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QUASILINEAR STABILIZATION OF THE TRANSVERSE INSTABILITY*

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April 2, 1969

ABSTRACT

The transverse instability is studied for the case of an infinite unmagnetized plasma with small electron temperature anisotropy. Growth rates and normal modes are found.

Relativistic corrections to the growth rate have a stabilizing effect. The distribution function satisfies a diffusion equation with an additional "flow term". The diffusion term causes the anisotropy to disappear and thus stabilizes the instability. The "flow term" is a perfect time derivative and makes no contribution to the net change in the distribution function between initial and final states which have negligible wave spectra.

0. INTRODUCTION

The Transverse Instability

The transverse instability is an instability in the low frequency transverse electromagnetic waves of a plasma, not in the high frequency transverse waves (modified light waves). It is driven by anisotropy either in temperature or in any of the even moments of the velocity distribution of either the electrons or the ions. The contribution of a group of resonant particles to the current is an essential part of the instability. Let α denote the electron temperature anisotropy, that is, the temperature difference divided by the mean temperature

$$\alpha = \Delta T/T \quad . \tag{0.1}$$

The instability occurs for very low wave numbers

$$k \lesssim \alpha^{\frac{1}{2}} \omega_e/c \sim (d/c)\, \alpha^{\frac{1}{2}} k_D \tag{0.2}$$

where ω_e is the electron plasma frequency, d is the mean electron thermal velocity, and k_D is the inverse of the electron Debye length. The instability has very slow growth rates

$$\gamma \sim \alpha\, kd \sim (d/c)\, \alpha^{3/2} \omega_e \quad . \tag{0.3}$$

For an unmagnetized plasma the instability is purely growing. The instability has been studied by various authors,[1-20] mainly for the case of a magnetized plasma.

Summary

An infinite unmagnetized plasma is considered. The electrons have a temperature maximum (or minimum) along the 3-axis and a temperature minimum (or maximum) in the plane perpendicular to the 3-axis. The case of small electron anisotropy is considered. According to Shapiro and Shevchenko[16] the validity of the quasilinear approximation requires

$$\gamma \ll kd \tag{0.4}$$

which for the transverse instability requires

$$\alpha \ll 1 \ . \tag{0.5}$$

We shall see that the nonlinear terms are of order $|\alpha|^{\frac{1}{2}}$ compared to the linear terms in the Vlasov equation. Ion anisotropy is assumed to be small enough so that its effects may be neglected. A set of ordered equations is found based on a multiple time scale expansion of the Vlasov and Maxwell equations in powers of $|\alpha|^{\frac{1}{2}}$. Growth rates for normal modes are found. The maximum growing modes have $\underset{\sim}{k}$ parallel to the axis of temperature minimum and $\underset{\sim}{E}_k$ parallel to the axis of temperature maximum. Relativistic corrections to the growth rate have a stabilizing effect.

The quasilinear stabilization is studied. The quasilinear approximation consists of a pair of coupled equations for the magnetic field, $\underset{\sim}{B}_k$, and the average particle distribution function, f_o.

We break f_o into two parts

$$f_o = f_o^{(o)} + f_o^{(2)} \tag{0.6}$$

where $f_o^{(o)}$ is defined to be the isotropic part of f_o at $t = 0$, and $f_o^{(2)}$ is the anisotropic part of f_o at $t = 0$. It is shown that $f_o^{(o)}$ is constant. Moreover, $f_o^{(2)}$ is shown to remain purely anisotropic for all time, that is the moments of 1, v^2, v^4, etc. with respect to $f_o^{(2)}$ remain zero throughout its evolution. We shall see that $f_o^{(2)}$ satisfies a diffusion equation with an additional "flow term" which is independent of $f_o^{(2)}$ and depends only on the magnetic wave spectrum at time t. The diffusion tensor is a positive operator. It thus produces dissipation and thereby has a stabilizing effect by tending to eliminate the anisotropy. It is shown that the only steady state solution is f_o isotropic. Each magnetic wave has a stabilizing effect on f_o through the flow term when the wave is growing, and a destabilizing effect when it is damping. However the flow term is a pure time derivative. The flow term makes no contribution to the <u>net</u> change in any even velocity moment between initial and final states which have negligible wave spectra. The net change in the velocity moments between such states is due entirely to the diffusion term.

I. ORDERING SCHEME

Ordering the Equations

The parameter in which we shall order our equations is the electron temperature anisotropy α

$$\alpha = (T_{\parallel} - T_{\perp})/T = (b^2 - a^2)/d^2 \tag{1.1}$$

where a and b are the perpendicular and parallel electron thermal velocities respectively. We let d be the thermal velocity which the electrons would have if their velocity distribution were isotropic

$$3d^2 = 2a^2(t = 0) + b^2(t = 0) \tag{1.2}$$

and T is the corresponding temperature. We shall assume that effects due to ion anisotropy may be neglected. The condition for the validity of this assumption will be given in the section on normal mode analysis.

We shall order the Maxwell and Vlasov equations. We define the space average of any quantity by

$$f_0(\underset{\sim}{v}, t) = \lim_{V \to \infty} \frac{1}{V} \int_V d^3r \, f(\underset{\sim}{r}, \underset{\sim}{v}, t) \, . \tag{1.3}$$

Henceforth in any equation in which V occurs, the limit $V \to \infty$ will be implied but not written. We define the deviation of any quantity by

$$\delta f(\underset{\sim}{r}, \underset{\sim}{v}, t) = f(\underset{\sim}{r}, \underset{\sim}{v}, t) - f_o(\underset{\sim}{v}, t) \, . \tag{1.4}$$

To simplify writing our equations we define two quantities for species

s

$$\delta R^s(\underset{\sim}{r}, t) = \frac{e_s}{m_s c}\left(\underset{\sim}{v} \times \delta \underset{\sim}{B}(\underset{\sim}{r}, t)\right) \cdot \frac{\partial}{\partial \underset{\sim}{v}} \tag{1.5}$$

$$\delta S^s(\underset{\sim}{r}, t) = \frac{e_s}{m_s} \delta \underset{\sim}{E}(\underset{\sim}{r}, t) \cdot \frac{\partial}{\partial \underset{\sim}{v}} \quad . \tag{1.6}$$

Normal mode analysis shows that the growing waves have growth rate

$$\gamma \sim \alpha\, kd \tag{1.7}$$

and have wave numbers k given by

$$kc \sim \alpha^{\frac{1}{2}}\, \omega_e \tag{1.8}$$

where ω_e is the electron plasma frequency. By $\alpha^{\frac{1}{2}}$ we shall mean the positive square root of the absolute value of α. Normal mode analysis further shows that $E_k(\text{longitudinal}) \sim E_k(\text{transverse})$. For typical spatial gradients and time scales we take $\nabla \sim k$ and $\partial/\partial t \sim \gamma$. We begin by defining the dimensionless parameter ϵ

$$\epsilon \sim \frac{1}{kd}\frac{e}{mc}\,\delta B \quad . \tag{1.9}$$

To estimate δR we must note that $\underset{\sim}{v} \times \partial/\partial \underset{\sim}{v}$ operating on a spherically symmetric function of velocity gives zero. The distribution function, f_o, is equal to a spherically symmetric part plus a nonsymmetric part which is of order $\alpha\, f_o$. Thus

$$\delta R\, \delta f \sim \epsilon\, kd\, \delta f \tag{1.10}$$

but

$$\delta R\, f_o \sim \alpha\, \epsilon kd\, f_o \quad . \tag{1.11}$$

We assume the two terms in Faraday's equation to be the same order

$$\frac{\partial}{\partial t}\,\delta\underset{\sim}{B} = -c\,\nabla \times \delta\underset{\sim}{E}$$

$$1 \;:\; 1 \;. \tag{1.12}$$

Using this we may estimate δE

$$\delta E \sim \alpha\left(\frac{d}{c}\right)\delta B \tag{1.13}$$

from which it follows that

$$\delta S\, f_o \sim \delta R\, f_o \tag{1.14}$$

but

$$\delta S\, \delta f \sim \alpha\, \delta R\, \delta f \;. \tag{1.15}$$

We estimate the relative magnitude of the two field terms in the Maxwell-Ampere equation, and assume that the current is the same order as the largest of these two terms.

$$\frac{\partial}{\partial t}\,\delta\underset{\sim}{E} - c\,\nabla \times \delta\underset{\sim}{B} + \sum_s 4\pi\, n_s e_s \int d^3v\, \underset{\sim}{v}\, \delta f^s = 0$$

$$\alpha^2\left(\frac{d}{c}\right)^2 \;:\; 1 \;:\; 1 \tag{1.16}$$

where n_s is the density of species s and

$$\omega_s^{\,2} = 4\pi\, n_s e_s^{\,2}/m_s \tag{1.17}$$

is the square of the plasma frequency for species s. From the above we conclude that

$$\delta f \sim \alpha \epsilon f_o \quad . \tag{1.18}$$

These estimates yield the following ordering for the Vlasov equation for δf

$$\frac{\partial}{\partial t} \delta f + \underset{\sim}{v} \cdot \nabla \delta f + \delta R f_o + \delta S f_o + \delta R \delta f + \delta S \delta f - \frac{1}{V} \int d^3 r (\delta R + \delta S) \delta f = 0$$

$$\alpha : 1 : 1 : 1 : \epsilon : \alpha\epsilon : (\epsilon : \alpha\epsilon) \tag{1.19}$$

and for Poisson's equation

$$\nabla \cdot \delta \underset{\sim}{E} = \sum_s 4\pi \, n_s \, e_s \int d^3 v \, \delta f^s$$

$$\alpha \left(\frac{d}{c}\right)^2 : 1 \quad . \tag{1.20}$$

This says that to lowest order in α the RHS is zero. We will find that this gives us an equation for $\underset{\sim}{E}_k$(longitudinal) in terms of $\underset{\sim}{B}_k$.

