Submitted to Physica; Also paper presented at the International Conference on the Science of Superconductivity, Stanford University, Stanford, Ca., August 26-29, 1969

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August 1969

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A New Theory of the Current-Voltage Characteristics of Weak Links

(Short title: Theory of i-v Characteristics of Weak Links)

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ABSTRACT

A new theory of the i-v characteristics of superconducting weak links of the SNS sandwich type is outlined. The theory is based on a simple model current-density equation and a careful assessment of the internal impedances of the high- and low-frequency sources present. All the important differences between the characteristics for SNS sandwiches and tunnel junctions are explained at least qualitatively and, to a large extent, quantitatively. A characteristic feature of the theory is the phase-locking of the rate of change of phase across the junction to any applied high-frequency signal. This feature is especially important in understanding the appearance of current steps at the "sub-harmonic" voltages $\frac{n\hbar\omega}{2em}$, and in explaining the extreme steepness of the steps observed. The model is compared with the theories for other types of weak link.

1. Introduction.

Weak links between superconductors include tunnel junctions,[1] which are rather well understood , superconducting-normal-superconducting (SNS) sandwiches[2] and point contacts.[3] The aim of this paper is to find a simple but adequate model of the SNS sandwich, but several of our ideas can be applied to weak links generally. All weak links have a small critical dc supercurrent at zero voltage. When a high frequency signal is applied to the link they also develop "supercurrent steps" in the low frequency i-v characteristic at the harmonic voltages $\frac{n\hbar\omega}{2e}$, corresponding to quantum processes in which n photons are absorbed and one condensed pair of electrons crosses the link. When large enough, all three types of link can contain quantised flux lines.[4-6]

The excited and unexcited i-v characteristics for a typical SNS sandwich[2] are shown in Fig. 4. We note here some of the characteristic features which any model must explain:

(i) In the unexcited characteristic the bulk of the critical supercurrent persists in the resistive state to values of current much greater than the critical current. There is no negative resistance region. (In tunnel junctions the dc supercurrent disappears at finite voltages and the system jumps to the normal dc characteristic.)

(ii) The current steps in the excited characteristic also seem to be ideal steps without negative resistance regions.

(iii) Additional small steps appear at the sub-harmonic voltages

$\frac{n\hbar\omega}{2em}$ (n photons absorbed as m pairs cross the link). This is also seen in point contacts but not in tunnel junctions. Anderson and Dayem[17] have made the attractive suggestion that in point contacts these steps are due to phase-locking of the flow of flux lines to the high frequency field, but no detailed mechanism of this phase-locking has been proposed.

(iv) One of us has shown[2] that the voltage on certain steps is independent of current to better than 1 in 10^8: The steps seem to be very little rounded by noise.

2. The Model.

Supercurrent can flow through an SNS junction by virtue of the proximity effect, but there seems to be no microscopic theory of the current flow at finite voltage. However, it is clear that the normal current carried by the excitations will be more important than it is in the tunnel junction. We have taken as a model the following very simple current density equation:

$$J = J_S + J_N = J_S \sin\phi + \sigma V, \qquad (1)$$

where ϕ is the phase difference between the superconductors at the point concerned.* The supercurrent term is like the Josephson supercurrent term for a tunnel junction. There are good theoretical[8] and experimental[2] reasons for believing that the supercurrent has this form in an SNS sandwich at zero voltage; we simply assume that the term is not strongly

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*We use a gauge in which the vector potential is parallel to the interfaces.

voltage sensitive. The normal term is an ordinary resistive term where σ is a conductance per unit area; we likewise assume that σ is not strongly voltage sensitive. This model is no doubt rather crude, but we shall see that it is adequate to explain most of the observed behaviour. We assume in addition the usual relations[1] between ϕ and the electric and magnetic fields:

$$\hbar\partial\phi/\partial t = 2eV \tag{2}$$

$$\hbar\partial\phi/\partial x = 2edB, \tag{3}$$

where d is the magnetic thickness of the barrier, typically 1μm. We have limited ourselves here to a one-dimensional case. The field B is wholly in the y-direction and the z-axis is normal to the junction.

