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THE CLEBSCH-GORDAN SERIES AND THE CLEBSCH-GORDAN COEFFICIENTS OF 0(2, 1) AND SU(1, 1)

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THE CLEBSCH-GORDAN SERIES AND THE CLEBSCH-GORDAN COEFFICIENTS OF O(2,1) AND SU(1,1)*

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ABSTRACT

The Clebsch-Gordan series of the O(2,1) group and its covering group SU(1,1) for all cases except that of the supplemental series are derived. The continuable Clebsch-Gordan coefficients (or equivalently, the Wigner coefficients) are explicitly expressed in terms of the generalized hypergeometric function ${}_3F_2$. The spectra in the decomposition of the product of the two principal series are discussed. The applications to the unitary irreducible representation of O(2,2) are also studied.

I. INTRODUCTION

Recently, much attention has been paid to the group-theoretical analysis of the scattering amplitude at zero or negative momentum transfer.[1,2] For the latter case, the amplitude exhibits an O(2,1) symmetry; Reggeons, which take the role of forces in relativistic S-matrix theory,[3] transform under this symmetry group O(2,1) as the basis vector[4] of its unitary irreducible representation (u.i. rep.). Thus the Clebsch-Gordan coefficient (the C-G coefficient) of O(2,1) plays the same role as that of O(3) for physical particles.

The C-G coefficient of O(2,1) for three positive discrete (or equivalently, negative discrete) series was worked out by Andrews and Gunson[5] and Sannikov.[6] Pukanszky[7] found the multiplicity of the irreducible components resulting from decomposition of the product of two u.i. rep. of O(2,1), but he did not work out the C-G coefficients explicitly. Ferretti and Verde[8] worked out the Clebsch-Gordan series for two continuous series with some restrictions on the magnetic quantum numbers, by using the Sommerfeld-Watson transform. However, their definition of the Wigner coefficient is not normalized. They also did not investigate the relationships between the C-G coefficients for various cases. Holman and Biedenharn[9] derived many C-G coefficients from the difference equation of second order obtained from their recursion relations. Thus their C-G coefficient is not continuable in the sense that it has different functional forms for various cases. Here we shall derive two continuable C-G coefficients which are orthonormal to each other for the case of three continuous series. For all other cases, one of them vanishes, and the other is identical to the C-G coefficient obtained from the C-G series.

In Sec. I, we introduce the definition of the C-G coefficient and the conventions and notations used in this paper. We reproduce the derivation of the Clebsch-Gordan series for two continuous series with positive magnetic quantum numbers by the method initiated by Andrews and Gunson[5] and developed by Ferretti and Verde.[8] We point out the differences between our results and theirs, and explain how they occur. In Sec. III, we study the symmetry properties and asymptotic behavior of the G-function, which is equivalent to the Wigner coefficient defined by Ferretti and Verde,[9] and which is simply related to the C-G coefficient. In Sec. IV, we work out all other C-G series and thus all other C-G coefficients for positive magnetic quantum numbers. From them, we find two C-G coefficients satisfying the properties stated in the end of the last paragraph above. In Sec. V, we calculate for all other cases the C-G series and thus the C-G coefficients. Finally, we show that the C-G coefficients are also valid for the group SU(1,1), the covering group of O(2,1).

The scattering amplitude at vanishing momentum transfer has larger symmetry.[2] If one restricts oneself to the ordinary helicity amplitude, one has O(4) symmetry[10] when the total energy is less than threshold energy, and O(2,2) symmetry[11] above threshold. The explicit expression of the u.i. rep. of O(2,2), suitable for this purpose, has not been worked out. In Sec. VI, we express explicitly the u.i. rep. of O(2,2) group in terms of the C-G coefficients of O(2,1). The transformation between two u.i. rep. of O(2,2), corresponding to two different bases, is discussed. In the final section, we summarize the results obtained.

II. DECOMPOSITION OF THE PRODUCT OF TWO CONTINUOUS SERIES FOR $\nu_i > \mu_i > 0$

The C-G coefficient[12] (or equivalently the Wigner coefficient[12]) of O(3) is obtained essentially from its recursion relations[13] or from the integrals[12] involving three representation functions like $d_{\nu\mu}{}^{j}(z)$ of O(3) in the integrand. The calculation of the C-G coefficient of O(2,1) is more complicated than that of O(3), even though the representation function $d_{\nu\mu}{}^{j}(z)$ of O(2,1) is a continuation in j of that of O(3). The method applicable to the latter is not directly applicable to the former. The differences are (a) the group O(2,1) is noncompact and has an infinite group manifold, and (b) the u.i. rep. of O(2,1) has three principal series: continuous, positive discrete, and negative discrete. Each principal series has a different range for the magnetic quantum numbers. The first difference prevents one from calculating the C-G coefficient directly from the integral

$$\int D^{j_1}_{\nu_1\mu_1}(g)\, D^{j_2}_{\nu_2\mu_2}(g)\, D^{j_3}_{\nu_3\mu_3}(g)\, dg \quad ,$$

since there are no general formulas for the integrals of products of three hypergeometric functions corresponding to the O(2,1) representation functions. Because of the second fact, there are no simple C-G coefficients for particular values of the magnetic quantum numbers, which are used as a starting point for the general case in the O(3) group. Therefore we use an indirect method, initiated by Andrews and Gunson[5] and developed by Ferretti and Verde.[8]

We begin by introducing notations and conventions. The angular momentum j is defined through the Casimir invariant Q of $O(2,1)$

$$Q = J_1^{\,2} + J_2^{\,2} - J_3^2 = -(j + \tfrac{1}{2})(j - \tfrac{1}{2})$$

where J_i is the $\underline{i}$th infinitesimal generator of $O(2,1)$. The quantum number j differs from the corresponding $O(3)$ quantum number by $\frac{1}{2}$; the definition used here has the advantage that the Legendre transformation involves replacing j_i by $-j_i$. The representation of $O(2,1)$ is given[6,14] by

$$d_{\nu\mu}^{\;\;j}(z) = \left[\frac{\Gamma(\frac{1}{2} - j + \nu)\ \Gamma(\frac{1}{2} + j + \nu)}{\Gamma(\frac{1}{2} - j + \mu)\ \Gamma(\frac{1}{2} + j + \mu)}\right]^{\frac{1}{2}} \left(\frac{2}{z + 1}\right)^{(\nu+\mu)/2} \left(\frac{z - 1}{2}\right)^{(\nu-\mu)/2}$$

$$\times\ \frac{1}{\Gamma(\nu - \mu + 1)}$$

$$\times\ {}_2F_1\!\left(\tfrac{1}{2} + j - \mu,\ \tfrac{1}{2} - j - \mu;\ \nu - j - \mu;\ v - \mu + 1;\ \tfrac{1}{2}(1 - z)\right) . \tag{1}$$

The principal sheet in the j plane is defined by requiring that $d_{\nu\mu}^{\;\;j}(z)$ be positive for large and positive j. Thus $d_{\nu\mu}^{\;\;j}(z)$ has cuts along the real axis whose positions depend on the relative values of ν and μ. With this convention, one has

$$d_{\nu\mu}^{-j}(z) = d_{\nu\mu}^{j}(z) \quad . \tag{2}$$

For $\nu > \mu > 0$, all the factors in (1) are finite, but some of these factors may be divergent for other cases. However, one may take a limit as j appraoches an integer or a half-integer, and by using the well-known transformations of hypergeometric functions, one finds that the product on the right-hand side of (1) is always finite. The results are

$$d^{j}_{-\nu-\mu}(z) = (-1)^{\nu-\mu} d_{\nu\mu}^{j}(z) \quad ,$$

$$d_{\mu\nu}^{j}(z) = (-1)^{\nu-\mu} d_{\nu\mu}^{j}(z) \quad , \tag{3}$$

and

$$d^{j}_{-\mu-\nu}(z) = d_{\nu\mu}^{j}(z) \quad .$$

Usually, these relations are quoted for $\nu > \mu > 0$ and used to extend the definition of the representation function $d_{\nu\mu}^{j}(z)$ to other cases. In the sense of the limiting process mentioned above, the relations (3) are valid for any integral ν and μ. When some factors in (1) are zero or infinite, it is always implied that one takes the limit as j approaches an integer or a half-integer.

Following Bargmann,[14] one has, for the continuous series,

$$\mathrm{Re}\ j = 0 \ , \qquad \nu,\mu = 0, \pm 1, \pm 2, \cdots \quad , \tag{4}$$

for the positive discrete series,

$$\begin{aligned} j &= \tfrac{1}{2}, \tfrac{3}{2}, \tfrac{5}{2}, \cdots \quad , \\ \nu,\mu &= j + \tfrac{1}{2}, j + \tfrac{3}{2}, \cdots \quad , \end{aligned} \tag{5}$$

and for the negative discrete series,

$$\begin{aligned} j &= \tfrac{1}{2}, \tfrac{3}{2}, \tfrac{5}{2}, \cdots \quad , \\ \nu,\mu &= -j - \tfrac{1}{2}, -j - \tfrac{3}{2}, \cdots \quad . \end{aligned} \tag{6}$$

The orthogonality relation is

$$\int_1^\infty dz\, d_{\nu\mu}^{j*}(z)\, d_{\nu\mu}^{j'}(z) = \delta(j, j')/\eta(j) \quad , \tag{7}$$

where

$$\begin{aligned} \delta(j,j') &= \delta(ij - ij') && \text{for continuous series } j \text{ and } j', \\ &= \delta_{jj'} && \text{for discrete series } j \text{ and } j', \\ &= 0 && \text{for one continuous series and one discrete series,} \end{aligned}$$

and

$$\eta(j) = 2j \tan \pi(j - \mu) \qquad \text{for the continuous series,}$$
$$= 2j \qquad \text{for the discrete series.} \tag{8}$$

The analytic continuation for $|1 - z| > 2$ of the representation function $d_{\nu\mu}{}^{j}(z)$ can be expressed as

$$d_{\nu\mu}{}^{j}(z) = a_{\nu\mu}{}^{j}(z) + a_{\nu\mu}{}^{-j}(z) \quad , \tag{9}$$

where $a_{\nu\mu}{}^{j}(z)$ is defined as

$$a_{\nu\mu}{}^{j}(z) \equiv - \frac{\pi}{\sin 2\pi j} \left[\frac{\Gamma(\frac{1}{2} - j + \nu)\, \Gamma(\frac{1}{2} + j + \nu)}{\Gamma(\frac{1}{2} - j + \mu)\Gamma(\frac{1}{2} + j + \mu)} \right]^{\frac{1}{2}} \left(\frac{z + 1}{z - 1} \right)^{-(\nu+\mu)/2}$$

$$\times \left(\frac{2}{z - 1} \right)^{j+\frac{1}{2}} \left[\Gamma(\tfrac{1}{2} - j - \mu)\, \Gamma(\tfrac{1}{2} - j + \nu)\, \Gamma(2j + 1) \right]^{-1}$$

$$\times \; {}_2F_1\left(\tfrac{1}{2} + j - \nu, \tfrac{1}{2} + j - \mu; 2j + 1; \frac{2}{1 - z} \right) . \tag{10}$$

For the discrete series, we have

$$a_{\nu\mu}{}^{-j}(z) = a_{\nu\mu}{}^{j}(z) \quad ,$$

i.e., (11)

$$d_{\nu\mu}{}^{j}(z) = 2a_{\nu\mu}{}^{j}(z) \quad .$$

In deriving (11), we have used the relation

$$F(a,b;\ c;\ z) = \frac{\Gamma(a - c + 1)\ \Gamma(b - c + 1)\ \Gamma(c)}{\Gamma(a)\ \Gamma(b)\ \Gamma(-c + 2)}\ z^{-c+1}$$

$$\times\ {}_2F_1(a - c + 1,\ b - c + 1;\ -c + 2;\ z)$$

for negative integral c.

The C-G coefficient $C(j_1,\ j_2,\ j_3;\ \nu_1,\ \nu_2)$ of $O(2,1)$, like that of $O(3)$, should satisfy the following conditions:

(a) Clebsch-Gordan series (C-G series):

$$d^{j_1}_{\nu_1\mu_1}(z)\ d^{j_2}_{\nu_2\mu_2}(z)$$

$$= \sum_{j_3} C(j_1,j_2,j_3;\nu_1,\nu_2)\ d^{j_3}_{\nu_3\mu_3}(z)\ C^*(j_1,j_2,j_3;\mu_1,\mu_2)\ , \qquad (12)$$

where $\sum$ means that one sums over all the discrete series and integrates over all the continuous series that occur in the reduction of the two principal series j_1 and j_2. From the conservation of magnetic quantum number, one has $\nu_3 = \nu_1 + \nu_2$, and $\mu_3 = \mu_1 + \mu_2$.

(b) Recursion relation:

$$\left[(\tfrac{3}{2} + j_1 + \nu_1)(\tfrac{3}{2} - j_1 + \nu_1)(\tfrac{3}{2} + j_2 - \nu_2)(\tfrac{3}{2} - j_2 - \nu_2)\right]^{\frac{1}{2}}$$

$$\times\ C(j_1, j_2, j_3; \nu_1 + 2, \nu_2 - 2) + \left[(\tfrac{3}{2} + j_1 + \nu_1)(\tfrac{3}{2} - j_1 + \nu_1)\right.$$

$$\left. + (\tfrac{1}{2} + j_2 - \nu_2)(\tfrac{1}{2} - j_2 - \nu_2) - (\tfrac{1}{2} - j_3 + \nu_3)(\tfrac{1}{2} + j_3 + \nu_3)\right]^{\frac{1}{2}}$$

$$\times\ C(j_1, j_2, j_3; \nu_1 + 1, \nu_2 + 1)$$

$$+ \left[(\tfrac{1}{2} + j_1 + \nu_1)(\tfrac{1}{2} - j_1 + \nu_1)(\tfrac{1}{2} + j_2 - \nu_2)(\tfrac{1}{2} - j_2 - \nu_2)\right]^{\frac{1}{2}}$$

$$\times\ C(j_1, j_2, j_3; \nu_1\ \nu_2) = 0 \tag{13}$$

for both continuous and discrete series.

(c) Orthogonality and normalization condition:

$$\sum_{\nu_1} C^*(j_1, j_2, j_3; \nu_1, \nu_2)\, C(j_1, j_2, j_3'; \nu_1, \nu_2) = \delta(j_3', j_3) \tag{14}$$

for fixed ν_3 and $\nu_2 = \nu_3 - \nu_1$. The summation $\sum$ for ν_1 means that one sums over all the possible values of ν_1 such that ν_1, ν_2, and ν_3 are in the spectra of the magnetic quantum numbers of the u.i. reps. j_1, j_2, and j_3 respectively, as stated in Eqs. (4), (5), and (6).

