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THE INTERACTIONS OF WAVES AND PARTICLES IN AN INHOMOGENEOUS ONE-DIMENSIONAL PLASMA

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July 29, 1969

ABSTRACT

When a one-dimensional plasma is confined by a static potential, the electric field fluctuations can be described in terms of normal modes. a mode interacts resonantly with a particle whose bounce frequency is an integral submultiple of the mode eigenfrequency. This leads to energy exchange and quasi-linear diffusion of particles. Kinetic equations are derived for the evolution of mode energy, due to wave emission and to damping or growth; and for the evolution of the particle distribution in action-space, due to radiation reaction and quasi-linear diffusion. These equations have an H-theorem, implying an approach to thermal equilibrium (in the absence of neglected effects). The mode-particle coupling coefficient is expressed explicitly, and is approximated for the limiting cases of high eigenfrequency and of short wave length.

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1. INTRODUCTION

The kinetic theory of the interaction of particles and waves in a uniform medium is now well understood,[1-4] at least when perturbation theory is valid. In a regime where collisions and wave-wave interactions may be neglected, one has coupled kinetic equations for particles and waves, representing the effect of wave emission and absorption by particles. This set of equations has an H-theorem,[5] leading to the conclusion that a system of interacting waves and particles approaches a state of thermal equilibrium,[6] characterized by a Maxwell distribution of particles, and Rayleigh-Jeans energy for the waves.

The present work extends these results to an inhomogeneous system. In the interests of simplicity,[7] we treat here only the one-dimensional case, wherein all the particles are confined by a static potential $\Phi(x)$. The inhomogeneous plasma supports a set of electrostatic normal modes, which interact resonantly with particles whose bounce-frequency is an integral sub-multiple of the wave-frequency.[8] This interaction produces, for the particles, an energy loss (from spontaneous wave emission) and a quasi-linear diffusion in action. For the wave energy, there is correspondingly a source (from emission) and a linear decay or growth rate. Again the kinetic equations satisfy an H-theorem; here, even in the one-dimensional case, one can in general deduce an approach to thermal equilibrium.

The methods used in this paper rely on the test-particle philosophy. We do not concern ourselves with mathematical rigor, since we believe that recent techniques, developed for the uniform case,[3,4,9,10] can be extended to the inhomogeneous case to justify our approach.

In the following section, the conductivity kernel is derived, for the linear response of the inhomogeneous Vlasov plasma. It is exhibited in a

form (2.16) which generalizes the Kubo relation[11] to the non-equilibrium case. An explicit expression (2.25) for the conductivity is then obtained, whose real part (2.27) exhibits the bounce-resonance phenomenon.

In section 3, the Green's function for the electric field equation (3.1) is expressed, first in terms of the normal modes of an associated eigenvalue problem (3.6), and then in terms of the actual normal modes, satisfying (3.4). The result, equation (3.17), is similar to the form postulated by Leavens and Love[12] on the basis of numerical calculations, and has explicit normalization (3.16) of the normal modes appearing in it.

In section 4, we study the case of nearly real eigenfrequencies, and obtain an expression (4.8) for the linear damping (or growth) rate in terms of a mode-particle coupling coefficient, defined in (4.7). Its explicit evaluation (4.9) shows that only bounce-resonant particles couple to a mode. In the limit of high frequency, it is a continuous function of particle energy (4.10).

In section 5, the spontaneous (incoherent) wave emission by particles is calculated, using the test-particle philosophy. The result, equation (5.8), is expressed in terms of the coupling coefficient introduced earlier. The wave kinetic equation is then immediately obtained (5.11).

To prepare for the particle kinetic equation in section 7, the electric field spectral density is studied in section 6, and is expressed (6.9) in terms of the normal modes and the wave energies, in the limit of infinitesimal damping rate.

The following section then derives the quasi-linear diffusivity of action (7.7), again in terms of the coupling coefficient. Including the drag due to emission, the particle kinetic equation (7.11) is obtained.

In section 8, the energy conservation law and the H-theorem are derived. The conclusion that thermal equilibrium is approached asymptotically hinges

on the properties of the coupling coefficient.

In section 9, we study the limiting case that the wave length of a normal mode is small compared to the size of the system. The coupling coefficient can then be expressed (9.8) in terms of local Landau resonance, and the action-diffusivity related (9.14) to the local velocity-diffusivity.

In section 10, the form of the coupled kinetic equations is derived by taking the classical limit of the quantum master equations. The quantum transition probability is related to the classical coupling coefficient.

In section 11, the five characteristic time scales of the problem are estimated.

2. THE CONDUCTIVITY KERNEL

The conductivity represents the linear response of the stationary inhomogeneous plasma to a time-dependent electric field. It is defined by the equivalent relations

$$j(x,t) = \int_0^\infty d\tau \int dx' \; \sigma\,(x,x';\,\tau)\; E\,(x',\,t-\tau), \tag{2.1}$$

$$j(x,\omega) = \int dx' \; \sigma\,(x,x';\,\omega)\; E\,(x';\,\omega); \tag{2.2}$$

where $E(x,t)$ is the linearly perturbing self-consistent electric field, $j(x,t)$ is the mean current density, $E(x,\omega)$ and $j(x,\omega)$ are their Fourier transforms:

$$E(x\;,\omega) \equiv \int_{-\infty}^{+\infty} dt\; e^{i\omega t}\; E(x,t); \tag{2.3}$$

and

$$\sigma(x,x';\omega) \equiv \int_0^\infty d\tau\; e^{i\omega\tau}\; \sigma(x,x';\,\tau) \tag{2.4}$$

is a one-sided Fourier transform, analytic in the upper half of the ω-plane. On the real ω-axis and in the lower half-plane, one is to use not (2.4) but its analytic continuation. The reality of $E(x,t)$ and $j(x,t)$ implies the reality of $\sigma(x,x';\ \tau)$, and the relation

$$\sigma^*(x,x';\ \omega) = \sigma(x,x';\ -\ \omega^*) \tag{2.5}$$

from the definition (2.4).

