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HAAG'S AND COLEMAN'S THEOREMS AT INFINITE MOMENTUM*

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SUMMARY

We show that neither Haag's theorem nor Coleman's theorem holds in the infinite-momentum reference frame: it is possible to formulate a non-trivial (not equivalent to a free field) relativistic field theory in Fock space at infinite momentum in which a charge that annihilates the vacuum is not necessarily independent of time. Both theorems fail because vacuum creation graphs vanish at infinite momentum (at least for a $\lambda\phi^3$ theory such as we consider in this paper).

I. INTRODUCTION

In recent years there has been some interest in looking at field theories in the limit in which the interacting system is moving with the velocity of light relative to the observer; this is the so-called $P = \infty$ reference frame (1). Doing physics in this limit has many interesting features, among which is the simplification of current algebra, since we can saturate the current commutator with states of the same infinite momentum and in many cases terms with more than one particle in the intermediate state vanish (2); moreover Lorentz invariance of a theory becomes in large part a two-dimensional Galilean invariance (3). This latter feature results in a considerable simplification of the structure of field theory. For our purposes in this paper we are concerned only with Weinberg's discovery that (at least in a scalar theory) all diagrams involving creation or annihilation from the vacuum vanish, and the realization by Bardakci and Halpern that the expansion of the field appropriate at infinite momentum is essentially a nonrelativistic one involving either creation or annihilation operators (but not both). We shall apply these elementary results to the consideration of two general theorems about fields, Haag's theorem and Coleman's theorem, at infinite momentum.

II. HAAG'S THEOREM

Haag's theorem states that if a quantum field theory satisfies certain general conditions that we would naively expect any quantum field theory to satisfy, it is completely equivalent to the free-field theory, i.e., there can be no interaction between the particles of the theory (4).

The conditions that are imposed on the theory are (5) (somewhat modified from Haag's original conditions): I--relativistic transformation properties of the field $\phi(x)$, II--unique, normalizable, invariant, vacuum state Φ_0 and no negative-energy states or states of spacelike momenta, III--canonical commutation relations at equal times, IV--being related to the free-field theory at a given time by a unitary transformation (both field $\phi(x)$ and conjugate momentum $\pi(x)$ so related). Greenberg then proved the equality of all vacuum expectation values of the field, for all times, with those of the free field, thus showing the equivalence of the two theories.

So far the impact of Haag's theorem has been to make us more careful with our constructions in field theory. The usual time-development operator $U(t_2,t_1)$ does not exist except in the special case $t_1 = -\infty$, $t_2 = +\infty$, where $U(+\infty, -\infty)$ is just the S-operator. And the physical vacuum defined by applying the time-development operator $U(0, -\infty)$ to the bare vacuum does not exist either. We are forced to use a physical vacuum which is out of Fock space (it contains an infinite number of particles), thus complicating the mathematical simplicity of Fock space representations of the operators.

III. HAAG'S THEOREM AT INFINITE MOMENTUM

As was mentioned first by Susskind (6), all the difficulties introduced into field theory by Haag's theorem disappear when we go to infinite momentum. Due to the nonrelativistic structure of field theory at infinite momentum, and in particular due to the absence of vacuum creation graphs, we have no trouble constructing an interacting field theory in which the physical vacuum is in Fock space (7) (and is in fact just the bare vacuum), the time-development operator is well defined, the canonical commutation relations are formally satisfied, and in fact all the hypotheses of Haag's theorem are satisfied.

A specific example of a field theory at infinite momentum which is free of Haag theorem difficulties is given in Bardakci and Halpern's paper for a $\lambda\phi^3$ theory. The interaction part of the Hamiltonian is given after the limit by:

$$(1) \qquad V = \int_{-\infty}^{+\infty} d\xi\, V(\xi) = \lambda \int_{-\infty}^{+\infty} d\xi \int d^2x_\perp \int_0^\infty \frac{d\eta}{(\eta)^{\frac{1}{2}}} \int_0^\infty \frac{d\eta'}{(\eta')^{\frac{1}{2}}} \int_0^\infty \frac{d\eta''}{(\eta'')^{\frac{1}{2}}}$$

$$\times \{\Psi(\tilde{x}_\perp,\eta,\xi)\, \Psi(\tilde{x}_\perp,\eta',\xi)\, \Psi^\dagger(\tilde{x}_\perp,\eta'',\xi)\, \delta(\eta+\eta'-\eta'')$$

$$+ \Psi(\tilde{x}_\perp,\eta,\xi)\, \Psi^\dagger(\tilde{x}_\perp,\eta',\xi)\, \Psi^\dagger(\tilde{x}_\perp,\eta'',\xi)\, \delta(\eta-\eta'-\eta'')\}$$

with:

$$(2) \qquad \Psi(\tilde{x}_\perp,\eta,\xi) = \int \frac{d^2q}{2\pi}\, e^{i\tilde{q}\cdot\tilde{x}_\perp}\, a(\tilde{q},\eta)\, e^{-i\left(\frac{q^2+m^2}{\eta}\right)\xi}$$