Finally, the above estimates yield the following ordering for the Vlasov equation for f_o

$$\frac{\partial}{\partial t} f_o = - \frac{1}{V} \int_V d^3 r (\delta R + \delta S) \delta f$$

$$1 : (\epsilon^2 : \alpha\epsilon^2) \quad . \tag{1.21}$$

This says that to lowest order in ϵ, f_o does not change on the γ^{-1} time scale. We will find that when f_o is expanded in powers of ϵ

$$f_o = f_o^{(o)} + f_o^{(1)} + f_o^{(2)} + \cdots \tag{1.22}$$

that $f_o^{(2)}$ does change on the γ^{-1} time scale. This change is caused by the $\delta R\ \delta f$ term.

Estimating How Large ϵ Becomes

From the energy appendix we have the following relations between energy densities

$$\frac{\partial}{\partial t}\mathcal{E}_{mag} \sim \frac{\partial}{\partial t}\mathcal{E}^{e}_{kin} \sim \alpha \frac{\partial}{\partial t}\mathcal{E}^{e}_{\perp kin} \quad . \tag{1.23}$$

Thus we have

$$\frac{\partial}{\partial t}\ \frac{1}{V}\int_V d^3r\ |\delta \underset{\sim}{B}|^2/8\pi \sim \alpha\ n\ m \frac{\partial}{\partial t} a^2 \quad . \tag{1.24}$$

We integrate and assume there is initially negligible energy in the wave

$$\left|\frac{e}{mc}\ \delta B\right|^2 \sim \frac{\alpha\ \omega_e^{\ 2}}{c^2}\left[a^2(t) - a^2(0)\right] \sim \frac{\alpha\ \omega_e^{\ 2}}{c^2}\ \alpha\ d^2 \sim \alpha\ k^2 d^2 \quad . \tag{1.25}$$

Comparing this with the definition of ϵ we see that

$$\epsilon \sim |\alpha|^{\frac{1}{2}} \quad . \tag{1.26}$$

Multiple Time Scale Expansion[21-24]

When we have a function having multiple time scales, $F(t, \alpha^{\frac{1}{2}}t, \alpha t, \cdots)$, we may introduce new variables, $t_o, t_1, t_2, \cdots$, such that

$$F(t, \alpha^{\frac{1}{2}}t, \alpha t, \cdots) = F(t_o, \alpha^{\frac{1}{2}}t_1, \alpha t_2, \cdots) \tag{1.27}$$

then

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t_o} + \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} + \cdots . \tag{1.28}$$

At the end of a calculation, we equate the various time scales by

$$t = t_o = t_1 = t_2 = \cdots . \tag{1.29}$$

Combining our various order estimates we may write

$$\delta f \sim \alpha^{3/2} f_o \tag{1.30}$$

$$\delta \underset{\sim}{B} \sim \alpha(nmd^2)^{\frac{1}{2}} \tag{1.31}$$

$$\delta \underset{\sim}{E} \sim \alpha^2(\frac{d}{c}) (nmd^2)^{\frac{1}{2}} \tag{1.32}$$

$$\frac{\partial}{\partial t} \sim \alpha^{3/2}(\frac{e}{mc}) (nmd^2)^{\frac{1}{2}} \tag{1.33}$$

$$kd \sim \alpha^{\frac{1}{2}}(\frac{e}{mc}) (nmd^2)^{\frac{1}{2}} . \tag{1.34}$$

We expand f_o' and $\partial/\partial t$, and the Fourier transforms of δf, $\delta \underset{\sim}{B}$, and $\delta \underset{\sim}{E}$ in powers of $\alpha^{\frac{1}{2}}$.

$$f_o = f_o^{(o)} + f_o^{(1)} + f_o^{(2)} + \cdots \tag{1.35}$$

$$f_k = f_k^{(3)} + f_k^{(4)} + \cdots \tag{1.36}$$

$$\underset{\sim}{B}_k = \underset{\sim}{B}_k^{(2)} + \underset{\sim}{B}_k^{(3)} + \cdots \tag{1.37}$$

$$\underset{\sim}{E}_k = \underset{\sim}{E}_k^{(4)} + \underset{\sim}{E}_k^{(5)} + \cdots \tag{1.38}$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t_3} + \frac{\partial}{\partial t_4} + \cdots \quad . \tag{1.39}$$

For initial conditions we choose

$$f_o^{(o)}(o) = \text{the isotropic part of } f_o(o) \quad . \tag{1.40}$$

Thus, $f_o^{(o)}(o)$ is defined to be an isotropic distribution, all of whose moments, $\langle 1 \rangle$, $\langle v^2 \rangle$, $\langle v^4 \rangle$, etc. are the same as those of $f_o(o)$.

$$f_o^{(2)}(o) = f_o(o) - f_o^{(o)}(o) \tag{1.41}$$

is the anisotropic part of $f_o(o)$. All other $f_o^{(n)}$ are taken to be zero at $t = 0$.

Substituting our expansions into the Fourier transforms of equations (1.12), (1.16), (1.19), (1.21), and (1.22), and collecting terms of the same order in α yields

$$\frac{\partial}{\partial t_3} \underset{\sim}{B}_k^{(2)} = - ic\underset{\sim}{k} \times \underset{\sim}{E}_k^{(4)} \tag{1.42}$$

$$-ic\underset{\sim}{k} \times \underset{\sim}{B}_k^{(2)} + \sum_s 4\pi n_s e_s \int d^3v \, \underset{\sim}{v} \, f_k^{s(3)} = 0 \tag{1.43}$$

$$i\underset{\sim}{k} \cdot \underset{\sim}{v} \, f_k^{(3)} + R_k^{(2)} f_o^{(2)} + S_k^{(4)} f_o^{(o)} = 0 \tag{1.44}$$

$$\sum_s 4\pi n_s e_s \int d^3v \, f_k^{s(3)} = 0 \tag{1.45}$$

$$\frac{\partial}{\partial t_3} f_o^{(o)} = 0 \tag{1.46}$$

$$\frac{\partial}{\partial t_3} f_o^{(1)} + \frac{\partial}{\partial t_4} f_o^{(o)} = 0 \tag{1.47}$$

$$\frac{\partial}{\partial t_3} f_o^{(2)} + \frac{\partial}{\partial t_4} f_o^{(1)} + \frac{\partial}{\partial t_5} f_o^{(o)} = -\frac{1}{V}\int \frac{d^3k}{(2\pi)^3} R_{-k}^{(2)} f_k^{(3)} . \tag{1.48}$$

II. THE QUASILINEAR EQUATIONS

The Evolution of $f_o^{(o)}$ and $f_o^{(1)}$

Equation (1.46) shows that $f_o^{(o)}$ is not a function of t_3. Integrate Eq. (1.47) on the t_3 time scale. Now $\partial f_o^{(o)}/\partial t_4$ is a constant on this time scale, hence $f_o^{(1)}$ will grow like

$$f_o^{(1)} = \text{constant} \cdot t_3 \quad . \tag{2.1}$$

Eventually $f_o^{(1)}$ would become comparable to $f_o^{(o)}$ contrary to the ordering assumptions. This unwanted secular behavior is suppressed by requiring

$$\frac{\partial}{\partial t_4} f_o^{(o)} = 0 \tag{2.2}$$

from which it follows that

$$\frac{\partial}{\partial t_3} f_o^{(1)} = 0 \quad . \tag{2.3}$$

Thus $f_o^{(o)}$ is not a function of t_4 and $f_o^{(1)}$ is not a function of t_3.

Let us integrate Eq. (1.48) on the t_3 time scale. Now $R_{-k}^{(2)} f_k^{(3)}$ is a function of t_3, however $\partial f_o^{(o)}/\partial t_5$ and $\partial f_o^{(1)}/\partial t_4$ are constants on the t_3 time scale. These would therefore lead to secular behavior in $f_o^{(2)}$. In order to suppress this secularity we must require

$$\frac{\partial}{\partial t_4} f_o^{(1)} + \frac{\partial}{\partial t_5} f_o^{(o)} = 0 \tag{2.4}$$

from which follows

$$\frac{\partial}{\partial t_3} f_o^{(2)} = -\frac{1}{V}\int \frac{d^3k}{(2\pi)^3} R_{-k}^{(2)} f_k^{(3)} \quad . \tag{2.5}$$

We integrate Eq. (2.4) on the t_4 time scale. Now $\partial f_o^{(o)}/\partial t_5$ is a constant on this time scale and would lead to secular behavior in $f_o^{(1)}$. To suppress this secularity we must require

$$\frac{\partial}{\partial t_5} f_o^{(o)} = 0 \tag{2.6}$$

from which follows

$$\frac{\partial}{\partial t_4} f_o^{(1)} = 0 \quad . \tag{2.7}$$

Thus $f_o^{(o)}$ is not a function of t_3, t_4, or t_5, and $f_o^{(1)}$ is not a function of t_3 or t_4. Repetition of this analysis with higher order equations leads to the conclusion that $f_o^{(o)}$ and $f_o^{(1)}$ are constant on all time scales. From our initial conditions it then follows that $f_o^{(1)}$ remains zero for all time.