In the Vlasov model, the phase-space density $f(x,p;\ t)$, for a single species, satisfies the equation

$$\partial f/\partial t + \left\{f,\ H\right\} = 0, \tag{2.6}$$

where the Hamiltonian is

$$H(x,p;\ t) = \mathcal{E}(x,p) + e\phi(x,t). \tag{2.7}$$

The unperturbed energy is

$$\mathcal{E}(x,p) = p^2/2m + e\Phi(x), \tag{2.8}$$

where $\Phi(x)$ is the confining potential, while $\phi(x,t)$ is the self-consistent perturbing potential. In the absence of ϕ, the stationary Vlasov equation is

$$\left\{f^o,\mathcal{E}\right\} = 0; \tag{2.9}$$

its general solution is

$$f^o(x,p) = f^o(\mathcal{E}). \tag{2.10}$$

(We shall not be concerned here with the self-consistency relations between $f^o(\mathcal{E})$ and $\Phi(x)$.)

In the presence of ϕ, the linearized Vlasov equation is

$$\partial\delta f/\partial t + \left\{\delta f, \mathcal{E}\right\} = -\left\{f^{o}, e\phi\right\}$$

$$= -f^{o}_{\mathcal{E}}\left\{\mathcal{E}, e\phi\right\}$$

$$= -f^{o}_{\mathcal{E}}\, ev\, E(x,t)$$

$$= -f^{o}_{\mathcal{E}}\int dx'\; j(x';\Gamma)\, E(x',t). \tag{2.11}$$

The subscript $\mathcal{E}$ denotes differentiation, and

$$j(x';\Gamma) \equiv e\; v(\Gamma)\; \delta\left[x' - x(\Gamma)\right] \tag{2.12}$$

is the current density at x', when a particle is at the phase-point $\Gamma \equiv (x,p)$. The solution of (2.11) is

$$\delta f(\Gamma;\, t) = -f^{o}_{\mathcal{E}}\int_{0}^{\infty} d\tau \int dx'\; j(x';\Gamma, -\tau)\; E(x',t-\tau), \tag{2.13}$$

where

$$j(x';\Gamma, -\tau) \equiv j\left[x';\Gamma'\,(\Gamma, -\tau)\right] \tag{2.14}$$

is the current density at x' and time $t-\tau$, when an unperturbed particle is at Γ at time t. [In (2.14), Γ' denotes the phase-point at $t-\tau$, for an unperturbed orbit through Γ at t.]

The mean current density is

$$j(x,t) \equiv \int d\Gamma\; j(x;\Gamma)\; \delta f(\Gamma\,;\, t), \tag{2.15}$$

where a sum over species will henceforth be implicit. Inserting (2.13), we obtain the conductivity kernel for (2.1):

$$\sigma(x,x';\, \tau) = -\int d\Gamma\;\; f^{o}_{\mathcal{E}}\; j(x;\Gamma)\; j(x';\Gamma, -\tau). \tag{2.16}$$

For the special case of a thermal distribution ($f^o_{\mathcal{E}} = -\beta f^o$), this reduces to

$$\sigma(x,x';\ \tau) = \beta \langle j(x,t)\ j(x't-\tau)\rangle_o\ , \tag{2.17}$$

an analogue of the Kubo relation[11] between conductivity and current fluctuations in equilibrium. Our equation (2.16) is its generalization to an inhomogeneous Vlasov system.

Examining the result (2.16), we see first that

$$\sigma(x,x';\ -\tau) = \sigma(x',\ x;\ \tau); \tag{2.18}$$

this follows from the invariance of $f^o_{\mathcal{E}}$ with respect to the time-translation $\Gamma \longrightarrow \Gamma'$. Next, the invariance of $f^o_{\mathcal{E}}$ with respect to time-reversal ($p \rightarrow -p$) leads to

$$\sigma(x,x';\ -\tau) = \sigma(x,x';\ \tau). \tag{2.19}$$

It follows from these two relations, that σ is a symmetric kernel:

$$\sigma(x,x';\ \tau) = \sigma(x',x;\ \tau), \tag{2.20}$$

$$\sigma(x,x';\ \omega) = \sigma(x',x;\ \omega). \tag{2.21}$$

(This important result is lost, if the unperturbed system includes untrapped particles, with f^o depending also on the sign of p.)

Without using the symmetry condition (2.21), we may derive a useful expression for the Hermitian part of σ at real ω, from (2.4) and (2.16):

$$\begin{aligned}\sigma'(x,x';\ \omega) &\equiv \frac{1}{2}\left[\sigma(x,x';\ \omega) + \sigma^*(x',x;\ \omega)\right] \\ &= -\frac{1}{2}\,T^{-1}\int d\Gamma\, f^o_{\mathcal{E}}\, j(x;\ \Gamma,\ \omega)\ j^*(x';\ \Gamma,\ \omega)\end{aligned} \tag{2.22}$$

where

$$j(x;\Gamma,\ \omega) \equiv \int_{-T/2}^{+T/2} dt\ \ e^{i\omega t}\ j(x;\Gamma,\ t), \tag{2.23}$$

and the limit $T \longrightarrow \infty$ is implied. For a thermal distribution, this becomes

$$\sigma'(x,x';\ \omega) = \frac{1}{2}\beta\, T^{-1} \langle j(x;\ \omega)\ j^*(x';\omega)\rangle_o\ , \tag{2.24}$$

a form of the dissipation-fluctuation theorem. Our result (2.22) is thus a generalization of it to the Vlasov non-equilibrium case.

Explicit evaluation of σ is accomplished by the introduction of action-angle variables, with $J \equiv \oint p\ dx$ in the unperturbed state, and $dw/dt = \nu(J) = d\mathcal{E}/dJ$. After some algebra we find

$$\sigma(x,x';\ \omega) = -\ ie^2\int dJ\ f_J^o \Big\{\big[\sin\omega\,\tau_1\ (x,x';\ J) - \sin\omega\,\tau_2(x,x';\ J)\big]$$

$$+\big[\cos\omega\,\tau_1\ (x,x';\ J) - \cos\omega\,\tau_2\ (x,x';\ J)\big]\ \cot\big[\omega/2\nu(J)\big]\Big\}, \tag{2.25}$$

where $\tau_1\ (x,x';\ J)$ is the direct route transit time between x and x' for an unperturbed particle of action J, while $\tau_2\ (x,x';\ J)$ is the transit time including a bounce off one end. This formula is valid in the complex ω-plane; on the real axis, approaching from above,

$$\cot\ (\omega/2\nu) \longrightarrow \mathcal{P}\Big[\cot\ (\omega/2\nu)\Big] - i\sum_{\ell=-\infty}^{+\infty} \delta(\ell - \omega/2\pi\nu) \tag{2.26}$$

Therefore on the real ω-axis, the real part of σ is

$$\sigma'(x,x';\ \omega) = -\ e^2\int dJ\ f_J^o\ (\cos\omega\,\tau_1 - \cos\omega\,\tau_2) \sum_{\ell} \delta\Big[\ell - \omega/\omega_b(J)\Big], \tag{2.27}$$

where $\omega_b\ (J) \equiv 2\pi\ \nu(J)$. We note that contributions to $\sigma'\ (\omega)$ (which is responsible for dissipation) come only from those J whose bounce-frequencies are submultiples of ω : $\omega_b\ (J) = \omega/\ell$.