$$(3) \qquad \Psi^\dagger(\tilde{x}_\perp,\eta,\xi) = \int \frac{d^2q}{2\pi}\, e^{-i\tilde{q}\cdot\tilde{x}_\perp}\, a^\dagger(\tilde{q},\eta)\, e^{i\left(\frac{q^2+m^2}{\eta}\right)\xi}$$

$$(4) \qquad [a(\tilde{q},\eta),\, a^\dagger(\tilde{q}',\eta')] = \delta^2(\tilde{q}-\tilde{q}')\,\delta(\eta-\eta')$$

where $a^\dagger(\tilde{q},\eta)$ and $a(\tilde{q},\eta)$ are creation and annihilation operators, $\tilde{q}$ is a two-dimensional momentum vector, $\eta = p_0+p_3$, and $\xi = \frac{1}{2}(t-z)$ is the "time" variable conjugate to the total Hamiltonian in the infinite - momentum reference frame. The Lorentz generators after the limit can also be given (see Bardakci and Halpern) and the theory is Lorentz invariant.

We see that due to the absence of vacuum creation and annihilation graphs the bare vacuum is an eigenvector of the total Hamiltonian and since it also has zero 3-momentum it is the physical vacuum (at any finite momentum the Hamiltonian contains vacuum creation terms which take the bare vacuum into a state containing particles, so the bare vacuum cannot be the physical vacuum). Thus the physical vacuum is in Fock space and is normalizable, eliminating the most serious difficulty raised by Haag's theorem. The absence of vacuum structure also allows us to define the time-development operator $U(\xi_2,\xi_1)$ without any difficulty; it is given by

$$(5) \qquad U(\xi_2,\xi_1) = \Xi \exp\left\{ i \int_{\xi_1}^{\xi_2} d\xi \;\; V(\xi) \right\}$$

where Ξ means time ordering with respect to ξ. Haag's proof that the canonical commutation relations cannot hold in any interacting field theory breaks down, since it also depends on the existence of vacuum creation terms; the commutation relations are given by (removing ξ dependence):

$$[\Psi(\tilde{x}_\perp,\eta),\ \Psi(\tilde{x}_\perp',\eta')] = 0 \tag{6}$$

$$[\Psi(\tilde{x}_\perp,\eta),\ \Psi^\dagger(\tilde{x}_\perp',\eta')] = \delta^{(2)}(\tilde{x}_\perp - \tilde{x}_\perp')\ \delta(\eta-\eta') \tag{7}$$

Similarly, Haag's proof that field theories of particles with different masses must have different vacua also breaks down at infinite momentum, since it depends on the expansion of the field in terms of both creation and annihilation operators, and the expansion of the field appropriate at infinite momentum contains only one or the other, as in eq. (2). The breakdown of his proof at infinite momentum can also be seen by taking the formal limit $\tilde{p} \to \infty$ in his equations (8).

IV. COLEMAN'S THEOREM

Coleman's theorem states that in a field theory satisfying the usual axioms and having no zero-mass particles, if the vacuum is invariant under the group generated by the charges $Q(t) = \int d^3x\, V_0(\tilde{x},t)$, i.e., the generator $Q(t)$ annihilates the vacuum: $Q(t)\,|0\rangle = 0$, then the Hamiltonian is also invariant, i.e., $Q(t)$ commutes with the Hamiltonian (9).

We shall demonstrate the failure of Coleman's theorem at infinite momentum by considering the specific case of the pion isospin charge $Q_a = \int d^3x\, V_a^{\,0} = \epsilon_{abc} \int d^3x\, \phi_b\, \partial_0\, \phi_c$ for pions of unequal masses $m_{\pi^+}, m_{\pi^0}, m_{\pi^-}$ (where a,b,c refer to the three possible values of I_z). We shall use the formalism of Bardakci and Halpern in performing the calculation, using the variables appropriate to calculation at infinite momentum. We shall demonstrate that this charge Q_a annihilates the vacuum but the current is not conserved (and hence the charge does not commute with the Hamiltonian). We find after, performing the limit procedure:

$$(8) \qquad V_a^{\,0}(\tilde{x}_\perp) = \rho_a = \epsilon_{abc} \int_0^\infty d\eta\, \Psi_b^{\dagger}(\tilde{x}_\perp,\eta)\, \Psi_c(\tilde{x}_\perp,\eta)$$

$$(9) \qquad V_a^{\,i}(\tilde{x}_\perp) = \rho_a^{\,i} = \frac{1}{2}\, \epsilon_{abc} \int_0^\infty \frac{d\eta}{\eta}\, \Psi_b^{\dagger}(\tilde{x}_\perp,\eta)\, (-i\, \overleftrightarrow{\partial}_i)\, \Psi_c(\tilde{x}_\perp,\eta)\ , \quad i = 1,2$$

with:

$$(10) \quad \Psi_a(\tilde{x}_\perp,\eta) = \int \frac{d^2q}{2\pi}\, e^{i\tilde{q}\cdot\tilde{x}_\perp}\, a_a(\tilde{q},\eta)$$

$$(11) \quad \Psi_a^\dagger(\tilde{x}_\perp,\eta) = \int \frac{d^2q}{2\pi}\, e^{-i\tilde{q}\cdot\tilde{x}_\perp}\, a_a^\dagger(\tilde{q},\eta)$$

$$(12) \quad [a_a(\tilde{q},\eta),\ a_b^\dagger(\tilde{q}',\eta')] = \delta^{(2)}(\tilde{q}-\tilde{q}')\ \delta(\eta-\eta')\ \delta_{ab}$$

$$(13) \quad [\Psi_a(\tilde{x}_\perp,\eta),\Psi_b^\dagger(\tilde{x}_\perp',\eta')] = \delta^{(2)}(\tilde{x}_\perp-\tilde{x}_\perp')\ \delta(\eta-\eta')\ \delta_{ab}$$

The current conservation condition is:

$$(14) \quad [H,\ \rho_a] + 2\ [P_i,\ \rho_a^{\,i}] = 0 \quad , \quad i = 1,2$$

where:

$$(15) \quad H = \sum_a \int d^2x_\perp d\eta\ \Psi_a^\dagger(\tilde{x}_\perp,\eta)\left(-\frac{\nabla_\perp^2}{\eta} + \frac{m_a^2}{\eta}\right)\Psi_a(\tilde{x}_\perp,\eta)$$

$$(16) \quad P_i = \sum_a \int d^2x_\perp d\eta\ \Psi_a^\dagger(\tilde{x}_\perp,\eta)\left(-i\frac{\partial}{\partial x_i}\right)\Psi_a(\tilde{x}_\perp,\eta)$$

Again, the Lorentz generators can be given and the theory is Lorentz invariant.

We see easily that ρ_a annihilates the vacuum (note that this is due to the nonrelativistic expansion of the field — and remember $\epsilon_{abc} \neq 0 \Longrightarrow b \neq c$). However, carrying out the calculation of the current conservation condition, we find that:

$$(17) \qquad [H, \rho_a] + 2[P_i, \rho_a^{\,i}] = \epsilon_{abc} \int \frac{d\eta}{\eta} \left(m_b^{\,2} - m_c^{\,2} \right) \int \frac{d^2q\, d^2q'}{(2\pi)^2}$$

$$\times \; e^{i(\tilde{q}'-\tilde{q})\cdot\tilde{x}_\perp} a_b^{\,\dagger}(\tilde{q},\eta) a_c(\tilde{q}',\eta) \neq 0 \quad \text{if} \quad m_b^{\,2} \neq m_c^{\,2} ,$$

thus verifying the conjectured breakdown of Coleman's theorem at infinite momentum (Coleman's theorem does not apply to this case at finite momentum since this charge does not annihilate the vacuum there— pair creation is allowed at finite momentum — although Coleman's theorem does apply to the case with equal masses).

V. SUMMARY

We have shown that Haag's and Coleman's theorems do not hold at infinite momentum. As much work in rigorous quantum field theory involves circumventing problems like Haag's theorem (10), it might be easier to formulate field theories directly at infinite momentum and extract rigorous consequences in that frame.

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REFERENCES AND FOOTNOTES

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1. S. WEINBERG, Phys. Rev. 150, 1313 (1966); S. FUBINI and G. FURLAN, Physics 1, 229 (1965); S.-J. CHANG and S.-K. MA, Phys. Rev. 180, 1506 (1969).

2. S. FUBINI, G. SEGRÈ, and J. D. WALECKA, Ann. Phys. 39, 381 (1966).

3. L. SUSSKIND, Phys. Rev. 165, 1535 (1968); K. BARDAKCI and M. B. HALPERN, Phys. Rev. 176, 1686 (1968).

4. R. HAAG, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. 29, No. 12 (1955); D. HALL and A. S. WIGHTMAN, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. 31, No. 5 (1957).

5. O. W. GREENBERG, Phys. Rev. 115, 706 (1959).

6. L. SUSSKIND, Lectures given at the 1968 Summer Institute for Theoretical Physics at the University of Colorado at Boulder (unpublished).

7. The ultraviolet divergences (as distinct from the normalization divergences associated with HAAG'S theorem) arising from the vacuum-vacuum graphs in ordinary field theory also disappear (for the simple reason that the vacuum-vacuum graphs disappear), although we can say nothing about other divergences in the theory.

8. See HAAG'S eq. (29).

9. S. COLEMAN, J. Math. Phys. 7, 787 (1966).

10. A. S. WIGHTMAN, Cargèse Lectures in Theoretical Physics, 1964; A. M. JAFFE, Phys. Rev. 158, 1454 (1967); M. GUENIN, Boulder Lectures in Theoretical Physics, Vol. IX A, 1966.

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