A further vital feature of our approach is that the source impedances of the dc and high frequency inputs must be included explicitly from the start. We have examined the impedances encountered in typical experiments, with the following conclusions. SNS sandwiches have very small internal impedances of the order of 10^{-7} Ω and no means of storing charge internally at frequencies below the plasma frequency. One can safely treat all practicable sources as <u>current</u> sources. In tunnel junctions the internal capacity is large enough to ensure that the effective high frequency source is a voltage source at the frequencies normally employed; the dc source impedance on the other hand may easily be made either greater or smaller than the junction resistance and is frequently made variable. In most point-contact experiments the high frequency source is a current source.

3. The Small Junction.

If a voltage source $V_o + V_1 \sin(\omega t)$ were applied to a junction small enough for the current flow to be uniform, we should deduce from (1) and (2) that

$$J = J_J \sin\left[\phi_o + \omega_o t - \frac{\omega_1}{\omega}\cos(\omega t)\right] + \sigma V, \tag{4}$$

where ω_o and ω_1 are $2eV_o/\hbar$ and $2eV_1/\hbar$. This result is very similar to Josephson's result[1] for tunnel junctions. The supercurrent only has a dc component at the harmonic voltages $n\hbar\omega/2e$. At these voltages there is a supercurrent "spike"; otherwise the i-v characteristic follows the "normal" relation $J_o = \sigma V_o$. There is no structure at the subharmonic voltages. But if, as we believe happens in practice, we have a current source $J = J_o + J_1 \sin(\omega t)$, we find the quite different result:

$$\sigma\hbar\dot{\phi}/2e = J_o + J_1 \sin(\omega t) - J_J \sin\phi. \tag{5}$$

This non-linear equation of motion has some unusual properties. Consider first the unexcited case ($J_1 = 0$). Typical solutions are shown in Fig. 1. A solution with finite average slope $\overline{d\phi/dt}$ corresponds to a finite dc voltage. The important point to notice is that the supercurrent does not oscillate sinusoidally as it would have done for a voltage source; the system spends more time in the forward supercurrent than in the backward supercurrent state. Consequently the dc supercurrent only disappears gradually as the voltage is raised. The solutions have a simple analytic form and the i-v characteristic can be calculated exactly[9] (see Fig. 1). Note the absence of a negative

resistance region.

Now consider the excited characteristic ($J_1 \neq 0$). The last two terms of Eq. (5) have lattice perodicity in the ϕ, t plane (Fig. 2). In general the average slope $\overline{d\phi/dt}$ for any trajectory is close to $2eJ_o/\sigma\hbar$ because the last two terms of Eq. (5) tend to average to zero. But when this slope is close to one of the rational values $n\omega/m$, the trajectory has a tendency to "lock-on" to the lattice periodicity. Consider the special case shown in the figure for which $J_1 = J_J$, and examine the trajectories locked onto the fundamental voltage, for which $\overline{d\phi/d(\omega t)} = 1$. In the special case chosen, the straight line of unit slope passing through the origin is actually a zero contour of the periodic part of $\dot{\phi}$. If started at the origin the system will move along this line when $J_o = \sigma\hbar\omega/2e$ (trajectory AA). There will also be, for some value of J_o, a corresponding trajectory CC passing close to the regions of maximum and minimum slope. The important point, which should be clear from the figure, is that the time average of $J_1 \sin(\omega t) - J_J \sin\phi$ is negative for such a trajectory because it spends longer in the negative slope than in the positive slope regions. It follows that J_o must be greater than $\sigma\hbar\omega/2e$ for this trajectory, to compensate. So we have two trajectories with the same long term slope (or dc voltage) but different dc currents: in other words, a vertical current step.

The same thing happens for all harmonic and subharmonic voltages $n\omega/m$, though for large n or m the limiting trajectories run very close to each other, and the steps will be small. Notice that since two solutions of Eq. (5) for different values of J_o can only cross once,

V_o must be a monotonically increasing function of J_o; there can be no negative resistance regions. We have determined a number of other analytic properties of the solutions of Eq. (5) and also solved it numerically for a wide range of parameters, obtaining harmonic and sub-harmonic steps. We have also analysed the case of finite source impedance. These details will be published elsewhere. It is worth emphasising that the subharmonic structure appears quite naturally in our model in exactly the same way as the harmonic structure does. It is linked essentially with the ac source impedance and only disappears when the source impedance is exactly zero.