These three conditions are sufficient to determine the C-G coefficient up to a phase factor which could be a function of the j_i, but they are not all necessary. If the C-G coefficient does not have multiplicity of order two, the first condition is enough. In order to remove those phase factors which depend on the j_i one must introduce the continuation condition for the C-G coefficient. That is to say, the C-G coefficient for all cases can be expressed by one analytic function. It is because the nonconstant phase factor in the C-G coefficient including the discrete (or alternatively, the continuous) series, when it is continued to the domain corresponding to the continuous (or alternatively, the discrete) series, is no longer a phase factor and thus should be omitted. The C-G coefficient for three continuous series has multiplicity two; one therefore requires the third condition to obtain the individual coefficients, as is explained later. The second condition may be taken as a consistency condition. Similarly, the second and the third conditions may be used to determine the C-G coefficient. We shall use the former method.

The C-G series for two continuous series j_1 and j_2 with positive magnetic quantum numbers has been worked out by Ferretti and Verde.[9] Since our expression is somewhat different from theirs, we derive it briefly in order to show how the difference occurs.

Using the Burchnall-Chaundy formula[15]

$$
{}_2F_1(a,\ b;\ c;\ x)\ {}_2F_1(\alpha,\ \beta;\ \gamma;\ x) = \sum_{n=0}^{\infty} \frac{(a)_n\ (b)_n\ (\gamma)_n}{n!\ (c)_n\ (c+\gamma+n-1)_n}
$$

$$
\times\ {}_3F_2(\alpha,\ 1-c-n,\ -n;\ \gamma,\ 1-a-n)\ {}_3F_2(\beta,\ 1-c-n,\ -n;\ \gamma, 1-b-n)
$$

$$
\times\ x^n\ {}_2F_1(a+\alpha+n,\ b+\beta+n;\ c+\gamma+2n;\ x)\ , \tag{15}
$$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$, etc., one obtains, for $\nu_i > \mu_i > 0$,

$$
a^{j_1}_{\nu_1\mu_1}(z)\ a^{j_2}_{\nu_2\mu_2}(z) = \sum_{n=0}^{\infty} c_n \frac{\pi \tan \pi\big((j_3)_n - \mu_3\big)}{2(j_3)_n}\ a^{(j_3)_n}_{\nu_3\mu_3}(z), \tag{16}
$$

where $(j_3)_n = \frac{1}{2} + j_1 + j_2 + n$, and j_1 and j_2 are in the continuous series, i.e., they are pure imaginary. The coefficients c_n is written in terms of the product of two G functions,

$$
c_n = -2(j_3)_n^2 \operatorname*{Res.}_{j_3=(j_3)_n} \{G(j,\ \nu)\ G(j,\ -\mu)\}\ , \tag{17}
$$

with the G function defined as

$$
\begin{aligned}
G(j,\ \nu) &\equiv G(j_1,\ j_2,\ j_3;\ \nu_1,\ \nu_2) \\
&\equiv \pi^{\frac{1}{2}}\ \alpha(j_1,\ \nu_1)\ \alpha(j_2,\ \nu_2)\ \alpha(-j_3,\ \nu_3)\omega(j_1,\ j_2,\ j_3) \qquad (18) \\
&\times\ K(j,\ \nu)\ F_{p\nu}(0.45)/\Gamma(\tfrac{1}{2} - j_2 + \nu_2)[\sin 2\pi\ j_2]^{\frac{1}{2}}\ ,
\end{aligned}
$$

where

$$\alpha(j, \nu) = [\Gamma(\tfrac{1}{2} - j + \nu)/\Gamma(\tfrac{1}{2} - j - \nu)]^{\frac{1}{2}}$$

$$\omega(j_1, j_2, j_3) = [\Gamma(\tfrac{1}{2} + j_1 + j_2 + j_3)\Gamma(\tfrac{1}{2} - j_1 + j_2 + j_3)$$
$$\times \; \Gamma(\tfrac{1}{2} + j_1 + j_2 - j_3)\,\Gamma(\tfrac{1}{2} - j_1 + j_2 - j_3)]^{\frac{1}{2}}$$

and

$$K(j, \nu) \equiv K(j_1, j_2, j_3; \nu_1, \nu_2)$$

$$= \left[\frac{\sin\pi(\tfrac{1}{2} + j_1 + \nu_1)\,\sin\pi(\tfrac{1}{2} + j_2 + \nu_2)\,\sin\pi(\tfrac{1}{2} + j_3 - \nu_3)}{\sin\pi(\tfrac{1}{2} + j_1)\,\sin\pi(\tfrac{1}{2} + j_2)\,\sin\pi(\tfrac{1}{2} + j_3)}\right]^{\frac{1}{2}} . \tag{19}$$

The Thomae-Whipple function $F_{p\nu}(0,45)$ is defined as

$$F_{p\nu}(0,45)$$
$$\equiv {}_3F_2(\tfrac{1}{2} + j_1+j_2-j_3, \tfrac{1}{2} - j_1+j_2-j_3, \tfrac{1}{2} + j_2-\nu_2; 1 + j_2-j_3+\nu_1; 1 + 2j_2)$$
$$\times \; [\Gamma(\tfrac{1}{2} + j_3 + \nu_3)\,\Gamma(1 + j_2 - j_3 + \nu_1)\,\Gamma(1 + 2j_2)]^{-1} ,$$

where ${}_3F_2$ is the generalized hypergeometric function[16,17] with unit argument. It is invariant under exchange of j_1 to $-j_1$ or j_3 to $-j_3$, or both. The G function has a one-over-square-root singularity at $j_3 = (j_3)_n$ via $\omega(j_1,j_2,j_3)$ so the product of two G-function has simple pole, and the coefficient c_n is the residue at $j_3 = (j_3)_n$. In deriving (16) we have used the relation

$$F_{p-\mu}(0)$$

$$= (-1)^{\frac{1}{2}+j_1+j_2-(j_3)_n} \frac{\Gamma(\frac{1}{2} - j_1 + \mu_1)\, \Gamma\left(\frac{1}{2} + (j_3)_n + \mu_3\right)}{\Gamma(\frac{1}{2} - j_1 - \mu_1)\, \Gamma\left(\frac{1}{2} + (j_3)_n - \mu_3\right)} F_{p\mu}(0) \quad ,$$

which can be proved from the definition of $F_{p\nu}(0)$ and the relation

$$\frac{\Gamma(\frac{1}{2} + j - \mu)\, \Gamma(\frac{1}{2} - j + \mu)}{\Gamma(\frac{1}{2} + j)\, \Gamma(\frac{1}{2} - j)} = \frac{\Gamma(\frac{1}{2} + j)\, \Gamma(\frac{1}{2} - j)}{\Gamma(\frac{1}{2} + j + \mu)\, \Gamma(\frac{1}{2} - j - \mu)} \quad .$$

One notes that the G function here is different from the Wigner coefficient defined by Ferretti and Verde by a phase factor $K(j_1, j_3, j_3; \nu_1, \nu_2)$. The sine functions in (18) and (19) and in the rest of this paper are only symbols to represent the inverse of the product of two gamma functions. Whenever one considers the phase factor for an expression involving sine functions, one must investigate the phase factor of the gamma functions through the relation

$$\sin \pi(\tfrac{1}{2} + j + n) = \frac{\pi}{\Gamma(\frac{1}{2} + j + n)\, \Gamma(\frac{1}{2} - j - n)} \quad . \tag{20}$$

This process fixes the phase factor of the expression uniquely. In this sense, one has, for $\mathrm{Re}\, j_i \geqslant 0$,

$$K(j_1, j_2, j_3; \nu_1, \nu_2) = 1 ,$$

$$K(-j_1, j_2, j_3; \nu_1, \nu_2) = (-1)^{\nu_1} ,$$

$$K(j_1, -j_2, j_3; \nu_1, \nu_2) = (-1)^{\nu_2} , \tag{21}$$

and

$$K(j_1, j_2, j_3; \nu_1, \nu_2) = (-1)^{\nu_3} .$$

From Eqs. (18), (19), and (21), it follows that

$$\begin{aligned} G(j_1,j_2,j_3; \nu_1,\nu_2) &= G(-j_1,j_2,j_3; \nu_1,\nu_2) = G(-j_1,j_2,-j_3; \nu_1,\nu_2) \\ &= G(j_1,j_2,-j_3; \nu_1,\nu_2) . \end{aligned} \tag{22}$$

These invariance properties are different by a phase factor from the Wigner coefficient in Ref. 8, because of the additional factor $K(j,\nu)$ in our definition. These properties are important to prove the positivity of the C-G series, as we discuss later.

The product $G(-j,\nu)\, G(-j,-\mu)$ has no poles at $j_3 = (j_3)_n$; one may replace (17) by

$$c_n = -2(2j_3)_n^2 \operatorname*{Res.}_{j_3=(j_3)_n}[G(j,\nu)\, G(j, -\mu) + G(-j, \nu)\, G(-j, -\mu)] . \tag{23}$$

Changing the summation in (16) into a contour integral and performing similar manipulations with

$$a^{-j_1}_{\nu_1\mu_1}(z) \quad a^{j_2}_{\nu_2\mu_2}(z), \quad a^{j_1}_{\nu_1\mu_1}(z)\ a^{-j_1}_{\nu_2\mu_2}(z), \text{ and } \quad a^{-j_1}_{\nu_1\mu_1}(z)\ a^{-j_2}_{\nu_2\mu_2}(z) ,$$

one has

$$d^{j_1}_{\nu_1\mu_1}(z)\ d^{j_2}_{\nu_2\mu_2}(z)$$

$$= i \oint_{\gamma_1+\gamma_2+\gamma_3+\gamma_4} dj_3(2j_3)\tan \pi(j_3-\mu_3)$$

$$\times\ [G(j,\ \nu)\ G(j,\ \ -\mu) + G(-j,\ \nu)\ G(-j,\ -\mu)]\ a^{j_3}_{\nu_3\mu_3}(z) \quad , \tag{24}$$

by use of (9), where the contours enclose the poles

$$\begin{aligned} j_3 &= \frac{1}{2} + j_1 + j_2 + n \ , \\ j_3 &= \frac{1}{2} + j_1 - j_2 + n \ , \\ j_3 &= \frac{1}{2} - j_1 + j_2 + n \ , \end{aligned} \tag{25}$$

and

$$j_4 = \frac{1}{2} - j_1 - j_2 + n \ , \qquad \text{for } n=0,1,2,\cdots ,$$

as shown in Fig. 1. Investigation of the asymptotic behavior in the j_3 plane shows that a Sommerfeld-Watson transform is possible. Hence, after deforming the entire contour onto the imaginary axis and picking up the pole terms, one has[8]

$$d^{j_1}_{\nu_1\mu_1}(z)\, d^{j_2}_{\nu_2\mu_2}(z)$$

$$= - \int_{0i}^{i\infty} i dj_3 (2j_3) \tan \pi(j_3 - \mu_3)$$

$$\times\ [G(j,\nu)\, G(j,-\mu) + G(-j,\nu)\, G(-j,-\mu)]\, d^{j_3}_{\nu_3\mu_3}(z)$$

$$+ \sum_{j_3=\frac{1}{2}}^{[\mu_3-\frac{1}{2}]} 2j_3 [G(j,\nu)\, G(j,-\mu) + G(-j,\nu)\, G(-j,-\mu)]\, d^{j_3}_{\nu_3\mu_3}(z) \quad . \tag{26}$$

This is the C-G series for the product of two continuous series j_1 and j_2. The two terms in the brackets in the first term of (26) cannot be separated into two factors; one depends on the magnetic quantum numbers ν_i and the other on the μ_i. The fact that they cannot reflects that the C-G coefficient for three continuous series has multiplicity of order two, as proved in the literature.[8] Thus the j_3 spectrum consists of two continuous series and one positive discrete series. Comparing (26) with (14), one has

$$C_1(j, \nu)\, C_1^*(j, \mu) + C_2(j, \nu)\, C_2^*(j, \mu)$$

$$= \eta(j_3)[G(j, \nu)\, G(j, -\mu) + G(-j, \nu)\, G(-j, -\mu)] \tag{27}$$

for the continuous series j_3, and

$$C(j, \nu)\, C^*(j, \mu) = \eta(j_3)[G(j, \nu)\, G(j, -\mu) + G(-j, \nu)\, G(-j, -\mu)] \tag{28}$$

for the positive discrete series j_3. Before identifying the C-G coefficient, we must study the properties of the G function.

III. THE G FUNCTION

Whipple and Thomae[18] investigated the relationships among the Thomae-Whipple functions $F_p(\ell;\, m,\, n)$, which are defined as

$$F_p(\ell;\, m,\, n) \;=\; [\Gamma(\alpha_{ghj})\, \Gamma(\beta_{m\ell})\, \Gamma(\beta_{n\ell})]^{-1}$$

$$\times\;\; {}_3F_2(\alpha_{gmn},\, \alpha_{hmn},\, \alpha_{jmn};\, \beta_{m\ell},\, \beta_{n\ell}) \qquad (29)$$

where g, h, j, ℓ, m, and n take $0, 1, 2, 3, 4$, and 5 permutatively. The parameters $\alpha_{\ell mn}$ and β_{mn} are defined as

$$\alpha_{\ell mn} \;=\; \frac{1}{2} + \gamma_\ell + \gamma_m + \gamma_n$$

and (30)

$$\beta_{mn} \;=\; 1 + \gamma_m - \gamma_n$$

for any γ_i, $i = 0,\cdots,5$, with the restriction

$$\sum_{i=0}^{5} \gamma_i \;=\; 0\;. \qquad (31)$$

The convergence condition for $F_p(\ell;\, m,\, n)$ is $R\ell(\alpha_{ghj}) > 0$. Thomae[18] showed that $F_p(\ell;\, m,\, n) = F_p(\ell;\, m',\, n')$ for any combination of ℓ, m, n, m', and n'. The Thomae-Whipple function $F_p(\ell;\, m,\, n)$ is thus independent of m and n and may be denoted by $F_p(\ell)$. Hence, there are ten representations for $F_p(\ell)$ obtained by permuting m and n; each has a different convergence domain and thus is useful for continuation. For our purposes, we express the γ_i in terms of the angular momenta j_i and the magnetic quantum numbers ν_i. These relations are

$$
\begin{aligned}
3\gamma_0 &= -3j_2 - 2\nu_1 - \nu_2 , \\
3\gamma_1 &= 3j_1 + \nu_1 + 2\nu_2 , \\
3\gamma_2 &= -3j_1 + \nu_1 + 2\nu_2 , \\
3\gamma_3 &= 3j_3 + \nu_1 - \nu_2 , \\
3\gamma_4 &= -2j_3 + \nu_1 - \nu_2 ,
\end{aligned}
\tag{32}
$$

and

$$3\gamma_5 = 3j_2 - 2\nu_1 - \nu_2 ,$$

where the j_i may be taken as complex numbers. The relationships between the set $(\alpha_{\ell mn}, \beta_{mn})$ and the set (j, ν) are set forth in Table I. In this connection, the Thomae-Whipple function $F_p(\ell)$ may be represented by $F_{p\nu}(\ell)$ to emphasize that it depends on the triplet (ν_1, ν_2, ν_3). By (32), we see that exchange of the indices 1 and 2 (or 3 and 4) is equivalent to the replacement of j_1 by $-j_1$ (or j_3 by $-j_3$). Hence $F_{p\nu}(0)$ is invariant under change of sign of j_1 or j_3 or both.