3. NORMAL MODES AND THE GREEN'S FUNCTION

The linear response of the plasma to an external current density

$j^e(x,t)$ is given by

$$\partial E(x,t)/\partial t + 4\pi\left[j(x,t) + j^e(x,t)\right] = 0,$$

or

$$E(x,\omega) + 4\pi\, i\omega^{-1}\int dx'\ \sigma(x,x';\ \omega)\ E(x',\ \omega) = -\ 4\pi\ i\omega^{-1}\ j^e(x,\omega). \qquad (3.1)$$

The solution of this equation is

$$E(x,\omega) = -\ 4\pi\ i\omega^{-1}\int dx'\ G(x,x';\ \omega)\ j^e(x',\ \omega), \qquad (3.2)$$

in terms of the Green's function, which satisfies

$$G(x,x';\ \omega) + 4\pi\ i\omega^{-1}\int dx''\ \ \sigma(x,x'';\ \omega)\ G(x'',x';\ \omega) = \delta(x - x'). \qquad (3.3)$$

In this section we derive a representation for G in terms of the normal modes, i.e, the solutions of (3.1) when $j^e \equiv 0$:

$$E_a(x) + 4\pi\ i\omega_a^{-1}\int dx'\ \sigma(x,x';\ \omega_a)\ E_a(x') = 0. \qquad (3.4)$$

We note that since σ is in general complex, the eigenfrequencies ω_a and the eigenfunctions $E_a(x)$ are also in general complex. However, the reality condition (2.5) shows that they occur in pairs $(\omega_a,\ \omega_a')$, with

$$\begin{aligned} \omega_a' &= -\ \omega_a^*\ , \\ E_a'\ (x) &= E_a^*\ (x). \end{aligned} \qquad (3.5)$$

We consider first the related eigenvalue problem, wherein ω appears merely as a fixed complex parameter:

$$\lambda_n\ (\omega)\ E_n(x;\omega) + 4\pi\ i\omega^{-1}\int dx'\ \sigma(x,x';\ \omega)\ E_n(x';\ \omega) = 0. \qquad (3.6)$$

Using the symmetry[13] of σ, we find that

$$\left[\ \lambda_n(\omega) - \lambda_m\ (\omega)\ \right]\int dx\ E_n(x;\ \omega)\ E_m(x;\ \omega) = 0. \qquad (3.7)$$

Thus the eigenfunctions $E_n(x;\ \omega)$ are orthogonal:

$$\int dx\ E_n(x;\ \omega)\ E_m(x;\ \omega) = 0 \quad (m \neq n). \tag{3.8}$$

By standard techniques, the Green's function is found to be

$$G(x,x';\ \omega) = \sum_n \frac{E_n(x;\ \omega)\ E_n(x';\ \omega)}{1 - \lambda_n(\omega)}\ N_n^{-1}(\omega), \tag{3.9}$$

where $N_n(\omega) \equiv \int dx\ E_n^2\ (x;\ \omega)$.

Now we assume that $\lambda_n(\omega)$ and $E_n(x;\ \omega)$ are analytic functions of ω. (It may be helpful here to think about the special case of a uniform plasma. Then the eigenfunctions of (3.6) are $E_k(x) = \exp(ikx)$, independent of ω, while the eigenvalues are $\lambda_k(\omega) = -\ 4\pi\ i\ \omega^{-1}\ \sigma(k,\omega)$. In this case at least they are analytic in ω.) By Cauchy's theorem, (3.9) may be written as

$$G(x,x';\ \omega) = (2\pi i)^{-1}\ \oint \frac{d\omega'}{\omega' - \omega}\ G(x,x';\ \omega'), \tag{3.10}$$

where the contour encircles only the pole at $\omega' = \omega$. We then deform the contour to exclude this pole, in order to encircle the singularities of the Green's function. [There is no singularity at infinity; as $\omega \rightarrow \infty$, $\sigma(x,x';\omega) \rightarrow i(e^2/m\omega)\ n(x)\ \delta(x - x')$, and $G(x,x';\ \omega) \rightarrow \delta(x - x')$.]

We note that $G(x,x';\ \omega)$ has simple poles at those ω for which $\lambda_n(\omega) = 1$. By comparing (3.6) and (3.4), we see that these are just the eigenfrequencies ω_a. (In general, there will be many ω_a for each λ_n, as we recognize from the case of the uniform plasma.) The possible singularities of the numerator and the possible branch points of $\lambda_n(\omega)$ are not known to us, and depend on the particular problem studied. We shall here arbitrarily exclude their contribution to (3.10).[14] We then obtain

$$G(x,x';\ \omega) = \sum_n \sum_a{}' \quad \frac{E_a(x)\ E_a(x')}{(\omega_a-\omega)(d\lambda_n/d\omega)_{\omega_a}}\ N_a^{-1} \tag{3.11}$$

where, for each n, the sum over a includes only that set of ω_a satisfying $\lambda_n(\omega_a) = 1$; where $E_a(x) \equiv E_n(x;\ \omega_a)$, and where $N_a \equiv \int dx\ E_a^{\ 2}(x)$.

