \pm (n-2), \; M \pm (n-4), \cdots, \; M \pm 1$ or $M$,

$$R_{s\lambda}{}^{M(n)}(t) = b_{s\lambda-M}{}^{M\alpha(n)} \langle s\lambda | \left(\frac{J_\pm}{2}\right)^{\pm(\lambda-M)} | sM \rangle \, \gamma_{sM}{}^{M(n)}(t)$$

$$= (-1)^n R_{s-\lambda}{}^{-M(n)} = (-1)^{M-\lambda} R_{s-\lambda}{}^{-M(n)} \quad ; \tag{47}$$

(ii) $\lambda = M \pm (n + k), \quad k \geqslant 0$ ,

$$R_{s\lambda}{}^{M(n)}(t) = b_{s\pm n}{}^{M\alpha(n)} \langle s\lambda | \left(\frac{J_\pm}{2}\right)^n | sM \pm k \rangle \, t^{\frac{1}{2}k} \, \gamma_{sM\pm k}{}^{M(n)}(t)$$

$$= (-1)^n R_{s-\lambda}{}^{-M(n)} = (-1)^{M\pm k-\lambda} R_{s-\lambda}{}^{-M(n)} \quad ; \tag{48}$$

(iii) $\lambda = M \pm (n-1),\ M \pm (n-3),\ M \pm (n-5), \cdots, M \pm 1$ or $M$,

$$R_{s\lambda}^{M(n)}(t) = \left[ {}^{+}r_{s\lambda}^{M\alpha(n)}\, \gamma_{sM+1}^{M(n)}(t) + {}^{o}r_{s\lambda}^{M\alpha(n)}\, \hat{\gamma}_{sM}^{M(n)}(t) + {}^{-}r_{s\lambda}^{M\alpha(n)}\, \gamma_{sM-1}^{M(n)}(t) \right] t^{\frac{1}{2}}$$

$$= (-1)^{n} R_{s-\lambda}^{-M(n)} = (-1)^{M+1-\lambda} R_{s-\lambda}^{-M(n)} \quad . \tag{49}$$

The $r_{s\lambda}^{M\alpha(n)}$ in (52) are uniquely determined functions of the indicated variables; their explicit forms are not necessary for our purposes. The $k$ appearing in (46) is the same as the power of $t$ in (47) to (49). In (47) to (49) we also have the constraint

$$\gamma_{s\lambda}^{\pm M(n)}(0) = \gamma_{s\lambda}^{\pm M}(0), \qquad \hat{\gamma}_{s\pm M}^{\pm M(n)}(0) = \hat{\gamma}_{s\pm M}^{\pm M(1)}(0) \quad , \tag{50}$$

which shows that the MKB is determined uniquely by the leading trajectory. There is, however, the additional parameter $\hat{\gamma}_{s\pm M}^{\pm M(1)}(0)$. Its occurrence is a special case of a more general formula to be discussed below.

The same trajectory constraint [(38) and (39)] is also obtained in this case, as it must be for consistency.

We now discuss the derivatives of the $\gamma^{\pm M(n)}$. As in the previous section we consider as an example $\lambda' = M$ and $\lambda \geqslant 0$. The $\underline{n}$th trajectory gives in order $s^{\alpha-N}$ a contribution to (42) of the form

$$\frac{1}{t^{\frac{1}{2}|\lambda-M|}} \; R_{s\lambda}{}^{M(n)}(t) \; \gamma_{s'M}{}^{M(n)}(t) \left(\frac{{}^4p_{42}p_{31}}{2y}\right)^{\alpha_n(t)+N} (2y)^N \quad .$$

Since $R_{s\lambda}{}^{M(n)}(t)$ behaves likes $\gamma_{s\lambda}{}^{M(n)}(t)\, t^k$, this contribution has the singularity $t^{-\frac{1}{2}(N+|\lambda-M|)+k}$. Therefore for the above ranges of the helicities the following derivatives are determined:

(i) $\lambda = M \pm (n - 2\ell)$ , $\ell = 0,1,\cdots, \left[\frac{N}{2}\right]$

first $N - \ell - 1$ derivatives;

(ii) $\lambda = M \pm (N + k)$ , $k \geqslant 0$ ,

first $N - 1$ derivatives;

(iii) $\lambda = M \pm (N - 2\ell - 1)$ , $\ell = 0,\cdots, \left[\frac{N-1}{2}\right]$ ,

first $N - \ell - 2$ derivatives.