Besides, there are three-term relations for $F_{p\nu}(\ell)$, which are collected in Bailey's and Slater's books.[18] Many relations between the G functions can be derived by means of them. One of the important relations[9] is

$$G(j,\ \nu) \ = \ a(j,\ \nu)\ G(j,\ -\nu) + b(j,\ \nu)\ G(-j,\ -\nu) \quad , \tag{33}$$

where the coefficients $a(j,\ \nu)$ and $b(j,\ \nu)$ are defined as

$$\begin{aligned}
a(j,\ \nu) \ &\equiv \ a(j_1,\ j_2,\ j_3;\ \nu_1,\ \nu_2) \\
&\equiv [\sin \pi(\tfrac{1}{2} + j_1 + j_2 - j_3) \sin \pi(\tfrac{1}{2} + j_1 - j_2 + j_3) \sin \pi(\tfrac{1}{2} + j_2 - \nu_2) \\
&\quad + \sin \pi(\tfrac{1}{2} + j_3 + \nu_3) \sin 2\pi j_2 \sin \pi(1 - j_2 + j_3 - \nu_1)] \\
&\quad \times [\sin \pi(\tfrac{1}{2} + j_3 - \nu_3) \sin \pi(\tfrac{1}{2} + j_1 - \nu_1) \sin \pi(2j_2 + 1)]^{-1} \\
&\quad \times \ K(k,\ -\nu)/K(j,\ +\nu) \ ,
\end{aligned} \tag{34}$$

and

$$\begin{aligned}
b(j,\ \nu) \ &\equiv \ b(j_1,\ j_2,\ j_3;\ \nu_1,\ \nu_2) \\
&= -[\sin \pi(\tfrac{1}{2} + j_1 + j_2 + j_3) \sin \pi(\tfrac{1}{2} + j_1 + j_2 - j_3) \\
&\times \sin \pi(\tfrac{1}{2} - j_1 + j_2 + j_3) \sin \pi(\tfrac{1}{2} - j_1 + j_2 - j_3) \\
&\times \sin \pi(\tfrac{1}{2} + j_2 + \nu_2) \sin \pi(\tfrac{1}{2} + j_2 - \nu_2)]^{\frac{1}{2}} \\
&\times [\sin 2\pi j_2 \sin 2\pi(1 + 2j_2) \\
&\times \sin \pi(\tfrac{1}{2} + j_1 + \nu_1) \sin \pi(\tfrac{1}{2} + j_1 - \nu_1) \sin \pi(\tfrac{1}{2} + j_3 - \nu_3) \\
&\times \sin \pi(\tfrac{1}{2} + j_3 + \nu_3)]^{-\frac{1}{2}} \quad .
\end{aligned} \tag{35}$$

For particular values of the j_i, the $a(j, \nu)$ and $b(j, \nu)$ may take simpler forms. One may easily show that

$$a(-j, \nu) = -a(j, \nu) , \qquad b(-j, \nu) = b(j,\nu) ,$$

$$a(j, -\nu) = a(j, \nu) , \qquad b(j, -\nu) = b(j, \nu) , \tag{36}$$

and

$$[a(j, \nu)]^2 + [b(j, \nu)]^2 = 1 .$$

The other important relation[10] is

$$G(j_2, j_1, j_3; \nu_2, \nu_1) = c(j, \nu) G(j, \nu) + d(j, \nu) G(-j, \nu) , \tag{37}$$

where

$$c(j, \nu) \equiv c(j_1, j_2, j_3; \nu_1, \nu_2)$$

$$= - \left[\frac{\sin \pi(\frac{1}{2} - j_1 + j_2 + j_3) \sin \pi(\frac{1}{2} - j_1 + j_2 - j_3)}{\sin 2\pi j_1 \sin 2\pi j_2}\right]^{\frac{1}{2}}$$

and

$$d(j, \nu) \equiv d(j_1, j_2, j_3; \nu_1, \nu_2)$$

$$= -\left[\frac{\sin \pi(\frac{1}{2} + j_1 + j_2 + j_3) \sin \pi(\frac{1}{2} - j_1 - j_2 + j_3)}{\sin 2\pi j_1 \sin 2\pi(1 + j_2)}\right]^{\frac{1}{2}}$$

$$K(j, +\nu)/K(j, -\nu) , \tag{38}$$

with $[c(j,\nu)]^2 + [d(j,\nu)]^2 = 1$. The relations (33) and (37) show that $G(j,\nu)\,G(j,-\mu) + G(-j,\nu)\,G(-j,-\mu)$ is invariant under the exchange of ν_i and μ_i or j_1 and j_2, or both. For particular values of the j_i, the three-term relations reduce to two-term relations. We collect some of them that will be useful later. For the case in which $\frac{1}{2} \pm j_3 - \nu_3$ equals a negative integer or zero, one has

$$a(j,\nu)\,G(j,-\nu) + b(j,\nu)\,G(-j,-\nu) = 0$$

and (39)

$$a(-j,\nu)\,G(-j,-\nu) + b(-j,\nu)\,G(j,-\nu) = 0 \;.$$

For the case in which $\frac{1}{2} + j_1 - \nu_1$ is negative integer or zero, one has

$$G(j_2, j_1, j_3; \nu_2, \nu_1)$$

$$= \left[\frac{\sin 2\pi j_2}{\sin \pi(\frac{1}{2} + j_1 - j_2 + j_3)\,\sin \pi(\frac{1}{2} - j_1 + j_2 + j_3)\,\sin 2\pi j_1}\right]^{\frac{1}{2}}$$

$$\frac{\sin \pi(\frac{1}{2} + j_3 - \nu_3)\,\sin \pi(\frac{1}{2} + j_1 + \nu_1)}{\sin \pi(\frac{1}{2} + j_2 - \nu_2)}\,G(j,-\nu)$$

and (40)

$$G(j_2, j_1, j_3; -\nu_2, -\nu_1)$$

$$= [\sin \pi(\tfrac{1}{2} + j_1 - \nu_1)\sin \pi(\tfrac{1}{2} + j_1 + \nu_1)\sin \pi(\tfrac{1}{2} + j_3 + \nu_3)\sin \pi(1+2j_2)]^{\frac{1}{2}}$$

$$\times\; [\sin \pi(\tfrac{1}{2} + j_1 + j_2 + j_3)\sin \pi(\tfrac{1}{2} + j_2 + j_1 - j_3)\sin \pi(\tfrac{1}{2} + j_2 + \nu_2)$$

$$\times\; \sin \pi(\tfrac{1}{2} + j_2 - \nu_2)\sin \pi(\tfrac{1}{2} + j_3 - \nu_3)\sin 2\pi j_1]^{-\frac{1}{2}}$$

$$\times\; \sin \pi(1 + j_1 + j_2 - \nu_3)\,G(-j,-\nu) \;.$$

For the case in which $\frac{1}{2} \pm j_2 - \nu_2$ equals a negative integer or zero, one has

$$G(j, \nu) = -\frac{\sin \pi(\frac{1}{2} - j_3 + \nu_3)}{\sin \pi(1 - j_1 - j_2 + \nu_3)} G(j, -\nu) \tag{41}$$

and

$$G(-j, -\nu) = e^{-\frac{\pi}{2}i} G(j, -\nu); \qquad G(-j, \nu) = e^{\frac{\pi}{2}i} G(j, \nu) . \tag{42}$$

From (42), we see that $G(j, \nu)\, G(j, -\mu)$ is invariant under change of signs of all j_i. The limits of the G functions for two or more angular momenta in discrete series can be obtained from the above relations.

We are now at a stage where we can find the asymptotic behavior of the G function in j_i or ν_i for other parameters fixed. (The ν_i are always taken as integers, half integers, or zeros. To derive the asymptotic behavior for large $|j_3|$, we may take, for example, $F_{p\nu}(0,24)$ for $F_{p\nu}(0)$ with $\mathrm{Re}(\frac{1}{2} + j_1 + j_2 + j_3) > 0$. The generalized hypergeometric function ${}_3F_2$ is related[19] to the hypergeometric function ${}_2F_1$, the asymptotic behavior of which can be obtained. For example, we have[20]

$${}_2F_1(\tfrac{1}{2} - j_1 + \nu_1, \tfrac{1}{2} - j_3 + \nu_3; 1 - j_1 + j_2 + \nu_3; s) \sim O[(-j_3 s)^{-\frac{1}{2}+j_1-\nu_1}] \tag{43}$$

for large $|j_3|$ and $\mathrm{Re}(j_3 s) > 0$, and with other parameters fixed. From (43), one obtains the asymptotic behavior in j_3 of the generalized hypergeometric function

$$ {}_3F_2(\tfrac{1}{2} - j_3+\nu_3, \tfrac{1}{2} - j_1+\nu_1, \tfrac{1}{2} - j_1+j_2-j_3; 1 - j_1+j_2+\nu_3, 1 + j_2-j_3+\nu_1) $$

$$ \sim \; O[(-j_3)^{-\frac{1}{2}+j_1-\nu_1}] \; . \tag{44} $$

Thus the asymptotic behavior in j_3 of the G function is

$$ G(j, \nu) \;\sim\; O[(j_3)^{\nu_3-2\nu_1-1}] \tag{45} $$

for larger $|j_3|$, $\mathrm{Re}\, j_3 > 0$ and $\mathrm{Re}(\frac{1}{2} + j_1 + j_2 + j_3) > 0$, and with other parameters fixed. Similarly, the asymptotic behaviors in j_1 or j_2 of the G function can be obtained:

$$ G(j, \nu) \;\sim\; O[(j_1)^{-\nu_2-\nu_3-1}] \tag{46} $$

for large $|j_1|$ and $\mathrm{Re}\, j_1 > 0$, and with other parameters fixed, and

$$ G(j; \nu) \;\sim\; O[(j_2)^{-\nu_1-\nu_3-1}] \tag{47} $$

for large $|j_2|$ and $\mathrm{Re}\, j_2 > 0$ and with other parameters fixed. In obtaining these formulas, we have used the asymptotic behavior (44) in j_3 of the G function in order to derive the C-G series for two continuous series j_1 and j_2. The asymptotic behaviors in the j_i are particularly important for performing the Sommerfeld-Watson transform in the j_i plane.

The asymptotic behaviors in ν_1 (or equivalently ν_2) or ν_3 can be obtained in a different way. For large positive ν_1, the functions

$F_{n\nu}(4.23)$ and $F_{n\nu}(3.24)$, defined respectively as $F_{p\nu}(4.23)$ and $F_{p\nu}(3.24)$, with j_i and ν_i replaced by $-j_i$ and $-\nu_i$, have the following simple asymptotic behaviors:

$$F_{n\nu}(4.23) \sim [\Gamma(\tfrac{1}{2} - j_1 + \nu_1)\,\Gamma(1 + j_1 - j_3 + \nu_1 - \nu_3)\Gamma(1 - 2j_3)]^{-1}$$

and (48)

$$F_{n\nu}(3.24) \sim [\Gamma(\tfrac{1}{2} - j_1 + \nu_1)\Gamma(1 + j_1 + j_3 + \nu_1 - \nu_3)\Gamma(1 + 2j_3)]^{-1}\,.$$

Using a three-term relation[18]

$$\frac{\sin \pi\beta_{43}}{\pi P(\alpha_{043})} F_{p\nu}(0) = \frac{F_{n\nu}(4.23)}{\Gamma(\alpha_{132})\,\Gamma(\alpha_{135})\,\Gamma(\alpha_{325})} - \frac{F_{n\nu}(3.24)}{\Gamma(\alpha_{142})\,\Gamma(\alpha_{145})\,\Gamma(\alpha_{425})}\,,$$

one can obtain, by manipulating gamma functions and by taking the Stirling approximation,

$$G(j;\,\nu) \sim \frac{1}{(\pi)^{\frac{1}{2}}}\,\alpha(-j_3,\nu_3)\,\omega(j_1,j_2,j_3)\left[\frac{\sin \pi(\frac{1}{2}+j_1+\nu_1)\sin \pi(\frac{1}{2}+j_2-\nu_2)}{\sin 2\pi\, j_2}\right]^{\frac{1}{2}}$$

$$\times \left[\frac{(\nu_1)^{-\frac{1}{2}+j_3}\,\Gamma(2j_3)}{\Gamma(\frac{1}{2} + j_3 + \nu_3)\,\Gamma(\frac{1}{2} + j_1 + j_3 + j_3)\,\Gamma(\frac{1}{2} - j_1 + j_2 + j_3)} \right.$$

$$\left. (j_3 \leftrightarrow -j_3)\right] \qquad (49)$$

for large positive ν_1 and $\mathrm{Re}(\frac{1}{2} - j_1 + \nu_1) > 0$. Similarly, one can obtain

$$G(j,\ \nu) \sim \frac{1}{(\pi)^{\frac{1}{2}}} \alpha(-j_3,\nu_3)\ \omega(j_1,j_2,j_3) \left[\frac{\sin \pi(\frac{1}{2} + j_2 + \nu_2)}{\sin 2\pi j_2 \ \sin \pi(\frac{1}{2} + j_1 - \nu_1)}\right]^{\frac{1}{2}}$$

$$\times \left[\frac{\sin \pi(1 + j_2 - j_3 + \nu_1)\,\Gamma(2j_3)(-\nu_1)^{-\frac{1}{2}+j_3}}{\Gamma(\frac{1}{2} + j_1 + j_2 + j_3)\ \Gamma(\frac{1}{2} - j_1 + j_2 + j_3)\ \Gamma(\frac{1}{2} + j_3 + \nu_3)}\right.$$

$$\left. + \ (j_3 \leftrightarrow -j_3)\right] \tag{50}$$

for large negative ν_1 and $\mathrm{Re}(\frac{1}{2} - j_1 - \nu_1) > 0$,

$$G(j,\ \nu) \sim \frac{1}{(\pi)^{\frac{1}{2}}} \alpha(j_1,\nu_1)\ \omega(j_1,j_2,j_3)$$

$$\times \left[\frac{\sin \pi(\frac{1}{2} + j_2 - \nu_2)\sin \pi(\frac{1}{2} + j_3 - \nu_3)}{\sin 2\pi j_2}\right]^{\frac{1}{2}}$$

$$\times \left[\frac{(\nu_3)^{-\frac{1}{2}+j_1}\ \Gamma(2j_1)}{\Gamma(\frac{1}{2} + j_1 + j_2 - j_3)\Gamma(\frac{1}{2} + j_1 + j_2 + j_3)\ \Gamma(\frac{1}{2} + j_1 - \nu_1)}\right.$$

$$\left. + \ (j_1 \leftrightarrow -j_1)\right] \tag{51}$$

for large positive ν_3 and $\mathrm{Re}(\frac{1}{2} + j_1 + j_2 - j_3) > 0$, and

$$G(j,\,\nu)\ \sim\ \frac{1}{(\pi)^{\frac{1}{2}}}\,\alpha(j_1,\,\nu_1)\,\omega(j_1,\,j_2,\,j_3)\left[\frac{\sin\pi(\frac{1}{2}+j_2-\nu_2)}{\sin 2\pi j_2\,\sin\pi(\frac{1}{2}+j_3+\nu_3)}\right]^{\frac{1}{2}}$$

$$\left[\frac{\sin\pi(1-j_1+j_2+j_3)\,\Gamma(2j_1)(-\nu_3)^{-\frac{1}{2}+j_1}}{\Gamma(\frac{1}{2}+j_1+j_2-j_3)\Gamma(\frac{1}{2}+j_1+j_2+j_3)\Gamma(\frac{1}{2}+j_1+\nu_1)}\right.$$

$$\left.+\ (j_1 \longleftrightarrow -j_1)\right] \tag{52}$$

for large negative ν_3 and $\mathrm{Re}(\frac{1}{2}+j_2-\nu_2)>0$. The asymptotic behavior in ν_2 is equivalent to that in $-\nu_1$. The asymptotic behaviors of the G functions, such as $G(-j,\,\nu)$ and $G(-j,\,-\nu)$, can be obtained by proper replacements in the expressions (49) through (52).