From (3.6) and (3.8), we see that

$$N_n(\omega)\,\lambda_n\,(\omega) = -\ 4\pi i\omega^{-1}\int dx\int dx'\sigma(x,x';\omega)\ E_n(x;\omega)\ E_n(x';\ \omega). \tag{3.12}$$

Differentiating, and using (3.6), we obtain

$$N_a\ \frac{d\lambda_n(\omega)}{d\omega}\Bigg|_{\omega_a} = -\ 4\pi\ i\int dx\int dx'\ E_a(x)\ E_a(x')\ \frac{\partial}{\partial\omega}\left[\omega^{-1}\ \sigma(x,x';\ \omega)\right]_{\omega_a}. \tag{3.13}$$

When we substitute this into (3.11), the result is

$$G(x,x';\omega) = \sum_a \frac{E_a(x)\ E_a(x')}{(\omega-\omega_a)\iint dxdx'\ E_a(x)\ E_a(x')\left[\partial\epsilon(x,x';\ \omega)/\partial\omega\right]_{\omega_a}}\ , \tag{3.14}$$

where the dielectric kernel

$$\epsilon(x,x';\ \omega) \equiv \delta(x - x') + 4\pi i\omega^{-1}\ \sigma(x,x';\ \omega) \tag{3.15}$$

has been introduced for the sake of familiarity. This form (3.14) is manifestly independent of the normalization of the normal modes. We shall now simplify the expression by choosing the normalization[15]

$$\omega_a \iint dxdx'\ E_a(x)\ E_a(x')\left[\ \partial\epsilon(x,x';\ \omega)/\partial\omega\right]_{\omega_a} = 1. \tag{3.16}$$

Then (3.14) reads

$$G(x,x';\ \omega) = \sum_a \frac{\omega_a}{\omega-\omega_a}\ E_a(x)\ E_a(x'). \tag{3.17}$$

We note that the reality conditions (3.5) ensure that G satisfies the same reality condition

$$G^*(x,x';\ \omega) = G(x,x';\ -\ \omega^*) \tag{3.18}$$

as a (3.5)

4. NEARLY REAL EIGENFREQUENCIES AND MODE-PARTICLE COUPLING

If the dielectric kernel is nearly real for real ω, i.e.,

$$\epsilon(x,x';\ \omega) \equiv \epsilon'(x,x';\ \omega) + i\ \epsilon''(x,x';\ \omega), \tag{4.1}$$

with $|\epsilon''| \ll |\epsilon'|$, perturbation theory may be used to obtain additional results. The zero-order eigenvalue equation (3.6), using only ϵ' (or σ''), is

$$\left[\lambda_n^{(o)}(\omega) - 1\right] E_n^{(o)}(x;\ \omega) + \int dx'\ \epsilon'(x,x';\ \omega)\ E_n^{(o)}(x';\ \omega) = 0. \tag{4.2}$$

Since ϵ' is Hermitian (being real and symmetric), the zero-order eigenvalues $\lambda_n^{(o)}(\omega)$ are real functions for real ω. Considering their analytic continuation, it follows that the zero-order eigenfrequencies $\omega_a^{(o)}$ [solutions of $\lambda_n^{(o)}(\omega) = 1$] occur in complex-conjugate pairs. Hence the system, to zero-order, either is unstable or has only real eigenfrequencies.

We proceed then to study the effect of the small ϵ'', in the latter case only. Equation (3.12) may be written, using (3.15), as

$$N_n(\omega)\left[1 - \lambda_n(\omega)\right] = \iint dx dx'\ \epsilon(x,x';\ \omega)\ E_n(x;\ \omega)\ E_n(x';\ \omega). \tag{4.3}$$

The perturbation in $\lambda_n(\omega)$, due to ϵ'', is thus

$$N_n^{(o)}(\omega)\ \lambda_n^{(1)}(\omega) = -\ i \iint dx dx'\ \epsilon''(x,x;\ \omega)\ E_n^{(o)}(x;\ \omega)\ E_n^{(o)}(x';\ \omega). \tag{4.4}$$

The first order perturbation in ω_a is then

$$\begin{aligned} \omega_a^{(1)} &= -\lambda_n^{(1)}(\omega_a^{(o)})\left[d\lambda_n^{(o)}/d\omega\right]^{-1} \\ &= -\ 4\pi i \iint dx dx' \sigma'(x,x';\ \omega_a^{(o)}) E_a^{(o)}(x)\ E_a^{(o)}(x'), \end{aligned} \tag{4.5}$$

where we have used (3.13). (These results are clearly generalizations of well-known formulas for the uniform plasma.) Thus the damping of the mode

is proportional to the real part of the conductivity.

With the form (2.22) for σ', the damping rate $\gamma_a \equiv -\operatorname{Im}\omega_a$ becomes

$$\gamma_a = -2\pi\int d\Gamma\, f^o_{\mathcal{E}}\, T^{-1}\left|\int dx\, j(x;\Gamma,\omega_a^{(o)})\, E_a^{(o)}(x)\right|^2 \tag{4.6}$$

[We have used the reality of $E_a^{(o)}(x)$, which follows from the reality of the integral equation (4.2).] It is now convenient to define a coupling coefficient, representing the resonant coupling of a particle of action J to the normal mode a:

$$\alpha_a(J) \equiv 2\pi T^{-1}\oint dw\left|\int dx\, j(x;J,w,\omega_a^{(o)})\, E_a^{(o)}(x)\right|^2. \tag{4.7}$$

The damping rate is then simply

$$\gamma_a = -\int dJ\, f^o_{\mathcal{E}}\, \alpha_a(J). \tag{4.8}$$

It is clear that a system with monotonically decreasing f^o is stable.

Explicit evaluation of $\alpha_a(J)$, using the particle orbit, yields

$$\alpha_a(J) = 4e^2\omega_b(J)\left[\int dx\, E_a^{(o)}(x)\sin\omega_a^{(o)}\tau(x,J)\right]^2\sum_{\ell=-\infty}^{+\infty}\delta[\ell-\omega_a^{(o)}/\omega_b(J)]. \tag{4.9}$$

where $\tau(x,J)$ is the transit time for a particle of action J to reach the position x from either turning point. For a given mode, this is a singular function of J, being infinite at the resonant values $\omega_b(J) = \omega_a^{(o)}/\ell$, and zero elsewhere.

