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QUANTUM STATISTICS OF A MULTICOMPONENT FULLY IONIZED GAS

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ABSTRACT

The quantum statistical theory of a non-relativistic fully ionized gas in thermal equilibrium is developed using the well-known linked-cluster expansion of the grand potential. A systematic analysis of the self-energy structures leads to a master-graph formulation of quantum statistics. This provides a much simpler derivation and improved version, with important differences, of the earlier work of Mohling and Grandy. In particular the analysis of the photon self-energy structures in now entirely different. As an application of the general theory the lowest order calculation of the photon self-energies and photon momentum distribution are presented. The results are compared with earlier work of others. Finally, explicit connections between the master-graph line factors and Green functions are outlined, and the consequences of such a connection are indicated.

I. INTRODUCTION

There exist several many-body theories, each having its own adherents and each having certain conceptual or calculational advantages over the others. Of course, all mathematically rigorous many-body theories must be, in some sense, equivalent. Thus, the applications of these theories to a given physical system differ only in their relative suitability for the problem, their mathematical sophistication or their appeal to the intuition. Owing to the inherent mathematical complexity of any many-body theory, diagrammatic techniques are usually incorporated.

In view of the preceding discussion, we do not purport to present in this paper a many-body theory which transcends all others; however, our formulation does have several novel features. We are interested in developing a theory of multicomponent, non-relativistic quantum electrodynamics at finite temperature. Thus, our formulation is developed so that, for example, mass-renormalization and removal of the infrared divergence can be accomplished in a straightforward manner. The Hamiltonian is rearranged by adding and subtracting a sum of arbitrary one-particle operators (called counterterms),

and it is important to note that these counterterms emerge in the theory in a mathematically very useful manner. Our diagrammatic expansion is subjected to a simple and complete self-energy analysis, and the counterterms can be used rigorously to cancel spurious, system-independent self-energies. Our formalism establishes certain connections between Green function techniques and the quantum statistical theory of Mohling[1] and Mohling, RamaRao, and Shea[2] which have the basic ideas of the Lee and Yang[3] method as their foundation. We note here that the final master graph formulation of Mohling[1] and MRS will be derived in a simpler way without the Lee-Yang method as a basis.

For a system with photons interacting with charged particles, certain photon self-energy structures [called (0,2) and (2,0) structures] can lead to important contributions to physical quantities. An important new feature of this paper is that we consistently take these quantities into account, both in our quantum statistical theory and in our developments involving Green functions.

It is our aim to make the presentation easily accessible to persons committed to another point of view. The necessity of keeping the length of the paper reasonable has required that detailed derivations not be reproduced whenever they have been presented elsewhere. However, it is hoped that the presentation is sufficiently self-contained so that the physical presence of referenced material is not crucial.

II. DEFINITION OF SYSTEM

It is our purpose to formulate a microscopic basis for calculating the properties of a non-relativistic fully-ionized gas of interacting charged particles and photons which are in thermal equilibrium; though it is not difficult to do, we do not include neutral components. The gas is multicomponent, and the consitituent particles are labeled by Greek letters $\alpha, \beta, \eta, \ldots$ (the symbol γ is reserved exclusively for photons). All particles (except photons, of course) are treated as point particles with mass, charge and spin, but with no internal degrees of freedom. The system is described by a Hamiltonian which we write first for a system of N particles with their radiation field (Gaussian units are used):

$$H = H_{rad} + \sum_{i=1}^{N} \frac{1}{2M_i^{(o)}} \left(\underset{\sim}{p}_i - \frac{eZ_i}{c} \underset{\sim}{A}_i \right)^2 + \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \frac{Z_i Z_j e^2}{r_{ij}} \tag{2.1}$$

$$= H_o + V_{1\gamma} + V_{1\gamma}^{\dagger} + V_{2\gamma} + V_{2\gamma}^{\dagger} + V_c = H_o + V_\gamma + V_c = H_o + V \tag{2.2}$$

where $\underset{\sim}{p}_i = \hbar \underset{\sim}{k}_i$ is the momentum $M_i^{(o)}$ the bare mass and Z_i the charge number (e is the magnitude of the electronic charge) of the ith particle; $\underset{\sim}{A}_i$ is the vector potential, in the Coulomb gauge, at the position of the ith particle (a constant external magnetic field can also be included in this vector potential); r_{ij} is the separation of the ith and jth particles; H_{rad} is the Hamiltonian of the free photon field. Since photon number is not conserved , as well as

for mathematical convenience, we perfer to adopt Fock space methods so that in the free-particle momentum representation the Hamiltonian in Eq. (2.1) involves the operators[4]

$$H_o = \sum_{\alpha} \sum_{\underset{\sim}{k}^{\alpha}} w^{(o)}(\underset{\sim}{k}^{\alpha})\, a^{\dagger}(\underset{\sim}{k}^{\alpha})\, a(\underset{\sim}{k}^{\alpha}) \tag{2.3}$$

$$V_{1\gamma} = \sum_{\substack{\alpha \\ \alpha \neq \gamma}} \sum_{\underset{\sim}{k}_1^{\alpha},\, \underset{\sim}{k}_2^{\alpha},\, \underset{\sim}{k}_3^{\gamma}} a^{\dagger}(\underset{\sim}{k}_1^{\alpha}) \langle \underset{\sim}{k}_1^{\alpha} | V_{1\gamma} | \underset{\sim}{k}_2^{\alpha}\, \underset{\sim}{k}_3^{\gamma} \rangle\, a(\underset{\sim}{k}_2^{\alpha})\, a(\underset{\sim}{k}_3^{\gamma}) \tag{2.4}$$

$$V_{2\gamma} = \sum_{\substack{\alpha \\ \alpha \neq \gamma}} \sum_{\underset{\sim}{k}_1^{\alpha},\, \underset{\sim}{k}_2^{\alpha},\, \underset{\sim}{k}_3^{\gamma},\, \underset{\sim}{k}_4^{\gamma}}$$

$$\times \Big[a^{\dagger}(\underset{\sim}{k}_1^{\alpha}) \langle \underset{\sim}{k}_1^{\alpha} | V_{2\gamma} | \underset{\sim}{k}_2^{\alpha}\, \underset{\sim}{k}_3^{\gamma}\, \underset{\sim}{k}_4^{\gamma} \rangle\, a(\underset{\sim}{k}_2^{\alpha})\, a(\underset{\sim}{k}_3^{\gamma})\, a(\underset{\sim}{k}_4^{\gamma})$$

$$+ \; a^{\dagger}(\underset{\sim}{k}_1^{\alpha})\, a^{\dagger}(\underset{\sim}{k}_3^{\gamma}) \langle \underset{\sim}{k}_1^{\alpha}\, \underset{\sim}{k}_3^{\gamma} | V_{2\gamma} | \underset{\sim}{k}_2^{\alpha}\, \underset{\sim}{k}_4^{\gamma} \rangle\, a(\underset{\sim}{k}_2^{\alpha}) a(\underset{\sim}{k}_4^{\gamma}) \Big] \tag{2.5}$$

$$V_c = \frac{1}{2} \sum_{\substack{\alpha\,\beta \\ \alpha \neq \gamma \\ \beta \neq \gamma}} \sum_{\underset{\sim}{k}_1^{\alpha},\, \underset{\sim}{k}_2^{\beta},\, \underset{\sim}{k}_3^{\alpha},\, \underset{\sim}{k}_4^{\beta}} a^{\dagger}(\underset{\sim}{k}_1^{\alpha})\, a^{\dagger}(\underset{\sim}{k}_2^{\beta})$$

$$\times \langle \underset{\sim}{k}_1^{\alpha}\, \underset{\sim}{k}_2^{\beta} | V_c | \underset{\sim}{k}_3^{\alpha}\, \underset{\sim}{k}_4^{\beta} \rangle\, a(\underset{\sim}{k}_4^{\beta})\, a(\underset{\sim}{k}_3^{\alpha}) \quad , \tag{2.6}$$

where for no external fields free-particle states are plane waves satisfying periodic boundary conditions so that

$$w^{(o)}(\underset{\sim}{k}^{\alpha}) = \begin{cases} \dfrac{\hbar^2(\underset{\sim}{k}^{\alpha})^2}{2M_{\alpha}^{(o)}} & \text{for } \alpha = \text{particles} \\ \\ \hbar\, ck & \text{for } \alpha = \gamma \text{ (photons)} \end{cases} \tag{2.7}$$

$$\langle \underset{\sim}{k}_1^{\alpha} | V_{1\gamma} | \underset{\sim}{k}_2^{\alpha}\, \underset{\sim}{k}_3^{\gamma} \rangle = -Z_{\alpha}(\hbar^2/M_{\alpha})(2\pi\alpha_o/\Omega\, k_3^{\gamma})^{1/2}(\underset{\sim}{k}_1^{\alpha} \cdot \hat{e}_3)$$

$$\times\, \delta_{\underset{\sim}{k}_1^{\alpha},\, (\underset{\sim}{k}_2^{\alpha} + \underset{\sim}{k}_3^{\gamma})}\, \delta_{m_1,\, m_2} \tag{2.8}$$

$$\langle \underset{\sim}{k}_1^{\alpha} | V_{2\gamma} | \underset{\sim}{k}_2^{\alpha}\, \underset{\sim}{k}_3^{\gamma}\, \underset{\sim}{k}_4^{\gamma} \rangle = Z_{\alpha}^2(\hbar^2/M_{\alpha})(\pi\alpha_o/\Omega)(k_3^{\gamma}\, k_4^{\gamma})^{-1/2}(\hat{e}_3 \cdot \hat{e}_4)$$

$$\times\, \delta_{\underset{\sim}{k}_1^{\alpha},\, (\underset{\sim}{k}_2^{\alpha} + \underset{\sim}{k}_3^{\gamma} + \underset{\sim}{k}_4^{\gamma})}\, \delta_{m_1,\, m_2} \tag{2.9}$$

$$\langle \underset{\sim}{k}_1^{\alpha}\, \underset{\sim}{k}_3^{\gamma} | V_{2\gamma} | \underset{\sim}{k}_2^{\alpha}\, \underset{\sim}{k}_4^{\gamma} \rangle = Z_{\alpha}^2(\hbar^2/M_{\alpha})(\pi\alpha_o/\Omega)(k_3^{\gamma}\, k_4^{\gamma})^{-1/2}(\hat{e}_3 \cdot \hat{e}_4)$$

$$\times\, \delta_{(\underset{\sim}{k}_1^{\alpha} + \underset{\sim}{k}_3^{\gamma}),\, (\underset{\sim}{k}_2^{\alpha} + \underset{\sim}{k}_4^{\gamma})}\, \delta_{m_1,\, m_2} \tag{2.10}$$

$$\langle \underset{\sim}{k}_1{}^{\alpha}\, \underset{\sim}{k}_2{}^{\beta} | V_c | \underset{\sim}{k}_3{}^{\alpha}\, \underset{\sim}{k}_4{}^{\beta} \rangle = (2\pi)^3\, (4\pi\, Z_\alpha Z_\beta e^2/\Omega\, q^2)$$

$$\times\ \delta_{(\underset{\sim}{k}_1{}^{\alpha} + \underset{\sim}{k}_2{}^{\beta}),\,(\underset{\sim}{k}_3{}^{\alpha} + \underset{\sim}{k}_4{}^{\beta})}\ \delta_{m_1,m_3}\delta_{m_2,m_4} \tag{2.11}$$

where $\delta_{a,b}$ is a Kronecker delta, $\alpha_o = e^2/\hbar c$ is the fine-structure constant, m_i is the spin projection quantum number, $\hat{e}_i$ is the polarization unit vector associated with the photon propagation vector $\underset{\sim}{k}_i{}^{\gamma}$ and $\underset{\sim}{q} = \underset{\sim}{k}_3 - \underset{\sim}{k}_1$ is the momentum transfer. In Eqs. (2.3) - (2.6) the notation $\underset{\sim}{k}$ includes spin degrees of freedom with each momentum state. The multicomponent feature of the description has been exhibited by a separate sum over particle species, and the particle-type and momentum summations are equivalent to the summation over the following values:

$$\underset{\sim}{k}_1{}^{\alpha},\ \underset{\sim}{k}_2{}^{\alpha},\ \ldots;\ \underset{\sim}{k}_1{}^{\beta},\ \underset{\sim}{k}_2{}^{\beta},\ \ldots;\ \ldots;\ \underset{\sim}{k}_1{}^{\gamma},\ \underset{\sim}{k}_2{}^{\gamma},\ \ldots\ . \tag{2.12}$$

Thus, at times the notation $\sum_{\underset{\sim}{k}_i}$ is used to mean summation over the eigenvalue spectrum in Eq. (2.12).