We now see that the extra parameter in (49) arises because for $N = 1$ and $\ell = 0$ no derivatives are determined in range (iii). The determined derivatives depend on the derivatives (with corresponding or lower order) of trajectories and residues for $n \leqslant N - 1$ and helicities in the range $\lambda - (N - n)$ to $\lambda + (N - n)$.

The above results may also be obtained from consideration of EE processes if coupling to channels of arbitrary high spin and the $t \equiv 0$ result (43) are assumed. In this case the basic equation (14) becomes

$$t^{-\frac{1}{2}|\lambda-\lambda'|} \sum_{P=\pm} \sum_{n=0} \sum_{\mu\mu'} d_{\mu\lambda}{}^{s}(\tfrac{\pi}{2})\, d_{\mu'-\lambda'}{}^{s'}(\tfrac{\pi}{2})\; E_{\bar{\beta}_{s\mu}}{}^{P(n)}(t)$$

$$\times\; E_{\bar{\beta}_{s'\mu'}}{}^{-P(n)} \left(\frac{{}^{4}p_{42}{}^{p}{}_{31}}{x}\right)^{\alpha_n{}^{P}(t)} (1+x)^{\frac{\mu+\mu'}{2}}$$

$$\times\; F\Big(\mu - \alpha_n{}^{P}(t),\ \mu' - \alpha_n{}^{P}(t),\ -2\alpha_n{}^{P}(t);\ -x\Big) \;=\; NS,$$

since $\sin\frac{\Theta_s}{2} \approx t^{\frac{1}{2}}$. Actually the EE processes provide a more economical derivation of the E residues but uniqueness is more difficult to demonstrate.

Since equal-mass channels in practice always consist of two particles of the same kind or a particle plus antiparticle we state here the requirements imposed by statistics and charge conjugation. These requirements are easily derived from the properties of helicity states given by Jacob and Wick[30] and take the form

$$\bar{\beta}_{s\mu} \;=\; \tau\xi\; \bar{\beta}_{s-\mu}\;, \tag{51}$$

where $\xi$ is given by[1]

$$\xi \;=\; \begin{cases} (-1)^{I-\frac{Y}{2}} & \text{particle-particle} \\ G(-1)^{I} & \text{particle-antiparticle.} \end{cases}$$

This requirement in conjunction with (9) will force some of the equal mass residues to vanish.[1,2]

## V. ZERO-MASS RESIDUES

The zero-mass residues are first derived by considering a ZU process. It is then shown that the residues thus obtained also satisfy all the conspiracy conditions for ZE and ZZ processes and therefore along with the equal- and unequal-mass residues give a complete solution. In this case, in contrast to the E and U residues, the MKB of the lower trajectories is not uniquely determined by the leading trajectory; only the power of $t$ and not its coefficient is determined.

We consider $H^s$ for the ZU process with $m_2 = m_4 = 0$. In this case $\sin\frac{\theta_s}{2} \approx t^{\frac{1}{2}}$ and the basic equation (14) becomes

$$t^{-\frac{1}{2}|\lambda-\lambda'|} \sum_{P=\pm} \sum_{n=0} P(-1)^n \; Z_{\bar{\beta}_{-\lambda_4\lambda_2}}{}^{P(n)}(t) \quad U_{R_{s'\lambda'}}{}^{P(n)}(t)$$

$$\times \left(\frac{4p_{42}p_{31}}{x}\right)^{\alpha_n^P(t)} (1 + x)^{\frac{-\nu-\lambda'}{2}}$$