Behaviors of the G functions, when the j_i take any of the three principal series, are tabulated in Tables II-IV.

IV. CLEBSCH-GORDAN COEFFICIENT FOR $\nu_i > \mu_i > 0$

Since the C-G coefficient for three continuous series has multiplicity of order two, we cannot determine it uniquely from (27) for this case. However, we are able to calculate two mutually orthogonal C-G coefficients.

From the recursion relation[21] between the generalized hypergeometric functions, one can prove that the G function $G(j, \nu)$ satisfies the recursion relation (13) of the C-G coefficients. Observing that the coefficients in the recursion relation are even functions of the j_i, one sees that $G(-j, \nu)$ also satisfies the recursion relation. Hence any linear combination of $G(j, \nu)$ and $G(-j, \nu)$, with its coefficients as functions of j_i and ν_3 only, satisfies this recurison relation, and so do $G(j, -\nu)$ and $G(-j, -\nu)$ by (33). This fact strongly suggests that the continuable C-G coefficients are linear combinations of $G(j, \nu)$ and $G(-j, \nu)$.

We now begin to check the orthogonality and normalization. For pure imaginary j_3, the condition (14) has to be replaced by

$$\lim_{\lambda\to 0^+} \sum_{\nu_1} C^*(j_1,j_2,j_3;\ \nu_1,\nu_2)\, C(j_1,j_2,j_3';\ \nu_1,\nu_2)\nu_1^{-\lambda}$$

$$= \delta(ij_3 - ij_3') \quad , \tag{53}$$

since $C^*(j, \nu)\, C(j, \nu)$ has oscillating terms for this case. From (49) and (50), one has

$$\lim_{\lambda\to 0^+} \sum_{\nu_1} G(j_1,j_2,j_3;\ \nu_1,\nu_2)\ G(j_1,j_2,j_3';\ -\nu_1-\nu_2)\eta(j_3)\ \nu_1^{-\lambda}$$

$$= \delta(ij_3 - ij_3') + \delta(ij_3 + ij_3')$$

and (54)

$$\lim_{\lambda\to 0^+} \sum_{\nu_1} G(j_1,j_2,j_3;\ \nu_1,\nu_2)\ G(-j_1,\ -j_2,-j_3;\ \nu_1,\nu_2)\ \eta(j_3)\ \nu_1^{-\lambda} = 0$$

for pure imaginary j_3. In deriving these relations (54), we have used the facts that (a) the inner product of two eigenfunctions with different eigenvalues j_3 and j_3' of the difference equation of second order vanishes, (b) the ordinary Riemann zeta function $\zeta(x)$ has a pole at $x = -1$ of unit residue, and (c) the singular part of the factor $(ij_3' - ij_3 + \lambda)^{-1}$ has the same effect as $\pi\delta(ij_3' - ij_3)$. However, the G function $G(j,\ \nu)$ is not the complex conjugate of $G(j,\ -\nu)$ for the case of three continuous series. One has to introduce new expressions which are linear combinations of $G(j,\ -\nu)$ and $G(j,\ \nu)$ and which are such that the orthogonality and normalization conditions are satisfied. One set of the candidates is the pair $[C_1(j,\ \nu),\ C_2(j,\ \nu)]$, with $C_1(j,\ \nu)$ and $C_2(j,\ \nu)$ defined as

$$C_1(j,\nu) \equiv C_1(j_1,j_2,j_3;\ \nu_1,\nu_2) \equiv [\eta(j_3)/b(j,\nu)]^{\frac{1}{2}}\ G(j,\ -\nu)$$

and (55)

$$C_2(j,\nu) \equiv C_1(j_1,j_2,j_3;\ \nu_1,\nu_2) \equiv [\eta(j_3)/b(j,\nu)]^{\frac{1}{2}}\ G(j,\nu) \quad .$$

From (33), (53), and (54), one can easily show that

$$\sum_{\nu_1} C_i^*(j_1,j_2,j_3;\ \nu_1,\nu_2)\ C_j(j_1,j_2,j_3';\ \nu_1,\nu_2)$$

$$= \delta_{ij}[\delta(ij_3' - ij_3) + \delta(ij_3' + ij_3)] \tag{56}$$

and

$$C_1(j,\nu)\ C_1^*(j,\mu) + C_2(j,\nu)\ C_2^*(j,\mu)$$

$$= \eta(j_3)[G(j,\ \nu)\ G(j,\ -\mu) + G(-j,\ \nu)\ G(-j,\ -\mu) \quad . \tag{57}$$

This pair of the C-G coefficients satisfies all three conditions stated in Sec. II. Nevertheless, unitary transformation in the $(C_1,\ C_2)$ space preserves the orthogonality, the normalization, and the quadratic form $C_1^*(j,\ \nu)\ C_1(j,\ \mu) + C_2^*(j,\ \nu)\ C_2(j,\ \mu)$. Infinite numbers of the pairs satisfy these conditions. One needs one more condition to fix the pair of the C-G coefficients. The continuation condition is just what we require. We have to find two C-G coefficients which are orthogonal to each other for the case of three continuous series; one of these must give the C-G coefficients when it is continued to the values of j_i

corresponding to other cases while the other must vanish. This can be achieved only after one works out the C-G coefficient for the other cases. In this section, we assume that ν_i and μ_i are integers or zeros with the restriction $\nu_i > \mu_i > 0$.

For positive discrete j_3, we expect that the right-hand side of expression (28) can be factorized. In this case, one has

$$
\begin{aligned}
&G(j,\ \nu) \sim Z^{\frac{1}{2}} , \qquad G(j,\ -\nu) \sim P^{\frac{1}{2}} , \\
&G(-j,\ \nu) \sim Z^{\frac{1}{2}} , \quad \text{and} \quad G(-j,\ -\nu) \sim P^{\frac{1}{2}} ,
\end{aligned}
\tag{58}
$$

where Z and P indicate zero and pole respectively. The superscript represents the order of the pole or zero. From (39) and (36) one can derive the equation

$$
\begin{aligned}
&\eta(j_3)[G(j,\ \nu)\ G(j,\ -\mu) + G(-j,\ \nu)\ G(-j,\ -\mu) \\
&\quad = \eta(j_3)\ G(j,\ -\nu)\ G(-j,\ -\mu)/b(j,\ \nu) \quad ,
\end{aligned}
\tag{59}
$$

where $b(j,\ \nu)$ can be shown to be real, i.e.,

$$
b(j,\ \nu) = [b(j,\ \nu)]^* \quad . \tag{60}
$$

From (22), one can identify the C-G coefficient

$$
C(j,\ \nu) = [\eta(j_3)/b(j,\ \nu)]^{\frac{1}{2}}\ G(j,\ -\nu) \quad . \tag{61}
$$

It is interesting to note that $[b(j,\ \nu)]^{-\frac{1}{2}}$ may be imaginary for some j_3 because of the factors like $[\sin \pi(\frac{1}{2} + j_3)]^{\frac{1}{2}}$. However, this factor is compensated by the factors $[\Gamma(\frac{1}{2} - j_3 - \nu_3)]^{\frac{1}{2}}$ and $K(j,\ \nu)$ in $G(j,\ \nu)$. Thus we have

$$C^*(j, \nu) = [\eta(j_3)/b(j, \nu)]^{\frac{1}{2}} G(-j, -\nu) \quad . \tag{62}$$

Comparing (55) with (61), we see that $C(j, \nu)$ and $C_1(j, \nu)$ have the same functional form except that the factor $\eta(j_3)$, which depends on whether the j_3 is in continuous series or in discrete series, is different in the two cases. The other C-G coefficients $C_2(j, \nu)$ vanishes for the discrete j_3 case.

In the case in which j_1 and j_2 are in the continuous series and the positive discrete series respectively, one can derive the C-G series by a similar method. The function $G(j, \nu)\, G(j, -\mu)$ has two series of poles in the right half j_3 plane, as shown in Fig. 2, instead of four series, as above. Thus one does not need to add a vanishing term $\underset{j_3=(j_3)_n}{\text{Res}} [G(-j, \nu)\, G(-j, -\mu)]$ to expression (17). Performing a Sommerfeld-Watson transform, one has

$$d^{j_1}_{\nu_1\mu_1}(z)\, d^{j_2}_{\nu_2\mu_2}(z) = -2\int_{0i}^{\infty i} i dj_3\, \eta(j_3)\, G(j, \nu)\, G(j, -\mu)\, d^{j_3}_{\nu_3\mu_3}(z)$$

$$+ \sum_{j_3=\frac{1}{2}}^{[\mu_3-\frac{1}{2}]} 2\eta(j_3)\, G(j, \nu)\, G(j, -\mu)\, d^{j_3}_{\nu_3\mu_3}(z) \quad . \tag{63}$$

In deriving (63), (42) is used. The j_3 spectrum is the same as in the above case. From (12), (41), and (63), one has

$$C(j,\nu)\, C^*(j,\mu) = -\frac{2\eta(j_3)\sin\pi(\frac{1}{2}-j_3+\nu_3)}{\sin\pi(1-j_1+j_2+\nu_3)}\, G(j,-\nu)\, G(j,-\mu) \quad , \tag{64}$$

from which one identifies the C-G coefficient

$$C(j,\nu) = C^*(j,\nu) \equiv \left[-\frac{2\eta(j_3)\sin\pi(\frac{1}{2}+j_3+\nu_3)}{\sin\pi(\frac{1}{2}-j_1+j_2+\nu_3)}\right]^{\frac{1}{2}} G(j,-\nu) \ . \tag{65}$$

If j_3 is in the discrete series, the factors like $[\sin\pi(\frac{1}{2}+j_3+\nu_3)]^{\frac{1}{2}}$ are compensated by $[\Gamma(\frac{1}{2}+j_3-\nu_3)]^{\frac{1}{2}}$ and $K(j,\nu)$, and the factor $(-1)^{-\frac{1}{2}+j_2+\nu_3}$ from $\sin\pi(\frac{1}{2}-j_2+j_2+\nu_3)$ is compensated by $[\Gamma(\frac{1}{2}-j_2-\nu_2)\sin 2\pi j_2]^{-\frac{1}{2}}$ and $K(j,\nu)$. It is obvious now that the presence of the factor $K(j,\nu)$ in (18) removes a phase factor that depends on the ν_i in the C-G coefficient and in the C-G series. The C-G coefficient is different from $C_1(j,\nu)$ and $C_2(j,\nu)$ in (55). However, the extra degree of freedom which we have observed in determining the C-G coefficient enables one to redefine it for the case of three continuous series so that it satisfies our continuation condition. We shall redefine it after working out the C-G coefficient for other combinations of j_1 and j_2.

In the cases in which j_1 and j_2 are in the discrete and the continuous series respectively, one cannot replace the summation in (16) by a contour integral, since there are two series of double poles in the j_3 plane, as shown in Fig. 3. One way to remove this difficulty is to exchange the roles of j_1 and j_2, so that one can use the previous method. One then obtains

$$d^{j_1}_{\nu_1\mu_1}(z)\ d^{j_2}_{\nu_2\mu_2}(z)$$

$$= -2\int_{0i}^{\infty i} i dj_3\ \eta(j_3)\ G(j_2,j_1,j_3;\ \nu_2,\nu_1)\ G(j_2,j_1,j_3;\ -\mu_2,-\mu_1)$$

$$\times\ d^{j_3}_{\nu_3\mu_3}(z) + \sum_{j_3=\frac{1}{2}}^{[\mu_3-\frac{1}{2}]} 2\eta(j_3) G(j_2,j_1,j_3;\ \nu_2,\nu_1) G(j_2,j_1,j_3;\ -\mu_2,-\mu_1)$$

$$\times\quad d^{j_3}_{\nu_3\mu_3}(z)\quad . \tag{66}$$

The j_3-spectrum is the same as in (63), as it should be. From (40), (66), and (35), one can identify the C-G coefficient

$$C(j,\ \nu) = [\eta(j_3)/b(j,\nu)]^{\frac{1}{2}}\ G(j,\ -\nu)\quad , \tag{67}$$

with

$$b(k,\ \nu) = \frac{\sin \pi(j_2 + j_3)\ \sin \pi(-j_2 + j_3)}{2e^{i\frac{\pi}{2}}\ \sin \pi(\frac{1}{2} + j_3)\ \sin \pi(\frac{1}{2} + j_1)\ \sin \pi j_2}\ ,$$

if considered in the j_3 plane.

A similar expression can be obtained if considered in the j_2 plane. As above, the factors $[\sin\pi(\frac{1}{2}+j_1)]^{-\frac{1}{2}}$ and $[\sin\pi(\frac{1}{2}+j_3)]^{-\frac{1}{2}}$ in (67) are compensated by the factors in $G(j, \nu)$, if j_3 is pure imaginary. This C-G coefficient is a continuation of $C_1(j, \nu)$, and $C_2(j, \nu)$ vanishes.