4. Large Junctions.

For larger junctions we find using (1), (2) and (3):

$$\lambda^2 \partial^2\phi/\partial x^2 = \sin\phi + (1/\omega_J)\, \partial\phi/\partial t, \qquad (6)$$

where λ is the Josephson penetration depth $(\hbar/2ed\mu_o J_J)^{1/2}$ and ω_J is the frequency associated with J_J, $2eJ_J/\sigma\hbar$. The time-independent solutions of Eq. (6) are familiar from tunnel-junction theory.[1,5,10] They correspond to edge currents with penetration depth λ and, within the junction, localised flux lines of width λ. The flux lines are regions through which ϕ changes by 2π, and they each contain one quantum of flux. At an edge current density equal to $2\lambda J_J$ the phase at the surface reaches π, the system becomes unstable, and flux lines enter the junction.

The interest of Eq. (6) is that it is an explicit model for the dynamics of flux-flow. We have computed numerical solutions for a number of cases. Typical flux-flow phenomena, such as the gradient of

flux-line density needed to maintain flux-flow and the non-uniformity of flux motion associated with the surface barrier are obvious in the results. Of particular interest is the dc current carried in this flux-flow situation. Some experimental results obtained by one of us[11] are compared with the results of the model calculations in Fig. 3. Both sets of curves refer to a junction of fixed width W for which W/λ is varied by varying the temperature. In both experiment and theory it is clear that the persistence of supercurrent at large currents depends on the value of W/λ. At large currents there is, of course, strong flux-flow, and the supercurrent at any instant is an oscillatory function of position which must be integrated across the specimen. In the theory, the persistence of supercurrent up to very large currents becomes marked for $W/\lambda \gtrsim 6$. This persistent supercurrent is associated with an uncompensated combination from the pair of fast-moving flux lines of opposite sign closest to the center of the junction which are about to annihilate one another. The comparison with experiment is confused by the fact that the experimental junction was square, not one-dimensional. In this case the flux almost certainly enters from the corners at the square at a comparatively low critical field, and the subsequent flux-flow pattern will be different. These facts make it difficult to determine the experimental value of λ precisely, and the figures attached to the curves are approximate. However, it is clear that in the experimental curves the persistent supercurrent is well established for values of W/λ considerably smaller than 6.

The agreement between our model results and the experiments is

as good as can be expected, bearing in mind the quite different junction configurations. It is likely that when our model is applied properly to two-dimensional junctions the persistent supercurrent will emerge from the integration over the junction area in a quite different way and at a different value of W/λ. Some of these possibilities will be discussed elsewhere.

We have also calculated the i-v characteristics for large junctions with high-frequency excitation. Illustrated in Fig. 4 is an attempt to fit some experimental data. Note that once again the specimen was square, and so not entirely comparable with the theory. In fact, the theoretical one-dimensional specimen was chosen to have the same value of W/λ and critical current (and thus roughly the same perimeter) as the real specimen. This means that it has about twice the area; this is the reason for the different dc resistances. The fit could probably be improved somewhat by reducing the amplitude of the rf in the calculation. With these points in mind, it is clear that our simple model gives a surprisingly good picture of the observed step structure.