The Equation for $f_o^{(2)}$

Since $f_o^{(o)}$ is constant and $f_o^{(1)}$ is zero, the lowest order piece of the distribution function that undergoes change is $f_o^{(2)}$. We solve Eq. (1.44) for $f_k^{(3)}$ and substitute this into Eq. (2.5).

$$\frac{\partial}{\partial t_3} f_o^{(2)} = -\frac{1}{V}\int \frac{d^3k}{(2\pi)^3} R_{-k}^{(2)} \frac{i}{\underset{\sim}{k}\cdot\underset{\sim}{v}} \left\{ R_k^{(2)} f_o^{(2)} + S_k^{(4)} f_o^{(o)} \right\} . \tag{2.8}$$

From Eqs. (1.5) and (1.6) we have

$$R_k^{(2)} = \frac{e}{mc}(\underset{\sim}{v} \times \underset{\sim}{B}_k^{(2)}) \cdot \frac{\partial}{\partial \underset{\sim}{v}} \tag{2.9}$$

$$S_k^{(4)} = \frac{e}{m}\left\{\underset{\sim}{E}_k^{(4)}(\text{trans}) + \underset{\sim}{E}_k^{(4)}(\text{long})\right\} \cdot \frac{\partial}{\partial \underset{\sim}{v}} \tag{2.10}$$

We may solve Eq. (1.42) for $\underset{\sim}{E}_k^{(4)}(\text{trans})$

$$\underset{\sim}{E}_k^{(4)}(\text{trans}) = -\frac{i}{k^2 c}\,\underset{\sim}{k} \times \frac{\partial}{\partial t_3}\,\underset{\sim}{B}_k^{(2)} . \tag{2.11}$$

We substitute these relations into Eq. (2.8) and consider the contribution of $\underset{\sim}{E}_k^{(4)}(\text{long})$ to the integrand. This contribution is

$$(\underset{\sim}{v} \times \underset{\sim}{B}_k^{(2)}) \cdot \frac{\partial}{\partial \underset{\sim}{v}} \frac{1}{\underset{\sim}{k}\cdot\underset{\sim}{v}}\,\underset{\sim}{k} \cdot \frac{\partial}{\partial \underset{\sim}{v}} f_o^{(o)} = (\underset{\sim}{v} \times \underset{\sim}{B}_k^{(2)}) \cdot \frac{\partial}{\partial \underset{\sim}{v}} \frac{1}{\underset{\sim}{k}\cdot\underset{\sim}{v}}\, 2\underset{\sim}{k}\cdot\underset{\sim}{v} \frac{\partial}{\partial v^2} f_o^{(o)}$$

$$= -2\,\underset{\sim}{B}_k^{(2)} \cdot (\underset{\sim}{v} \times \frac{\partial}{\partial \underset{\sim}{v}}) \frac{\partial}{\partial v^2} f_o^{(o)} = 0 \tag{2.12}$$

where we have made use of the fact that $f_o^{(o)}$ is spherically symmetric, and therefore a function of v^2. Thus $\underset{\sim}{E}_k^{(4)}(\text{long})$ does not affect the evolution of $f_o^{(2)}$. Equation (2.8) then becomes

$$\frac{\partial}{\partial t_3} f_o^{(2)} = -\frac{1}{V}\frac{e^2}{m^2c^2}\int \frac{d^3k}{(2\pi)^3}\,(\underset{\sim}{v} \times \underset{\sim}{B}_{-k}^{(2)}) \cdot \frac{\partial}{\partial \underset{\sim}{v}} \frac{1}{\underset{\sim}{k}\cdot\underset{\sim}{v}}$$

$$\left\{ i(\underset{\sim}{v} \times \underset{\sim}{B}_k^{(2)}) \cdot \frac{\partial}{\partial \underset{\sim}{v}} f_o^{(2)} + \frac{1}{k^2}(\underset{\sim}{k} \times \frac{\partial}{\partial t_3}\underset{\sim}{B}_k^{(o)}) \cdot \frac{\partial}{\partial \underset{\sim}{v}} f_o^{(o)} \right\} . \tag{2.13}$$

Note that the first term is odd in k and thus only the pole contributes to this term. The second term is even in k and thus only the principal value is nonzero.

Ordering the Relativistic Vlasov Equation

It is interesting to see the effect that relativistic corrections have on the linear growth rate. In order to calculate this we will need the relativistic version of Eq. (1.44). If we define the tensor

$$I_v = (1 - v^2/c^2)^{\frac{1}{2}} (I - \underset{\sim}{v}\underset{\sim}{v}/c^2) \tag{2.14}$$

then the relativistic Vlasov equation may be written[25]

$$\left\{\frac{\partial}{\partial t} + \underset{\sim}{v}\cdot \nabla + \frac{e}{m}\frac{\partial}{\partial \underset{\sim}{v}}\cdot I_v \cdot (\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B}/c)\right\} f = 0 \quad . \tag{2.15}$$

We use the relation

$$\frac{\partial}{\partial \underset{\sim}{v}}\cdot I_v = I_v \cdot \frac{\partial}{\partial \underset{\sim}{v}} - \frac{5}{c^2}\underset{\sim}{v}(1 - v^2/c^2)^{\frac{1}{2}} \tag{2.16}$$

to find that

$$\frac{\partial}{\partial \underset{\sim}{v}}\cdot I_v \cdot (\underset{\sim}{v} \times \underset{\sim}{B}) = -(1 - v^2/c^2)^{\frac{1}{2}}(\underset{\sim}{v} \times \frac{\partial}{\partial \underset{\sim}{v}})\cdot \underset{\sim}{B} \tag{2.17}$$

whereupon the relativistic Vlasov equation may be written

$$\left\{\frac{\partial}{\partial t} + \underset{\sim}{v}\cdot \nabla + \frac{e}{m}\frac{\partial}{\partial \underset{\sim}{v}}\cdot I_v \cdot \underset{\sim}{E} - \frac{e}{mc}(1 - v^2/c^2)^{\frac{1}{2}}\, \underset{\sim}{B}\cdot(\underset{\sim}{v} \times \frac{\partial}{\partial \underset{\sim}{v}})\right\} f = 0 \quad . \tag{2.18}$$

We order the Fourier transform of Eq. (2.18) in α but not in v/c.

$$i\underset{\sim}{k}\cdot\underset{\sim}{v}\, f_k^{(3)} + \frac{e}{m}\frac{\partial}{\partial \underset{\sim}{v}}\cdot I_v \cdot \underset{\sim}{E}_k^{(4)}\, f_o^{(o)} - \frac{e}{mc}(1 - v^2/c^2)^{\frac{1}{2}}\, \underset{\sim}{B}_k^{(2)}\cdot(\underset{\sim}{v} \times \frac{\partial}{\partial \underset{\sim}{v}}) f_o^{(2)}$$

$$= 0 \quad . \tag{2.19}$$

The Equation for $\underset{\sim}{B}_k^{(2)}$

We solve Eq. (2.19) for $f_k^{(3)}$ and substitute this into Eq. (1.43).

$$-ic\underset{\sim}{k} \times \underset{\sim}{B}_k^{(2)} + \sum_s \omega_s^2 \int d^3v\, \underset{\sim}{v}\, \frac{i}{\underset{\sim}{k}\cdot\underset{\sim}{v}} \left\{ \frac{\partial}{\partial \underset{\sim}{v}}\cdot I_v \cdot \underset{\sim}{E}_k^{(4)}\, f_o^{s(o)} \right.$$

$$\left. - \frac{1}{c}(1 - v^2/c^2)^{\frac{1}{2}}\, \underset{\sim}{B}_k^{(2)} \cdot (\underset{\sim}{v} \times \frac{\partial}{\partial \underset{\sim}{v}})\, f_o^{s(2)} \right\} = 0 \quad . \tag{2.20}$$

Let us consider the $\underset{\sim}{E}_k^{(4)}$(long) contribution. The $\underset{\sim}{k}\cdot\underset{\sim}{v}$ in the denominator is canceled because $f_o^{(o)}$ is a function of v^2. Thus we are left with an integral of the form

$$\int d^3v\, \underset{\sim}{v} \text{ function of } v^2 \tag{2.21}$$

which is identically zero. Thus $\underset{\sim}{E}_k^{(4)}$(long) does not contribute to the equation for the evolution of $\underset{\sim}{B}_k^{(2)}$.