In the case in which both j_1 and j_2 are in the discrete series, one has, from (16) and (42),

$$d^{j_1}_{\nu_1\mu_1}(z)\; d^{j_2}_{\nu_2\mu_2}(z)$$

$$= 2 \sum_{j_3=\frac{1}{2}+j_1+j_2}^{[\mu_3-\frac{1}{2}]} \eta(j_3)\, G(j,\nu)\, G(j,-\mu)\, d^{j_3}_{\nu_3\mu_3}(z) \quad , \tag{68}$$

after discarding the vanishing terms. The j_3 spectrum in this case is well known.[7,9] The C-G coefficient can be considered as the limiting case of (65) or (67).

In summary, we have obtained all the C-G coefficients for $\nu_i > \mu_i > 0$. The C-G coefficients for the three continuous series has multiplicity two. These two orthogonal C-G coefficients $G(j, \nu)$ and $C_1(j, \nu)$ are defined in (55). Except for continuous j_1 and discrete j_2, the C-G coefficient for other cases is the continuation of $C_1(j, \nu)$. Since the linear combination of $C_1(j, \nu)$ and $C_2(j, \nu)$ obtained by unitary transformation for the case of three continuous series is also a C-G coefficient, we can find an expression such that the C-G coefficient for all cases is equal to its continuation. With some calculations, we obtain the following two C-G coefficients $C(j, \nu)$ and $C'(j, \nu)$:

$$C(j, \nu) = [\eta(j_3)]^{\frac{1}{2}}\{[D(j,\nu)]^{-\frac{1}{2}}G(j,-\nu) + [D(j,\nu)]^{\frac{1}{2}}G(j,\nu)\}/\sqrt{2}\, b(j,\nu)$$

and (69)

$$C'(j,\nu) = [\eta(j_3)]^{-\frac{1}{2}}\{-[D(j,\nu)]^{\frac{1}{2}}G(j,-\nu) + [D(j,\nu)^{-\frac{1}{2}}G(j,\nu)\}/\sqrt{2}\, b(j,\nu) ,$$

where $D(j,\nu)$ is defined by

$$D(j, \nu) = -b(j, \nu) + \{[b(j, \nu)]^2 + 1\}^{\frac{1}{2}} . \tag{70}$$

One can see that $C(j, \nu)$ and $C'(j,\nu)$ are orthogonal for three-continuous-series. For all other cases, $C(j, \nu)$ reduces to the C-G coefficient obtained from the C-G series, and $C'(j, \nu)$ vanishes. except the two discrete j_1 and j_2 case in which $C(j, \nu)$ and $C'(j, \nu)$ are degenerate. Hence, $C(j, \nu)$ and $C'(j, \nu)$ are the required C-G coefficients.

V. CLEBSCH-GORDAN COEFFICIENT FOR OTHER CASES

In the preceding section we have worked out the C-G coefficients with the restriction that $\nu_i > \mu_i > 0$. In this section, we calculate the C-G coefficient for arbitrary ν_i and μ_i. Finally we extend our results to the double-valued u.i. rep. of SU(1,1).

Each representation function $d^j_{\nu\mu}(z)$ has four kinds of representations, as in (3). Hence, first of all, one must decide which one should be used in applying Burchnall-Chaundy formula (15). For convenience, we always choose the expressions (1) and (10) for the representation function of the rotation along the y axis sandwiched by the state vectors with magnetic quantum number ν_i and μ_i, irrespective of the relative values and the relative signs of ν_i and μ_i.

From (21) one can easily see that the discrete spectrum for j_3 is determined by poles of the integrand $j_3 \tan \pi j_3\, G(j, \nu)\, G(j, -\mu)\, a^{j_3}_{\nu_3\mu_3}(z)$. If j_1 is in the continuous series and j_2 in the discrete series, the functions $F_{p\nu}(0)$ and $F_{p-\nu}(0)$ are finite for any ν_i and μ_ν, as they are both in the continuous series. If j_1 is in the discrete series and j_2 in the continuous series, one must derive the C-G series by exchanging the roles of j_1 and j_2 in order to remove double poles which occur in the integrand as for the case $\nu_i > \mu_i > 0$. Hence, for all the cases except that of two discrete series, the functions $F_{p\nu}(0)$ and $F_{p-\nu}(0)$, as well as $\omega(j_1, j_2, j_3)$, are finite. The order of zeros or poles of the G functions can be found in Tables II through IV. For the last case, $F_{p\nu}(0)$, $F_{p-\nu}(0)$, and $\omega(j_1, j_2, j_3)$ behave differently for various relative values and signs of the j_i and ν_i. We discuss this case in more detail.

The derivation of the C-G series can be carried out as for $\nu_i > \mu_i > 0$. The finiteness of the expression and the discrete spectrum for j_3 in the C-G series can be determined by using Tables II through V.

Previous discussions on the normalization and orthogonality condition and unitary transformation are still valid for arbitrary ν_i and μ_i. In the following, we study the C-G coefficients for any ν_i and μ_i in four cases.

A. j_1 and j_2 Continuous

The j_3 spectra for any ν_i and μ_i are given in Table VI. The two orthonormal C-G coefficients for three continuous series are the same as in (55). For discrete j_3, the C-G coefficient is defined as

$$C(j, \nu) \equiv C(j_1, j_2, j_3; \nu_1, \nu_2) \equiv [\eta(j_3)/b(j,\nu)]^{\frac{1}{2}} G(j, -\nu\zeta_A) , \tag{71}$$

where ζ_A is determined by $\nu_3\zeta_A = |\nu_3|$. These C-G coefficients are analytic continuations of one of the two C-G coefficients in (55).

B. j_1 Continuous, j_2 Discrete

The j_3 spectra for any ν_i and μ_i are summarized in Table VII. Some spectra in Table VII are missing, since there is no discrete series for j_2 with two magnetic quantum numbers of different signs. We note that there is no positive-discrete series in the decomposition of the product of the continuous and negative-discrete series, and no negative-discrete series for continuous and positive-discrete series, even though this discrete series is not forbidden by the conditions of u.i. rep. of O(2,1); one can see from Table III that in this case $G(j, \nu)\, G(j, -\mu)$ or $G(-j, \nu)\, G(-j, -\mu)$ vanishes as a double zero. This phenomenon of missing spectra also occurs in the decomposition of two positive-discrete (or negative-discrete) series; there is no continuous spectrum for j_3.

The explicit expression of the C-G coefficient for any ν_i and μ_i is

$$C(j,\nu) = [-2\eta(j_3) \sin\pi(\tfrac{1}{2} + j_3 + \nu_3)/\sin\pi(\tfrac{1}{2} - j_1 + j_2 + \nu_3)]^{\frac{1}{2}}$$

$$\times\ G(j, -\nu\zeta_B) \quad . \tag{72}$$

where ζ_B is defined through $\nu_2\zeta_B = |\nu_2|$. This C-G coefficient is identical to one of the two C-G coefficients in (55), if the latter are continued in j_i to the region corresponding to this case.

C. j_1 Discrete, j_2 Continuous

The j_3 spectra are given in Table VIII. As in case B, there is no positive-discrete spectrum of j_3 for the combination of one negative-discrete series and one continuous series (or no negative-discrete for continuous and positive-discrete). It is necessary to cope with the similar missing discrete spectrum in case B, since by exchanging the roles of j_1 and j_2 case (C) becomes case (B). The C-G coefficient is defined by

$$C(j, \nu) = [\eta(j_3)/b(j, \nu)]^{\frac{1}{2}} G(j, -\nu\zeta_c) , \tag{73}$$

where ζ_c is determined by $\nu_1\zeta_c = |\nu_1|$. Once again, this C-G coefficient is the analytic continuation of one of the two C-G coefficients in (55).

D. j_1, j_2 Discrete

For $\nu_i > \mu_i > 0$, we have worked out the C-G series (68). For many other ν_i and μ_i, there are no j_3 spectra, continuous and discrete, as shown in Table IX, because of the condition on the signs of the magnetic quantum numbers for the u.i. rep. of O(2,1). The derivations of the C-G series for the present case are much more complicated for the reasons stated earlier in this section. We have divided this case into four subcases, according to the signs of ν_i and μ_i.

If j_1 is in the negative and j_2 in the positive discrete series, one obtains, from (11), (15), and (16),

$$d^{j_1}_{\nu_1\mu_1}(z)\, d^{j_2}_{\nu_2\mu_2}(z) = \sum_{j=j_1+j_2+\frac{1}{2}} G(j,\nu)\, G(j,-\mu)\, (8j_3)\, a^{j_3}_{\nu_3\mu_3}(z) \quad . \tag{74}$$

It is easily checked by using Table IX that each term under the summation is finite except for $\nu_3 < -\frac{1}{2} - j_1 - j_2$, in which the terms with $j_3 < -\nu_3$ vanish. Equation (74) is therefore not a decomposition into u.i. reps. One can transform it into the required form by performing a Sommerfeld-Watson transform. By converting the summation into an integral, as shown in Fig. 4, one obtains the C-G series, similar to that in (63). The determination of the discrete spectrum depends on the relative values among the j_i and the ν_i. By using Tables V and IX we can get the following results.

For $j_1 > j_2$, one has two classes.

(a) For $-\nu_3$, $-\mu_3 > 0$, one has one negative-discrete spectrum for j_3 extending from $j_3 = -\frac{1}{2}$ to $j_3 = -\frac{1}{2} - j_1 + j_2$, and one continuous series.

(b) For other cases, there are only continuous spectra. Similarly, for $j_2 > j_1$, one has one continuous series and one positive-discrete spectrum running from $j_3 = \frac{1}{2}$ to $j_3 = +\frac{1}{2} - j_1 + j_2$ for the case ν_3, $\mu_3 > 0$ and only one continuous spectrum for other cases.

From the above reasoning, one sees that there are no negative discrete spectra in the decomposition of the product of one negative discrete series and one positive discrete series if the angular momentum of the latter is larger than that of the former and no positive-discrete spectrum if the angular momentum is less than the former. In a paper on the duality theorem for the SU(1,1) group, Tatsuuma[22] obtained similar results.

If j_1 is in the positive and j_2 in the negative discrete series one can obtain similar results. This can be verified directly by using Tables V and VIII. It is interesting to note that this subcase becomes the same as the above if one exchanges the role of j_1 and j_2 in (74) and finally in (63).

We have worked the subcase in which both j_1 and j_2 are in the positive discrete series [see Eq. (68)]. In a similar manner, one can obtain the C-G series for the two negative-discrete series. Again there is only one negative discrete spectrum.

For all the subcases in case (D), the C-G coefficients are the limits of those in cases B and C, as one of j_1 and j_2 becomes a half-integer.

The C-G coefficients so far obtained are for the one-valued representations i.e., all the ν_i and μ_i take integral values or zeros. We shall show that one can extend them to the double-valued representations, i.e., ν_i and μ_i take half-integral values. One notes, however, that at least one pair of ν_i and μ_i are taken as integers or zeros because of conservation of magnetic quantum numbers.

It is well known that double-valued representation belongs to the u.i. rep. of SU(1,1), the covering group of O(2,1).

From Bargmann's paper,[14] one has the double-valued representations of SU(1,1) for continuous series,

$$\mathrm{Re}\ j = 0\ , \qquad \nu,\ \mu = \pm\frac{1}{2}, \pm\frac{3}{2}, \cdots, \tag{4$'$}$$

for positive discrete series

$$\begin{aligned} j &= 1,2,\cdots, \\ \nu,\ \mu &= j+\frac{1}{2},\ j+\frac{3}{2},\cdots\ , \end{aligned} \tag{5$'$}$$

and for negative discrete series,

$$\begin{aligned} j &= 1,2,\cdots, \\ \nu,\ \mu &= -j-\frac{1}{2},\ -j-\frac{3}{2},\cdots\ . \end{aligned} \tag{6$'$}$$

We observe several facts. (a) The representation function $d_{\nu\mu}{}^{j}(z)$ defined in (1) and the G function defined in (18) are products of gamma functions, hypergeometric functions, or generalized hypergeometric functions, the argument of which are quantities like $2j_i$, $\frac{1}{2} \pm j_i \pm \nu_i$, $\frac{1}{2} \pm j_1 \pm j_2 \pm j_3$, $1 \pm j_i \pm j_j \pm \nu_k$ etc. These quantities behave as integers or zeros when the corresponding j_i are in the discrete u.i. reps, whether they are one-valued or double-valued. (b) The poles and the zeros, if any, of a gamma function, a hypergeometric function, or a generalized hypergeometric function

occur only when its arguments take negative integral values or zeros.
(c) The function $K(j, \nu)$ in the G functions is a phase factor even though some of the gamma functions in it have arguments different from the quantities mentioned above. In other words, the function $K(j, \nu)$ does not contribute to the pole structure of the G function. From these four facts, one sees that one can derive the C-G series and thus the C-G coefficients of SU(1,1) in the same way as those of O(2,1). Only one change must be made: The range of the positive (or negative) discrete spectrum of j_3, if j_3 is integral, is changed to begin from 1 (or -1) instead of $\frac{1}{2}$ (or $-\frac{1}{2}$). Therefore, our results are valid for the group SU(1,1) also.

VI. UNITARY REPRESENTATION OF THE O(2,2) GROUP

In this section, we use the explicit expression of the C-G coefficient to express the u.i.rep. of the O(2,2) group.

Let us define J_i and K_i as the infinitesimal generators of O(2,2), which keeps invariant the quadratic form $x_0^2 + x_3^2 - x_1^2 - x_2^2$. They satisfy the following commutation relations:

$$
\begin{aligned}
[J_2, J_3] &= iJ_1 , & [K_2, K_3] &= iJ_1 , \\
[J_3, J_1] &= -iJ_2 , & [K_3, K_1] &= iJ_2 , \\
[J_1, J_2] &= -iJ_3 , & [K_1, K_2] &= -iJ_3 , \\
[J_2, K_3] &= iK_1 , & [K_2, J_3] &= iK_1 , \\
[J_3, K_1] &= iK_2 , & [K_3, J_1] &= iK_2 , \\
[J_1, K_2] &= -iK_3 , & [K_1, J_2] &= -iK_3 .
\end{aligned}
\tag{75}
$$

We can easily see that any three noncommuting infinitesimal generators form the Lie Algebra of an O(2,1) group. The commutation relations in (75) are somewhat complicated. However, we may obtain simpler commutation by introducing new Lie Algebra as linear combinations of the J_i and the K_i. Defining

$$A_i = \frac{1}{2}(J_i + K_i)$$

and (76)

$$B_i = \frac{1}{2}(J_i - K_i) \qquad \text{for } i=1,2,3 ,$$

one has, from (76) and (77),

$$
\begin{aligned}
[A_2, A_3] &= iA_1 , & [B_2, B_3] &= iB_1 , \\
[A_3, A_1] &= iA_2 , & [B_3, B_1] &= iB_2 , \\
[A_1, A_2] &= -iA_3 , & [B_1, B_2] &= -iB_3 ,
\end{aligned} \tag{77}
$$

and

$$[A_i, B_j] = 0 \qquad \text{for } i,j=1,2,3.$$

The generators A_i and B_i separately form a Lie Algebra of O(2,1). In other words, O(2,2) is the product group of two O(2,1) groups, i.e.,

$$O(2,2) = O(2,1) \otimes O(2,1) \quad . \tag{78}$$

The group O(4) has similar structure, i.e., $O(4) = O(3) \otimes O(3)$. This similarity is one of the reasons that lead to the conjecture that the u.i.rep. of O(4) is a continuation of that of O(2,2), as well as that the u.i.rep. of O(3) is that of O(2,1).

