Submitted to Physical Review Letters

UCRL-18820
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March 27, 1969

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SPIN-MATRIX POLYNOMIALS AND THE VENEZIANO FORMULA*

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March 27, 1969

We consider a partial wave expansion for $\pi\pi \to \pi\omega$ in terms of the spin-matrix polynomials $W_n(z) = \Gamma(z + n/2 + 1/2)/\Gamma(z - n/2 + 1/2)$ rather than the conventional Legendre polynomials. This leads naturally to Veneziano's formula when his supplementary condition $\alpha(s) + \alpha(t) + \alpha(u) = 2$ is invoked.

The appearance of the beta function $B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x + y)$ in Veneziano's representation of the $\pi\pi \to \pi\omega$ scattering amplitude[1] is interesting. A general representation of the scattering amplitude in which this functional form appears naturally could conceivably be of fundamental significance. We will start from a partial wave expansion in the s-channel

$$A(s,t,u) = \frac{1}{2} \sum_{\ell} (2\ell + 1) \left| a_\ell(s)[P_\ell(z) + P_\ell(-z)] + (s \longleftrightarrow t,u) \right. , \tag{1}$$

for the symmetric factor in Veneziano's amplitude. Instead of an expansion in terms of Legendre polynomials, we wish to consider one in the spin-matrix polynomials defined by

$$W_n(z) = \Gamma(z + n/2 + 1/2)/\Gamma(z - n/2 + 1/2)$$

which are polynomials of degree n in the variable z for integer n. Weber and Williams[2] have shown that

$$\nabla^k W_n(k/2 - 1/2) = n!\, \delta_{kn} \quad , \tag{2}$$

if k and n are both even or both odd, where ∇ is the difference operator

$$\nabla g(z) = g(z) - g(z - 1) \; .$$

The function $W_n(z)$ is called a spin-matrix polynomial because of its advantages in expansions of functions of $z = \underset{\sim}{s}\cdot\hat{\underset{\sim}{\theta}}$, where s_i are the three $(2s + 1) \times (2s + 1)$ spin matrices for spin s. In particular $W_{2(s+n)+1}(z)$ vanishes for $n \geqslant 0$ as a consequence of the Cayley-Hamilton theorem.

The Legendre polynomials in Eq. (1) can be expressed as an expansion in the spin-matrix polynomials in the form

$$P_\ell(z) + P_\ell(-z) = \sum_{n=0}^{\ell} b_n[W_n(2\rho z) + W_n(-2\rho z)] \tag{3}$$

where ρ can be an arbitrary function of s. We now apply Eq. (2) with respect to the variable $2\rho z$ with k even to get

$$b_k = \nabla^k \left[P_\ell\left(\frac{k - 1}{4\rho}\right) + P_\ell\left(-\frac{k - 1}{4\rho}\right)\right]\Big/2k! \quad . \tag{4}$$

The coefficients b_n for odd n drop out in Eq. (3) because of the symmetry property $W_n(-z) = (-)^n W_n(z)$, so there will be no damage done if

Eq. (4) defines b_n for all non-negative integer values of n. The partial wave expansion then takes the form

$$\frac{1}{2}\sum_{\ell}(2\ell+1)a_\ell(s)[P_\ell(z) + P_\ell(-z)] = \sum_{n=0}^{\infty} \frac{1}{n!} c_n(s)[W_n(2\rho z) + W_n(-2\rho z)] \tag{5}$$

where $c_n(s)$ is defined by

$$c_n(s) = \frac{1}{4}\sum_{\ell=0}^{\infty}(2\ell+1)\, a_\ell(s)\nabla^n\left[P_\ell\left(\frac{n-1}{4\rho}\right) + P_\ell\left(-\frac{n-1}{4\rho}\right)\right], \tag{6}$$

if it is safe to interchange the two sums. We have extended the range of the sum over ℓ here downward to zero because ∇^n will reduce a polynomial of degree ℓ to zero if $n > \ell$. We also take Eq. (6) as the definition of $c_n(s)$ for odd values of n since they drop out of the amplitude anyway. The form of Eq. (6) is interesting in that it relates $c_n(s)$ to discrete values of the entire Legendre series. We now consider the Sommerfeld-Watson transform of the right-hand side of Eq. (5). This is given by