Multiply Eq. (2.20) by $ic\underset{\sim}{k}\times$. Substitute $\underset{\sim}{E}_k^{(4)}$(trans) from Eq. (2.11). Introduce the dimensionless velocity

$$\underset{\sim}{u} = \underset{\sim}{v}/d \tag{2.22}$$

define the small parameter

$$\beta = d^2/c^2 \tag{2.23}$$

and let $f_o(\underset{\sim}{u})$ denote

$$f_o(\underset{\sim}{u}) = d^3 f_o(\underset{\sim}{v}) \quad . \tag{2.24}$$

Then Eq. (2.20) may be written

$$\left\{ -k^2c^2 I_k + \sum_s \omega_s^2 J_2^s + \sum_s \frac{\omega_s^2}{kd_s} J_1^s \frac{\partial}{\partial t_3} \right\} \cdot \underset{\sim}{B}_k^{(2)} = 0 \tag{2.25}$$

where

$$I_k = I - \hat{\underset{\sim}{k}}\,\hat{\underset{\sim}{k}} \tag{2.26}$$

$$J_1 = i \int d^3u \frac{1}{\underset{\sim}{k}\cdot\underset{\sim}{u}} \underset{\sim}{k} \times \underset{\sim}{u} \frac{\partial}{\partial \underset{\sim}{u}} \cdot I_v \times \hat{\underset{\sim}{k}} f_o^{(o)}(\underset{\sim}{u}) \tag{2.27}$$

$$J_2 = \int d^3u \frac{1}{\underset{\sim}{k}\cdot\underset{\sim}{u}} \underset{\sim}{k} \times \underset{\sim}{u} (1 - \beta u^2)^{\frac{1}{2}} (\underset{\sim}{u} \times \frac{\partial}{\partial \underset{\sim}{u}}) f_o^{(2)}(u) \quad . \tag{2.28}$$

The Eqs. (2.13) and (2.25) for the evolution of $f_o^{(2)}$ and $\underset{\sim}{B}_k^{(2)}$ form a closed set.

The Equation for $\underset{\sim}{E}_k^{(4)}$(long)

We solve Eq. (1.44) for $f_k^{(3)}$ and substitute this into Eq. (1.45). Substituting R and S from Eqs. (1.5) and (1.6) yields

$$\sum_s \omega_s^2 \int d^3v \frac{1}{\underset{\sim}{k}\cdot\underset{\sim}{v}} \left\{ \frac{1}{c}(\underset{\sim}{v} \times \underset{\sim}{B}_k^{(2)}) \cdot \frac{\partial}{\partial \underset{\sim}{v}} f_o^{s(2)} + \underset{\sim}{E}_k^{(4)} \cdot \frac{\partial}{\partial \underset{\sim}{v}} f_o^{s(o)} \right\} = 0 . \tag{2.29}$$

Consider the integral

$$\underset{\sim}{H}^s = \int d^3v \frac{1}{\underset{\sim}{k}\cdot\underset{\sim}{v}} \frac{\partial}{\partial \underset{\sim}{v}} f_o^{s(o)} = 2 \int d^3v \, \underset{\sim}{v} \frac{1}{\underset{\sim}{k}\cdot\underset{\sim}{v}} \frac{\partial}{\partial v^2} f_o^{s(o)} . \tag{2.30}$$

Now $f_o^{(o)}$ is isotropic, therefore $\underset{\sim}{H}$ must be proportional to $\underset{\sim}{k}$ since it is the only vector in $\underset{\sim}{H}$. Thus

$$\underset{\sim}{H}^s = (\hat{\underset{\sim}{k}} \cdot \underset{\sim}{H}^s)\, \hat{\underset{\sim}{k}} = \left(\frac{2}{k} \int d^3v \frac{\partial}{\partial v^2} f_o^{s(o)}\right) \hat{\underset{\sim}{k}} = h^s \hat{\underset{\sim}{k}} . \tag{2.31}$$

Consequently $\underset{\sim}{E}_k^{(4)}(\text{trans})$ does not contribute to Eq. (2.29) which may be written

$$\sum_s \omega_s^2 h^s E_k^{(4)}(\text{long}) = \sum_s \frac{\omega_s^2}{c} \underset{\sim}{H}_3^s \cdot \underset{\sim}{B}_k^{(2)} \tag{2.32}$$

where

$$\underset{\sim}{H}_3^s = \int d^3v \frac{1}{\underset{\sim}{k}\cdot\underset{\sim}{v}} \left(\underset{\sim}{v} \times \frac{\partial}{\partial \underset{\sim}{v}}\right) f_o^{s(2)} . \tag{2.33}$$

Thus $\underset{\sim}{E}_k^{(4)}(\text{long})$ is given once $\underset{\sim}{B}_k^{(2)}$ is known. Also, $\underset{\sim}{E}_k^{(4)}(\text{trans})$ is given in terms of $\underset{\sim}{B}_k^{(2)}$. However as we have seen $\underset{\sim}{E}_k^{(4)}(\text{long})$ is not needed to find the evolution of $f_o^{(2)}$ and $\underset{\sim}{B}_k^{(2)}$. The electric field has the role of a dependent variable. We shall

investigate the polarization of the electric field in the section on normal mode analysis, and thereafter we shall not concern ourselves with the electric field.

Henceforth the superscript (2) of $\underset{\sim}{B}_k^{(2)}$ and the subscript 3 of $\partial/\partial t_3$ will be implied but not written.

III. NORMAL MODE ANALYSIS

Coordinate Axes

Let $\hat{\underset{\sim}{3}}$ be a unit vector parallel to the axis of temperature symmetry. Let the subscript $\perp$ denote the component of a vector lying in the plane perpendicular to $\hat{\underset{\sim}{3}}$. Let $\hat{\underset{\sim}{1}}$ be a unit vector in the k, 3-plane and let $\hat{\underset{\sim}{2}}$ be a unit vector perpendicular to $\underset{\sim}{k}$ and $\hat{\underset{\sim}{3}}$. Define

$$\hat{\underset{\sim}{n}} = \hat{\underset{\sim}{k}} \times \hat{\underset{\sim}{2}} \quad . \tag{3.1}$$

Then $(\hat{\underset{\sim}{1}}, \hat{\underset{\sim}{2}}, \hat{\underset{\sim}{3}})$ and $(\hat{\underset{\sim}{k}}, \hat{\underset{\sim}{2}}, \hat{\underset{\sim}{n}})$ form two right handed orthogonal bases. We define velocity components

$$x = \hat{\underset{\sim}{k}} \cdot \underset{\sim}{u} \qquad y = \hat{\underset{\sim}{2}} \cdot \underset{\sim}{u} \quad \text{and} \quad z = \hat{\underset{\sim}{n}} \cdot \underset{\sim}{u} \quad . \tag{3.2}$$

Let θ denote the angle between the k-vector and the 3-axis, and denote

$$\mu = \hat{\underset{\sim}{k}} \cdot \hat{\underset{\sim}{3}} = \cos\theta \tag{3.3}$$

$$\xi = \sin\theta \quad . \tag{3.4}$$

Expansion of a BiMaxwellian in Powers of α

We shall find the normal model vectors and growth rates from Eq. (2.25) for the special case of a bimaxwellian f_o. We consider a bimaxwellian distribution function and expand the anisotropic part of the exponent

$$f_B(\underset{\sim}{u}) = (2\pi)^{-3/2} (d^3/a^2b) \exp -\frac{1}{2} \{u_\perp^2 d^2/a^2 - u_3^2 d^2/b^2\}$$

$$= f_M(d^3/a^2b) \{1 + \frac{1}{2} u_\perp^2(1 - d^2/a^2) + \frac{1}{2} u_3^2(1 - d^2/b^2) + \cdots\} \tag{3.5}$$

where f_M denotes the maxwellian

$$f_M(\underset{\sim}{u}) = (2\pi)^{-3/2} \exp - \frac{1}{2} u^2 \quad . \tag{3.6}$$

It follows from energy considerations that

$$(1 - d^2/a^2) = - \frac{1}{3} \alpha + O(\alpha^2) \tag{3.7}$$

$$(1 - d^2/b^2) = \frac{2}{3} \alpha + O(\alpha^2) \tag{3.8}$$

$$(d^3/a^2b) = 1 + O(\alpha^4) \tag{3.9}$$

therefore

$$f_B = f_M \{1 + \frac{\alpha}{6}(3u_3^2 - u^2) + O(\alpha^2)\} \quad . \tag{3.10}$$

From Eq. (3.10) we have to order α

$$f_o^{(o)} = f_M \tag{3.11}$$

$$f_o^{(2)} = f_M \frac{\alpha}{6}(3u_3^2 - u^2) \quad . \tag{3.12}$$

The moments of 1, u^2, u^4, etc. with respect to $f_o^{(2)}$ as given by Eq. (3.12) are all zero. Therefore $f_o^{(2)}$ is the purely anisotropic part and $f_o^{(o)}$ is the purely isotropic part of f_o just as our definition requires.