$$\times \; F(-\nu - \alpha_n^P(t), -\lambda' - \alpha_n^P(t); -2\alpha_n^P(t); -x) = NS \quad , \quad (52)$$

where $\nu = \lambda_4 + \lambda_2$ and we have assumed $m_3 < m_1$ for definiteness. From the relations

$$\frac{4p_{42}p_{31}}{x} = s + \frac{1}{2}\left(t - (m_1^2 + m_3^2) - \mathcal{J}_{31}\right) \quad ,$$

$$x = \frac{\mathcal{J}_{31}}{s + \frac{1}{2}\left(t - (m_1^2 + m_3^2) - \mathcal{J}_{31}\right)} \quad ,$$

and the formulae (26) and (28) for the $n = 0$ residues, it is easily seen that the leading trajectory by itself satisfies (52). Since $U_R P(n) \propto t^{-\frac{n}{2}} U_R P(0)$ it is clear that $Z_{\bar{\beta}} P(n) \propto t^{\frac{n}{2}} Z_{\bar{\beta}} P(0)$. The proportionality constant, however, is undetermined. Hence,

$$Z_{R_{s\lambda}} P(n)(t) = e^{i\frac{\pi}{4}(1-P)} \left[ t^{\frac{1}{2}|\lambda-M|} \gamma_{s\lambda}{}^{M(n)}(t) + P\eta_\lambda \, t^{\frac{1}{2}|\lambda+M|} \gamma_{s\lambda}{}^{-M(n)}(t) \right]$$

$$\times (2p_{ij})^n \quad , \tag{53}$$

where

$$\gamma_{s\lambda}{}^{M(n)}(t) = \gamma_{s-\lambda}{}^{-M(n)}(t) \quad .$$

There is no relationship between the $\gamma^{\pm M(n)}(0)$.

For the ZE process an equation similar to (52) holds except with an additional sum over $\mu$ and a $d_{\mu'-\lambda'}{}^{s'}(\frac{\pi}{2})$. In this case, $x \approx t^{\frac{1}{2}} s^{-1}$ and therefore successive terms in the expansion of the hypergeometric function have the form $(\mu t^{\frac{1}{2}})^k$. It is easily seen that these give no violation of (52) and thus the leading trajectory alone satisfies the equation. Similarly each conspirator for $n > 0$ satisfies (52) by itself. The ZZ processes are trivially satisfied by (53). We therefore conclude that (53) is consistent with arbitrary reactions.

Finally, we make two obvious remarks. If the two zero-mass particles are of the same kind or antiparticles, Eq. (51) must be imposed. At $t \equiv 0$ for ZE processes the result of the Lorentz family (or a single trajectory) is to give the single power behavior $s^{\alpha_{31}}$.

## VI. DISCUSSION

We first demonstrate that our results for the MKB of the equal- and unequal-mass residues are the same as those given by Cosenza, Sciarrino, and Toller[2] (Eqs. 5.6 to 5.13 of II).

In the unequal-mass case a straightforward calculation shows that the coefficient of $t^{\frac{1}{2}|\lambda \pm M|}$ is the same as that given in (36) and allows us to identify

$$\gamma_{\lambda_i \lambda_j}{}^{M}(0) = \sum_{j'} {}^{U}C_{j'\lambda}{}^{M\alpha} \; \gamma_{-\lambda_i \lambda_j}{}^{(U)j'}(p,0) \; ,$$

where ${}^{U}C_{j'\lambda}{}^{M\alpha}$ is a complicated function which we do not specify here. The equal-mass case is more complicated and we discuss it in somewhat greater detail. For example, for $M \neq 0$ we transcribe the formulae of Ref. 2 to our notation to find

$$E_{\bar{\beta}_{s\mu}} P^{(n)}(t) = {}^{E}C_{s}{}^{M\alpha} \sum_{\hat{s}s'} \gamma_{\hat{s}s}{}^{(E)s'+1}(t) \, t^{\frac{1}{2}\hat{s}} \; C(ss'\hat{s};\, \mu-\mu)(-1)^{s'+\mu}$$

$$\times \left(1 + \eta\eta_i\eta_{\bar{j}} \, e^{-i\pi v} \, (-1)^{M-\mu+\hat{s}+\eta}\right) C_{s'}{}^{\alpha} \; K_{s'\mu}{}^{M\alpha(n)} \; . \tag{54}$$