One can see directly from Eqs. (2.4) and (2.5), for example, that in interaction terms involving photons the number of photon creation operators may differ from the number of photon annihilation operators. This feature leads to important self-energy structures with zero photon momentum transfer. This is the reason for the occurrence of (0,2) and (2,0) photon self-energy structures in Secs. IV and V.

III. QUANTUM STATISTICAL DEVELOPMENTS

Before developing a calculational technique, we must specify the type of physical quantities which we wish to be able to calculate. We are interested in thermodynamic functions, momentum distributions, quasi-particle energies, correlation functions, electromagnetic properties and so forth. In this paper we concentrate on thermodynamic functions and momentum distribution. For example, if f is the grand potential and Ω the volume, then the pressure P, the average particle number of α-type particles $\langle N_\alpha \rangle$, the particle density ρ, the average energy $\langle E \rangle$ and the average entropy $\langle S \rangle$ can be calculated as follows:

$$P = \beta^{-1} \partial(\Omega f)/\partial\Omega \tag{3.1}$$

$$\langle N_\alpha \rangle = \beta^{-1} \partial(\Omega f)/\partial g_\alpha \tag{3.2}$$

$$\rho = \langle N \rangle/\Omega = \sum_\alpha \langle N_\alpha \rangle/\Omega = \beta^{-1} \sum_\alpha \partial f/\partial g_\alpha \tag{3.3}$$

$$\langle E \rangle = G - \partial(\Omega f)/\partial\beta \tag{3.4}$$

$$\langle S \rangle = \partial(\beta^{-1}\Omega f)/\partial T \tag{3.5}$$

where $G = \sum_\alpha \langle N_\alpha \rangle g_\alpha$ is the Gibbs potential and $[G,H] = 0$ (note: $g_\gamma = 0$).

The basic definition of the grand potential is

$$\Omega f(\beta, g_\alpha, g_\beta, \cdots, \Omega) = \ln \mathrm{Tr}\{\exp[\beta(G - H)]\} \tag{3.6}$$

where H is the total Hamiltonian and Tr indicates the trace in Fock space. The momentum distribution $\langle n_\alpha(\underset{\sim}{k}) \rangle$ (which is the average

number of α-type particles with momentum $\underset{\sim}{k}$) is defined

$$\langle n_\alpha(\underset{\sim}{k})\rangle = \mathrm{Tr}\{a^\dagger(\underset{\sim}{k}^\alpha)\, a(\underset{\sim}{k}^\alpha)\, \exp(-\Omega f)\exp[\beta(G - H)]\} \tag{3.7}$$

$$= -\beta^{-1}[\delta/\delta w(\underset{\sim}{k}^\alpha)]\, \Omega f \quad , \tag{3.8}$$

where in Eq. (3.8) the functional derivative is performed for fixed potential in the Hamiltonian. Our subsequent development will result in diagrammatic expansions for Ωf, $\langle n_\alpha(\underset{\sim}{k})\rangle$ and one-particle Green functions (defined in Sec. 5) and interrelationships of these expansions.

Thus, we start with Eq. (3.6) which we re-write in the quasi-interaction representation by means of the operator $W(\beta)$, defined

$$e^{-\beta K_o}\, W(\beta) \equiv e^{-\beta K} \tag{3.9}$$

where

$$K_o = H_o + U - G \tag{3.10}$$

and

$$K = H - G = K_o + V - U \tag{3.11}$$

with the one-particle operator U defined[5]

$$U \equiv \sum_{\underset{\sim}{k}} u(\underset{\sim}{k})\, a^\dagger(\underset{\sim}{k})\, a(\underset{\sim}{k}) + \sum_{\underset{\sim}{k}} S(\underset{\sim}{k})\, a^\dagger(\underset{\sim}{k})\, a(\underset{\sim}{k})$$

$$\equiv \sum_{\underset{\sim}{k}} C(\underset{\sim}{k})\, a^\dagger(\underset{\sim}{k})\, a(\underset{\sim}{k}) \tag{3.12}$$

The free-particle energies in H_o become renormalized by U so that $H_o + U$ has the quasi-particle energies as follows:

$$K_o = \sum_{\alpha} \sum_{\underset{\sim}{k}^{\alpha}} a^{\dagger}(\underset{\sim}{k}^{\alpha})\, a(\underset{\sim}{k}^{\alpha})\, E(\underset{\sim}{k}^{\alpha}) \tag{3.13}$$

where

$$E(\underset{\sim}{k}^{\alpha}) = w^{(o)}(\underset{\sim}{k}^{\alpha}) + u_{\alpha}(\underset{\sim}{k}) + S_{\alpha}(\underset{\sim}{k}) - g_{\alpha} \quad . \tag{3.14}$$

The counterterm $S_{\alpha}(\underset{\sim}{k})$ is introduced <u>specifically</u> to achieve mass renormalization for charged particles; thus, we set

$$S_{\gamma}(\underset{\sim}{k}) = 0$$

$$S_{\alpha}(\underset{\sim}{k}) = \left(\frac{\hbar^2 k^2}{2M_{\alpha}} \right) D_{\alpha} \tag{3.15}$$

where

$$D_{\alpha} = 1 - M_{\alpha}/M_{\alpha}^{(o)} \neq D_{\alpha}(k) \tag{3.16}$$

and M_{α} is the observed mass. The quantity $u_{\alpha}(\underset{\sim}{k})$ will be discussed in Sec VI. With Eqs. (2.7) and (3.15) we see that for charged particles

$$W_\alpha^{(o)}(\underset{\sim}{k}) + S_\alpha(\underset{\sim}{k}) = \hbar^2 k^2/2M_\alpha \tag{3.17}$$

which is the correct free-particle energy.

Now, Eq. (3.9) permits us to re-write Eq. (3.6) in the more useful form

$$\Omega f = \ln \mathrm{Tr}\, \rho_o W(\beta) \tag{3.18}$$

$$= \Omega f_o + \ln \mathrm{Tr}\, W(\beta) \tag{3.19}$$

where

$$\rho_o = e^{-\beta K_o} \tag{3.20}$$

and

$$\Omega f_o = \ln \mathrm{Tr}\, \rho_o \quad . \tag{3.21}$$

It is well known that $W(\beta)$ satisfies a Bloch-type differential equation which is equivalent to an integral equation whose iterative solution is the following Dyson expansion:

$$W(\beta) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^\beta ds_1 \cdots \int_0^\beta ds_n\, P_T\{H_1(s_1) \cdots H_1(s_n)\} \tag{3.22}$$

where P_T is the temperature ordering operator which orders the variables $s_1, \cdots, s_n$ in increasing order from right to left. It is to be noted that Eq. (3.22) is equivalent to the quantum Ursell expansion. If Eq. (3.22) is substituted into Eq. (3.18) and the finite temperature version of Wick's theorem[6] is utilized, then one obtains an expansion of Ωf which is most efficiently stated in terms of connected diagrams.[7] Thus, we write

$$\Omega f = \Omega f_o + \sum \begin{bmatrix} \text{all connected} \\ (0,0) \text{ graphs} \end{bmatrix} \tag{3.23}$$

where the rules for connected (μ, ν) graphs are given in Appendix A, and here $\mu = 0$, $\nu = 0$. Similarly, in terms of connected graphs the momentum distribution is

$$\langle n_\alpha(\underset{\sim}{k}) \rangle = \nu_\alpha(\underset{\sim}{k}) + \nu_\alpha(\underset{\sim}{k}) \sum \begin{bmatrix} \text{all connected} \\ (1,1) \text{ graphs} \end{bmatrix} \tag{3.24}$$

where

$$\nu(\underset{\sim}{k}^\alpha) = \nu_\alpha(\underset{\sim}{k}) = \{\exp \beta[E(\underset{\sim}{k}^\alpha)] - \epsilon_\alpha\}^{-1} \tag{3.25}$$

It is important to note that throughout this paper we adopt forms of the diagrammatic rules that facilitate a comparison of this and other developments. Here, we note that the line factors in connected (μ,ν) graphs are free-particle Green functions [see Eq. (A.1) in Appendix A].

IV. SELF-ENERGY ANALYSIS

In connected (μ, ν) graphs, graphical structures which can be removed from the rest of the diagram by cutting two internal lines are called self-energy structures. Self-energy structures are always connected (1,1), (0,2) or (2,0) graphs and it is for this reason that in Appendix A we give the rules for connected (μ, ν) graphs, where $\mu + \nu = 2$, as well as $(\mu, \nu) = (0,0)$. For example, in Figs. 2-a(3), 2-b(4) and 2-c(4) we observe the occurrence of the self-energy structure in Fig. 2-a(1) for $\eta = \gamma$, where it is important to note that (0,2) and (2,0) structures always have photon external lines. Similarly, all of the diagrams in Fig. 2 can occur as self-energy structures in other appropriate graphs. In connected (μ, ν) graphs it is possible to sum the entire class of self-energy structures and to relegate the effect of the summation of these structures to line factors. Moreover, it is possible to enlarge this summation procedure so as to encompass graphical structures called improper graphs which we discuss next.

An improper graph is a connected (1,1), (0,2) or (2,0) graph which can be separated into two graphs by cutting one internal line, where each of the graphs is a (1,1), (0,2) or (2,0) structure. A proper graph is a connected (1,1), (0,2) or (2,0) graph which is not improper. Thus, we wish to perform a partial summation which will result in proper graphs with no self-energy parts.

The summations referred to in the preceding paragraphs will now be performed. First, we define the functions $(\mu + \nu = 2)$

$$L_{\mu\nu}(t_2, t_1, \underset{\sim}{k}^{\alpha}) = \sum \begin{bmatrix} \text{all different connected} \\ (\mu, \nu) \text{ L-graphs} \end{bmatrix}_{\underset{\sim}{k}^{\alpha}} \tag{4.1}$$

where a (μ,ν) L-graph is defined to be a graph with the same structure as a given (μ,ν) graph with the following exceptions:

(a) No vertex factors are associated with the temperature labels t_2, t_1 of the lines. Such vertices do not contribute to the order of the diagram.