The product of the two u.i.rep. of O(2,1) is a u.i.rep. of O(2,2). In this representation, the basis vectors $|a,b; \lambda_a, \lambda_b\rangle$ are eigenvector of A_3, B_3, and the two Casimir operators $\underset{\sim}{A}^2$ and $\underset{\sim}{B}^2$, where $\underset{\sim}{A}^2$ and $\underset{\sim}{B}^2$, are defined as

$$\underset{\sim}{A}^2 = A_1^{\ 2} + A_2^{\ 2} - A_3^{\ 2}$$

and (79)

$$\underset{\sim}{B}^2 = B_1^{\ 2} + B_2^{\ 2} - B_3^{\ 2} \quad .$$

The Casimir operators $\underset{\sim}{A}^2$ and $\underset{\sim}{B}^2$ have the eigenvalues $-(a + \frac{1}{2})(a - \frac{1}{2})$ and $-(b + \frac{1}{2})(b - \frac{1}{2})$. The physical interpretation of A_3 and B_3 is not clear. The basis vector $|a, b; \lambda_a \lambda_b\rangle$ is normalized by the condition

$$\langle a', b'; \lambda_a', \lambda_b' | a, b; \lambda_a \lambda_b\rangle = \delta(a', a)\, \delta(b', b)\, \delta_{\lambda_a' \lambda_a} \delta_{\lambda_b' \lambda_b} \quad . \tag{80}$$

The group element g of $O(2,2)$ can be uniquely parameterized by

$$g = e^{-iA_3\phi_A} e^{-iA_2\Theta_A} e^{-iA_3\psi_A} e^{-iB_3\phi_B} e^{-iB_2\Theta_B} e^{-iB_3\psi_B} \equiv g_A g_B \quad , \tag{81}$$

with the group parameters restricted in the domains

$$0 < \phi_A, \phi_B, \psi_A, \psi_B < 2\pi \qquad \text{and} \qquad 0 < \Theta_A, \Theta_B < \infty \quad . \tag{82}$$

The Haar measure of the group $O(2,2)$ for this parameterization is

$$dg = (2\pi)^{-4}\, d\phi_A\, d\cosh\Theta_A\, d\psi_A\, d\phi_B\, d\cosh\Theta_B\, d\psi_B \equiv dg_A\, dg_B \quad . \tag{83}$$

The corresponding u.i.rep. $D^{ab}_{\lambda_a\mu_a\lambda_b\mu_b}(g)$ defined by the equation

$$\begin{aligned} D^{ab}_{\lambda_a\mu_a\lambda_b\mu_b}(g) &\equiv \langle a, b; \lambda_a, \lambda_b | U(g) | a, b; \mu_a, \mu_b\rangle \\ &= D^a_{\lambda_a\mu_a}(g_A)\, D^b_{\lambda_b\mu_b}(g_B) \end{aligned} \tag{84}$$

satisfies the normalization and orthogonality condition

$$\int D^{ab\ *}_{\lambda_a\mu_a\lambda_b\mu_b}(g)\ D^{a'b'}_{\lambda'_a\mu'_a\lambda'_b\mu'_b}(g)\ dg$$

$$= \eta(a)^{-1}\eta(b)^{-1}\delta(a',a)\ \delta(b',b)\ \delta_{\lambda'_a\lambda_a}\ \delta_{\lambda'_b\lambda_b}\ \delta_{\mu'_a\mu_a}\ \delta_{\mu'_b\mu_b} \quad . \qquad (85)$$

We are particularly interested here in the u.i.rep. of O(2,2) whose basis vector $|a, b; j, \lambda\rangle$ is an eigenvector of $\underset{\sim}{J}^2$, J_3, $\underset{\sim}{A}^2$, and $\underset{\sim}{B}^2$. The corresponding parameterization of the group element g' can be uniquely expressed as

$$g' = e^{-i\phi J_3}\ e^{-i\beta K_3}\ e^{-i\theta J_2}\ e^{-i\alpha K_2}\ e^{-i\psi J_3}\ e^{-i\gamma K_3}$$

$$\equiv u_z(\phi)\ a_z(\beta)\ u_y(\theta)\ a_y(\alpha)\ u_z(\psi)\ a_z(\gamma) \quad , \qquad (86)$$

where the new parameters ϕ, θ, ψ, α, β, and γ are related to the old ones by

$$\phi = \frac{1}{2}(\phi_A + \phi_B)\ , \quad \theta = \frac{1}{2}(\theta_A + \theta_B)\ , \quad \psi = \frac{1}{2}(\psi_A + \psi_B)\ ,$$

and (87)

$$\alpha = \frac{1}{2}(\phi_A - \phi_B)\ , \quad \beta = \frac{1}{2}(\theta_A - \theta_B)\ , \quad \gamma = \frac{1}{2}(\psi_A - \psi_B)\quad .$$

From (87), one can calculate the Jacobian, which equals $\frac{1}{8}$. Hence the Haar measure dg' for the second parameterization is equal to $\frac{1}{8}$ that of the first one. The domains of the second set of the parameters can be obtained from (82) and (87).

The basic vector $|a, b; j, \lambda\rangle$ is normalized by the condition

$$\langle a', b'; j', \lambda'|a, b; j, \lambda\rangle = \delta(a', a)\, \delta(b'\, b)\, \delta(j', j)\, \delta_{\lambda'\lambda} \ . \tag{88}$$

It is related to the basis vector $|a, b; \lambda_a, \lambda_b\rangle$ via the equation

$$|a, b; j, \lambda\rangle = \sum_{\lambda_a} C(a, b, j; \lambda_a, \lambda - \lambda_a)|a, b; \lambda_a, \lambda - \lambda_a\rangle \ , \tag{89}$$

where the summation for λ_a has the same meaning as described in (14). From (12), one can obtain the inverse relation

$$|a, b; \lambda_a, \lambda_b\rangle = \sum_{j} C(a, b, j; \lambda_a, \lambda_b)|a, b; j, \lambda_a + \lambda_b\rangle \ , \tag{90}$$

where the meaning of the summation for j is specified in (12). From (89) and (90), one can relate the two u.i.reps. of O(2,2) by the equation

$$D^{ab}_{\lambda_a'\lambda_a\lambda_b'\lambda_b}(g)$$

$$= \sum_{jj'} C(a, b, j'; \lambda_a', \lambda_b')\, D^{ab}_{j'\lambda'j\lambda}(g')\, C(a, b, j; \lambda_a, \lambda_b) \ , \tag{91}$$

where $\lambda = \lambda_a + \lambda_b$, and $\lambda' = \lambda_a' + \lambda_b'$. The u.i.rep. $D^{ab}_{j'\lambda j\lambda}(g')$ is defined by

$$D^{ab}_{j'\lambda' j\lambda}(g') = \langle a, b; j'\lambda' | U(g') | a, b; j\lambda \rangle .$$

From (91) and (14), one can derive the inverse relation of Eq. (91). By means of Eq. (91) or its inverse relation, one can obtain the orthogonal relation for $D^{ab}_{j'\lambda' j\lambda}(g')$ from that of $D^{ab}_{\lambda_a'\lambda_b'\lambda_a\lambda_b}(g)$. That is,

$$\int D^{ab}_{j\lambda k\mu}(g')\, D^{a'b'*}_{j'\lambda' k'\mu'}(g')\, dg'$$

$$= \eta(a)^{-1}\eta(b)^{-1}\delta(a', a)\, \delta(b', b)\, \delta(j', j)\, \delta(k', k)\, \delta_{\lambda'\lambda}\, \delta_{\mu'\mu} .$$

From the properties of the C-G coefficients and the representation function $D^{ab}_{\lambda_a\mu_a\lambda_b\mu_b}(g)$, one can calculate the orthogonality relations for the representation function of the subgroups of O(2,2). From (87), (89), and (90) one can express $D^{ab}_{j'\lambda' j\lambda}(g)$ in terms of the representation functions of its one-parameter subgroups,

$$D^{ab}_{j'\lambda' j\lambda}(g')$$

$$= \sum_{kk'\mu} e^{-i\phi\lambda'}\, D^{ab}_{j\lambda' k}[a_z(\alpha)]\, d_{\lambda'\mu}{}^{k}(\theta)\, D^{ab}_{k\mu k'\lambda}[a_y(\beta)]\, D^{ab}_{k'\lambda g}[a_z(\gamma)] e^{-i\psi\lambda} .$$

The representation functions $D^{ab}_{j'\lambda j}[a_z(\alpha)]$ and $D^{ab}_{k\mu k'\lambda}[a_y(\beta)]$ can be explicitly calculated by means of Eqs. (80), (89), and (90). The expression for $D^{ab}_{j'\lambda j}[a_z(\alpha)]$ is particularly simple, i.e.,

$$D^{ab}_{j'\lambda j}[a_z(\alpha)]$$

$$= \sum_{\mu} c(a,\ b,\ j';\ \mu,\ \lambda - \mu)\ e^{i(\lambda-2\mu)\alpha}\ c^{*}(a,\ b,\ j;\ \mu,\ \lambda - \mu) \quad ,$$

which has the same form as that of O(4).

VII. CONCLUSION

The C-G coefficient of O(2,1) and SU(1,1) defined in this paper, when it is continued in the j_i into the domain corresponding to the O(3) group, is equal to that[12,13] of O(3) except for a phase factor. Strictly speaking, the Wigner coefficient,[12,23] defined by

$$W(j_1, j_2, j_3; \nu_1, \nu_2) = [\eta(j_3)]^{-\frac{1}{2}} c(j_1, j_2, j_3; \nu_1, \nu_2) ,$$

is a continuable quantity, rather than the C-G coefficient, since the Plancherel measures of O(2,1) and SU(1,1) for the discrete and continuous series differ by a factor $\tan \pi(j_3 - \mu_3)$.

The general continuable expressions of the two C-G coefficients are defined in (69). The simple expressions for particular cases are defined in (61), (65), and (67). The j_3 spectra are tabulated in Tables VI through IX. The pole structures of the related G functions for some or all of the j_i in the discrete series are collected in Tables II, III, IV, and X.

As a final remark, one notes that our SU(1,1) representation functions $d_{\nu\mu}^{\ j}(z)$ and $a_{\nu\mu}^{\ j}(z)$ of the first and the second kinds are related to Andrews and Gunson's $d_j^{\ \nu\mu}(z)$ and $e_j^{\ \nu\mu}(z)$ by the equations

$$d_j^{\ \nu\mu}(z) = e^{i\pi(\nu-\mu)} d_{\nu\mu}^{\ j}(z)$$

and

$$e_j^{\ \nu\mu}(z) = \pi e^{-i\pi(\mu-\nu)} \cot \pi(\tfrac{1}{2} + j - \mu) a_{\nu\mu}^{\ j}(z) .$$

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Table I. The relationships between the set $(\alpha_{\ell,m,n}, \beta_{mn})$ and the set (j, ν).

$\alpha_{012} = \frac{1}{2} - j_2 + \nu_2$	$\alpha_{024} = \frac{1}{2} - j_1 - j_2 - j_3$	$\alpha_{123} = \frac{1}{2} + j_3 + \nu_1 + \nu_2$	$\alpha_{145} = \frac{1}{2} + j_1 + j_2 + j_3$
$\alpha_{013} = \frac{1}{2} + j_1 - j_2 + j_3$	$\alpha_{025} = \frac{1}{2} - j_1 - \nu_1$	$\alpha_{124} = \frac{1}{2} - j_3 + \nu_3$	$\alpha_{234} = \frac{1}{2} - j_1 + \nu_1$
$\alpha_{014} = \frac{1}{2} + j_1 - j_2 - j_3$	$\alpha_{034} = \frac{1}{2} - j_2 - \nu_2$	$\alpha_{125} = \frac{1}{2} + j_2 + \nu_2$	$\alpha_{235} = \frac{1}{2} - j_1 + j_2 + j_3$
$\alpha_{015} = \frac{1}{2} + j_1 - \nu_1$	$\alpha_{035} = \frac{1}{2} + j_3 - \nu_3$	$\alpha_{134} = \frac{1}{2} + j_1 + \nu_1$	$\alpha_{245} = \frac{1}{2} - j_1 + j_2 - j_3$
$\alpha_{023} = \frac{1}{2} - j_1 - j_2 + j_3$	$\alpha_{045} = \frac{1}{2} - j_3 - \nu_3$	$\alpha_{135} = \frac{1}{2} + j_1 + j_2 + j_3$	$\alpha_{345} = \frac{1}{2} + j_2 - \nu_2$
$\beta_{01} = 1 - j_1 - j_2 - \nu_3$	$\beta_{05} = 1 - 2j_2$	$\beta_{15} = 1 + j_1 - j_2 + \nu_3$	$\beta_{34} = 1 + 2j_3$
$\beta_{02} = 1 + j_1 - j_2 - \nu_3$	$\beta_{12} = 1 + 2j_1$	$\beta_{23} = 1 - j_1 - j_3 + \nu_2$	$\beta_{35} = 1 - j_2 + j_3 + \nu_1$
$\beta_{03} = 1 - j_2 - j_3 - \nu_1$	$\beta_{13} = 1 + j_1 - j_3 + \nu_2$	$\beta_{24} = 1 - j_1 + j_3 + \nu_2$	$\beta_{45} = 1 - j_2 - j_3 + \nu_1$
$\beta_{04} = 1 - j_2 + j_3 - \nu_1$	$\beta_{14} = 1 + j_2 + j_3 + \nu_2$	$\beta_{25} = 1 - j_1 - j_2 + \nu_3$	

Table II. Behavior of the G functions for all j_i taking the values correpsonding to the i.u.rep. of O(2,1) with $\nu_1 > 0$, $\nu_2 > 0$, and $\nu_3 > 0$. The symbols F, Z, and P indicate finiteness, zero, and pole respectively. The superscripts on Z and P represent the order of zeros and poles respectively.