From (6) and (11) we then find

$$E_{R_{s\lambda}} P^{(n)}(t) = {}^{E}C_{s}{}^{M\alpha} \sum_{\hat{s}s'} \gamma_{\hat{s}s}{}^{(E)s'+1}(t) \, t^{\frac{1}{2}\hat{s}} \sum_{\mu} d_{\mu\lambda}{}^{s} \left(\tfrac{\pi}{2}\right)$$

$$\times \; C(ss'\hat{s},\, -\mu\mu) \left(1 + P\eta_{\mu}(-1)^{M-s+\hat{s}+\eta}\right) C_{s'}{}^{\alpha} \; K_{s'\mu}{}^{M\alpha(n)} \; .$$

From (45) and the identity

$$\sum_{\mu} (-1)^{s'+\mu}\, C(ss'\hat{s};\, \mu-\mu)\, d_{\mu\lambda}{}^{s}(\tfrac{\pi}{2})\, d_{\mu\nu}{}^{s'}(\tfrac{\pi}{2})$$

$$= (-1)^{s'+\nu}\, C(ss'\hat{s};\, \lambda-\nu)\, d_{o\lambda-\nu}{}^{\hat{s}}(\tfrac{\pi}{2}) \quad ,$$

we obtain

$$E_{R_{s\lambda}} P^{(n)}(t) = E_{C_s} M\alpha \sum_{\hat{s}s'} \sum_{\nu} r_{\hat{s}s}(E)\, s'+1(t)\, C'_{s'}{}^{\alpha}\; t^{\frac{1}{2}\hat{s}} (-1)^{s'+\nu}$$

$$\times\, C(ss'\hat{s};\, \lambda-\nu)\, d_{0\lambda-\nu}{}^{\hat{s}}(\tfrac{\pi}{2})$$

$$\times \Big[ b_{s'\nu-M}{}^{M\alpha(n)} \langle s'\nu | \left(\frac{J_{\pm}}{2}\right)^{\pm(\nu-M)} | s'M\rangle + P\eta_{-M}(-1)^{\hat{s}}\, b_{s'\nu+M}{}^{-M\alpha(n)}$$

$$\times\, \langle s'\nu | \left(\frac{J_{+}}{2}\right)^{\pm(\nu+M)} | s'M\rangle \Big] \quad . \tag{55}$$

We now extract the dominant behavior for small $t$ of the first term in (55). For the values (i) of $\lambda$ discussed in Sec. IV, we may take $\hat{s} = 0$; the Clebsch-Gordan coefficient then gives $s = s'$ and the form of (55) is the same as that of (47). For the values (ii) we must take $\hat{s} = k = \lambda - \nu$; using the explicit form of the Clebsch-Gordan coefficient we may verify that the form of (55) is the same as (48). Furthermore we identify

$$\gamma_{sM+k}{}^{M}(0) = {}^{E}C_{s}{}^{M\alpha} \sum_{s'=s-k}^{s+k} \gamma_{ks}{}^{(E)s'+1} \; C_{s'}{}^{\alpha}(-1)^{s'+M}$$

$$\times \; C(ss'k;\, M-k,\, -M) \;\; d_{ok}{}^{k}(\tfrac{\pi}{2}) \quad .$$

For the values (iii) we must take $\hat{s} = 1$; again the form is the same as (49), with $\gamma_{1s}^{(E)s'+1}(s' = s \pm 1,\, s)$ taking the place of our $\gamma_{sM\pm1}{}^{M}(0)$ and $\hat{\gamma}_{sM}{}^{M(1)}(0)$. Similarly the second term in (55) corresponds to the second term in (46). We have therefore shown that the residues of Ref. 2 have the most general MKB.