(b) There is no integration over the external temperature labels t_2, t_1. (4.2)

(c) There is no summation over the momentum labels of external lines

Next, we define the quantities which we shall eventually regard as line factors

$$G_{\mu,\nu}(t_2, t_1, \underset{\sim}{k}^{\alpha}) = \delta(t_2^{(-)} - t_1)\delta_{\mu,\nu} + \epsilon_{\alpha} L_{\mu,\nu}(t_2, t_1, \underset{\sim}{k}^{\alpha}) \tag{4.3}$$

where throughout this section $\mu + \nu = 2$ and always (0,2) and (2,0) structures have photon external lines. Finally, we define the functions

$$M_{\mu,\nu}(t_2,t_1,\underset{\sim}{k}^{\alpha}) = \sum \left[\begin{array}{l}\text{all different proper}\\ \text{connected } (\mu,\nu) \text{ L-graphs}\end{array}\right]_{\underset{\sim}{k}^{\alpha}} \tag{4.4}$$

The functions defined in Eqs. (4.1), (4.3) and (4.4) are related by the following set of simple coupled integral equations:[2]

$$L_{1,1}(t_2,t_1,\underset{\sim}{k}^{\alpha}) = \int_0^{\beta} ds\, [G_{1,1}(t_2,s,\underset{\sim}{k}^{\alpha})\, M_{1,1}(s,t_1,\underset{\sim}{k}^{\alpha}) + \delta_{\alpha,\gamma}\, G_{2,0}(t_2,s,\underset{\sim}{k}^{\alpha})\, M_{0,2}(t_1,s,\underset{\sim}{k}^{\alpha})] \tag{4.5}$$

$$L_{0,2}(t_2,t_1,\underset{\sim}{k}^{\alpha}) = \int_0^{\beta} ds\, [G_{0,2}(t_2,s,\underset{\sim}{k}^{\alpha})\, \overline{M}_{1,1}(s,t_1,-\underset{\sim}{k}^{\alpha}) + \delta_{\alpha,\gamma}\, G_{1,1}(s,t_2,\underset{\sim}{k}^{\alpha})\, M_{0,2}(s,t_1,\underset{\sim}{k}^{\alpha})] \tag{4.6}$$

$$L_{2,0}(t_2,t_1,k^\alpha) = \int_0^\beta ds\, [G_{2,0}(t_2,s,\underset{\sim}{k}^\alpha)\, \overline{M}_{1,1}(t_1,s,-\underset{\sim}{k}^\alpha)$$

$$+ \delta_{\alpha,\gamma}\, G_{1,1}(t_2,s,\underset{\sim}{k}^\alpha)\, M_{2,0}(s,t_1,\underset{\sim}{k}^\alpha)] \tag{4.7}$$

In general quantities associated with $-\underset{\sim}{k}$ lines will be different from the corresponding ones for $+\underset{\sim}{k}$ lines. Thus, we always place a bar over quantities associated with $-\underset{\sim}{k}$ lines [note, for example, Eqs. (4.6) and (4.7)].

The preceding self-energy analysis is based on integral equations which involve $M_{\mu,\nu}(t_2,t_1,\underset{\sim}{k}^\alpha)$ in a very intimate manner; however, up to now we have considered only self-energy structures whose external momenta are not summed. But indeed there exist proper graphs which can be separated into two disjoint graphs by cutting two internal lines; thus, we should complete our analysis for internal lines by including self-energy structures whose momenta are summed.

This program is easily accomplished if we introduce master (μ,ν) graphs which are <u>irreducible</u> connected (μ,ν) graphs whose internal lines have as line factors $G_{\mu,\nu}(t_2,t_1,\underset{\sim}{k}^{\alpha})$. An irreducible graph is a graphical structure which cannot be separated into two or more (μ,ν) graphs $(\mu + \nu = 2 \text{ or } 4)$ by cutting one or two internal lines. In terms of master graphs all self-energy structures are contained in the line factors. In summary, this basic result is expressed

$$M_{\mu,\nu}(t_2,t_1,\underset{\sim}{k}^{\alpha}) = \epsilon_\alpha c_\alpha(\underset{\sim}{k})[\theta(t_2 - t_1) + \epsilon_\alpha \nu_\alpha(\underset{\sim}{k})]\delta_{\mu,\nu} + K_{\mu,\nu}(t_2,t_1,\underset{\sim}{k}^{\alpha}) , \tag{4.8}$$

where

$$K_{\mu,\nu}(t_2,t_1,\underset{\sim}{k}^{\alpha}) = \sum \begin{bmatrix} \text{all different} \\ \text{master}(\mu,\nu)\ \text{L-graphs} \end{bmatrix}_{\underset{\sim}{k}^{\alpha}} . \tag{4.9}$$

All of our subsequent developments will be based on master (μ,ν) graphs. For example, in calculating the momentum distribution, where

$$\langle n_\alpha(\underset{\sim}{k})\rangle = \nu_\alpha(\underset{\sim}{k}) \int_0^\beta dt\, G_{1,1}(\beta,t,\underset{\sim}{k}^{\alpha}) \tag{4.10}$$

[see Eqs. (3.24), (4.1) and (4.2)], we are to understand that $G_{1,1}(t_2,t_1,\underset{\sim}{k}^{\alpha})$ is evaluated in terms of master graphs.

Although our self-energy analysis is complete it is useful, for the purpose of applications, to perform a rearrangement of Eqs. (4.6) - (4.8) so that the integral equations for $G_{\mu,\nu}(t_2,t_1,\underset{\sim}{k}^{\gamma})$ are partially decoupled. This rearrangement is readily accomplished by the introduction of the following three new functions:

$$\overline{G}(t_2,t_1,-\underset{\sim}{k}^{\gamma}) = \delta(t_2^{(-)}-t_1) + \overline{L}(t_2,t_1,-\underset{\sim}{k}^{\gamma}), \tag{4.11}$$

$$\overline{L}(t_2,t_1,-\underset{\sim}{k}^{\gamma}) = \int_0^{\beta} ds\ \overline{G}(t_2,s,-\underset{\sim}{k}^{\gamma})\ \overline{M}_{1,1}(s,t_1,-\underset{\sim}{k}^{\gamma}) , \tag{4.12}$$

$$Q(t_2,t_1,\underset{\sim}{k}^{\gamma}) = u_{\gamma}(\underset{\sim}{k})[\theta(t_2 - t_1) + \nu'_{\gamma}(\underset{\sim}{k})] + K_{1,1}(t_2,t_1,\underset{\sim}{k}^{\gamma})$$
$$+ \int_0^{\beta} ds_1 ds_2 K_{2,0}(t_2,s_1,\underset{\sim}{k}^{\gamma})\ \overline{G}(s_2,s_1,-\underset{\sim}{k}^{\gamma}) K_{0,2}(t_1,s_2,\underset{\sim}{k}^{\gamma}). \tag{4.13}$$

In terms of the three new functions in Eqs. (4.11) - (4.13), the integral equations in Eqs. (4.6) - (4.8) become

$$L_{1,1}(t_2,t_1,\underset{\sim}{k}^{\gamma}) = \int_0^{\beta} ds\ G_{1,1}(t_2,s,\underset{\sim}{k}^{\gamma})\ Q(s,t_1,\underset{\sim}{k}^{\gamma}) \tag{4.14}$$

$$L_{0,2}(t_2,t_1,\underset{\sim}{k}^{\gamma}) = \int_0^{\beta} ds_1 ds_2 M_{0,2}(s_2,s_1,\underset{\sim}{k}^{\gamma})\, G_{1,1}(s_2,t_2,\underset{\sim}{k}^{\gamma})\, \overline{G}(s_1,t_1,-\underset{\sim}{k}^{\gamma}), \tag{4.15}$$

$$L_{2,0}(t_2,t_1,\underset{\sim}{k}^{\gamma}) = \int_0^{\beta} ds_1 ds_2 G_{1,1}(t_2,s_2,\underset{\sim}{k}^{\gamma})\, \overline{G}(t_1,s_1,-\underset{\sim}{k}^{\gamma})\, M_{2,0}(s_2,s_1,\underset{\sim}{k}^{\gamma}), \tag{4.16}$$

It is now possible to formulate a general procedure for selecting the counterterms. For example, in Eq. (4.14) one calculates, to any desired order, the quantity $Q_{1,1}(t_2,t_1,\underset{\sim}{k}^{\alpha})$. Then, $\epsilon_\alpha u_\alpha(\underset{\sim}{k})$ can be chosen to be any term in $Q_{1,1}(t_2,t_1,\underset{\sim}{k}^{\alpha})$ which is multiplied by $[\theta(t_2 - t_1) + \epsilon_\alpha \nu_\alpha(\underset{\sim}{k})]$ and which is otherwise independent of t_2 and t_1. For selecting $u_\gamma(\underset{\sim}{k})$ it is more useful to apply this same procedure to Eq. (4.13). Indeed, one attempts to select the counterterms to meet one or more of the following criteria: (1) well-behaved line factors are obtained; (2) mass-renormalization is achieved; (3) a highly convergent iterative theory is obtained; and (4) the theory is somehow simplified. Of course, one must establish the effects on

the theory after the counterterms have been selected. Also, it is to be noted that the selection procedure discussed above usually leads to integral equations since the line factors themselves are functionals of the counterterms.

Up to this point we have given the prescription for calculating only the momentum distribution. It is not at all difficult to derive an expression for the grand potential in terms of (0,0) master graphs. Not only is this relation available elsewhere,[2] but also we take the point of view that (0,0) graphs are unnecessary; thus in the next section on Green functions we express the grand potential in terms of (1,1) graphs.

At this point it is useful to point out that one important accomplishment of the preceding development is that we have presented a highly simplified version of the formulation of others. Moreover, our derivation incorporates, by means of counterterms, the so-called Λ-transformation from the beginning, thereby avoiding the complicated procedure of performing the Λ-transformation on the theory containing no counterterms. It is also important to note that our line factors are the same as those of MRS only after the Λ-transformation has been performed.[8,9]

V. CONNECTION WITH GREEN FUNCTIONS

In the preceding section we formulated a theory in which the line factors $G_{\mu,\nu}(t_2, t_1, \underset{\sim}{k}^{\alpha})$ play a fundamental role. Whether these quantities have any deeper, physical meaning has been an open question. Thus, it is our objective in this section to establish formal relations between the line factors $G_{\mu,\nu}(t_2, t_1, \underset{\sim}{k}^{\alpha})$ and one-particle Green functions. This not only clarifies the meaning of the line factors, but also it enables us to use well-known properties of Green functions to deduce important properties of the line factors.

The single-particle Green function is defined as follows:

$$S_{1,1}(t_2, t_1, \underset{\sim}{k}^{\alpha}) = \mathrm{Tr}\{\hat{\rho}\, P_T'[a(\underset{\sim}{k}^{\alpha}, t_2) a^{\dagger}(\underset{\sim}{k}^{\alpha}, t_1)]\} \quad . \tag{5.1}$$

It is also necessary to introduce for photons <u>anomalous</u> Green functions, defined as follows:

$$S_{2,0}(t_2, t_1, \underset{\sim}{k}^{\alpha}) = \delta_{\alpha,\gamma}\, \mathrm{Tr}\{\hat{\rho}\, P_T'[a^{\dagger}(\underset{\sim}{k}^{\alpha}, t_2) a^{\dagger}(-\underset{\sim}{k}^{\alpha}, t_1)]\} \quad , \tag{5.2}$$

$$S_{0,2}(t_2, t_1, \underset{\sim}{k}^{\alpha}) = \delta_{\alpha,\gamma}\, \mathrm{Tr}\{\hat{\rho}\, P_T'[a(\underset{\sim}{k}^{\alpha}, t_2) a(-\underset{\sim}{k}^{\alpha}, t_1)]\} \quad . \tag{5.3}$$

In Eqs. (5.1), (5.2) and (5.3) $\hat{\rho} = \exp(-\Omega f)\exp(-\beta K)$; P_T' is the temperature ordering operator

[cf. Eq. (3.22)], but now defined to give a factor of ϵ_α for each odd permutation of operators for α-type particles required to restore the original order; the creation and annihilation operators are defined in the Heisenberg representation, where

$$a(\underset{\sim}{k}^\alpha, t) = \exp(+tK)\, a(\underset{\sim}{k}^\alpha)\, \exp(-tK) \tag{5.4}$$

$$a^\dagger(\underset{\sim}{k}^\alpha, t) = \exp(+tK)\, a^\dagger(\underset{\sim}{k}^\alpha)\, \exp(-tK); \tag{5.5}$$

and κt is an inverse temperature variable. From the definition in Eqs. (5.1) - (5.3) and Eqs. (5.4) and (5.5) we have the important result that

$$S_{\mu,\nu}(t_2, t_1, \underset{\sim}{k}^\alpha) = S_{\mu,\nu}(t_2 - t_1, \underset{\sim}{k}^\alpha) \quad , \tag{5.6}$$

where $(\mu,\nu) = (1,1)$ $(2,0)$ and $(0,2)$.