5. Noise.

It is interesting to ask why the current steps are so steep. The answer is that the phase-locking mechanism is self-correcting against small noise fluctuations. This is most easy to see in the small junction case discussed in § 3. If, in Fig. 2, the system starts with a value of J_o appropriate to trajectory BB but at arbitrary phase, it will rapidly converge onto BB, or the equivalent trajectory in the nearest shaded region. The shaded regions are in fact areas containing stable trajectories. There is another set of trajectories

in the unshaded regions corresponding to the same values of J_o, which are unstable. The system tends to move away from them. Thus small noise fluctuations from a stable trajectory will be corrected automatically, and such fluctuations will not affect the average slope $\overline{d\phi/dt}$. This explains why the steps are so steep. Larger noise fluctuations may however, carry the system beyond the nearest unstable trajectory. If this happens it will then converge onto the next stable trajectory. The situation is analogous to the diffusion of atoms by hopping through a lattice or the Browmian motion of a particle in a tilted sinusoidal potential. This last analogy is in fact exact, and we have used it to calculate the theoretical rounding of the steps.* On account of its very low impedance, the noise picked up by the SNS junction from external sources is small compared with the Johnson noise developed in the junction itself. The calculation shows that the phase drift due to hopping from one trajectory to another is indeed very small and accounts adequately for the observations. For most other types of junctions, external noise is dominant and the steps are comparatively broad.

6. Conclusions.

Our simple model gives at least qualitative descriptions of all the features noted in § 1. The agreement may well be improved when it becomes possible to compare theory and experiment for specimens of identical geometry. For large specimens our model offers an explicit description of flux-flow. As Anderson and Dayem[5] anticipated for the

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*A somewhat similar argument has been given by Stephen.[12] We shall compare it with ours elsewhere.

parallel case of point contacts, the model shows that the current steps in the excited characteristic may be thought of as due to a phase-locked interaction between the flux-flow and the external field. In small junctions phase-locking between $\phi(t)$ and the external fields is still present but cannot usefully be regarded as involving flux-flow. It is the phase-locking which explains the lack of effect of noise on the steps.

Acknowledgments

We would like to thank M. J. Stephen for sending us results prior to publication.

Part of this work was done under the auspices of the U. S. Atomic Energy Commission.

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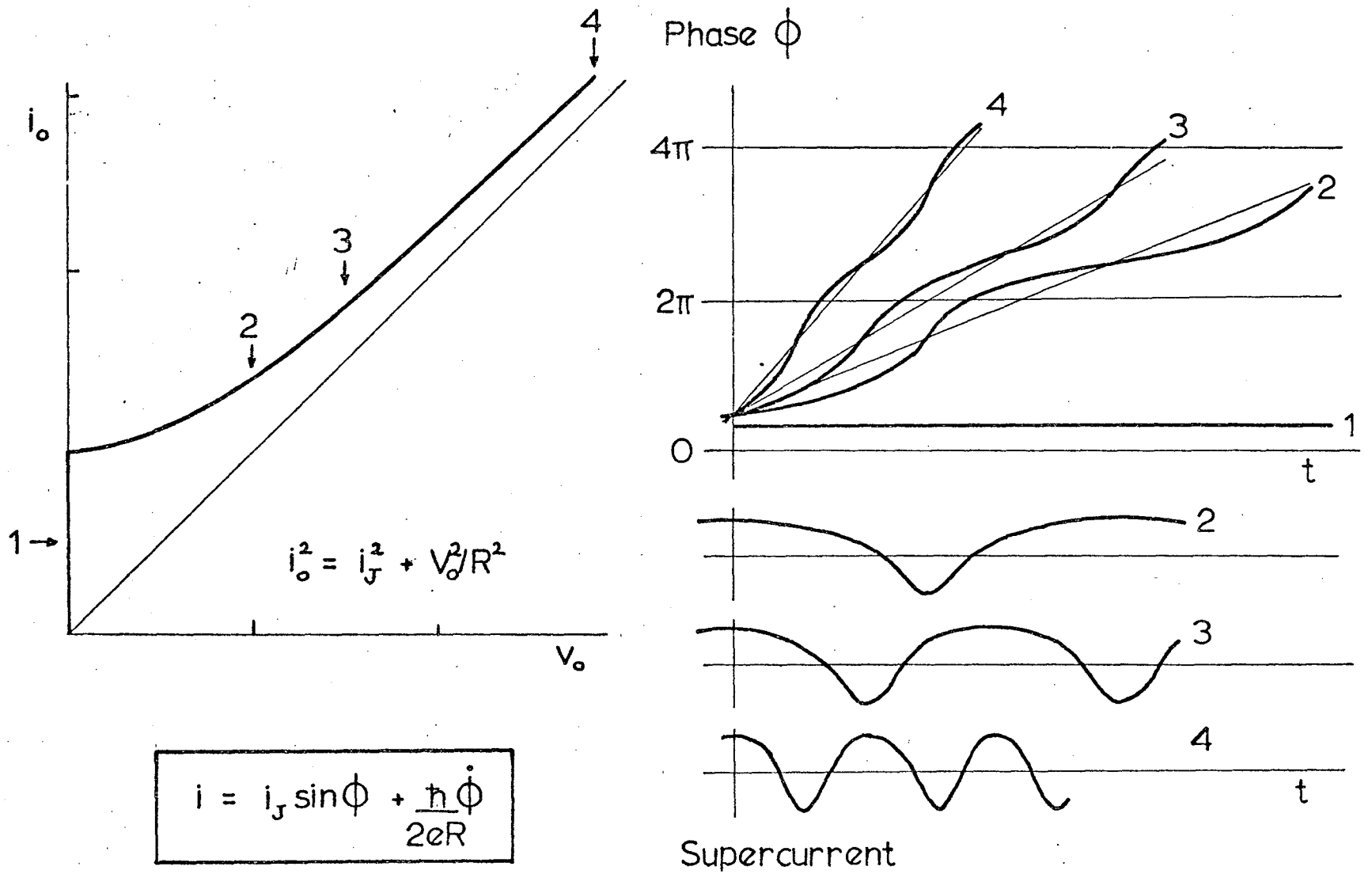
Figure Captions