Evaluation of J_1 and J_2

Using the fact that $f_o^{(o)}$ is a function of u^2 only, we may write in general

$$J_1 = -i \int d^3u \; \frac{1}{\underset{\sim}{k}\cdot\underset{\sim}{u}}(\underset{\sim}{k} \times \underset{\sim}{u})(\hat{\underset{\sim}{k}} \times \underset{\sim}{u}) \left\{ (1 - \beta u^2)^{3/2} \, 2 \, \frac{\partial}{\partial u^2} - 5\beta(1 - \beta u^2)^{\frac{1}{2}} \right\} f_o^{(o)}(u^2) \quad . \tag{3.13}$$

Now J_1 is a symmetric tensor and thus must have the form

$$J_1 = j_1 \, I + j_o \, \hat{\underset{\sim}{k}} \, \hat{\underset{\sim}{k}} \tag{3.14}$$

but since $\hat{\underset{\sim}{k}} \cdot J_1$ is zero, J_1 must be of the form

$$J_1 = j_1 \, I_k \quad . \tag{3.15}$$

We calculate half the trace of J_1. Only the residue contributes to the x-integration which we calculate by passing under the pole. We denote

$$w^2 = y^2 + z^2 \tag{3.16}$$

then

$$j_1 = \frac{\pi}{2} \int dy \, dz \, w^2 \left\{ (1 - \beta w^2)^{3/2} \, 2 \, \frac{\partial}{\partial w^2} - 5\beta(1 - \beta w^2)^{\frac{1}{2}} \right\} f_0^{(o)}(w^2) \quad . \tag{3.17}$$

Integration by parts gives

$$j_1 = -\pi^2 \int_0^\infty dw^2 \, (1 - \beta w^2)^{\frac{1}{2}} \, f_o^{(o)}(w^2) \quad . \tag{3.18}$$

Thus j_1 is in general a negative constant. If we use the maxwellian $f_o^{(o)}$ from Eq. (3.11) and expand (3.18) in powers of β then

$$j_1 = -\left(\frac{\pi}{2}\right)^{\frac{1}{2}} \{1 - \beta + \cdots\} \quad . \tag{3.19}$$

Using the fact that $f_o^{(2)}$ is a function of u^2 and u_3^2 only, we may write

$$\underset{\sim}{u} \times \frac{\partial}{\partial \underset{\sim}{u}} = (\underset{\sim}{u} \times \hat{\underset{\sim}{3}}) \frac{\partial}{\partial u_3}^{*} \tag{3.20}$$

where the asterisk * denotes that u^2 is held constant during the u_3-differentiation. Thus J_2 may be written

$$J_2 = \int d^3u \, \frac{1}{\underset{\sim}{k} \cdot \underset{\sim}{u}} (\underset{\sim}{k} \times \underset{\sim}{u})(\underset{\sim}{u} \times \hat{\underset{\sim}{3}})(1 - \beta u^2)^{\frac{1}{2}} \frac{\partial}{\partial u_3}^{*} f_o^{(2)}(u^2, u_3^2) \quad . \tag{3.21}$$

The tensor J_2 is proportional to the distribution function anisotropy (in the general sense), and is proportional to α for the special case of a bimaxwellian distribution. Using the fact that the rest of the integrand is even in y, the tensor $(\hat{\underset{\sim}{k}} \times \underset{\sim}{u})(\underset{\sim}{u} \times \hat{\underset{\sim}{3}})$ may be written

$$(\hat{\underset{\sim}{k}} \times \underset{\sim}{u})(\underset{\sim}{u} \times \hat{\underset{\sim}{z}}) = \{y^2(\xi\, \hat{\underset{\sim}{n}}\, \hat{\underset{\sim}{k}} - \mu\, \hat{\underset{\sim}{n}}\, \hat{\underset{\sim}{n}}) + (\xi\, xz - \mu\, z^2)\hat{\underset{\sim}{z}}\, \hat{\underset{\sim}{z}}\} \quad . \tag{3.22}$$

Thus J_2 has the form

$$J_2 = j_2\, \hat{\underset{\sim}{z}}\, \hat{\underset{\sim}{z}} + j_n\, \hat{\underset{\sim}{n}}\, \hat{\underset{\sim}{n}} + \hat{\underset{\sim}{n}}\, \hat{\underset{\sim}{k}} \text{ term.} \tag{3.23}$$

We see that the 2,n-submatrix is diagonal for general $f_o^{(2)}$. It follows from this that the normal mode vectors for Eq. (2.25) are

$$\hat{\underset{\sim}{B}}_k = \hat{\underset{\sim}{z}} \quad \text{and} \quad \hat{\underset{\sim}{B}}_k = \hat{n} \quad . \tag{3.24}$$

If we use $f_o^{(2)}$ from Eq. (3.12) corresponding to a bimaxwellian f_o then J_2 becomes, disregarding the $\hat{\underset{\sim}{n}}\, \hat{\underset{\sim}{k}}$ term

$$J_2 = \alpha\{-\mu^2\, \hat{\underset{\sim}{n}}\, \hat{\underset{\sim}{n}} + (\xi^2 - \mu^2)\, \hat{\underset{\sim}{z}}\, \hat{\underset{\sim}{z}}\} \int d^3u\, z^2(1 - \beta u^2)^{\frac{1}{2}}\, f_M(u^2) \quad . \tag{3.25}$$

We expand (3.25) in powers of β to obtain.

$$J_2 = \alpha\{-\mu^2\, \hat{\underset{\sim}{n}}\, \hat{\underset{\sim}{n}} + (1 - 2\mu^2)\, \hat{\underset{\sim}{z}}\, \hat{\underset{\sim}{z}}\}\left(1 - \frac{5}{2}\beta + \cdots\right). \tag{3.26}$$

The Normal Mode Equation

We may use Eqs. (3.19) and (3.26) to estimate the values of J_1 and J_2 in Eq. (2.25) for $\underset{\sim}{B}_k$. Using these values we may now state the condition for which the effect of ion anisotropy on the dispersion relation may be neglected. This is

$$\omega_e^{\,2}\, \alpha_e \gg \omega_i^{\,2}\, a_i \quad . \tag{3.27}$$

This condition is equivalent to

$$\alpha_i \ll \frac{m_i}{m_e} \alpha_e \tag{3.28}$$

which is very easily satisfied. We henceforth disregard ion anisotropy.

We suppress the k-dependence and write

$$\underset{\sim}{B}_k = B_2 \hat{\underset{\sim}{2}} + B_n \hat{\underset{\sim}{n}} \ . \tag{3.29}$$

Then Eq. (2.25) becomes for general f_o

$$\frac{\partial}{\partial t} B_{2,n} = \gamma_{2,n} B_{2,n} \tag{3.30}$$

where the growth rate is

$$\gamma_{2,n} = kd^* \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \{j_{2,n} - k^2c^2/\omega_e{}^2\} \tag{3.31}$$

and we have defined

$$\frac{1}{d^*} = -\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \left[j_1{}^e \ \frac{1}{d_e} + j_1{}^i \ \frac{1}{d_i}\right] \ . \tag{3.32}$$

We note that if the rather mild restriction

$$T_i \gg \frac{m_e}{m_i} T_e \tag{3.33}$$

is satisfied then the ion contribution to d^* is negligible. And if $f_o{}^{(o)}$ is maxwellian then $d^* = d_e$. In any case d^* is a constant throughout the evolution of the plasma.

Polarization

If we substitute Eqs. (3.11) and (3.12) into Eqs. (2.31) and (2.33) we obtain

$$h^s = -1/kd_s^{\,2} \tag{3.34}$$

$$\underset{\sim}{H}_3^{\,s} = i\left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{\alpha_s}{kd_s} \xi\mu \, \hat{\underset{\sim}{2}} \tag{3.35}$$

thus if we define

$$\eta = \sum_s \omega_s^{\,2}/d_s^{\,2} \tag{3.36}$$

$$\zeta = \sum_s \omega_s^{\,2} \alpha_s/d_s \, c \quad . \tag{3.37}$$

Then $E_k^{(4)}(\text{long})$ may be written

$$E_k^{(4)}(\text{long}) = -i\left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{\zeta}{\eta} \xi\mu \, (\hat{\underset{\sim}{2}} \cdot \underset{\sim}{B}_k) \tag{3.38}$$

where if we neglect the ion contribution we can estimate the magnitude of ζ/η as

$$\zeta/\eta = \alpha_e \, d_e/c \quad . \tag{3.39}$$

For comparison we have

$$\underset{\sim}{E}_k^{(4)}(\text{trans}) = -\frac{i}{kc} \gamma_k \, (\hat{\underset{\sim}{k}} \times \underset{\sim}{B}_k) \quad . \tag{3.40}$$

We see that mode n is purely transverse. For mode 2 we see that generally $E_k^{(4)}(\text{long})$ and $E_k^{(4)}(\text{trans})$ are the same order of magnitude. This mode becomes transversely polarized at $\theta = 0°$ and $90°$, which corresponds to maximum growth for $\alpha < 0$ $(T_{\parallel} < T_{\perp})$ and $\alpha > 0$ $(T_{\parallel} > T_{\perp})$ respectively.