As noted in Appendix A connected (μ,ν) graphs involve single-particle Green functions; thus, on the basis of intuition, one would expect the single-particle and anomalous Green functions to have a close relation to connected (μ,ν) graphs. This is indeed the case. Moreover, using the Dyson expansion of Eq. (3.22), the finite temperature form of Wick's theorem and a self-energy analysis similar to that of Sec. IV, one can formulate a diagrammatic representation for $S_{\mu,\nu}(t_2, t_1, \underset{\sim}{k}^\alpha)$ and thereby arrive at the following relations between the Green functions of Eqs. (5.1) - (5.3) and the line factors of Eq. (4.3):

$$S_{1,1}(t_2,t_1,\underset{\sim}{k}^{\alpha}) = \exp[-t_2\, E(\underset{\sim}{k}^{\alpha})]\, \exp[t_1\, E(\underset{\sim}{k}^{\alpha})] \int_0^{\beta} ds\, G_{1,1}(t_2,s,\underset{\sim}{k}^{\alpha})$$

$$\times\, [\theta(s - t_1) + \epsilon_{\alpha}\nu_{\alpha}(\underset{\sim}{k})] \tag{5.7}$$

$$S_{2,0}(t_2,t_1,\underset{\sim}{k}^{\alpha}) = \exp[-t_2\, E(\underset{\sim}{k}^{\alpha})]\, \exp[-t_1\, \overline{E}(-\underset{\sim}{k}^{\alpha})]\, G_{2,0}(t_2,t_1,\underset{\sim}{k}^{\alpha}) \tag{5.8}$$

$$S_{0,2}(t_2,t_1,\underset{\sim}{k}^{\alpha}) = \exp[t_2\, E(\underset{\sim}{k}^{\alpha})]\, \exp[t_1\, \overline{E}(-\underset{\sim}{k}^{\alpha})] \int_0^{\beta} ds_2 ds_1\, G_{0,2}(s_2,s_1,\underset{\sim}{k}^{\alpha})$$

$$\times\, [\theta(s_2 - t_2) + \epsilon_{\alpha}\nu_{\alpha}(\underset{\sim}{k})][\theta(s_1 - t_1) + \epsilon_{\alpha}\overline{\nu}_{\alpha}(-\underset{\sim}{k})] \tag{5.9}$$

Although one can write integral equations for the $S_{\mu,\nu}(t_2,t_1,\underset{\sim}{k}^{\alpha})$ analogous to those for $G_{\mu,\nu}(t_2,t_1,\underset{\sim}{k}^{\alpha})$ in Sec. IV, it is perhaps more instructive to substitute Eqs. (4.3) and (4.5) -(4.7) into Eqs. (5.7) - (5.9) to obtain

$$S_{1,1}(t_2,t_1,\underset{\sim}{k}^{\alpha}) = S_0(t_2,t_1,\underset{\sim}{k}^{\alpha})$$

$$+ \int_0^{\beta} ds_2 ds_1 \; S_{1,1}(t_2,s_2,\underset{\sim}{k}^{\alpha}) \; R_{1,1}(s_2,s_1,\underset{\sim}{k}^{\alpha}) \; S_0(s_1,t_1,\underset{\sim}{k}^{\alpha})$$

$$+ \int_0^{\beta} ds_2 ds_1 \; S_{2,0}(t_2,s_2,\underset{\sim}{k}^{\alpha}) \; R_{0,2}(s_1,s_2,\underset{\sim}{k}^{\alpha}) \; S_0(s_1,t_1,\underset{\sim}{k}^{\alpha}) \tag{5.10}$$

$$S_{2,0}(t_2,t_1,\underset{\sim}{k}^{\alpha}) = \int_0^{\beta} ds_2 ds_1 \; S_{2,0}(t_2,s_1,\underset{\sim}{k}^{\alpha}) \; \overline{R}_{1,1}(s_2,s_1,-\underset{\sim}{k}^{\alpha}) \; \overline{S}_0(t_1,s_2,-\underset{\sim}{k}^{\alpha})$$

$$+ \int_0^{\beta} ds_2 ds_1 \; S_{1,1}(t_2,s_1,\underset{\sim}{k}^{\alpha}) \; R_{2,0}(s_1,s_2,\underset{\sim}{k}^{\alpha}) \; \overline{S}_0(t_1,s_2,-\underset{\sim}{k}^{\alpha}) \tag{5.11}$$

$$S_{0,2}(t_2,t_1,\underset{\sim}{k}^{\alpha}) = \int_0^{\beta} ds_2 ds_1 \; S_{0,2}(t_2,s_2,\underset{\sim}{k}^{\alpha}) \; \overline{R}_{1,1}(s_2,s_1,-\underset{\sim}{k}^{\alpha}) \; \overline{S}_0(s_1,t_1,-\underset{\sim}{k}^{\alpha})$$

$$+ \int_0^{\beta} ds_2 ds_1 \; S_{1,1}(s_2,t_2,\underset{\sim}{k}^{\alpha}) \; R_{0,2}(s_2,s_1,\underset{\sim}{k}^{\alpha}) \; \overline{S}_0(s_1,t_1,-\underset{\sim}{k}^{\alpha}) \tag{5.12}$$

The functions $R_{\mu,\nu}(t_2,t_1,\underset{\sim}{k}^{\alpha})$ are related to the $M_{\mu,\nu}(t_2,t_1,\underset{\sim}{k}^{\alpha})$ of Eq. (4.8) as follows:

$$M_{1,1}(t_2,t_1,\underset{\sim}{k}^{\alpha}) = \exp[t_2\, E(\underset{\sim}{k}^{\alpha})]\, \exp[-t_1\, E(\underset{\sim}{k}^{\alpha})] \times \int_0^{\beta} ds\, R_{1,1}(s,t_1,\underset{\sim}{k}^{\alpha})\, S_0(t_2,s,\underset{\sim}{k}^{\alpha}) \tag{5.13}$$

$$M_{2,0}(t_2,t_1,\underset{\sim}{k}^{\alpha}) = \exp[t_2\, E(\underset{\sim}{k}^{\alpha})]\, \exp[t_1\, \overline{E}(-\underset{\sim}{k}^{\alpha})] \times \int_0^{\beta} ds_2 ds_1\, R_{2,0}(s_2,s_1,\underset{\sim}{k}^{\alpha})\, S_0(t_2,s_2,\underset{\sim}{k}^{\alpha})\, \overline{S}_0(t_1,s_1,-\underset{\sim}{k}^{\alpha}) \tag{5.14}$$

$$M_{0,2}(t_2,t_1,\underset{\sim}{k}^{\alpha}) = \exp[-t_2\, E(\underset{\sim}{k}^{\alpha})]\, \exp[-t_1\, \overline{E}(-\underset{\sim}{k}^{\alpha})]\, R_{0,2}(t_2,t_1,\underset{\sim}{k}^{\alpha}) \tag{5.15}$$

The connection between the Green functions $S_{\mu,\nu}(t_2,t_1,\underset{\sim}{k}^{\alpha})$ and the line factors $G_{\mu,\nu}(t_2,t_1,\underset{\sim}{k}^{\alpha})$ is now established. Next we wish to explore some consequences of such a connection. One of the more important observations, based on Eqs. (5.6) and (5.7) - (5.9), is

$$G_{1,1}(t_2,t_1,\underset{\sim}{k}^{\alpha}) = G_{1,1}(t_2 - t_1,\underset{\sim}{k}^{\alpha}), \qquad (5.16)$$

which is useful as a check on calculations of line factors. Also, it is not difficult to establish, for a homogeneous system, that

$$G_{2,0}(t_2,t_1,\underset{\sim}{k}^{\alpha}) = \exp\{(t_2 + t_1)[E(\underset{\sim}{k}^{\alpha}) + \overline{E}(-\underset{\sim}{k}^{\alpha})]\}\int_0^{\beta} ds_2 ds_1\, G_{0,2}(s_2,s_1,\underset{\sim}{k}^{\alpha})$$

$$\times\ [\theta(s_2 - t_2) + \epsilon_{\alpha}\nu_{\alpha}(\underset{\sim}{k})][\theta(s_1 - t_1) + \epsilon_{\alpha}\overline{\nu}_{\alpha}(-\underset{\sim}{k})]$$

$$(5.17)$$

This relation can be used as a consistency check on the separate calculations of $G_{0,2}(t_2,t_1,\underset{\sim}{k}^{\alpha})$ and $G_{2,0}(t_2,t_1,\underset{\sim}{k}^{\alpha})$. A special, important case of Eq. (5.17) is

$$G_{2,0}(\beta,\ \beta,\ \underset{\sim}{k}^{\gamma}) = [1 + \nu_{\gamma}(\underset{\sim}{k})][1 + \overline{\nu}_{\gamma}(-\underset{\sim}{k})]\ \exp\{\beta[E(\underset{\sim}{k}^{\gamma}) + \overline{E}(-\underset{\sim}{k}^{\gamma})]\}$$

$$\times \int_0^{\beta} ds_2 ds_1\, G_{0,2}(s_2,s_1,\underset{\sim}{k}^{\gamma}) \qquad (5.18)$$

We mentioned in Sec. IV that we could calculate the grand potential in terms of (0,0) master graphs, but we would use instead an alternate procedure which permits the evaluation of the grand potential in terms of (1,1) master graphs. We discuss this procedure.[7]

First, one constructs the <u>scaled</u> single-particle Green function

$$S_{1,1}^{\lambda}(t_2,t_1,\underset{\sim}{k}^{\alpha}) = \mathrm{Tr}\{\exp(-\Omega f^{\lambda})\ \exp(-\beta K^{\lambda})$$

$$\times\ P_T'[a_{\lambda}(\underset{\sim}{k}^{\alpha},t_2)\ a_{\lambda}^{\dagger}(\underset{\sim}{k}^{\alpha},t_1)]\} \qquad (5.19)$$

where

$$K^{\lambda} = H_0 + \lambda V_{\gamma} + \lambda^2 V_c \quad . \qquad (5.20)$$

The scaled Green function in Eq. (5.19) is obtained from the normal Green function in Eq. (5.1) with the replacement of K by K^{λ} everywhere. It is tedious, but straightforward, to establish the identity

$$\Omega f = \Omega f_0 - \beta \sum_{\substack{\alpha \\ \alpha \neq \gamma}} \sum_{\underset{\sim}{k}^{\alpha}} \int_0^1 d\lambda\ \lambda^{-1} \lim_{t_1 \to t_2^+} \left[\frac{\partial}{\partial t_2} + w^{(0)}(\underset{\sim}{k}^{\alpha}) \right]$$

$$\times\ \epsilon_{\alpha} S_{1,1}^{\lambda}(t_2,\ t_1,\ \underset{\sim}{k}^{\alpha}) \qquad (5.21)$$

Hence from Eqs. (5.7) and (5.21) we see that a knowledge of scaled master (1,1) graphs in sufficient for the determination of the grand potential, and the explicit calculation of master (0,0) graphs is unnecessary.