However, if the eigenfrequency $\omega_a^{(o)}$ is much larger than typical bounce frequencies, these singularities are spaced very close together. A slight amount of coarse-graining in J, representing almost any higher-order effect, allows us then to replace the sum over ℓ by an integral over ℓ, whence the

summation equals one. Thus for $\omega_b(J) \ll \omega_a^{(o)}$, we have the approximation

$$\alpha_a(J) = 4e^2\omega_b(J)\left[\int dx\ E_a^{(o)}(x)\ \sin\ \omega_a^{(o)}\ \tau(x,J)\right]^2. \tag{4.10}$$

5. WAVE EMISSION AND THE WAVE KINETIC EQUATION

The rate of spontaneous emission of wave energy may be calculated by considering the particles as uncorrelated current sources [9,10]. We therefore first calculate the rate at which an arbitrary external current $j^e(x,t)$ does work on the plasma, in the linear-response approximation. The rate of increase of plasma energy is

$$\dot{W} = -\ T^{-1}\int dt \int dx\ E(x,t)\ j^e(x,t), \tag{5.1}$$

where $E(x,t)$ is the mean field, given by (3.1). Since the rate of wave emission is independent of the wave damping or growth rate, if the latter is small, we take the formal limit of $\gamma_a \longrightarrow 0$, for modes with small $|\gamma_a|$, and discard strongly damped modes. Then the poles of the Green's function (3.17) are at $\omega_a^{(o)}$, and lie on the real ω-axis. We may then write (5.1) as

$$\dot{W} = -\ T^{-1}\int \frac{d\omega}{2\pi}\int dx\ E(x,\omega)\ j^{e*}(x,\omega), \tag{5.2}$$

where the ω-contour lies just above the real axis. By (3.2)and (3.17), this is

$$\dot{W} = 4\pi i\ T^{-1}\int \frac{d\omega}{2\pi\omega}\int dx \int dx'\ G(x,x';\ \omega)\ j^{e*}(x,\omega) j^e(x',\omega). \tag{5.3}$$

$$= 4\pi i\ T^{-1}\int \frac{d\omega}{2\pi}\ \sum_a\ \frac{\omega_a^{(o)}}{\omega}\ \frac{1}{\omega-\omega_a(o)}\ \left|\int dx\ j^e(x,\omega)\ E_a^{(o)}(x)\right|^2. \tag{5.4}$$

The contribution of the poles at $\omega_a^{(o)}$ yields

$$\dot{W} = 4\pi\ T^{-1}\ \sum_a \left|\int dx\ j^e(x,\omega_a)\ E_a(x)\right|^2, \tag{5.5}$$

where we have included the pair $(\omega_a^{(o)}, -\omega_a^{(o)})$ for each term in the sum, and let the superscript zero be implicit. There is no pole at $\omega=0$, because j_e vanishes at $\omega=0$. The contribution of the contour between poles vanishes, because the integrand is odd in ω.

The test particle theorem[9] now allows us to obtain the mean rate of wave emission by replacing $j^e(x,\omega)$ by the sum of contributions from uncorrelated unperturbed particles:

$$\langle j^e(x,\omega)\, j^{e*}(x',\omega)\rangle \longrightarrow \int dJ\, f^o(J) \oint dw\, j(x;\, J,w,\omega)\, j^*(x';\, J,w,\omega). \tag{5.6}$$

The wave emission is thus a sum of terms:

$$\dot{W} = \sum_a \dot{W}_a\,, \tag{5.7}$$

with the emission of mode a given by

$$\dot{W}_a = 2\int dJ\, f^o(J)\, \alpha_a(J), \tag{5.8}$$

in terms of the coupling coefficient (4.7). The appearance of $\alpha_a(J)$ in both emission (5.8) and damping (4.8) may be thought of as "detailed balance."

Including both emission and damping (or growth), the wave kinetic equation is

$$dW_a/dt = \dot{W}_a - 2\gamma_a W_a. \tag{5.9}$$

If f^o is changing slowly compared to W_a, and if γ_a is positive, a quasi-stationary state is reached for W_a given by

$$\begin{aligned} W_a &= \frac{1}{2}\dot{W}_a/\gamma_a \\ &= \left[\int dJ\, f^o(J)\, \alpha_a(J)\right] \Big/ \left[\int dJ(-f^o_{\mathcal{E}})\, \alpha_a(J)\right]\,; \end{aligned} \tag{5.10}$$

then W_a varies in time adiabatically with f^o.

In any case, equation (5.9) may be written, with (4.8) and (5.8), as

$$dW_a/dt = 2\int dJ\ \alpha_a(J)\ (f^o + W_a\ f^o_{\mathcal{E}}). \tag{5.11}$$

This form may be used in particular for an unstable mode ($\gamma_a < 0$), so long as higher order effects may be neglected, and when f^o varies on the same scale as W_a, so long as their variation is sufficiently slow ($\partial/\partial t \ll \omega_a$) to retain the validity of the formulas for γ_a and $\dot{W}_a$.

Since energy must be conserved, it is clear that f^o must vary when W_a does; the particle kinetic equation analogous to (5.11) is derived in Section 7.

6. THE SPECTRAL DENSITY

To determine the particle kinetic equation in terms of the wave energies in the next section, it is necessary to express the field correlation function

$$S(x,x';\ \tau) \equiv \langle\ E(x,t)\ E(x',t-\tau)\rangle, \tag{6.1}$$

in terms of the set $\left\{ W_a \right\}$. We assume a (quasi-) steady state, so that (6.1) is independent of t. By the Wiener-Khinchin theorem, its Fourier transform:

$$S(x,x';\ \omega) \equiv \int_{-\infty}^{+\infty} d\tau\ e^{i\omega\tau}\ S(x,x';\ \tau) \tag{6.2}$$

is the spectral density, expressible in terms of the Fourier transform of the random field:

$$S(x,x';\ \omega) = T^{-1}\ \langle\ E(x,\omega)\ E^*(x',\omega)\ \rangle\ . \tag{6.3}$$

Again keeping only the contribution of modes with small $|\gamma_a|$, and taking the limit $\gamma_a \longrightarrow 0$, we have

$$E(x,\omega) = \sum_a E_a^{(o)}(x)\left[A_a\,\delta(\omega - \omega_a^{(o)}) + A_a^*\,\delta(\omega + \omega_a^{(o)})\right], \tag{6.4}$$

where A_a is the complex random amplitude of mode a, the sum being over positive eigenfrequencies. Dropping the supercript again, we substitute (6.4) into (6.3):

$$S(x,x';\,\omega) = (2\pi)^{-1} \sum_a \langle |A_a|^2 \rangle E_a(x)E_a(x')\left[\delta(\omega-\omega_a) + \delta(\omega+\omega_a)\right]. \tag{6.5}$$