$$\sum_{n=0}^{\infty} \frac{1}{n!} c_n(s)[W_n(2\rho z) + W_n(-2\rho z)]$$

$$= \frac{1}{-i2\pi}\int_C dn\, \frac{c(n,s)}{\Gamma(n+1)}\left[\frac{\Gamma(2\rho z + \frac{n}{2} + \frac{1}{2})}{\Gamma(2\rho z - \frac{n}{2} + \frac{1}{2})} + \frac{\Gamma(-2\rho z + \frac{n}{2} + \frac{1}{2})}{\Gamma(-2\rho z - \frac{n}{2} + \frac{1}{2})}\right]\frac{\pi}{\sin \pi n},$$

where the contour C passes around the positive real axis of the complex n-plane in the clockwise direction. We now distort the contour past the

leading pole of $c(n,s)$, which we suppose to be located at $n(s) = \alpha(s) - 1$, picking up the contribution

$$\beta(s)\,\Gamma\big(1 - \alpha(s)\big)\left[\frac{\Gamma\big(2\rho z + \alpha(s)/2\big)}{\Gamma\big(2\rho z - \alpha(s)/2 + 1\big)} + \frac{\Gamma\big(-2\rho z + \alpha(s)/2\big)}{\Gamma\big(-2\rho z - \alpha(s)/2 + 1\big)}\right] .$$

We can easily interpret this contribution to the amplitude $A(s,t,u)$ if we assume linear trajectories. We have $z = (t - u)/4k_ik_f$ so taking ρ to be bk_ik_f, where b is the slope, we get

$$\begin{aligned} 2\rho z &= -\alpha(u) - \alpha(s)/2 + (3a + b\Sigma)/2 \quad , \\ &= \alpha(t) + \alpha(s)/2 - (3a + b\Sigma)/2 \quad , \end{aligned}$$

where a is the intercept. Our contribution from the pole is then

$$\beta(s)\,\Gamma\big(1 - \alpha(s)\big)\left[\frac{\Gamma\big(\sigma/2 - \alpha(u)\big)}{\Gamma\big(1 + \sigma/2 - \alpha(s) - \alpha(u)\big)} + \frac{\Gamma\big(\sigma/2 - \alpha(t)\big)}{\Gamma\big(1 + \sigma/2 - \alpha(s) - \alpha(t)\big)}\right] \tag{7}$$

where $\sigma = 3a + b\Sigma = \alpha(s) + \alpha(t) + \alpha(u)$. Evidently $\sigma/2$ has to be an integer if we are to avoid poles at nonintegral values of the trajectory function in the t and u channels. Experimentally $\sigma/2$ is close to unity. The supplementary condition $\sigma/2 = 1$ was discussed by Veneziano and also by Mandelstam.[3] Taking $\beta(s)$ to be constant, say $\bar{\beta}/2\pi$, our symmetrized contribution to $A(s,t,u)$ becomes

$$\begin{aligned} A(s,t,u) = \frac{\bar{\beta}}{\pi}\,\{&B\big(1 - \alpha(s),\ 1 - \alpha(t)\big) + B\big(1 - \alpha(s),\ 1 - \alpha(u)\big) \\ &+ B\big(1 - \alpha(t),\ 1 - \alpha(u)\big)\} \end{aligned}$$

which coincides with Veneziano's proposal.

Evidently there are several steps in the deduction given here that will need to be filled in with some arguments more rigorous than mere enthusiasm. Pending the outcome of these investigations, it seems fair to say that the use of the spin-matrix polynomials instead of the Legendre polynomials can probably reduce the dynamics inherent in the Veneziano formula to just the pole in the complex n-plane.

It is a pleasure to thank Professor G. F. Chew for the hospitality of the Lawrence Radiation Laboratory.

FOOTNOTES AND REFERENCES

* This work was done under the auspices of the U.S. Atomic Energy Commission.

† AEC Postdoctoral Fellow.

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