VI. APPLICATION TO PHOTON MOMENTUM DISTRIBUTION

For the purpose of illustrating the formalism developed in the preceding sections we now calculate, to low order, the photon momentum distribution. This calculation will prove to have several interesting features. For example, the removal of the infrared divergence results in a renormalized photon energy-momentum relation. Also, we obtain an approximate, non-iterative solution to the integral equation for the line factor.

In this example, we calculate the three graphs in Fig. 3, where the external lines are for photons. The momentum sums are evaluated in the high-temperature, low-density limit. Thus, for the first graph in Fig. 3, we have

$$K_{1,1}(t_2,t_1,\underset{\sim}{k}^{\gamma}) = \sum_{\alpha} \sum_{\underset{\sim}{k}_1^{\alpha}} \int_0^{\beta} ds \left\langle \begin{matrix} \underset{\sim}{k}_1 & \underset{\sim}{k} \\ \underset{\sim}{k}_1 & \underset{\sim}{k} \end{matrix} \right\rangle_{\substack{t_1 \\ \alpha\ \gamma}}^{st_2} G_{1,1}(t_1,s,\underset{\sim}{k}_1^{\alpha})$$

$$\simeq - \frac{\hbar^2 \omega_p^{\ 2}}{2\hbar ck} [\theta(t_2 - t_1) + \nu_{\gamma}(\underset{\sim}{k})] , \qquad (6.1)$$

where

$$\omega_p^{\ 2} = \sum_{\alpha} \omega_p^{\ 2}(\alpha) \qquad (6.2)$$

and

$$\omega_p^{\,2}(\alpha) = 4\pi\rho_\alpha Z_\alpha^{\,2} e^2/M_\alpha \tag{6.3}$$

and ρ_α is the number of α-type particles per unit volume. For the second graph in Fig. 3, we have

$$K_{2,0}(t_2,t_1,\underset{\sim}{k}^\gamma) = \sum_\alpha \sum_{\underset{\sim}{k}_1^\alpha} \int_0^\beta ds_2 ds_1 \; {}^{s_2 t_2 t_1}\left\langle \begin{matrix} \underset{\sim}{k}_1 & \underset{\sim}{k} & -\underset{\sim}{k} \\ \underset{\sim}{k}_1 & & \\ \alpha & \gamma & \gamma \end{matrix} \right\rangle^{s_1} G_{1,1}(s_1,s_2,\underset{\sim}{k}_1^\alpha)$$

$$\simeq \frac{\hbar^2 \omega_p^{\,2}}{2\hbar c k} \left[w'_\gamma(\underset{\sim}{k}) + \overline{w}_\gamma(-\underset{\sim}{k})\right]^{-1}$$

$$\times \left\{\left[\theta(t_2 - t_1) + \nu_\gamma(\underset{\sim}{k})\right] \exp t_1\left[w_\gamma(\underset{\sim}{k}) + \overline{w}_\gamma(-\underset{\sim}{k})\right]\right.$$

$$\left. + \left[\theta(t_1 - t_2) + \overline{\nu}_\gamma(-\underset{\sim}{k})\right] \exp t_2\left[w_\gamma(\underset{\sim}{k}) + \overline{w}_\gamma(-k_\gamma)\right]\right.$$

$$\tag{6.4}$$

where $w'_\gamma(\underset{\sim}{k}) = \hbar ck + u_\gamma(\underset{\sim}{k})$. Finally, for the last graph in Fig. 3, we have

$$K_{0,2}(t_2,t_1,\underset{\sim}{k}^{\gamma}) = \sum_{\alpha} \sum_{\underset{\sim}{k}_1^{\alpha}} \int_0^{\beta} ds \left\langle \begin{matrix} \underset{\sim}{k}_1 \\ \underset{\sim}{k}_1 \\ \alpha \end{matrix} \quad \begin{matrix} \\ \underset{\sim}{k} \\ \gamma \end{matrix} \quad \begin{matrix} \\ -\underset{\sim}{k} \\ \gamma \end{matrix} \right\rangle_{t_2}^{s}$$

$$\times\ G_{1,1}(t_1,s,\underset{\sim}{k}_1^{\alpha})\ \delta(t_2 - t_1) \simeq \frac{\hbar^2\omega_p^{\ 2}}{2\hbar ck}$$

$$\times\ \delta(t_2 - t_1)\ \exp\{-t_2[w'_{\gamma}(\underset{\sim}{k}) + \overline{w}_{\gamma}(-\underset{\sim}{k})] \tag{6.5}$$

One should note that Eqs. (6.4) and (6.5) satisfy the relation in Eq. (5.17).

Now, in order to calculate the photon momentum distribution in Eq. (4.10) with $\alpha = \gamma$, we must first calculate $G_{1,1}(\beta,t,\underset{\sim}{k}^{\gamma})$. To do this we use Eq. (4.3) with Eq. (4.14). Thus, we evaluate the approximate expression

$$Q(t_2,t_1,\underset{\sim}{k}^{\gamma}) \simeq u_{\gamma}(\underset{\sim}{k})[\theta(t_2 - t_1) + \nu_{\gamma}(\underset{\sim}{k})] + K_{1,1}(t_2,t_1,\underset{\sim}{k}^{\gamma}) \tag{6.6}$$

$$+ \int_0^{\beta} ds\ K_{2,0}(t_2,s,\underset{\sim}{k}^{\gamma})\ K_{0,2}(t_1,s,\underset{\sim}{k}^{\gamma})$$

$$\simeq [\theta(t_2 - t_1) + \nu_{\gamma}(\underset{\sim}{k})] \left\{u_{\gamma}(\underset{\sim}{k}) - \frac{\hbar^2\omega_p^{\ 2}}{2\hbar ck} + \frac{[\hbar^2\omega_p^{\ 2}/2\hbar ck]^2}{w'_{\gamma}(\underset{\sim}{k}) + \overline{w}_{\gamma}(-\underset{\sim}{k})}\right\}$$

$$+ [\theta(t_1 - t_2) + \overline{\nu}_{\gamma}(-\underset{\sim}{k})]\ \frac{[\hbar^2\omega_p^{\ 2}/2\hbar ck]^2}{w'_{\gamma}(\underset{\sim}{k}) + \overline{w}_{\gamma}(-\underset{\sim}{k})}$$

$$\times\ \exp\{(t_2 - t_1)[w'_{\gamma}(\underset{\sim}{k}) + \overline{w}_{\gamma}(-\underset{\sim}{k})]$$

This expression is simplified considerably if we choose

$$u_\gamma(\underset{\sim}{k}) \simeq \frac{\hbar^2\omega_p{}^2}{2\hbar ck} - \frac{[\hbar^2\omega_p{}^2/2\hbar ck]}{w'_\gamma(\underset{\sim}{k}) + \overline{w}_\gamma(-\underset{\sim}{k})} . \qquad (6.7)$$

To lowest order we take

$$\overline{w}_\gamma(-\underset{\sim}{k}) \simeq \hbar ck + \hbar^2\omega_p{}^2/2\hbar ck . \qquad (6.8)$$

Thus, with Eq. (6.7) we have the result

$$w'_\gamma(\underset{\sim}{k}) \simeq [\hbar ck)^2 + (\hbar\omega_p)^2]^{1/2} . \qquad (6.9)$$

It is important to note that with $u_\gamma(\underset{\sim}{k}) = 0$ Eq. (6.6) is divergent for $k \to 0$, and the divergence is a manifestation of the so-called infrared divergence. Now, with the choice in Eq. (6.9), Eq. (6.6) becomes

$$Q(t_2, t_1, \underset{\sim}{k}^\gamma) \simeq \frac{[\hbar^2\omega_p{}^2/2\hbar ck]^2}{w'_\gamma(\underset{\sim}{k}) + \overline{w}_\gamma(-\underset{\sim}{k})} [\theta(t_1 - t_2) + \overline{v}_\gamma(-\underset{\sim}{k})]$$

$$\times \exp\{(t_2 - t_1)[w'_\gamma(\underset{\sim}{k}) + \overline{w}_\gamma(-\underset{\sim}{k})]\} ,$$

$$(6.10)$$

Thus, we have succeeded in removing the infrared divergence, at least to lowest order, by the identification of the counterterm in Eq. (6.7). Also, we point out that Eq. (6.9) has been obtained by others by subjecting the Hamiltonian in Eq. (2.2) to a Bogoliubov transformation with the same consequences as here: the infrared divergence is removed.[10,11]

The calculation of the photon momentum distribution is based on Eq. (4.10) with Eqs. (4.3) (4.14) and (6.10). It is immediately obvious that Eq. (4.3) with Eq. (4.14) does not possess an iterative solution when Eq. (6.10) is used (see Appendix C). However, we can use the developments in Appendix C to obtain the result

$$G_{1,1}(t_2,t_1,\underset{\sim}{k}^{\gamma}) = \delta(t_2 - t_1) + A[\theta(t_1 - t_2) + M] \exp[(t_2 - t_1)(B - A)] \tag{6.11}$$

where

$$A = \frac{[\hbar^2\omega_p{}^2/2\hbar c\, k]^2}{w'_{\gamma}(\underset{\sim}{k}) + \overline{w}_{\gamma}(-\underset{\sim}{k})}$$

$$B = w'_{\gamma}(\underset{\sim}{k}) + \overline{w}_{\gamma}(-\underset{\sim}{k})$$

$$M = \left\{\exp \beta \left[w_{\gamma}'(\underset{\sim}{k})\right] - 1\right\}^{-1} = \nu_{\gamma}(\underset{\sim}{k}) \tag{6.12}$$

It should be noted that A in Eq. (6.12) diverges for $k \to 0$, so that an iterative solution to Eq. (4.3) with Eqs. (4.14) and (6.10) would be unacceptable if one existed. Thus, we use Eq. (6.11) in order to calculate the photon momentum distribution in Eq. (4.10). We obtain

$$\langle n_\gamma(\underset{\sim}{k})\rangle = \nu_\gamma(\underset{\sim}{k}) + J(\underset{\sim}{k})[1 + 2\nu_\gamma(\underset{\sim}{k})] , \tag{6.13}$$

where

$$\nu_\gamma(\underset{\sim}{k}) = \left[e^{\beta w'_\gamma(\underset{\sim}{k})} - 1\right]^{-1}$$

$$J(\underset{\sim}{k}) = \frac{\overline{w}_\gamma(-\underset{\sim}{k}) - w'_\gamma(\underset{\sim}{k})}{2w'_\gamma(\underset{\sim}{k})}$$

with $w'_\gamma(\underset{\sim}{k})$ given by Eq. (6.9) and $\overline{w}_\gamma(-\underset{\sim}{k})$ given by Eq. (6.8).

It is of interest to examine Eq. (6.13) in the limits of large and small k. For large k we find

$$\langle n_\gamma(\underset{\sim}{k})\rangle \to \nu_\gamma(\underset{\sim}{k}) + \left(\frac{\hbar\omega_p}{2\hbar ck}\right)^4 [1 + 2\,\nu_\gamma(\underset{\sim}{k})] \qquad (6.14)$$

which approaches

$$\nu_\gamma^{(0)}(\underset{\sim}{k}) = [\exp(\beta\hbar ck) - 1]^{-1} \qquad (6.15)$$

as k increases without limit; thus, Eq. (6.14) is essentially the free photon distribution function. On the other hand, for small k we obtain

$$\langle n_\gamma(\underset{\sim}{k})\rangle \to \nu_\gamma(\underset{\sim}{k}) + \frac{\hbar\omega_p}{4\hbar ck}[1 + 2\nu_\gamma(\underset{\sim}{k})] \qquad (6.16)$$

The expression in Eq. (6.13) is a new result. In particular, as is clear from Eq. (6.16), this momentum distribution is divergent for $k \to 0$. This result disagrees with the earlier result of Hwang and Grandy,[11,12] who find that the photon momentum distribution approaches a system dependent limit as $k \to 0$. The reason for this discrepancy is mainly because these authors do not include the (0,2) and (2,0) photon self-energy structures in their calculation. Physically, a singularity in $\langle n_\gamma(k)\rangle$ as $k \to 0$ is not at all surprising (the Planck distribution also diverges for vanishing photon momentum).