Discussion of the Dispersion Relation

If we disregard relativistic corrections and assume a bimaxwellian f_o, then the growth rate for mode 2 may be written

$$\gamma_2 = kd^* \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \{\alpha_e(1 - 2\mu^2) - k^2c^2/\omega_e^2\} \quad . \tag{3.41}$$

For $\alpha > 0$ $(T_{\parallel} > T_{\perp})$ this mode grows for

$$45° \leq \theta \leq 90° \tag{3.42}$$

and

$$0 \leq k^2 \leq k_2^2(\text{crit}, \theta) = \alpha(1 - 2\mu^2)\omega_e^2/c^2 \quad . \tag{3.43}$$

Maximum growth occurs at

$$\theta = 90° \tag{3.44}$$

and

$$k^2 = \frac{1}{3} k_2^2(\text{crit}, 90°) \quad . \tag{3.45}$$

For $\alpha < 0$ $(T_{\parallel} < T_{\perp})$ this mode grows for

$$0° \leq \theta \leq 45° \tag{3.46}$$

and

$$0 \leq k^2 \leq k_2^{\,2}(\text{crit}, \theta) \quad . \tag{3.47}$$

In this case maximum growth occurs at

$$\theta = 0^\circ \tag{3.48}$$

and

$$k^2 = \frac{1}{3} k_2^{\,2}(\text{crit}, 0^\circ) \quad . \tag{3.49}$$

Likewise, the growth rate for mode n may be written

$$\gamma_n = -kd^* \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \{\alpha_e \mu^2 + k^2c^2/\omega_e^{\,2}\} \quad . \tag{3.50}$$

For $\alpha > 0$ $(T_{||} > T_\perp)$ this mode is damped for all k. For $\alpha < 0$ $(T_{||} < T_\perp)$ we have growth for

$$0^\circ \leq \theta \leq 90^\circ \tag{3.51}$$

and

$$0 \leq k^2 \leq k_n^{\,2}(\text{crit}, \theta) = -\alpha \mu^2 \omega_e^{\,2}/c^2 \quad . \tag{3.52}$$

Maximum growth occurs at

$$\theta = 0^\circ \tag{3.53}$$

and

$$k^2 = \frac{1}{3} k_n^{\,2}(\text{crit}, 0^\circ) \quad . \tag{3.54}$$

In each case, maximum growth occurs for $\underset{\sim}{k}$ parallel to the axis of temperature minimum and $\underset{\sim}{E}_k$ parallel to the axis of temperature maximum.

The relativistic calculation shows that the growth rate has corrections of order d^2/c^2. For both modes 2 and n the relativistic terms decrease the growth rate for a given k-value and reduce the region of unstable waves. Thus relativistic terms have a stabilizing effect on the transverse instability.

IV. QUASILINEAR STABILIZATION

Form of the Equation for $f_o^{(2)}$

Equation (2.13) for $f_o^{(2)}$ may be written in the form

$$\frac{\partial}{\partial t} f_o^{(2)} = \frac{\partial}{\partial \underset{\sim}{u}} \cdot D \cdot \frac{\partial}{\partial \underset{\sim}{u}} f_o^{(2)} + \left(\frac{\partial}{\partial \underset{\sim}{u}} \cdot \underset{\sim}{N}\right)\left(\frac{\partial}{\partial u^2} f_o^{(o)}\right) \tag{4.1}$$

where

$$D = -\frac{1}{V}\frac{e^2}{m^2c^2}\int \frac{d^3k}{(2\pi)^3}\frac{1}{d}\frac{i}{\underset{\sim}{k}\cdot\underset{\sim}{u}}(\underset{\sim}{u} \times \underset{\sim}{B}_k^*)(\underset{\sim}{u} \times \underset{\sim}{B}_k) \tag{4.2}$$

$$\underset{\sim}{N} = -\frac{1}{V}\frac{e^2}{m^2c^2}\int \frac{d^3k}{(2\pi)^3}\frac{2}{k^2d^2}\frac{1}{\underset{\sim}{k}\cdot\underset{\sim}{u}}(\underset{\sim}{u} \times \underset{\sim}{B}_k^*)(\underset{\sim}{k} \times \frac{\partial}{\partial t}\underset{\sim}{B}_k)\cdot u \ . \tag{4.3}$$

We shall refer to the first term on the RHS of Eq. (4.1) as the diffusion term and the second term as the flow term. The diffusion term causes a change in $f_o^{(2)}$ which is proportional to $f_o^{(2)}$. Whereas the flow term depends only on the state of the magnetic wave spectrum at time t since $f_o^{(o)}$ is constant throughout the evolution of the plasma. If there were only a single magnetic wave present, then D and $\underset{\sim}{N}$ would be singular functions of u. We assume that there is a continuous spectrum of waves so that D and $\underset{\sim}{N}$ are smooth functions of u.

The factor $\underset{\sim}{u} \times \partial/\partial\underset{\sim}{u}$ appears on the left in both the diffusion and flow terms. It follows that the moments

$$\frac{\partial}{\partial t}\langle u^n\rangle^{(2)} = \frac{\partial}{\partial t}\int d^3u\, u^n f_o^{(2)} = 0 \ . \tag{4.4}$$

Therefore since the moments of $1, u^2, u^4$, etc. with $f_o^{(2)}$ are initially zero, they remain zero forever.

$$\langle u^n \rangle^{(2)} = 0 \quad . \tag{4.5}$$

Thus $f_o^{(2)}$ remains purely anisotropic throughout its evolution.

Equations for the Moments of $f_o^{(2)}$

The moments

$$\langle u_3^{\,m} u^n \rangle^{(2)} = \int d^3u \; u_3^{\,m} u^n f_o^{(2)}(u^2, u_3^{\,2}) \tag{4.6}$$

are all independent and together specify $f_o^{(2)}$. Since $f_o^{(2)}$ is an even function of u_3 and u, we may restrict our attention to m and n even integers. The functions $1, u^2, u_3^{\,2}, u^4, u^2u_3^{\,2}, \cdots$ can be combined to give a set of polynomials, $p_m(u^2, u_3^{\,2})$, which are orthonormal relative to a positive weighting function, $f_w(u^2)$.

$$\int d^3u \; p_M \, p_m \, f_w = \delta_{mM} \quad . \tag{4.7}$$

The choice for the weighting function will be made later. We expand $f_o^{(2)}$ in the above polynomials.

$$f_o^{(2)} = \sum_m \alpha_m(t) \, p_m(u^2, u_3^{\,2}) \, f_w(u^2) \quad . \tag{4.8}$$

We take the moment of p_M with respect to Eq. (4.1) for $f_o^{(2)}$. The result may be written in the form

$$\frac{\partial}{\partial t}\alpha_M = \frac{1}{V}\frac{e^2}{m^2c^2}\int\frac{d^3k}{(2\pi)^3}\left\{\frac{1}{kd}\sum_m \alpha_m(\underset{\sim}{B}_k{}^* \cdot I^{Mm}\cdot \underset{\sim}{B}_k) + \frac{2}{k^2d^2}(\underset{\sim}{B}_k{}^* \cdot I^M \cdot \frac{\partial}{\partial t}\underset{\sim}{B}_k)\right\} \tag{4.9}$$

where

$$I^M = \int d^3u \frac{1}{\underset{\sim}{k}\cdot\underset{\sim}{u}}(\hat{\underset{\sim}{3}} \times \underset{\sim}{u})(\underset{\sim}{u} \times \underset{\sim}{k})\left(\frac{\partial}{\partial u_3{}^*}p_M\right)\frac{\partial}{\partial u^2}f_o{}^{(o)} \tag{4.10}$$

$$I^{Mm} = \int d^3u \frac{i}{\hat{\underset{\sim}{k}}\cdot\underset{\sim}{u}}(\hat{\underset{\sim}{3}} \times \underset{\sim}{u})(\hat{\underset{\sim}{3}} \times \underset{\sim}{u})\left(\frac{\partial}{\partial u_3{}^*}p_M\right)\left(\frac{\partial}{\partial u_3{}^*}p_m\right)f_w \quad . \tag{4.11}$$

We consider the contribution of the flow term to the moment equation. Since the remainder of the integrand is even in y, the tensor $(\hat{\underset{\sim}{3}} \times \underset{\sim}{u})(\underset{\sim}{u} \times \hat{\underset{\sim}{k}})$ in I^M reduces to

$$\{-\mu y^2\,\hat{n}\,\hat{n} + (\xi xz - \mu z^2)\,\hat{2}\,\hat{2} + \hat{k}\,\hat{n}\text{ term}\} \quad . \tag{4.12}$$

Thus the 2, n-submatrix of I^M is diagonal and constant. Hence we may write

$$(\underset{\sim}{B}_k{}^* \cdot I^M \cdot \frac{\partial}{\partial t}\underset{\sim}{B}_k) = \frac{1}{2}\frac{\partial}{\partial t}(\underset{\sim}{B}_k{}^* \cdot I^M \cdot \underset{\sim}{B}_k) \quad . \tag{4.13}$$

Thus the flow term in Eq. (4.9) is a pure time derivative. Therefore the flow term makes no contribution to the <u>net</u> change in any even velocity moment between initial and final states which have negligible wave spectra. The net change in the velocity moments between such states

is due entirely to the diffusion term. Assuming that α_M is of order α, negligible wave spectra corresponds to a magnetic wave energy density satisfying

$$\mathcal{E}_{mag} \ll \alpha^2 \mathcal{E}_{kin} \quad . \tag{4.14}$$

Since D and $\underset{\sim}{N}$ were assumed to be smooth functions, $f_o^{(2)}$ is a continuous function and so may be completely represented by its moments. Thus we may say that the flow term in Eq. (4.1) is a pure time derivative although this is not at all obvious from Eq. (4.3).