The total electric energy in the modes is

$$\begin{aligned} W^E &\equiv (8\pi)^{-1}\int dx\,\langle E^2(x,t)\rangle \\ &= (8\pi)^{-1}\int dx\; S\;(x,x;\;\tau=0) \\ &= (16\pi^2)^{-1}\int dx \int d\omega\;\; S(x,x;\;\omega) \\ &= (16\pi^3)^{-1}\;\sum_a \langle |A_a|^2 \rangle N_a; \end{aligned} \tag{6.6}$$

we have used (6.1), the inverse of (6.2), (6.5), and the definition of N_a below (3.11). By an extension of the method of Klimontovich[2], it may be shown that the ratio of electric energy to total energy in a wave is

$$W_a^E/W_a = N_a/1, \tag{6.7}$$

where the one represents the normalization integral (3.16). Since, by (6.6), the electric energy of mode a is $W_a^E = (16\pi^3)^{-1}\langle\, |A_a|^2\rangle\, N_a$, its total energy is thus

$$W_a = (16\pi^3)^{-1}\,\langle\, |A_a|^2\rangle \quad . \tag{6.8}$$

Eliminating the amplitude A_a from (6.5), we obtain the desired expression for the spectral density:

$$S(x,x';\ \omega) = 8\pi^2 \sum_a W_a\ E_a(x)\ E_a(x')\Big[\delta(\omega - \omega_a) + \delta(\omega + \omega_a)\Big]. \tag{6.9}$$

We note that the correlation function is even in τ:

$$S(x,x';\ \tau) = 8\pi \sum_a W_a\ E_a(x)\ E_a(x')\ \cos\ \omega_a\ \tau. \tag{6.10}$$

7. THE PARTICLE KINETIC EQUATION

The particle distribution f^o varies in time from two effects: a quasi-linear diffusion in action due to resonant energy exchange with waves; and a loss of energy due to discrete-particle wave emission. We study the former effect first, from the nonlinear Vlasov equation:

$$\partial f(J,w;\ t)/\partial t = -\ \partial_J(\dot{J}f)\ -\ \partial_w(\dot{w}f) \tag{7.1}$$

Averaging with respect to w, and averaging over fluctuations, this becomes

$$\partial f^o(J;\ t)/\partial t = -\ \partial_J \oint dw \langle \delta\dot{J}\ \delta f\rangle \tag{7.2}$$

The factor $\delta\dot{J}$ is

$$\begin{aligned}\delta\dot{J}(J,w,t) &= -\ \partial\Big[e\ \phi(x,t)\Big]/\partial w \\ &= -\ \partial(e\ \phi)/\partial x\ \cdot\ \partial x(J,w)/\partial w \\ &= eE(x,t)\ v(J,w)\ \nu^{-1}(J) \\ &= \nu^{-1}(J)\int dx\ E(x,t)\ j(x;\Gamma),\end{aligned} \tag{7.3}$$

where $E(x,t)$ is the fluctuating field. For the factor δf in (7.2), we may use the linear expression (2.13). Equation (7.2) then reads

$$\partial f^{o}(J;\ t) = \partial_J \left[D(J) f_J^{\ o} \right], \tag{7.4}$$

a diffusion equation, with the action-diffusivity

$$D(J) = \nu^{-2}(J) \int_0^\infty d\tau \int dx \int dx'\ S(x,x';\ \tau) \oint dw\ j(x;\Gamma)\ j(x';\Gamma,\ -\tau). \tag{7.5}$$

Since the correlation function (6.10) is even in τ, and the w-integral is invariant with respect to time-translation of its argument, the diffusivity may be written:

$$D(J) = \frac{1}{2}\ \nu^{-2}(J)\ T^{-1} \int \frac{d\omega}{2\pi} \int dx \int dx'\ S(x,x';\ \omega)\ j^{*}(x;\ \Gamma,\ \omega)\ j(x';\ \Gamma,\ \omega). \tag{7.6}$$

With the form (6.9) for the spectral density, this becomes

$$D(J) = 2\ \nu^{-2}(J) \sum_a \alpha_a(J)\ W_a \tag{7.7}$$

in terms of the mode-particle coupling coefficients (4.7), and the wave energies.

The effect of wave emission may be included by considering the $\dot{J}$ of (7.1) to include, not only the fluctuation effect (7.3), but also the drag due to the self-consistent field of a discrete particle. In analogy to (7.3), we have

$$\dot{J}(J,\ w,\ t) = \nu^{-1}(J) \int dx\ E(x,t)\ j(x;\ J,w,t), \tag{7.8}$$

where $E(x,t)$ is now given by (3.2) [with j^e in (3.2) replaced by the discrete-particle current $j(x';\ J,w,\omega)$]. We average with respect to w and t, and obtain by manipulations identical to those leading from (5.1) to (5.5), the result

$$\langle \dot{J} \rangle\ (J) = -2\ \nu^{-1}(J) \sum_a \alpha_a(J). \tag{7.9}$$

The complete kinetic equation is now

$$\partial f^{o}(J;\ t)/\partial t = -\ \partial_J(\langle \dot{J} \rangle\ f^{o}) + \partial_J\left[D(J)\ f^{o}_J\right]. \tag{7.10}$$

Incorporating the formulas (7.7) for the diffusivity, and (7.9) for the drag, this reads

$$\partial f^{o}(J;\ t)/\partial t = \partial_J\left\{2\nu^{-1}(J) \sum_a \alpha_a(J)\left[f^{o}(J) + W_a f^{o}_{\mathcal{E}}\right]\right\}. \tag{7.11}$$

The Markov assumption has been made implicitly, by the use of (2.13) for δf, and in the calculation of $\langle \dot{J} \rangle$.

8. ENERGY CONSERVATION AND THE H-THEOREM

The coupled kinetic equations for particles and waves (7.11) and (5.11) have two important properties. First, the total energy of particles and waves,

$$W(t) \equiv \int dJ\ f^{o}(J;t)\ \mathcal{E}\ (J) + \sum_a W_a(t), \tag{8.1}$$

is independent of time. Secondly, the entropy of the system of particles and waves,[16]

$$\mathcal{S}(t) \equiv -\int dJ\ f^{o}(J;\ t)\ \ell n\ f^{o}(J;\ t) + \sum_a \ell n\ W_a(t), \tag{8.2}$$

is a monotonically increasing function of time.

The proof that $dW/dt = 0$ is trivial. It is also straightfoward to show that

$$d\mathcal{S}/dt = 2\sum_a \int dJ\ \alpha_a(J)\ \nu(J)\left[f^{o}(J)\ W_a\right]^{-1}\left[f^{o}(J) + W_a\ f^{o}_{\mathcal{E}}\right]^2. \tag{8.3}$$

Since the integrand is never negative, it follows that $d\mathcal{S}/dt \geq 0$.