VII. DISCUSSION

The main results of this paper will now be summarized.

(1) We have outlined a quantum statistical theory of a nonrelativistic, fully ionized gas in thermal equilibrium.

(2) We have also established explicitly the relation between the present work and the Green function technique in many-body theory. In this connection the results of Sec. V are not only valid for the model system considered here, but also are true for other many-body systems as well.

(3) The momentum distribution of the photons has been evaluated in lowest order. It is found that this quantity behaves as $1/k$ for $k \to 0$, thereby exhibiting a singular behavior which is physically reasonable. This result disagrees with the earlier work of Hwang and Grandy,[11] and the source of the disagreement has been discussed at the end of Sec. VI.

(4) We have not investigated the effects of the Coulomb interaction, in which case one needs to sum the ring diagrams. Such a summation can be performed easily in the present framework and results in a much simpler version of the earlier work by Mohling and Grandy.[4]

APPENDIX A. RULES FOR CONNECTED (μ,ν) GRAPHS AND MASTER (μ,ν) GRAPHS

A Pth order connected (μ,ν) graph is defined to be a graphical structure consisting of P_1 interaction vertices and P_2 Δ-vertices (see Fig. 1), with $P = P_1 + P_2$, which are entirely interconnected by m directed internal solid lines; attached to the entire structure are μ outgoing external solid lines and ν incoming external solid lines. If $(\mu,\nu) \neq (0,0)$, then the incoming particle (but not photon) lines refer to the same set of particles as the outgoing lines. Photon lines representing strictly zero momentum are to be excluded from diagrams, since such photons correspond to vacuum interactions. Each vertex is assigned a different temperature label $s_1, s_2, \cdots s_P$, and each outgoing external line attaches to a point with a temperature label $t_1, \cdots, t_\mu$. Each line is assigned a different particle label and momentum label, and the external line momentum labels are fixed or pregiven. External lines with different momentum labels or directions are treated as distinguishable. Each topologically distinct arrangement of lines and vertices gives a different graph-see Rule 4 below, The rules for interconnecting the P vertices and for associating with the graph the appropriate analytic expression are given below.

1. To each vertex assign a vertex factor, as provided by Fig. 1 and Appendix B, and form a product of these vertex functions. Note that momentum is conserved at each vertex.

2. To each line directed from t_1 to t_2 and labeled with momentum $\underset{\sim}{k}$ and particle label α assign a line factor

$$S_0(t_2,t_1,\underset{\sim}{k}^\alpha) = \exp[-(t_2 - t_1)\, E(\underset{\sim}{k}^\alpha)][\theta(t_2 - t_1) + \epsilon_\alpha \nu_\alpha(\underset{\sim}{k})], \tag{A.1}$$

where $E(\underset{\sim}{k}^\alpha)$ is defined in Eq. (3.14) and $\nu_\alpha(\underset{\sim}{k})$ is defined in Eq. (3.25). The step function is defined $\theta(x) = 1$ if $x > 0$, $\theta(x) = 0$ if $x \leq 0$.

3. Assign to the graph an overall factor of $\prod_\alpha (\epsilon^{P_\alpha})$ where P_α is the parity of the relative permutation of the bottom and top row momenta of α-type particles in the product of vertex functions.

4. Assign a factor S^{-1} to the entire graph, where the symmetry number S is defined to be the total number of permutations of the m integers assigned to the internal lines which lead to a topologically equivalent graph. Two connected (μ,ν) graphs are topologically different if their structures (including particle-type labels and line directions) are topologically inequivalent.

5. Each internal particle momentum is summed over all states of <u>all</u> particles and each internal photon momentum is summed over all photon states-see Eq. (2.12).

6. Integrate over the P temperature variables $s_1, \cdots, s_P$ from 0 to β.

It is important to realize that Eq. (A.1) is the basic contracted product which is produced in the application of Wick's theorem. Moreover, one observes that this contraction is precisely the single-particle Green function or propagator for the non-interacting case. In practice, manipulations of the theory are facilitated if we eliminate the line factor in Eq. (A.1) by redefining the vertex functions so that this factor is included. Thus, consider a vertex with temperature label t_1; to this vertex assign a factor $[\theta(t' - t_1) + \epsilon_\alpha \nu_\alpha(\underset{\sim}{k}_i)] \exp[-(t' - t_1) E(\underset{\sim}{k}_i^{\alpha})]$ for <u>each</u> outgoing line (which has momentum label $\underset{\sim}{k}_i^{\alpha}$ and which goes to a temperature label t'); moreover, for each outgoing line assign a factor $\exp[t' E(\underset{\sim}{k}_i^{\alpha})]$; finally, for each incoming line (with momentum label $\underset{\sim}{k}_j^{\alpha}$) assign a factor $\exp[-t_1 E(\underset{\sim}{k}_j^{\alpha})]$.

The summation of all (μ,ν) self-energy structures leads to master (μ,ν) graphs. Thus, we give next the rules for master (μ,ν) graphs.

A <u>P</u>th-order master (μ,ν) graph is a collection of P cluster vertices (but no Δ-vertices), defined in Appendix B, which are entirely interconnected by m internal <u>solid</u> lines and to which are attached μ outgoing external <u>solid</u> lines and ν incoming external <u>solid</u> lines. Each external solid line carries a single arrow, and each internal solid line carries two arrows--one at each end. At the head of each arrow there is a dot. If the arrow points toward a vertex this dot is identical with the vertex. Three different types of internal solid lines are possible, depending upon whether the two arrows point in the

same direction, point toward each other or point away from each other. A master (μ,ν) graph is irreducible in the sense that the cutting of any two of its internal lines must not produce two (or three) disconnected graphs, at least one of which is a $(1,1)$, $(0,2)$ or $(2,0)$ graph. Corresponding to each master (μ,ν) graph there is prescribed an analytic term according to the following rules:

(1) To each external solid line assign a pregiven momentum with a particle label; if $(\mu,\nu) \neq (0,0)$ the incoming particle (not photon) lines refer to the same set of particles as the outgoing lines. External lines with different momentum labels or directions are treated as distinguishable.

(2) Two master (μ,ν) graphs are different if their topological structures (including arrow directions, particle-type labels and external lines, but not including the momentum labels of internal arrows and the temperature labels which will be assigned below) are different.

(3) To each arrow of the m internal solid lines assign a different integer i $(i = 1,2,\cdots,2m)$ and a corresponding momentum k_i^{α} (the possible choices of α will be fixed by the associated cluster vertices). Assign a different temperature variable t_j to each of the P cluster vertices and to each of the dots which occur at the head ends of internal arrows that

point away from vertices. To each dot of the outgoing external solid lines assign the temperature variable β.

(4) Assign to the entire graph a factor S^{-1}, where S is the symmetry number. The symmetry number is defined to be the total number of permutations of the $2m$ integers (assigned to the arrows of the internal lines) which leave the graph topologically unchanged (including the positions of these integers with respect to the arrows).

(5) Associate with the entire graph the appropriate product of P cluster vertices with the momentum variable assignments of rules (1) and (3). Assign to the graph an overall sign factor $\prod_\alpha \epsilon^{P_\alpha}$, where P_α is the parity of the permutation of the bottom row momenta of α-type particles in the vertex functions with respect to the corresponding ones in the top row.

(6) To each <u>internal</u> solid line with arrow labels i and j assign a line factor and a momentum conserving Kronecker delta as follows:

$\delta_{\underline{k}_i,\underline{k}_j}\, G_{1,1}(t,s,\underline{k}_i^{\alpha})$ when the arrows point in the same direction,

$\delta_{\underline{k}_i,-\underline{k}_j}\, G_{0,2}(t,s,\underline{k}_i^{\alpha})\, \delta_{\alpha,\gamma}$ when the arrows point toward each other,

$\delta_{\underline{k}_i,-\underline{k}_j}\, G_{2,0}(t,s,\underline{k}_i^{\alpha})\, \delta_{\alpha,\gamma}$ when the arrows point away from each other,

where the temperature labels t and s correspond to those assigned by rule (3). In each case, the arrow labeled i

points toward the dot labeled t. The Kronecker delta implies conservation of particle-type.

(7) Finally, sum over the 2m internal momenta and integrate from 0 to β over the temperature variables t_j assigned in rule (3).

APPENDIX B. VERTEX FUNCTIONS

In this appendix we give the expressions for the vertex functions associated with the vertex symbols of connected (μ,ν) graphs and master (μ,ν) graphs. One should note that the vertex functions account for the interaction terms in the Hamiltonian, along with certain other statistical factors.

First, we list the vertex functions associated with connected (μ,ν) graphs; the symbols corresponding to these vertex functions are given in Fig. 1. It should be pointed out that the temperature labels for these vertex functions are superfluous; however, the addition of temperature labels maintains a desirable continuity in our notation. The vertex functions of connected (μ,ν) graphs are the following:

$$
{}^{t_1 t_2}\begin{pmatrix} \underset{\sim}{k}_1 & \underset{\sim}{k}_2 \\ \underset{\sim}{k}_3 & \underset{\sim}{k}_4 \end{pmatrix}_t
\begin{matrix} \\ \\ \alpha \quad \beta \end{matrix}
\begin{aligned}
&= -\left[\langle \underset{\sim}{k}_1, \underset{\sim}{k}_2 | V_c | \underset{\sim}{k}_3, \underset{\sim}{k}_4 \rangle + \epsilon_\alpha \langle \underset{\sim}{k}_1, \underset{\sim}{k}_2 | V_c | \underset{\sim}{k}_4, \underset{\sim}{k}_3 \rangle\right] && \text{for } \alpha = \beta \\
&= -\epsilon_\alpha \epsilon_\beta \langle \underset{\sim}{k}_1, \underset{\sim}{k}_2 | V_c | \underset{\sim}{k}_3, \underset{\sim}{k}_4 \rangle && \text{for } \alpha \neq \beta \\
&= -2\epsilon_\alpha \langle \underset{\sim}{k}_1, \underset{\sim}{k}_2{}^{\gamma} | V_{2\gamma} | \underset{\sim}{k}_3, \underset{\sim}{k}_4{}^{\gamma} \rangle && \text{for } \begin{cases} \alpha = \text{charged particle} \\ \beta = \gamma \text{ (photon)} \end{cases}
\end{aligned}
\tag{B.1}
$$

$$\overset{t_1}{\underset{\alpha\ \beta}{\left(\begin{matrix}\underset{\sim}{k}_3 \\ \underset{\sim}{k}_1\ \underset{\sim}{k}_2\end{matrix}\right)_t}} = -\epsilon_\alpha \langle \underset{\sim}{k}_3 | V_{1\gamma} | \underset{\sim}{k}_1, \underset{\sim}{k}_2{}^\gamma \rangle \tag{B.2}$$

$$\overset{t_1 t_2}{\underset{\alpha\ \gamma}{\left(\begin{matrix}\underset{\sim}{k}_1\ \underset{\sim}{k}_2 \\ \underset{\sim}{k}_3\end{matrix}\right)_t}} = -\epsilon_\alpha \langle \underset{\sim}{k}_1, \underset{\sim}{k}_2{}^\gamma | V^\dagger_{1\gamma} | \underset{\sim}{k}_3 \rangle \tag{B.3}$$