We consider now the diffusion term. Since the remainder of the integrand is even in y, the tensor $(\hat{\underset{\sim}{z}} \times \underset{\sim}{u})(\hat{\underset{\sim}{z}} \times \underset{\sim}{u})$ in I^{Mm} reduces to

$$\{y^2(\mu\hat{\underset{\sim}{n}} - \xi\,\hat{\underset{\sim}{k}})(\mu\hat{\underset{\sim}{n}} - \xi\hat{\underset{\sim}{k}}) + \mu^2 z^2\, \hat{\underset{\sim}{z}}\,\hat{\underset{\sim}{z}}\} \quad . \tag{4.15}$$

The 2, n-submatrix of I^{Mm} is diagonal. Thus no cross terms between mode 2 and mode n occur in the equation for $f_o^{(2)}$. Only the pole contributes to I^{Mm}. We do the x-integration and retain only the 2, n-submatrix.

$$I^{Mm} = -\pi\,\mu^2\,\xi^{-2} \int dy\, dz\, \{y^2\, \hat{\underset{\sim}{n}}\,\hat{\underset{\sim}{n}} + z^2\, \hat{\underset{\sim}{z}}\,\hat{\underset{\sim}{z}}\} \left(\frac{\partial}{\partial z^*} p_M(w^2, \xi^2 z^2)\right)$$

$$\left(\frac{\partial}{\partial z^*} p_m(w^2, \xi^2 z^2)\right) f_w(w^2) \tag{4.16}$$

where now the * denotes that w^2 is held constant during the z-differentiation. And we have used the relation

$$\left[\frac{\partial}{\partial u_3^*} p(u^2, u_3^2)\right]_{x=0} = \frac{1}{\xi} \frac{\partial}{\partial z^*} p(w^2, \xi^2 z^2) \quad . \tag{4.17}$$

We note that the components of the tensors I^{MM} for all M are negative. Thus the α_M component of the diffusion terms has a stabilizing effect in the equation for $\partial\alpha_M/\partial t$.

Moment Expansion of the Growth Rate Formula

If we substitute Eq. (4.8) for $f_o^{(2)}$ into Eq. (3.21) and ignore relativistic corrections then J_2 may be written

$$J_2 = \int d^3u \frac{1}{\underset{\sim}{k}\cdot\underset{\sim}{u}} (\hat{\underset{\sim}{z}} \times \underset{\sim}{u})(\underset{\sim}{u} \times \underset{\sim}{k}) \sum_M \alpha_M \left(\frac{\partial}{\partial u_3^*} p_M\right) f_w \quad . \tag{4.18}$$

Thus if we choose

$$f_w = -2 \frac{\partial}{\partial u^2} f_o^{(o)} \tag{4.19}$$

(where the minus sign makes f_w positive for single humped $f_o^{(o)}$), then J_2 becomes

$$J_2 = - \sum_M 2\, \alpha_M I^M \quad . \tag{4.20}$$

Using this expression we may write Eqs. (3.30) and (3.31) for $\underset{\sim}{B}_k$ as

$$\frac{\partial}{\partial t} \underset{\sim}{B}_k = - kd^* \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \{I_k\, k^2c^2/\omega_e^2 + \sum_M 2\, \alpha_M I^M\} \cdot \underset{\sim}{B}_k \quad . \tag{4.21}$$

The same functions $I^M(\theta)$ appear in the growth rate formula and in the flow term. We see from Eq. (4.21) that α_M will contribute to growing waves for mode j at angle θ if

$$\alpha_M I_j^M(\theta) < 0 \quad . \tag{4.22}$$

However the contribution of these waves at angle θ to the flow term for $\partial\alpha_M/\partial t$ is proportional to

$$I_j^M(\theta) \frac{\partial}{\partial t} |B_j(\theta)|^2 \quad . \tag{4.23}$$

Thus these waves have a stabilizing effect (reduce $|\alpha_M|$) when they are growing and a destabilizing effect when they are damping.

Steady State Solutions

A steady state solution to Eq. (4.1) would have to satisfy

$$\frac{\partial}{\partial \underset{\sim}{u}} \cdot D \cdot \frac{\partial}{\partial \underset{\sim}{u}} f_o^{(2)} = 0 \quad . \tag{4.24}$$

Denote

$$\underset{\sim}{\phi} = \frac{\partial}{\partial \underset{\sim}{u}} f_o^{(2)} \quad . \tag{4.25}$$

Multiply Eq. (4.24) by $f_o^{(2)}$, integrate with respect to u, and integrate the first $\partial/\partial \underset{\sim}{u}$ by parts. Then

$$\int d^3u \; \underset{\sim}{\phi} \cdot D \cdot \underset{\sim}{\phi} = 0 \tag{4.26}$$

which implies that

$$\underset{\sim}{\phi} \cdot D \cdot \underset{\sim}{\phi} = 0 \tag{4.27}$$

for all u since the quadratic form will be shown to be positive definite.

Consider D. Perform the $\underset{\sim}{k} \cdot \hat{\underset{\sim}{u}}$ integration in Eq. (4.2). Then D becomes

$$D = \frac{1}{V} \frac{e^2}{m^2 c^2} \frac{1}{2ud} \frac{1}{(2\pi)^2} \int_{k \perp u} d^2k \, (\underset{\sim}{u} \times \underset{\sim}{B}_k^*)(\underset{\sim}{u} \times \underset{\sim}{B}_k) \quad . \tag{4.28}$$

Evidently D is a real and symmetric tensor. Furthermore the quadratic form, $\underset{\sim}{\phi} \cdot D \cdot \underset{\sim}{\phi}$ is positive definite. Denote

$$\underset{\sim}{\psi} = \underset{\sim}{u} \times \frac{\partial}{\partial \underset{\sim}{u}} f_o^{(2)} = (\underset{\sim}{u} \times \hat{\underset{\sim}{3}}) \frac{\partial}{\partial u_3^*} f_o^{(2)} \quad . \tag{4.29}$$

Then we may write

$$\underset{\sim}{\phi} \cdot D \cdot \underset{\sim}{\phi} = \left\{ \frac{1}{V} \frac{e^2}{m^2 c^2} \frac{1}{2ud} \frac{1}{(2\pi)^2} \psi^2 \right\} \int_{k \perp u} d^2k \, |\hat{\underset{\sim}{\psi}} \cdot \underset{\sim}{B}_k|^2 \quad . \tag{4.30}$$

Now for u parallel to $\hat{1}$, $\hat{\psi}$ equals $\hat{2}$. Therefore $\underset{\sim}{\phi} \cdot D \cdot \underset{\sim}{\phi}$ is proportional to

$$\int dk_2 \, dk_3 \, |B_2|^2 \quad . \tag{4.31}$$

And for u parallel to $\hat{2}$, $\hat{\psi}$ equals $\hat{1}$. Therefore $\underset{\sim}{\phi} \cdot D \cdot \underset{\sim}{\phi}$ is proportional to

$$\int dk_\perp \, dk_3 \, (k_3/k)^2 \, |B_n|^2 \quad . \tag{4.32}$$

Therefore for $\underset{\sim}{\phi} \cdot D \cdot \underset{\sim}{\phi}$ to be zero for all u requires that the fields for both mode 2 and mode n are zero or that $f_o^{(2)}$ is zero. If we consider also the growth rate formula it is evident that both the fields and $f_o^{(2)}$ must be zero. This then is the only steady state solution.

$$f_o(\text{steady state}) = f_o^{(o)} \quad . \tag{4.33}$$

We have not shown that the plasma, if left to itself, actually approaches the above steady state. However this is highly plausible because of the steady dissipative effect of the diffusion term.

Physical Description of the Stabilization Process

Consider the initial value problem where $T_{||}$ is larger than $T_\perp$. Waves grow up corresponding to every k-vector lying near the perpendicular plane. Consequently every particle resonates with some of the waves, that is, has its velocity vector perpendicular to the k-vector of the wave. Let the angular width of the spectrum about the perpendicular plane be Δ. Particles whose velocity vector lies within an angle Δ of the parallel axis will interact with more waves than particles whose velocity vector lies outside this cone. This causes particles with large $v_{||}/v_\perp$ to diffuse more rapidly than particles with small $v_{||}/v_\perp$. This reduces $T_{||}$ and increases $T_\perp$ thereby causing stabilization.