It is clear from (8.3) that a necessary and sufficient condition for a steady state is that

$$f^{o}\ (J) + W_a\ f^{o}_{\mathcal{E}} = 0 \tag{8.4}$$

for all J, a which are coupled by $\alpha_a(J) \neq 0$. In general, we expect that a given mode a couples to many J, which in turn couple to other modes, and so on. Then condition (8.4) must be valid for all J and a (for which our basic assumptions---no collisions, small $|\gamma_a|$, etc.---are valid). It follows that the only steady state of the kinetic equations is given by

$$\partial \ln f^o/\partial \mathcal{E} = - W_a$$

for all J, a; i.e., by

$$\partial f^o/\partial \mathcal{E} = - \beta f^o, \; W_a = \beta^{-1}, \tag{8.5}$$

which is a thermal distribution of particles and waves.

It is a well-known fact of statistical mechanics that the entropy is maximized (subject to energy conservation) by a thermal distribution. It follows then from (8.3) that the system tends monotonically and asymptotically to a thermal distribution.

Thus, if the conditions for our formalism remain valid, a system approaches thermal equilibrium eventually, from an initial stable or unstable state. This result does not hold in the one-dimensional uniform case, where there is a one-to-one correspondence of resonant particles and waves; but it is true in the two-or three-dimensional case,[5,17] where a particle couples to many waves, and a wave to many particles.

9. LOCAL LANDAU RESONANCE

In the expression (4.9) for $\alpha_a(J)$ there appears the integral

$$I_a(J) \equiv \int dx \; E_a(x) \sin \omega_a \tau(x,J). \tag{9.1}$$

When the wave length of $E_a(x)$ is short compared to the scale length for the confining potential, this integral may be evaluated approximately by the method of stationary phase. In general, we may express $E_a(x)$ as

$$E_a(x) = \mathcal{A}_a(x) \sin \int^x k_a(x')\, dx', \tag{9.2}$$

where $\mathcal{A}_a(x)$ and $k_a(x)$ are real and positive, while the lower limit of the integral is at some node of $E_a(x)$. (Explicit expressions for $\mathcal{A}_a(x)$ and $k_a(x)$ may be found in the work of other authors,[18] based on the extension of WKB techniques to integral equations.) Substituting (9.2) into (9.1), we have

$$I_a(J) = \mathrm{Re}\, \frac{1}{2} \sum_{\pm} (\mp 1) \int dx\, \mathcal{A}_a(x) \exp i\left[\int^x k_a(x')\, dx' \mp \omega_a\, \tau(x,J)\right]. \tag{9.3}$$

The dominant contribution is from the neighborhood of the point (or points, in general) of stationary phase (occuring for the upper sign only):

$$k_a(x) = \omega_a / v(J,x), \tag{9.4}$$

where $v(J,x)$ is the positive velocity of particle J at x. Denoting the point of stationary phase by $x_a(J)$, we see that at this point the Landau resonance condition $\omega_a = k_a v$ is satisfied. Expanding the exponent in (9.3) to second order in $x - x_a(J)$, we find

$$I_a(J) = (\pi/2|\mu|)^{\frac{1}{2}}\, \mathcal{A}_a\left[x_a(J)\right] \cos\left\{\int^{x_a(J)} k_a(x')\, dx' - \omega_a \tau\left[x_a(J), J\right] \pm \tfrac{1}{4}\pi\right\}, \tag{9.5}$$

where

$$\mu \equiv \left\{ v^{-1}(J,x)\, \partial_x \left[v(J,x)\, k_a(x) \right] \right\}_{x_a(J)}, \tag{9.6}$$

the two signs in (9.5) refer to the sign of μ, and a sum over the stationary points (if more than one) is implied.

Upon squaring (9.5) and substituting into (4.9), we have

$$\alpha_a(J) = 2\pi e^2 \omega_b(J) |\mu|^{-1} \mathcal{A}_a^{\,2}\left[x_a(J)\right] \cos^2\{\ldots\} \sum_{\ell} \delta\left[\ell - \omega_a/\omega_b(J)\right]$$

$$= 2\pi e^2 \omega_b(J) \int dx\; v(J,x)\; \mathcal{A}_a^{\,2}(x) \cos^2\left\{\int^x k_a(x')\, dx' - \omega_a\, \tau(x,J) \pm \frac{\pi}{4}\right\} \cdot$$

$$\cdot\; \delta\left[\omega_a - k_a(x)\; v(J,x)\right] \sum_{\ell} \delta\left[\ell - \omega_a/\omega_b(J)\right]. \qquad (9.7)$$

Upon coarse-graining in J, the $\cos^2$ becomes $\frac{1}{2}$, the sum over ℓ becomes one, while $\mathcal{A}_a^2(x)$ equals $2\langle E_a^{\,2}(x)\rangle$, averaged over one local wave length:

$$\alpha_a(J) = 2\pi e^2 \omega_b(J) \int dx\; v(J,x)\; \langle E_a^{\,2}(x)\rangle\; \delta\left[\omega_a - k_a(x)\; v(J,x)\right] . \qquad (9.8)$$

From (6.7), the electric energy density in mode a, averaged over a local wave length, is

$$\mathcal{W}_a^{\,E}(x) = W_a \langle E_a^{\,2}(x)\rangle. \qquad (9.9)$$

We substitute (9.9) into (9.8), and then (9.8) into (7.7), obtaining

$$D(J) = 8\pi^2 e^2\, \nu^{-1}(J) \sum_a \int dx\; v(J,x)\, \mathcal{W}_a^{\,E}(x)\; \delta\left[\omega_a - k_a(x)\; v(J,x)\right]; \qquad (9.10)$$

the action-diffusivity is expressed as an integral over local Landau resonances.