$$\overset{t_1}{\underset{\alpha\ \gamma\ \gamma}{\left(\begin{matrix}\underset{\sim}{k}_4 \\ \underset{\sim}{k}_1\ \underset{\sim}{k}_2\ \underset{\sim}{k}_3\end{matrix}\right)_t}} = -\epsilon_\alpha [\langle \underset{\sim}{k}_4 | V_{2\gamma} | \underset{\sim}{k}_1, \underset{\sim}{k}_2{}^\gamma \underset{\sim}{k}_3{}^\gamma \rangle + \langle \underset{\sim}{k}_4 | V_{2\gamma} | \underset{\sim}{k}_1, \underset{\sim}{k}_3{}^\gamma \underset{\sim}{k}_2{}^\gamma \rangle]$$

$$= -2\epsilon_\alpha \langle \underset{\sim}{k}_4 | V_{2\gamma} | \underset{\sim}{k}_1, \underset{\sim}{k}_2{}^\gamma \underset{\sim}{k}_3{}^\gamma \rangle \tag{B.4}$$

$$\overset{t_1 t_2 t_3}{\underset{\alpha\ \beta\ \gamma}{\left(\begin{matrix}\underset{\sim}{k}_1\ \underset{\sim}{k}_2\ \underset{\sim}{k}_3 \\ \underset{\sim}{k}_4\end{matrix}\right)_t}} = -\epsilon_\alpha [\langle \underset{\sim}{k}_1, \underset{\sim}{k}_2{}^\gamma \underset{\sim}{k}_3{}^\gamma | V^\dagger_{2\gamma} | \underset{\sim}{k}_4 \rangle + \langle \underset{\sim}{k}_1, \underset{\sim}{k}_3{}^\gamma \underset{\sim}{k}_2{}^\gamma | V^\dagger_{2\gamma} | \underset{\sim}{k}_4 \rangle]$$

$$= -2\epsilon_\alpha \langle \underset{\sim}{k}_1, \underset{\sim}{k}_2{}^\gamma \underset{\sim}{k}_3{}^\gamma | V^\dagger_{2\gamma} | \underset{\sim}{k}_4 \rangle \tag{B.5}$$

The explicit expressions for the matrix elements in Eqs. (B.1)-(B.5) will be included in the vertex functions for master (μ,ν) graphs.

Next, we give the analytic expressions for the vertex functions of master (μ,ν) graphs:

$$
{}^{t_1 t_2}\left\langle \begin{matrix} \underset{\sim}{k}_1 & \underset{\sim}{k}_2 \\ \underset{\sim}{k}_3 & \underset{\sim}{k}_4 \end{matrix} \right\rangle_{t}^{\alpha\ \beta} = [\Theta(t_1 - t) + \epsilon_\alpha \nu'_\alpha(\underset{\sim}{k}_1)][\Theta(t_2 - t) + \epsilon_\beta \nu'_\beta(\underset{\sim}{k}_2)]
$$

$$
\times \left(\begin{matrix} \underset{\sim}{k}_1 & \underset{\sim}{k}_2 \\ \underset{\sim}{k}_3 & \underset{\sim}{k}_4 \end{matrix} \right)_{\alpha\ \beta} e^{t[E_\alpha(\underset{\sim}{k}_1)+E_\beta(\underset{\sim}{k}_2)-E_\alpha(\underset{\sim}{k}_3)-E_\beta(\underset{\sim}{k}_4)]}
$$

(B.6)

$$
{}^{t_1 t_2}\left\langle \begin{matrix} \underset{\sim}{k}_1 & \underset{\sim}{k}_2 \\ \underset{\sim}{k}_3 & \underset{\sim}{k}_4 \end{matrix} \right\rangle_{t}^{\alpha\ \gamma} = - \frac{2\pi\hbar^2 \alpha_0 \epsilon_\alpha Z_\alpha^{\ 2}}{\Omega M_\alpha/(1 - D_\alpha)} \frac{\hat{e}_2 \cdot \hat{e}_4}{(k_2 k_4)^{\frac{1}{2}}} [\Theta(t_1 - t) + \epsilon_\alpha \nu'_\alpha(\underset{\sim}{k}_1)]
$$

$$
\times [\Theta(t_2 - t) + \nu'_\gamma(\underset{\sim}{k}_2)]\ e^{t[E_\alpha(\underset{\sim}{k}_1)+E_\gamma(\underset{\sim}{k}_2)-E_\alpha(\underset{\sim}{k}_3)-E_\gamma(\underset{\sim}{k}_4)]}
$$

$$
\times\ \delta_{(\underset{\sim}{k}_1+\underset{\sim}{k}_2),(\underset{\sim}{k}_3+\underset{\sim}{k}_4)}\ \delta_{m_1,m_3}
$$

(B.7)

$$
{}^{t_1}\left\langle \begin{matrix} \underset{\sim}{k}_3 \\ \underset{\sim}{k}_1 \quad \underset{\sim}{k}_2 \end{matrix} \right\rangle_t^{\alpha\ \gamma} = \frac{\hbar^2\ \epsilon_\alpha Z_\alpha}{M_\alpha/(1 - D_\alpha)} \left(\frac{2\pi\alpha_0}{\Omega}\right)^{\frac{1}{2}} \frac{\underset{\sim}{k}_3 \cdot \hat{e}_2}{(k_2)^{\frac{1}{2}}} \left[\Theta(t_1 - t) + \epsilon_\alpha \nu'_\alpha(\underset{\sim}{k}_3)\right]
$$

$$
\times\ e^{t[E_\alpha(\underset{\sim}{k}_3) - E_\alpha(\underset{\sim}{k}_1) - E_\gamma(\underset{\sim}{k}_2)]}\ \delta_{\underset{\sim}{k}_3,(\underset{\sim}{k}_1+\underset{\sim}{k}_2)}\ \delta_{m_1,m_3} \tag{B.8}
$$

$$
{}^{t_1 t_2}\left\langle \begin{matrix} \underset{\sim}{k}_1 \quad \underset{\sim}{k}_2 \\ \underset{\sim}{k}_3 \end{matrix} \right\rangle_t^{\alpha\ \gamma} = \frac{\hbar^2\ \epsilon_\alpha Z_\alpha}{M_\alpha/(1 - D_\alpha)} \left(\frac{2\pi\alpha_0}{\Omega}\right)^{\frac{1}{2}} \frac{\underset{\sim}{k}_3 \cdot \hat{e}_2}{(k_2)^{\frac{1}{2}}} \left[\Theta(t_1 - t) + \epsilon_\alpha \nu'_\alpha(\underset{\sim}{k}_1)\right]
$$

$$
\times\ \left[\Theta(t_2 - t) + \nu'_\gamma(\underset{\sim}{k}_2)\right]\ e^{t[E_\alpha(\underset{\sim}{k}_1) + E_\gamma(\underset{\sim}{k}_2) - E_\alpha(\underset{\sim}{k}_3)]}\ \delta_{\underset{\sim}{k}_3,(\underset{\sim}{k}_1+\underset{\sim}{k}_2)}\ \delta_{m_1,m_3} \tag{B.9}
$$

$$
{}^{t_1}\left\langle \begin{matrix} \underset{\sim}{k}_4 \\ \underset{\sim}{k}_1 \quad \underset{\sim}{k}_2 \quad \underset{\sim}{k}_3 \end{matrix} \right\rangle_t^{\alpha\ \gamma\ \gamma} = -\frac{2\pi\hbar^2 \alpha_0 \epsilon_\alpha Z_\alpha{}^2}{\Omega M_\alpha/(1 - D_\alpha)} \frac{\hat{e}_2 \cdot \hat{e}_3}{(k_2 k_3)^{\frac{1}{2}}} \left[\Theta(t_1 - t) + \epsilon_\alpha \nu'_\alpha(\underset{\sim}{k}_4)\right]
$$

$$
\times\ e^{t[E_\alpha(\underset{\sim}{k}_4) - E_\alpha(\underset{\sim}{k}_1) - E_\gamma(\underset{\sim}{k}_2) - E_\gamma(\underset{\sim}{k}_3)]}\ \delta_{\underset{\sim}{k}_4,(\underset{\sim}{k}_1+\underset{\sim}{k}_2+\underset{\sim}{k}_3)}\ \delta_{m_1,m_4} \tag{B.10}
$$

$$
{}^{t_1 t_2 t_3}\left\langle \begin{matrix} \underset{\sim}{k}_1 & \underset{\sim}{k}_2 & \underset{\sim}{k}_3 \\ \underset{\sim}{k}_4 & & \end{matrix} \right\rangle_{t}^{\alpha\ \gamma\ \gamma} = -\frac{2\pi\hbar^2\alpha_0\epsilon_\alpha Z_\alpha^{\,2}}{\Omega M_\alpha/(1-D_\alpha)}\ \frac{\hat{e}_2\cdot\hat{e}_3}{(k_2k_3)^{\frac{1}{2}}}\,[\Theta(t_1-t)+\epsilon_\alpha\nu'_\alpha(\underset{\sim}{k}_1)]
$$

$$
\times\ [\Theta(t_2-t)+\nu'_\gamma(\underset{\sim}{k}_2)][\Theta(t_3-t)+\nu'_\gamma(\underset{\sim}{k}_3)]\ e^{t[E_\alpha(\underset{\sim}{k}_1)+E_\gamma(\underset{\sim}{k}_2)+E_\gamma(\underset{\sim}{k}_3)-E_\alpha(\underset{\sim}{k}_4)]}
$$

$$
\times\ \delta_{\underset{\sim}{k}_4,(\underset{\sim}{k}_1+\underset{\sim}{k}_2+\underset{\sim}{k}_3)}\ \delta_{m_1,m_4}\ . \qquad \text{(B.11)}
$$

In Eqs. (B.7)-(B.11) m_i is the spin projection (the Kronecker deltas conserve momentum and spin), $\hat{e}_i$ is the polarization unit vector and $\alpha_0 = e^2/\hbar c$ is the fine-structure constant.

The bracket symbol in Eq. (B.6) is defined as follows:

$$
\begin{pmatrix} \underset{\sim}{k}_1 & \underset{\sim}{k}_2 \\ \underset{\sim}{k}_3 & \underset{\sim}{k}_4 \end{pmatrix}_{\alpha\ \beta} = -\,[\langle \underset{\sim}{k}_1,\underset{\sim}{k}_2|V_c|\underset{\sim}{k}_3,\underset{\sim}{k}_4\rangle + \epsilon_\alpha\langle \underset{\sim}{k}_1,\underset{\sim}{k}_2|V_c|\underset{\sim}{k}_4,\underset{\sim}{k}_3\rangle] \qquad \text{for } \alpha=\beta
$$

$$
= -\,\epsilon_\alpha\epsilon_\beta\langle \underset{\sim}{k}_1^{\,\alpha},\underset{\sim}{k}_2^{\,\beta}|V_c|\underset{\sim}{k}_3^{\,\alpha},\underset{\sim}{k}_4^{\,\beta}\rangle \qquad \text{for } \alpha\neq\beta \qquad \text{(B.12)}
$$

In Eqs. (B.6) and (B.12) α and β are both charged particles and

$$
\langle \underset{\sim}{k}_1^{\,\alpha},\underset{\sim}{k}_2^{\,\beta}|V_c|\underset{\sim}{k}_3^{\,\alpha},k_4^{\,\beta}\rangle = \frac{4\pi Z_\alpha Z_\beta e^2}{\Omega q^2}\ \delta_{(\underset{\sim}{k}_1+\underset{\sim}{k}_2),(\underset{\sim}{k}_3+\underset{\sim}{k}_4)}\ \delta_{m_1,m_3}\ \delta_{m_2,m_4}\ , \qquad \text{(B.13)}
$$

where V_c corresponds to the Coulomb interaction between two particles, one of charge $Z_\alpha e$ and the other of charge $Z_\beta e$; $q = \underset{\sim}{k}_3 - \underset{\sim}{k}_1$ is the momentum transfer.