j_1	j_2	j_3	$G(j, \nu)$	$G(j, -\nu)$	$G(-j, \nu)$	$G(-j, -\nu)$
conti.	conti.	conti.	F	F	F	F
conti.	conti.	discrete	$Z^{\frac{1}{2}}$	$P^{\frac{1}{2}}$	$Z^{\frac{1}{2}}$	$P^{\frac{1}{2}}$
conti.	discrete	conti.	F	F	F	F
conti.	discrete	discrete	$Z^{\frac{1}{2}}$	$P^{\frac{1}{2}}$	$Z^{\frac{1}{2}}$	$P^{\frac{1}{2}}$
discrete	conti.	conti.	$Z^{\frac{1}{2}}$	$P^{\frac{1}{2}}$	$Z^{\frac{1}{2}}$	$P^{\frac{1}{2}}$
discrete	conti.	discrete	Z	P	Z	P
discrete	discrete	conti.	---	---	---	---
discrete	discrete	discrete	F	F	F	F

Table III. Behavior of the G functions for all j_k taking the values corresponding to the i.u.rep. of O(2,1) with $\nu_1 > 0$, $-\nu_2 > 0$, and $\nu_3 > 0$. The symbols and superscripts have the same meanings as in Table II.

j_1	j_2	j_3	$G(j, \nu)$	$G(j, -\nu)$	$G(-j, \nu)$	$G(-j, -\nu)$
conti.	conti.	conti.	F	F	F	F
conti.	conti.	discrete	$Z^{1/2}$	$P^{1/2}$	$Z^{1/2}$	$P^{1/2}$
conti.	discrete	conti.	F	F	F	F
conti.	discrete	discrete	$Z^{1/2}$	$Z^{3/2}$	$Z^{1/2}$	$Z^{3/2}$
discrete	conti.	conti.	$Z^{1/2}$	$P^{1/2}$	$Z^{1/2}$	$P^{1/2}$
discrete	conti.	discrete	Z	P	Z	P
discrete	discrete	conti.	$Z^{1/2}$	$P^{1/2}$	$Z^{1/2}$	$P^{1/2}$
discrete	discrete	discrete	---	---	---	---

Table IV. Behavior of the G functions for all j_i taking the values corresponding to the u.i.rep. of O(2,1) with $-\nu_1$, $\nu_2 > 0$, and $\nu_3 > 0$. The symbols and superscripts have the same meanings as in Table II.

j_1	j_2	j_3	$G(j, \nu)$	$G(j, -\nu)$	$G(-j, \nu)$	$G(-j, -\nu)$
conti.	conti.	conti.	F	F	F	F
conti.	conti.	discrete	$Z^{\frac{1}{2}}$	$P^{\frac{1}{2}}$	$Z^{\frac{1}{2}}$	$P^{\frac{1}{2}}$
conti.	discrete	conti.	F	F	F	F
conti.	discrete	discrete	$Z^{\frac{1}{2}}$	$P^{\frac{1}{2}}$	$Z^{\frac{1}{2}}$	$P^{\frac{1}{2}}$
discrete	conti.	conti.	$P^{\frac{1}{2}}$	$Z^{\frac{1}{2}}$	$P^{\frac{1}{2}}$	$Z^{\frac{1}{2}}$
discrete	conti.	discrete	F	F	F	F
discrete	discrete	conti.	$P^{\frac{1}{2}}$	$Z^{\frac{1}{2}}$	$P^{\frac{1}{2}}$	$Z^{\frac{1}{2}}$
discrete	discrete	discrete	---	---	---	---

Table V. Behavior of $a^{j_3}_{\nu_3\mu_3}(z)$ when j_3 is taken as positive integer. The symbols and superscripts have the same meanings as in Table II.

		$\nu_3 > 0$		$-\nu_3 > 0$	
		$\nu_3 > j_3 > 0$	$j_3 > \nu_3 > 0$	$-\nu_3 > j_3 > 0$	$j_3 > -\nu_3 > 0$
$\mu_3 > 0$	$\mu_3 > j_3 > 0$	F	$Z^{\frac{1}{2}}$	F	$Z^{\frac{1}{2}}$
	$j_3 > \mu_3 > 0$	$Z^{\frac{1}{2}}$	Z	$Z^{\frac{1}{2}}$	Z
$-\mu_3 > 0$	$-\mu_3 > j_3 > 0$	F	$Z^{\frac{1}{2}}$	F	$Z^{\frac{1}{2}}$
	$j_3 > -\mu_3 > 0$	$Z^{\frac{1}{2}}$	Z	$Z^{\frac{1}{2}}$	Z

Table VI. The j_3 spectra from the decomposition of the product of two continuous series j_1 and j_2 for $\nu_3 > 0$. The positive discrete spectrum runs from $j_3 = \frac{1}{2}$ to $j_3 = m + \frac{1}{2}$, the negative from $j_3 = -\frac{1}{2}$ to $j_3 = -\frac{1}{2} - m$, where m is the smaller of $|\mu_3|$ and $|\nu_3|$. The symbols c and d indicate continuous and discrete spectra, respectively. Similar results can be obtained for $\nu_3 < 0$.

	$\nu_1, \nu_2, \nu_3 > 0$	$-\nu_1, \nu_2, \nu_3 > 0$	$\nu_1, -\nu_2, \nu_3 > 0$
$\mu_1, \mu_2, \mu_3 > 0$	c, d	c, d	c, d
$-\mu_1, \mu_2, \mu_3 > 0$	c, d	c, d	c, d
$\mu_1, -\mu_2, \mu_3 > 0$	c, d	c, d	c, d
$-\mu_1, -\mu_2, -\mu_3 > 0$	c	c	c
$\mu_1, -\mu_2, \mu_3 > 0$	c	c	c
$-\mu_1, \mu_2, -\mu_3 > 0$	c	c	c

Table VII. The j_3 spectra from the decomposition of the product of one continuous series j_1 and one discrete series j_2. The range of the discrete spectrum of j_3 is the same as in Table VI. The symbols c and d indicate continuous and discrete spectra, respectively. Similar results can be obtained for $\nu_3 > 0$.

	$\nu_1, \nu_2, \nu_3 > 0$	$-\nu_1, \nu_2, \nu_3 > 0$	$\nu_1, -\nu_2, \nu_3 > 0$
$\mu_1, \mu_2, \mu_3 > 0$	c, d	c, d	---
$-\mu_1, \mu_2, \mu_3 > 0$	c, d	c, d	---
$\mu_1, -\mu_2, \mu_3 > 0$	---	---	c
$-\mu_1, -\mu_2, -\mu_3 > 0$	---	---	c
$+\mu_1, -\mu_1, -\mu_3 > 0$	---	---	c
$-\mu_1, \mu_2, -\mu_3 > 0$	c	c	---

Table VIII. The j_3 spectra from the decomposition of the product of one discrete series j_1 and one continuous series j_2 for $\nu_3 > 0$. The range of the discrete spectrum of j_3 is the same as in Table VI. The symbols c and d indicate continuous and discrete spectra, respectively. Similar results can be obtained for $\nu_3 < 0$.

	$\nu_1, \nu_2, \nu_3 > 0$	$-\mu_1, \nu_2, \nu_3 > 0$	$\nu_1, -\nu_2, \nu_3 > 0$
$\mu_1, \mu_2, \mu_3 > 0$	c, d	---	c, d
$-\mu_1, \mu_2, \mu_3 > 0$	---	c	---
$\mu_1, -\mu_2, \mu_3 > 0$	c, d	---	c, d
$-\mu_1, -\mu_2, -\mu_3 > 0$	---	c	---
$\mu_1, -\mu_2, -\mu_3 > 0$	c	---	c
$-\mu_1, \mu_2, -\mu_3 > 0$	---	c	---

Table IX. The j_3 spectrum from the decomposition of the product of two discrete series j_1 and j_2 for $\nu_3 > 0$. The range of the positive (negative) discrete spectrum) extends from $j_3 = \frac{1}{2}$ to $|\frac{1}{2} - j_1 + j_2|$. The symbols c and d indicate continuous and discrete spectra, respectively. The star * indicates that the negative discrete spectrum occurs only when the angular momentum of the negative discrete series is less than that of the positive discrete series, and vice versa. A similar result can be obtained for $\nu_3 < 0$.

	$\nu_1, \nu_2, \nu_3 > 0$	$-\nu_1, \nu_2, \nu_3 > 0$	$\nu_1, -\nu_2, \nu_3 > 0$
$\mu_1, \mu_2, \mu_3 > 0$	d	----	---
$-\mu_1, \mu_2, \mu_3 > 0$	---	c, d*	---
$\mu_1, -\mu_2, \mu_3 > 0$	---	---	c, d*
$-\mu_1, -\mu_2, -\mu_3 > 0$	---	---	---
$\mu_1, -\mu_2, -\mu_3 > 0$	---	---	---
$-\mu_1, \mu_2, -\mu_3 > 0$	---	---	---

Table X. Behavior of the G functions for the case in which j_1 and j_2 are in the negative and the positive discrete series, respectively. The symbols and superscripts have the same meanings as in Table I. Behavior of the G functions for the case in which j_1 and j_2 are in the positive and the negative discrete series, respectively, can be obtained by exchanging the roles of j_1 and j_2.

	$\nu_3 > \frac{1}{2} + j_1 + j_2$		$\frac{1}{2} + j_1 + j_2 > \nu_3 > \lvert\frac{1}{2} - j_1 + j_2\rvert$		$\lvert\frac{1}{2} - j_1 + j_2\rvert > \nu_3 > -\lvert\frac{1}{2} - j_1 + j_2\rvert$				$-\lvert\frac{1}{2} - j_1 + j_2\rvert > \nu_3 > \frac{1}{2} - j_1 - j_2$		$-\frac{1}{2} - j_1 - j_2 > \nu_3$	
					$\nu_3 > 0$		$-\nu_3 > 0$					
	$j_3 > \nu_3$	$j_3 < \nu_3$	$j_3 > \nu_3$	$j_3 < \nu_3$	$j_3 > \nu_3$	$j_3 < \nu_3$	$j_3 > -\nu_3$	$j_3 < -\nu_3$	$j_3 > -\nu_3$	$j_3 < -\nu_3$	$j_3 > -\nu_3$	$j_3 < -\nu_3$
(a) G(j, ν) with $j_1 > j_2$												
$j_3 \geq \frac{1}{2} + j_1 + j_2$	$P^{\frac{1}{2}}$	F	$P^{\frac{1}{2}}$	---	$P^{\frac{1}{2}}$	---	$P^{\frac{1}{2}}$	---	$P^{\frac{1}{2}}$	---	$P^{\frac{1}{2}}$	Z
$\frac{1}{2} + j_1 + j_2 > j_3 \geq \lvert\frac{1}{2} - j_1 + j_2\rvert$	---	$Z^{\frac{1}{2}}$	F	$Z^{\frac{1}{2}}$	F	---	F	---	F	$Z^{\frac{1}{2}}$	---	$Z^{\frac{1}{2}}$
$\lvert\frac{1}{2} - j_1 + j_2\rvert > j_3 > -\lvert\frac{1}{2} - j_1 + j_2\rvert$	---	Z	---	Z	$Z^{\frac{1}{2}}$	Z	$Z^{\frac{1}{2}}$	F	---	F	---	F
(b) G(j, -ν) with $j_1 > j_2$ and G(j, ν) with $j_1 < j_2$												
$j_3 \geq \frac{1}{2} + j_1 + j_2$	$P^{\frac{1}{2}}$	F	$P^{\frac{1}{2}}$	---	$P^{\frac{1}{2}}$	---	$P^{\frac{1}{2}}$	---	$P^{\frac{1}{2}}$	---	$P^{\frac{1}{2}}$	F
$\frac{1}{2} + j_1 + j_2 > j_3 \geq \lvert\frac{1}{2} - j_1 + j_2\rvert$	---	$Z^{\frac{1}{2}}$	F	$Z^{\frac{1}{2}}$	F	---	F	---	F	$Z^{\frac{1}{2}}$	---	$Z^{\frac{1}{2}}$
$\lvert\frac{1}{2} - j_1 + j_2\rvert > j_3 > -\lvert\frac{1}{2} - j_1 + j_2\rvert$	---	F	---	F	$Z^{\frac{1}{2}}$	F	$Z^{\frac{1}{2}}$	F	---	F	---	F
(c) G(j, -ν) with $j_1 < j_2$												
$j_3 \geq \frac{1}{2} + j_1 + j_2$	$P^{\frac{1}{2}}$	Z	$P^{\frac{1}{2}}$	---	$P^{\frac{1}{2}}$	---	$P^{\frac{1}{2}}$	---	$P^{\frac{1}{2}}$	---	$P^{\frac{1}{2}}$	F
$\frac{1}{2} + j_1 + j_2 > j_3 \geq \lvert\frac{1}{2} - j_1 + j_2\rvert$	---	$Z^{\frac{1}{2}}$	F	$Z^{\frac{1}{2}}$	F	---	F	---	F	$Z^{\frac{1}{2}}$	---	$Z^{\frac{1}{2}}$
$\lvert\frac{1}{2} - j_1 + j_2\rvert > j_3 > -\lvert\frac{1}{2} - j_1 + j_2\rvert$	---	F	---	F	$Z^{\frac{1}{2}}$	F	$Z^{\frac{1}{2}}$	Z	---	Z	---	Z

FIGURE CAPTIONS

Fig. 1. The contour of the integral with $\mathrm{Im}\ j_1 > \mathrm{Im}\ j_2 > 0$ for the case in which j_1 and j_2 are both in the continuous series. The contours γ_1 and γ_2 enclose the contributing poles of $G(j, \nu)G(j, -\mu)$, and those for γ_3 and γ_4 enclose the poles of $G(-j, \nu)\ G(-j, -\mu)$. These four contours become a single one by being connected at infinity, as shown in this figure. Similar figures can be obtained for the cases