Fig. 1 The trajectories $\phi(t)$ for a small SNS junction in the absence of excitation for a number of different dc currents. The thin line on the i-v characteristic is the normal part of the current.

Fig. 2 Schematic contours of $J_1 \sin(\omega t) - J_J \sin\phi$ for the special case $J_1 = J_J$, and trajectories corresponding to points on the fundamental current step at $2eV = \hbar\omega$. Trajectories in the shaded regions are stable, others are unstable.

Fig. 3 Experimental and theoretical i-v characteristics for an SNS sandwich of fixed size as the characteristic length λ is varied. The experiment refers to a square junction. The theory refers to a one-dimensional junction. The current scale has been chosen so that the critical currents agree with experiment for small W/λ.

Fig. 4 Experimental excited and unexcited i-v characteristics for a typical square junction with attempted theoretical fit. The theory refers to a physically similar one-dimensional junction with the same critical current and reduced width. Its perimeter is about the same as the experimental perimeter; its area and hence its normal conductance are therefore about double the experimental values. Steps are visible for this specimen at some of the voltages nV_ω/m where m is 1, 2, 3, or 4.



$$i = i_J \sin\phi + \frac{\hbar}{2eR}\dot{\phi}$$

Fig. 1

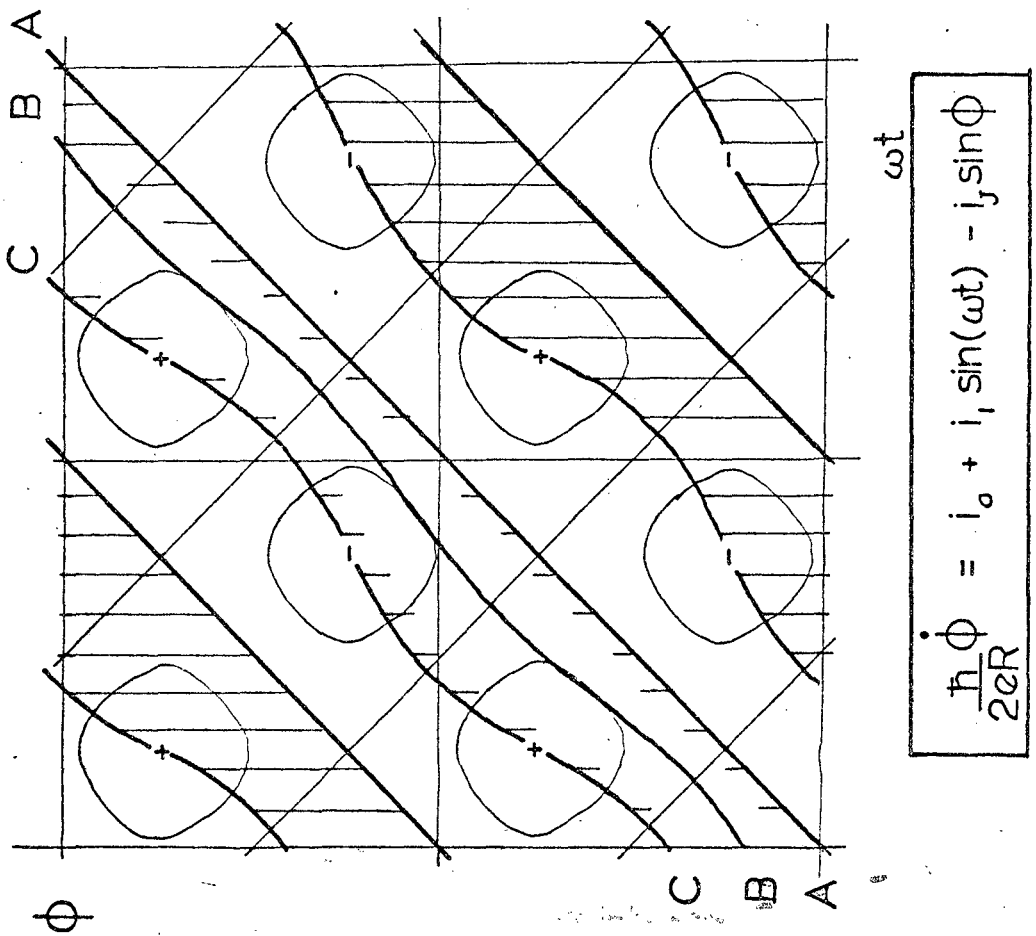
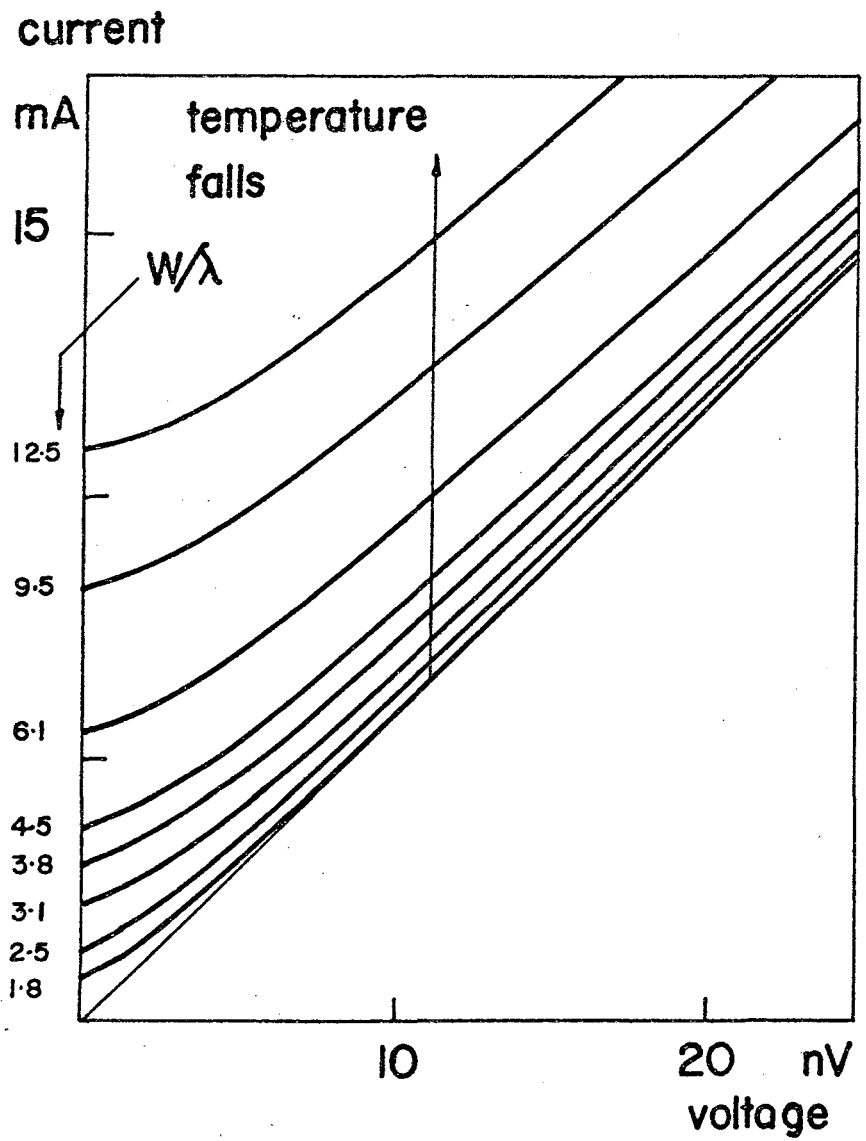
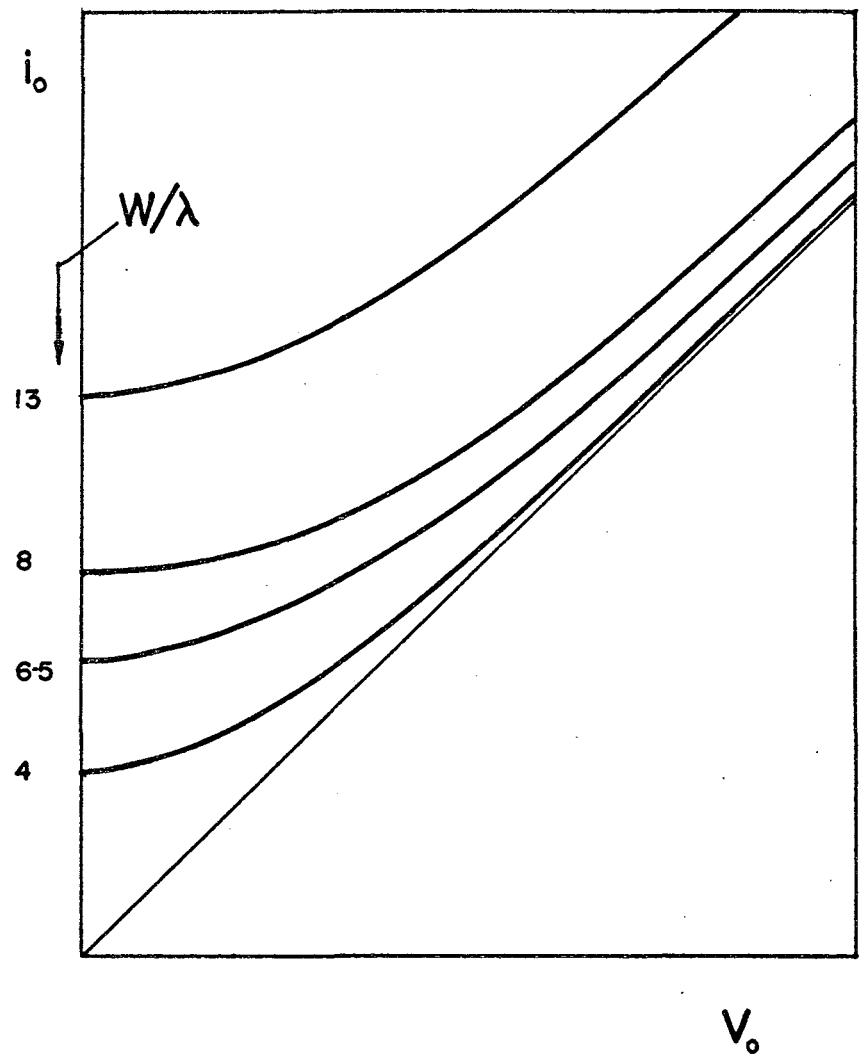