APPENDIX C. AN INTEGRAL EQUATION AND ITS SOLUTION

The following integral equation occurs quite frequently in calculations of the type discussed in Sec. VI:

$$G_0(t_2,t_1,\underset{\sim}{k}) = \delta(t_2 - t_1) + \int_0^\beta ds\{A[\theta(t_1 - s) + N] \exp[(s - t_1)B] \times G_0(t_2,s,\underset{\sim}{k})\} , \tag{C.1}$$

where A, N, and B are independent of s, but may be functions of β and $\underset{\sim}{k}$. It is to be noted that it is not possible to iterate Eq. (C.1) to arbitrary order, since the iterations become independent of the integration variable. Moreover, the function A could be divergent for certain values of β and k, and the successive terms in an iterative solution would be divergent to higher and higher orders. It is quite straightforward to verify that the following function is a solution of Eq. (C.1):

$$G_0(t_2,t_1,\underset{\sim}{k}) = \delta(t_2 - t_1) + A[\theta(t_1 - t_2) + M] \exp[(t_2 - t_1)(B-A)] , \tag{C.2}$$

where

$$M = [\exp(-\beta A)\ (1 + N^{-1})\ -1]^{-1} . \tag{C.3}$$

Use of the solution Eq. (C.2) is made in Sec. VI.

FOOTNOTES AND REFERENCES

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FIGURE CAPTIONS

Fig. 1. Symbols for the interaction vertices and the Δ-vertex. The vertex functions are defined in Appendix B.

Fig. 2. (a) Examples of connected (1,1) graphs.

(b) Examples of connected (0,2) graphs.

(c) Examples of connected (2,0) graphs.

Fig. 3. The master graphs used in the lowest-order calculation of the photon momentum distribution.

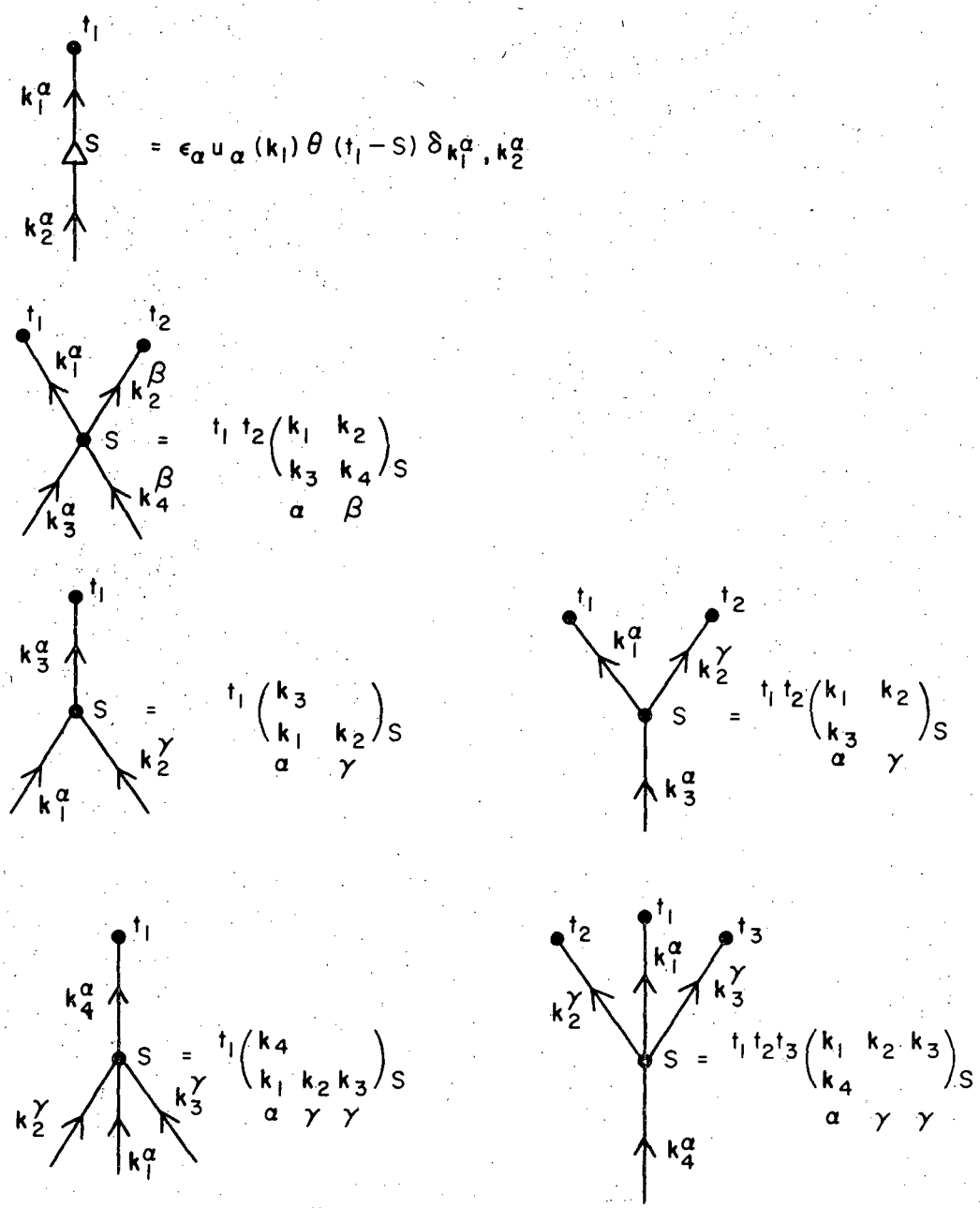

t_1
k_1^α
S
k_2^α
$= \epsilon_\alpha u_\alpha(k_1)\,\theta(t_1 - S)\,\delta_{k_1^\alpha, k_2^\alpha}$
t_1 t_2
k_1^α k_2^β k_3^α k_4^β
$= t_1 t_2 \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix}_S$
α β
k_3^α k_1^α k_2^γ
$= t_1 \begin{pmatrix} k_3 & \\ k_1 & k_2 \end{pmatrix}_S$
α γ
k_1^α k_2^γ k_3^α
$= t_1 t_2 \begin{pmatrix} k_1 & k_2 \\ k_3 & \end{pmatrix}_S$
α γ
k_4^α k_2^γ k_1^α k_3^γ
$= t_1 \begin{pmatrix} k_4 & \\ k_1 & k_2\ k_3 \end{pmatrix}_S$
α γ γ
t_2 t_3 k_1^α k_2^γ k_3^γ k_4^α
$= t_1 t_2 t_3 \begin{pmatrix} k_1 & k_2\ k_3 \\ k_4 & \end{pmatrix}_S$
α γ γ
XBL6911-6282

Fig. 1

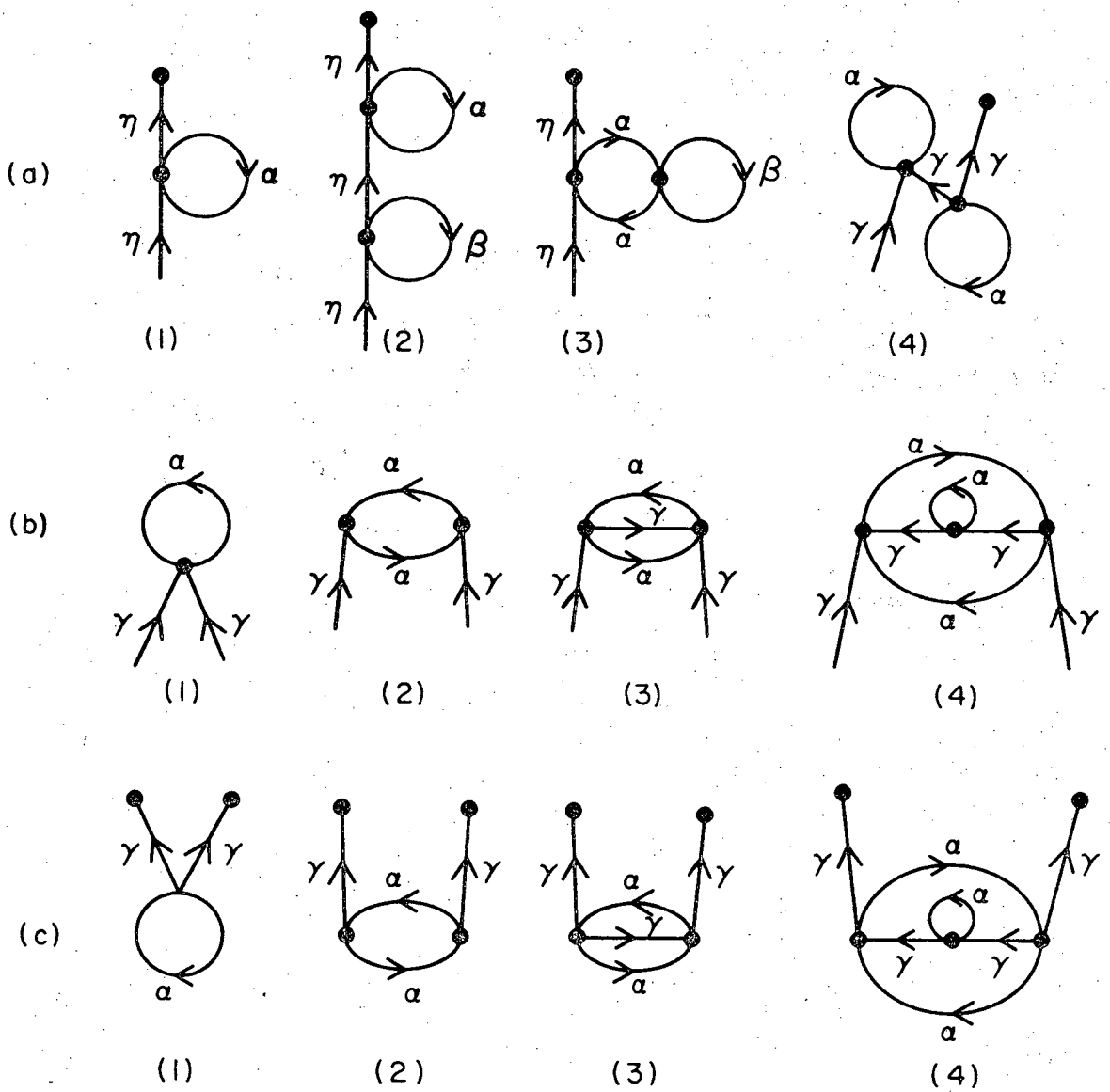
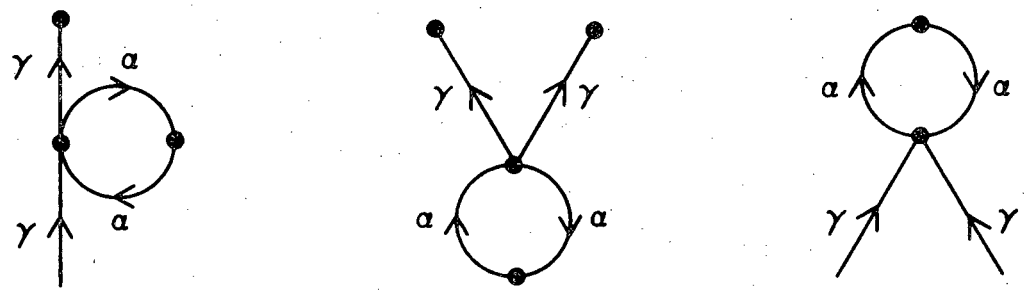


Fig. 2



XBL6911-6283

Fig. 3

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