It may be further expressed in terms of the local velocity-diffusivity. In a uniform medium, the velocity-diffusivity is

$$D(v) = \frac{1}{2}\,(e/m)^2 \iint \frac{dk\; d\omega}{(2\pi)^2}\; S(k,\omega) \int_{-\infty}^{+\infty} d\tau\; \exp i(kv - \omega)\,\tau$$

$$= 8\pi^2 (e/m)^2 \iint \frac{dk\; d\omega}{(2\pi)^2}\; \mathcal{W}^E(k,\omega)\; \delta(\omega - kv) \qquad (9.11)$$

To make the transition to a set of modes, we let

$$(2\pi)^{-2}\, \mathcal{W}^E\; (k,\omega) \longrightarrow \sum_a \mathcal{W}_a^{\,E}\; \delta(k - k_a)\; \frac{1}{2}\left[\delta(\omega - \omega_a) + \delta(\omega + \omega_a)\right] . \qquad (9.12)$$

With this replacement, the velocity-diffusivity becomes

$$D(v) = 4\pi^2 (e/m)^2 \sum_a \mathcal{W}_a^E \Big[\delta(\omega_a - k_a v) + \delta(\omega_a + k_a v)\Big] . \tag{9.13}$$

Comparing (9.10) and (9.13), where $\mathcal{W}_a^E$, k_a, and v are now functions of x, we see that

$$D(J) = 2m^2 \nu^{-1}(J) \int dx\, v(J,x)\, D\Big[v(J,x);\, x\Big]. \tag{9.14}$$

This relation between action - and local velocity-diffusivities is just as one would expect from elementary considerations. Since $\delta J = \nu^{-1} \delta\mathcal{E} = \nu^{-1} mv\, \delta v$, we expect that $D(J) = \nu^{-2} m^2 \langle v^2 D(v)\rangle$, with averaging over a complete bounce: $\langle \ldots \rangle \equiv \nu\, 2 \int (dx/v) (\ldots)$. This leads directly to (9.14).

10. THE QUANTUM PICTURE

Our understanding of the kinetic equations may be enhanced by considering their derivation from the corresponding set of postulated quantum master equations.[1] These are

$$\partial f(J;\, t)/\partial t = \int dJ' \int dJ'' \Big[\, \delta(J' - J) - \delta(J'' - J)\Big] \sum_a \mathcal{P}(J'' \longleftrightarrow J',\, a) \;\; x$$

$$x \Big[f(J'')\, (N_a + 1) - f(J')\, N_a\Big]\, \delta(\mathcal{E}_{J''} - \mathcal{E}_{J'} - \mathcal{E}_a) \tag{10.1}$$

and

$$dN_a/dt = \int dJ \int dJ'\, \mathcal{P}(J \longleftrightarrow J',\, a) \;\; \Big[\, f(J)\, (N_a + 1) - f(J')\, N_a\Big] \;\; x$$

$$x\, \delta(\, \mathcal{E}_J - \mathcal{E}_{J'} - \mathcal{E}_a\,), \tag{10.2}$$

where $\mathcal{E}_a \equiv \hbar\omega_a$ is the energy of a normal mode quantum, $N_a \equiv W_a/\mathcal{E}_a$ is the mean number of quanta in the system, and $\mathcal{P}(J'' \longleftrightarrow J',\, a)$ is a postulated transition probability (on the energy shell) for emission of a quantum of mode

a by particle J", or for its absorption by particle J'.

If, in these equations, we take the limit $\hbar \to 0$, we obtain (after a bit of algebra)

$$\partial f(J;\, t)/\partial t = \partial_J \left\{ \nu^{-2}(J) \sum_a \hbar\omega_a \, \rho(J \longleftrightarrow J,\, a)[f(J) + W_a f_{\mathcal{E}}] \right\} , \qquad (10.3)$$

$$dW_a/dt = \hbar\omega_a \int dJ \; \nu^{-1} \rho(J \longleftrightarrow J,\, a) \; [f(J) + W_a f_{\mathcal{E}}]. \qquad (10.4)$$

Comparing these equations with (7.11) and (5.11), we see that they are identical if we make the correspondence

$$\hbar\omega_a \; \nu^{-1} \; \rho(J \longleftrightarrow J,\, a) = 2 \; \alpha_a(J) \qquad (10.5)$$

between the quantum transition probability and the classical coupling coefficient.

11. TIME SCALES

There are five characteristic times in this problem: (1) the particle bounce-time, $\nu^{-1}(J)$; (2) the normal mode oscillation period, ω_a^{-1}; (3) the normal mode decay (or growth) time, $|\gamma_a^{-1}|$; (4) the radiation reaction time, $\tau_{\dot{J}} \equiv J/|\langle \dot{J} \rangle|$; and (5) the quasi-linear evolution time for f^o, $\tau_f \equiv J^2/D(J)$. From equations (4.8), (7.9), and (7.7), we may estimate the latter three as

$$|\gamma_a^{-1}| \sim \mathcal{E}/N_p \, \alpha_a \, , \qquad (11.1)$$

$$\tau_{\dot{J}} \sim \mathcal{E}/N_W \, \alpha_a \, , \qquad (11.2)$$

$$\tau_f \sim (\mathcal{E}/W_a) \, \tau_{\dot{J}} \, ; \qquad (11.3)$$

where $\mathcal{E}$ is a typical particle energy, α_a a typical coupling coefficient, N_p the total number of resonant particles, N_W the total number of resonant modes,

and W_a a typical mode energy.

Our analysis is valid only if the former two times, ν^{-1} and ω_a^{-1}, are short compared to the latter three. From (11.1), we see that this hinges on the number of resonant particles being not too large, or on the resonant coupling being not too large. Typically we expect that $N_p \gg N_W$, so that τ_J is long compared to $|\gamma_a^{-1}|$. In the unstable case, we expect that $N_p \mathcal{C}$ and $N_W W_a$ are of the same order; then $|\gamma_a^{-1}|$ and τ_f are comparable. In the stable regime however, eventually $\mathcal{C}$ and W_a are of the same order, so τ_J and τ_f are comparable.

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FOOTNOTES AND REFERENCES

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6. For a one-dimensional uniform system, this conclusion is invalid, since there is then a one-to-one correspondence of waves and resonant particles.
7. In a future paper, we plan to discuss the generalizations to three dimensions, to unconfined particles, and to electromagnetic waves. We expect that these generalizations will be straightforward.
8. In the degenerate case of a parabolic potential the bounce frequency is independent of energy, the wave frequency is never an integral multiple of it, and no resonant interaction occurs.
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13. When σ is asymmetric, a formulation in terms of adjoint functions can be used instead.
14. A possible justification is the numerical analysis of Leavens and Love,[12] which is well fit by the normal-mode representation.

15. We assume that this expression, which is proportional to wave energy [see Eq. (6.7)], is positive for all zero-order modes. If it is not, appropriate modifications must be made.

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