A
B
C
ϕ
ωt
$\frac{\hbar}{2eR}\dot{\phi} = i_o + i_1 \sin(\omega t) - i_j \sin\phi$

Fig. 2



Experiment



Theory

Fig. 3

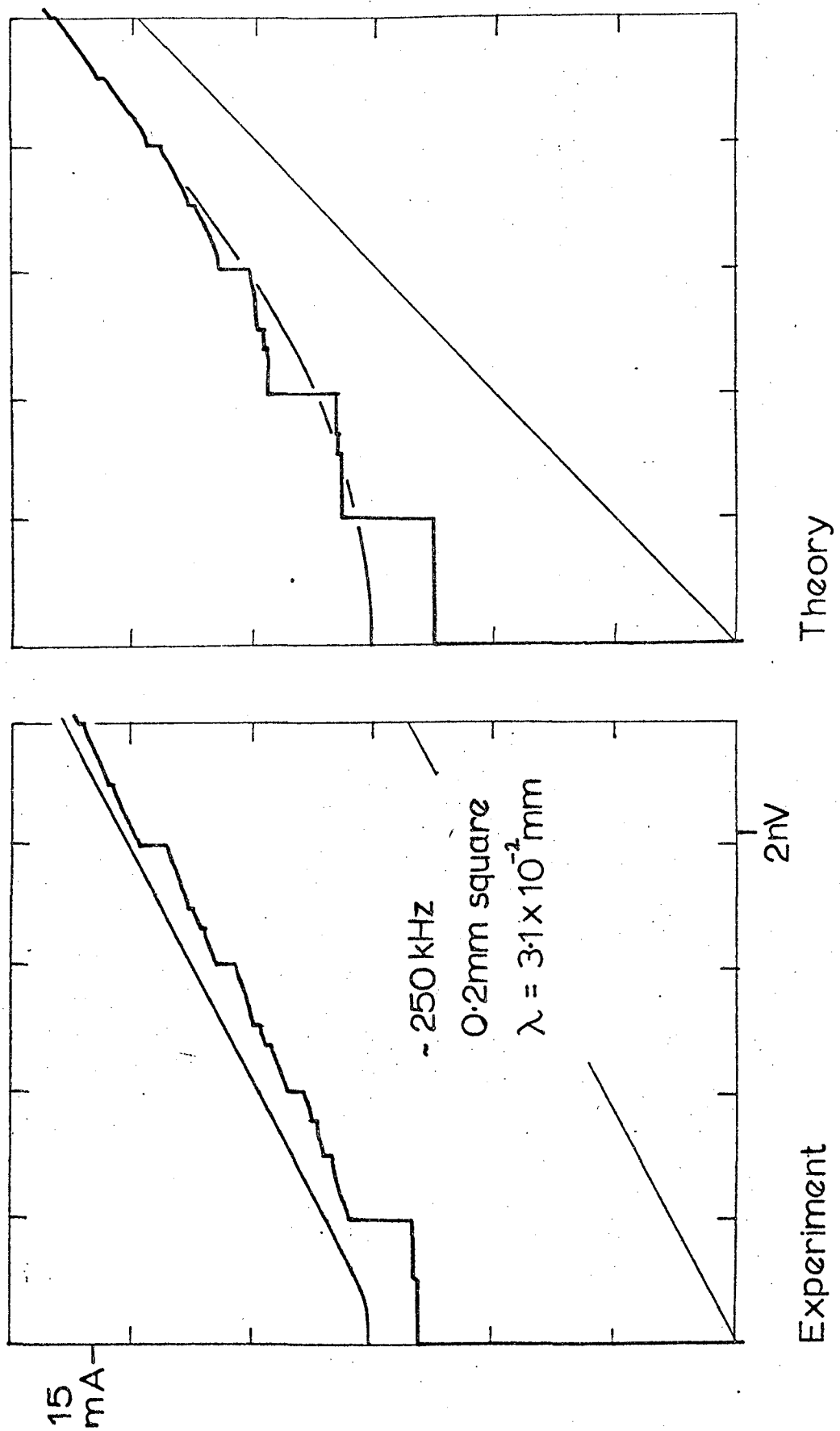


Fig. 4

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