As stated in Section I, we may therefore conclude that the MKB given here satisfies all the conspiracy conditions (to leading order in $t$) for EE, EU, and UU processes. The parity-splitting term $a_n(0)$ which was not considered in Ref. 2, may also be shown to satisfy the conspiracy conditions by using the fact that the dominant term $\alpha_n(0)$ does so. In Section V it was shown that the zero-mass residues satisfy all the conspiracy conditions for ZE, ZU, and ZZ processes. Thus the MKB given above has all the required properties.

We have not investigated in detail the derivatives of the $\gamma^{\pm M(n)}$ obtained in Ref. 2. For the U residues a detailed knowledge of the functions $V_{jm}{}^{M\lambda(n)}(\zeta)$ is required. For the E residues, however, it is not difficult to show that the solution of Ref. 2 has precisely the maximum number of independent residue parameters as determined in Sec. IV. However, these parameters are linear combinations of derivatives of our $\gamma^{\pm M(n)}$ and we have not shown that the

relationships are linearly independent. Nevertheless it seems very likely to us that the solution of Ref. 2 is the most general for parallel trajectories.

A very compact model for the E and U residues has been given by Bitar and Tindle.[8] It assumes parallel trajectories and makes use of a group theoretical trick to show that the conspiracy conditions are satisfied. This model does not have the most general MKB. For example, whereas for the U residues the powers of $t$ are the same as above, there is only one independent residue $\gamma_{s\lambda}(0)$ [which may be taken as $\gamma_{sM}^{M}(0)$] instead of the possible $2s + 1$. Also $H^s$ has the particular form of a single $O(3,1)$ representation at $t = 0$ which is a result of a particular choice for the nonleading terms.

To our knowledge the parameterization by Klein[19] is the most general given thus far. He has given a power series expansion for the residues in the case without parity conservation and with parallel trajectories. The U residues have the general MKB given above. He also pointed out, as we have also seen above, that the leading $t$ behavior of the residues is independent of the parallelness or nonparallelness of the trajectories, and the $t \equiv 0$ behavior of residues is simply proportional to $(p_{ij})^{\alpha_o(0)}$. Furthermore he has given a discussion of parity splitting for trajectories $\alpha_n^P(t)$ and obtained a more general form of (38) and (39).

Recently Durand, Fishbane, and Simmons[20] have given power series expansion for residues and nonparallel trajectories for spinless particles and $M = 0$ which is alleged to be the most general possible. The U

residues and the trajectory functions have only the first N-1 derivatives determined and are hence the most general. However, the E residues also have the first N-1 derivatives determined as compared with our minimum of $\left[\frac{N-2}{2}\right]$. One can see that the E residues are indeed not the most general by comparing with Ref. 2 for the case of parallel trajectories. See also footnote 12.

It appears likely to us that the requirement that the parameterization of the residues and trajectories be consistent with factorization and coupling to unequal-mass channels forces it to be exactly the same as that obtained in Bethe-Salpeter models with general kernels and one[32] O(3,1) representation at $t \equiv 0$. Therefore the most practical method of obtaining a general parameterization may be to use the group-theoretical O(3,1)-symmetry-breaking off-mass-shell models developed by Domokos and Suranyi[33] and Frazer, Halpern, Lipinski, and Snider.[34]

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## FOOTNOTES AND REFERENCES

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25. This differs from the results of Ref. 14, where the behavior $Y(\lambda) = M$ is given.

26. These results have first been given by: Eq. (23), Ref. 8 and 19; Eq. (24), Ref. 15 (see, however, footnote 25); Eq. (25), Ref. 14.

27. Equation (36) without the consideration of parity has been given in Ref. 19. Equations (38) and (39) have the general form given in Ref. 19 and agree with the results obtained by several authors for $M = \frac{1}{2}, 1$.

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