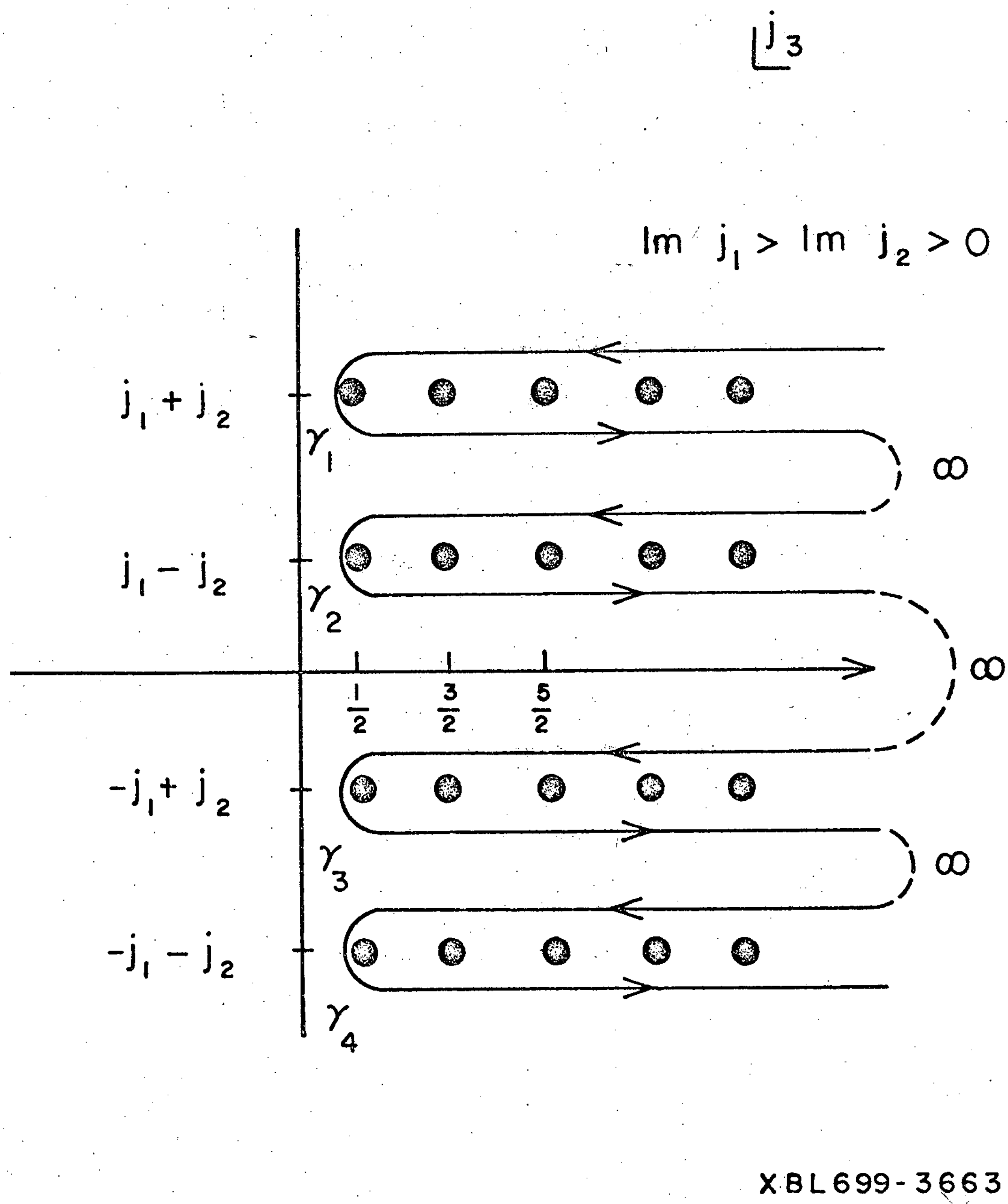
$\mathrm{Im}\ j_1 > -\mathrm{Im}\ j_2 > 0$, $\mathrm{Im}\ j_1 > \pm\ j_2 > 0$, and

$0 < \pm\ \mathrm{Im}\ j_1 < \pm\ \mathrm{Im}\ j_2$.

Fig. 2. The contour of the integral with $\mathrm{Im}\ j_1 > 0$ for the case in which j_1 and j_2 are in the continuous and the discrete series, respectively. The contours γ_1 and γ_2 enclose the contributing poles of $G(j, \nu)\ G(j, -\mu)$ in the right half plane. These two contours become a single one by being connected at infinity, as shown in this figure. A similar figure can be obtained for $\mathrm{Im}\ j_1 < 0$.

Fig. 3. The poles of $G(j, \nu)\ G(j, -\mu)$ with $\mathrm{Im}\ j_2 > 0$ in the j_3 plane for the case j_1 and j_2 in the discrete and the continuous series. The circle indicates a simple pole; the triangle indicates a double pole. A similar figure can be obtained for $\mathrm{Im}\ j_2 < 0$.

Fig. 4. The contour of the integral for the case in which j_1 and j_2 are in the negative and the positive discrete series. The starting point for the contour can be determined from Tables V and X. This figure is also valid for the case in which j_1 and j_2 are in the positive and the negative discrete series, respectively.



j_3
$\mathrm{Im}\ j_1 > \mathrm{Im}\ j_2 > 0$
$j_1 + j_2$
γ_1
∞
$j_1 - j_2$
γ_2
∞
$\frac{1}{2}$ $\frac{3}{2}$ $\frac{5}{2}$
$-j_1 + j_2$
γ_3
∞
$-j_1 - j_2$
γ_4
XBL699-3663

Fig. 1

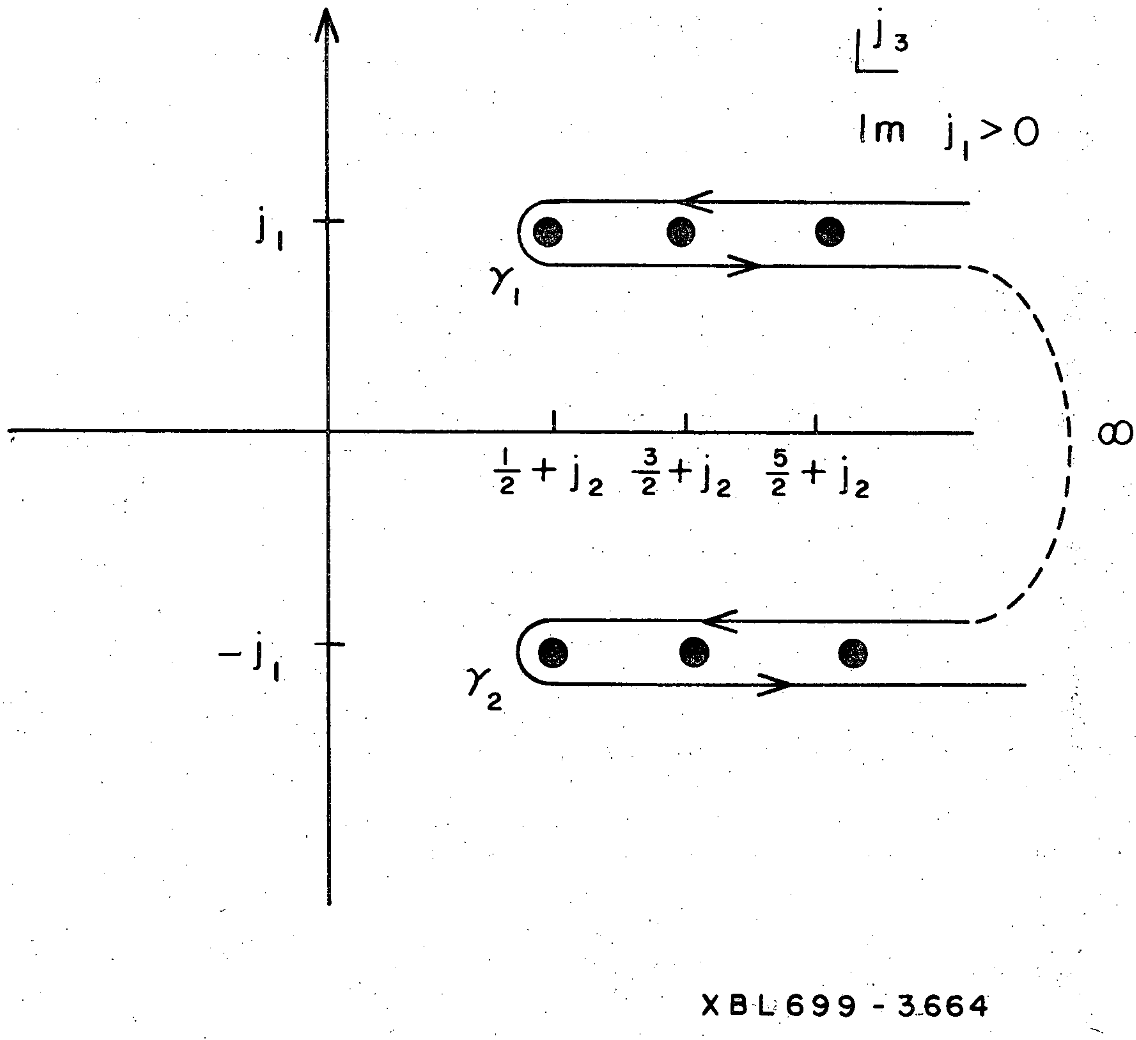
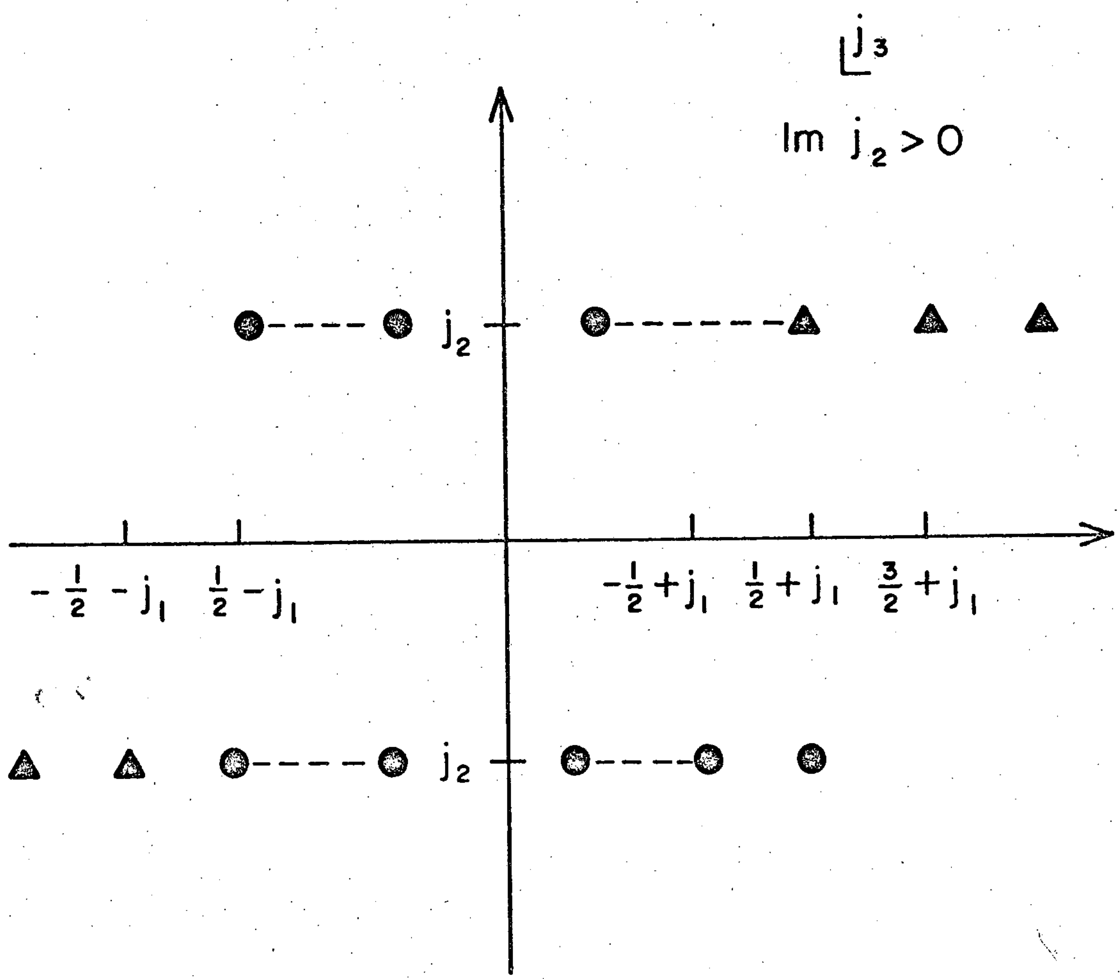

j_3
Im $j_1 > 0$
j_1
γ_1
∞
$\frac{1}{2}+j_2$ $\frac{3}{2}+j_2$ $\frac{5}{2}+j_2$
$-j_1$
γ_2
XBL 699 - 3664

Fig. 2



XBL699-3666

Fig. 3

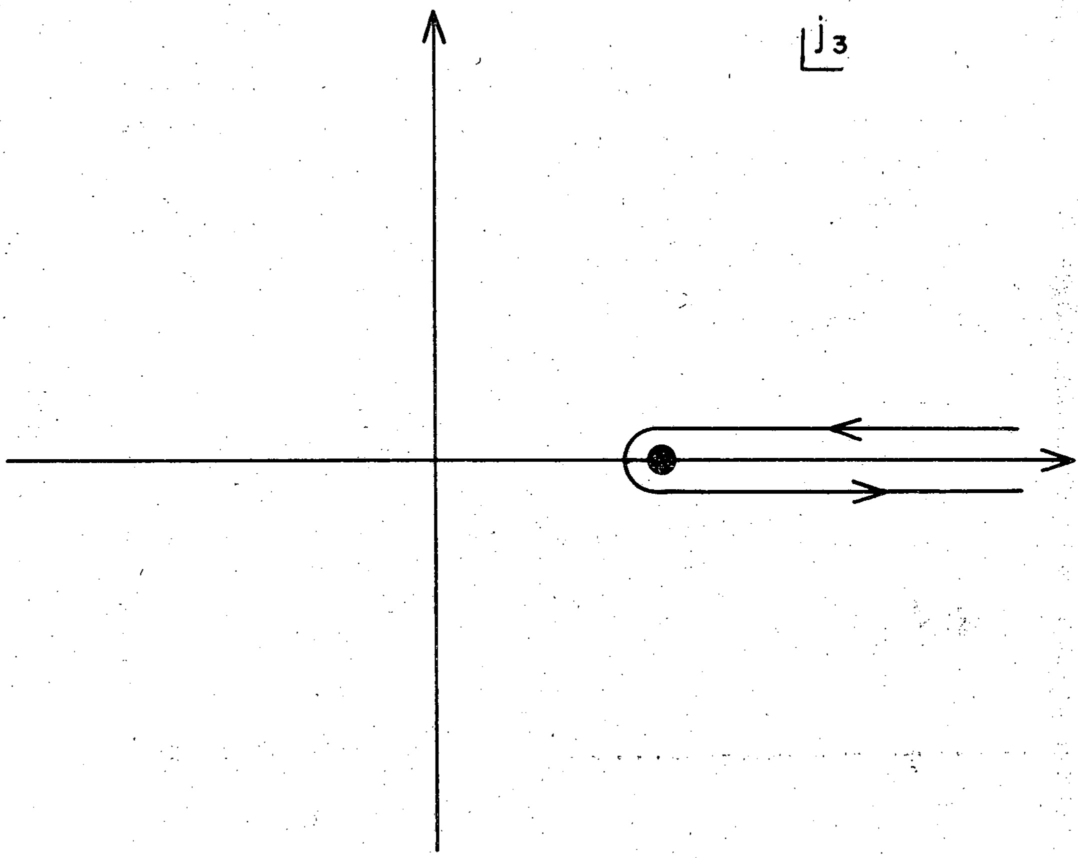


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Fig. 4

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