Numerical Calculation

In the authors thesis[26] a numerical calculation was made assuming f_o to be bimaxwellian throughout its evolution. This is equivalent to retaining only the polynomial

$$p_3 = (3u_3^2 - u^2)/(12)^{\frac{1}{2}} \tag{4.34}$$

in the expansion (4.8). Comparing with Eq. (3.12) we see that

$$\alpha_3 = \alpha/\sqrt{3} \ . \tag{4.35}$$

The initial conditions were $T_{\parallel} > T_{\perp}$ and negligible initial wave energy. The evolution of α may be broken into three distinct phases: an initial phase, a rapid phase, and an asymptotic phase. During the initial phase the wave spectrum grows up peaked about the perpendicular plane. Whereas particle diffusion and flow occurs very slowly at first, and hence little change occurs in the anisotropy. Eventually the amplitude of the wave spectrum becomes so large that α begins to decrease rapidly. This is due mainly to the flow term which is larger than the diffusion term at first. The temperature anisotropy decreases rapidly until slightly more than half of the wave spectrum (with larger wave numbers) is damping, and slightly less than half is still growing. This reduces the flow term and changes its sign. The flow term then partially cancels the diffusion term. This ends the rapid phase. Particle diffusion then proceeds ever more slowly as temperature isotropy is approached. Asymptotic analysis shows that the anisotropy decreases with time like $t^{-2/3}$. Simultaneously the wave spectrum shifts to lower and lower wave numbers, and eventually decays. The asymptotic phase is characterized by the wave spectrum shifting so as to always be about half damping and half growing.

APPENDICES

A. Energy Relations

We define the kinetic, magnetic, and electric energy densities by

$$\mathcal{E}^{s}_{kin} = \int d^3v \, \frac{1}{2} \, n_s \, m_s \, v^2 \, f_o^{\,s} \tag{A.1}$$

$$\mathcal{E}^{s}_{\perp\,kin} = \int d^3v \, \frac{1}{2} \, n_s \, m_s \, v_\perp^{\,2} \, f_o^{\,s} \tag{A.2}$$

$$\mathcal{E}_{mag} = \frac{1}{V} \int d^3r \, |\delta \underset{\sim}{B}|^2/8\pi \tag{A.3}$$

$$\mathcal{E}_{elec} = \frac{1}{V} \int d^3r \, |\delta \underset{\sim}{E}|^2/8\pi \quad . \tag{A.4}$$

Energy conservation follows immediately from the Vlasov and Maxwell equations.

$$\frac{\partial}{\partial t} \left(\mathcal{E}_{kin} + \mathcal{E}_{mag} + \mathcal{E}_{elec} \right) = 0 \quad . \tag{A.5}$$

But since

$$\delta \underset{\sim}{E} \sim \alpha \frac{d}{c} \, \delta \underset{\sim}{B} \tag{A.6}$$

it follows that

$$\mathcal{E}_{elec} \sim \alpha^2 \left(\frac{d}{c} \right)^2 \mathcal{E}_{mag} \quad . \tag{A.7}$$

Now from the Vlasov Eq. (1.21)

$$\frac{\partial}{\partial t} \mathcal{E}^{s}_{\perp kin} = - \int d^3v \; \frac{1}{2} n_s \, m_s \, v_\perp^{\,2} \; \frac{1}{V} \int d^3r \;\; \delta R^s \, \delta f^s \neq 0 \; . \tag{A.8}$$

But on the other hand

$$\frac{\partial}{\partial t} \mathcal{E}^{s}_{kin} = - \int d^3v \; \frac{1}{2} n_s \, m_s \, v^2 \; \frac{1}{V} \int d^3r \; \delta S^s \, \delta f^s \tag{A.9}$$

since the $\underset{\sim}{v} \times \partial/\partial \underset{\sim}{v}$ in δR gives zero when integrated with v^2. Now

$$\delta S \; \delta f \sim \alpha \; \delta F \; \delta f \tag{A.10}$$

therefore it follows that

$$\frac{\partial}{\partial t} \mathcal{E}^{s}_{kin} \sim \alpha \; \frac{\partial}{\partial t} \mathcal{E}^{s}_{\perp kin} \; . \tag{A.11}$$

Now from the definition of α we have than an amount of energy, $\delta \mathcal{E}_{\perp kin} \sim \alpha \, \delta \, \mathcal{E}_{kin}$, flows from parallel motion into perpendicular motion. However from Eq. (A.11) it follows that only $\alpha \, \delta \mathcal{E}_{\perp kin} \sim \alpha^2 \, \delta \, \mathcal{E}_{kin}$ of it flows into the magnetic field.

Assume that the ions are isotropic and so do not participate in the energy flow. We defined d to be the mean thermal velocity at $t = 0$, when the fields were assumed to have negligible energy. The perpendicular and parallel thermal velocities were denoted by a and b. Equation (A.11) may be written

$$\delta \mathcal{E}^{e}_{kin} \sim \alpha \, \delta \mathcal{E}^{e}_{\perp kin} \tag{A.12}$$

which may be written in terms of the electron thermal velocities as

$$\{2(a^2 - d^2) + (b^2 - d^2)\} \sim \alpha(a^2 - d^2) \sim \alpha^2 a^2 \quad . \tag{A.13}$$

Let us define $\bar{\alpha}$ by

$$(a^2 - d^2) = -\frac{1}{3}\bar{\alpha}\, d^2 \quad . \tag{A.14}$$

Then from Eq. (A.13) it follows that

$$(b^2 - d^2) = \frac{2}{3}\bar{\alpha}\, d^2 + O(\alpha^2 a^2) \quad . \tag{A.15}$$

But from the definition of α

$$\alpha\, d^2 = b^2 - a^2 = \bar{\alpha}\, d^2 + O(\alpha^2 a^2) \quad . \tag{A.16}$$

Thus to lowest order

$$\bar{\alpha} = \alpha \tag{A.17}$$

whence

$$a^2 = d^2\{1 - \frac{1}{3}\alpha + O(\alpha^2)\} \tag{A.18}$$

$$b^2 = d^2\{1 + \frac{2}{3}\alpha + O(\alpha^2)\} \quad . \tag{A.19}$$

B. Irreversibility

We have used the Vlasov and Maxwell equations to describe our plasma. We have performed no phase or ensemble averaging. And yet we have obtained a tensor D which is a positive operator, and hence which produces dissipation. From where did this irreversibility arise? It came from the assumption that D is a smooth function of u. This restricted us to a continuous spectrum of waves. Physically, the velocity space diffusion occurs because of the different phases of the magnetic fields which the electron experiences as it travels through these waves. If we had considered the case of only a single magnetic wave, then the electron would have a coherent phase relationship with this wave. Then D would be a singular function of u and Eq. (4.1) would described the exact, reversible electron trajectories.

C. Pole Prescription

Let us denote

$$\phi = -\delta R^{(2)} f_o^{(2)} - \delta S^{(4)} f_o^{(0)} \tag{C.1}$$

then the ordered Vlasov Eq. (1.19) may be written

$$\left(\frac{\partial}{\partial t_1} + \underset{\sim}{v} \cdot \nabla\right) \delta f^{(3)} = \phi \ . \tag{C.2}$$

Previously we dropped the $\partial/\partial t_1$ term because δf and ϕ have no t_1 dependence. We retain this term in order to find the $\underset{\sim}{k} \cdot \underset{\sim}{v}$ pole prescription. Fourier transform in space and in the t_1 variable

$$i(\underset{\sim}{k} \cdot \underset{\sim}{v} - \omega) f_{k\omega}^{(3)}(t_2\ t_3 \cdots) = \phi_{k\omega}(t_2\ t_3 \cdots) \ . \tag{C.3}$$

Whence

$$f_k^{(3)}(t_1\ t_2 \cdots) = \int \frac{d\omega}{2\pi} \frac{i}{\omega - \underset{\sim}{k} \cdot \underset{\sim}{v}} e^{-i\omega t_1} \phi_{k\omega}(t_2 \cdots) \ . \tag{C.4}$$

It is well known for initial value problems that the ω-contour lies in some UHP above all the poles. In order to have a real ω-contour we may write (C.4) as

$$f_k^{(3)}(t_1\ t_2 \cdots) = \lim_{\epsilon \to 0} \int \frac{d\omega}{2\pi} \frac{i}{\omega - \underset{\sim}{k} \cdot \underset{\sim}{v} + i\epsilon} e^{-i\omega t_1} \phi_{k\omega}(t_2 \cdots) \ . \tag{C.5}$$

Now

$$\phi_{k\omega}(t_2 \cdots) = \int dt_1 \, e^{i\omega t_1} \phi_k(t_1 \, t_2 \cdots) \quad . \tag{C.6}$$

We now use the fact that δf and ϕ are not functions of t_1 (whose absence we indicate with a carat). Then

$$\phi_{k\omega}(t_2 \cdots) = 2\pi \, \delta(\omega) \, \phi_k(\hat{t}_1 \, t_2 \cdots) \quad . \tag{C.7}$$

Substituting this into (C.5) gives

$$f_k^{(3)}(\hat{t}_1 \, t_2 \cdots) = \lim_{\epsilon \to 0} \frac{-i}{\underset{\sim}{k} \cdot \underset{\sim}{v} - i\epsilon} \, \phi_k(\hat{t}_1 \, t_2 \cdots) \quad . \tag{C.8}$$

This is the same as Eq. (1.44), but in addition it has the pole prescription for the k and v integrations. Namely, the k and v contours go under the pole.

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FOOTNOTES AND REFERENCES

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