Submitted to J. of Mathematical Physics

UCRL-19466
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January 8, 1970

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UCRL-19466

RADIATION TRANSPORT ALONG CURVED RAY PATHS*

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January 8, 1970

ABSTRACT

The transport of radiation in a turbulent, refracting medium is studied. It is shown that the conventional transport equation must be generalized. Path integrals are taken along curved ray trajectories. When these ray paths have torsion a rotation of the polarization vectors needs to be taken into account. Two derivations of the transport equation are given. One is phenomenological and one is based on Maxwell's equations. Some discussion is given of cross polarization of radar backscatter.

I. INTRODUCTION

Earlier papers[1,2,3] in this series have presented a derivation from Maxwell's equations of the radiation transport equation for scattering by a turbulent medium.[4,5] A transport equation of conventional form was derived making two kinds of approximations. The first of these amounted to treating the scattering from a single "turbulent eddy," or a single correlated cluster of scatterers, in the distorted wave Born approximation (DWBA). The second approximation made was the assumption that coherent propagation in the refracting medium could be treated in the eikonal approximation and, furthermore, that the ray paths of the eikonal approximation could be replaced by straight lines.

It was observed in I that the restriction to straight line ray paths is quite unnecessary for deriving the transport equation. In the present paper we shall drop the restriction to straight ray paths. The resulting transport equation differs from the conventional one in the appearance of a rotation operator acting on the polarization indices and depending on the radius of torsion of the ray paths.

To make our discussion definite, let us consider the physical situation studied in I and illustrated in Fig. 1. The scattering medium is of finite extent and surrounded by empty space. The source of the radiation is at a great distance from the scatterer. Thus the incident radiation at the scatterer can be considered to be a plane wave, with wave number vector (say) $\underset{\sim}{k}$. The detector is also at some distance from the scatterer. Evidently, this particular choice of boundary conditions is incidental for the derivation of the transport equation.

In I the scatterer was assumed to be a plasma. This is evidently easily generalized to other scattering systems by replacing the Thomson

scattering amplitudes by those appropriate for the system of interest.

The eikonal approximation, although not required for the formal derivation of a transport equation, seems necessary to obtain the conventional form with its usual geometrical interpretation. In Sec. II some required properties of the eikonal approximation will be reviewed. A phenomenological derivation of the transport equation, generalized for curved ray paths, is given in Sec. III. In Sec. IV the same equation is derived from Maxwell's equations. [The corresponding quantum form can be obtained, as was done in Ref. 4).] Several applications, including a discussion of cross polarization for radar backscatter, are given in the final sections.

Before considering the generalized transport equation let us review briefly the conventional form of this and the derivation given in I and III. In the classical theory the quantity

$$I(\underset{\sim}{x}, \hat{\underset{\sim}{p}}, \omega)\, d\Omega_{\hat{\underset{\sim}{p}}}\, d\omega \tag{1.1}$$

represents the flow of radiant energy per unit area, per unit time, having angular frequency ω within $d\omega$, and propagating parallel to $\hat{\underset{\sim}{p}}$ within the element of solid angle $d\Omega_{\hat{\underset{\sim}{p}}}$. When the shift in frequency due to scattering may be neglected ω appears as just a parameter in the transport equation. If in addition the bandwidth of the radiation is sufficiently narrow that frequency dispersion in the scattering may be neglected, the transport equation may be expressed in terms of the quantity

$$I(\underset{\sim}{x}, \hat{\underset{\sim}{p}}) \equiv \int_0^\infty I(\underset{\sim}{x}, \hat{\underset{\sim}{p}}, \omega)\, d\omega\,. \tag{1.2}$$

It is for this case that the derivation in I was given. The equation for $I(\underset{\sim}{x}, \hat{\underset{\sim}{p}}, \omega)$, including frequency shift, was obtained in III.

To describe polarization it is necessary to generalize (1.1). As described above, we suppose that an incident plane wave with wave number vector $\underset{\sim}{k}$ illuminates the scattering medium. At a point along a ray path we suppose the tangent vector is $\hat{\underset{\sim}{p}}$. Two more unit vectors are then defined as

$$\hat{\underset{\sim}{e}}_{\hat{\underset{\sim}{p}}}(2) = C(\hat{\underset{\sim}{p}})\, \hat{\underset{\sim}{p}} \times \hat{\underset{\sim}{k}} ,$$

$$\hat{\underset{\sim}{e}}_{\hat{\underset{\sim}{p}}}(1) = \hat{\underset{\sim}{e}}_{\hat{\underset{\sim}{p}}}(2) \times \hat{\underset{\sim}{p}} . \tag{1.3}$$

For curved ray paths we take $\hat{\underset{\sim}{p}}$ as the local tangent in Eqs. (1.3).

The electric field vector for a wavelet propagating along the given ray path is therefore of the form

$$\underset{\sim}{E} = E_{\hat{\underset{\sim}{p}}}(1)\, \hat{\underset{\sim}{e}}_{\hat{\underset{\sim}{p}}}(1) + E_{\hat{\underset{\sim}{p}}}(2)\, \hat{\underset{\sim}{e}}_{\hat{\underset{\sim}{p}}}(2) , \tag{1.4}$$

and the generalized intensity is [here $(i,j) = 1,2$]

$$I_{ij}(\underset{\sim}{x}, \hat{\underset{\sim}{p}}, \omega) = \text{const} \times [E^*_{\hat{\underset{\sim}{p}}}(i) E_{\hat{\underset{\sim}{p}}}(j)] . \tag{1.5}$$

The "const" here is chosen so that

$$I = I_{11} + I_{22} .$$

[The constant will be given an explicit form in Eq. (3.6) below.]

The radiation transport equation obtained in I was of conventional form:[6]

$$\frac{d}{ds} I_{ij}(\underset{\sim}{x}, \hat{\underset{\sim}{p}}) + \frac{1}{\ell} I_{ij}(\underset{\sim}{x}, \hat{\underset{\sim}{p}}) = B_{ij}(\underset{\sim}{x}, \hat{\underset{\sim}{p}}) , \tag{1.6a}$$

where

$$B_{ij}(\underset{\sim}{x}, \hat{\underset{\sim}{p}}) \equiv \sum_{s,t=1}^{2} \int d\Omega_{\hat{\underset{\sim}{p}}'} (ij \,|M|\, st)\, I_{st}(\underset{\sim}{x}, \hat{\underset{\sim}{p}}') . \tag{1.7a}$$

Here the derivative in (1.6a) is taken along the straight line parallel to $\hat{\underset{\sim}{p}}$, $\ell^{-1}(\underset{\sim}{x})$ is the absorption coefficient, and $(ij \mid M(\hat{\underset{\sim}{p}}, \hat{\underset{\sim}{p}}') \mid st)$ is the scattering function. In more compact matrix notation we write Eqs. (1.6a) and (1.7a) in the form

$$\frac{d}{ds} \underset{\sim}{I}(\underset{\sim}{x}, \hat{\underset{\sim}{p}}) + \frac{1}{\ell} \underset{\sim}{I}(\underset{\sim}{x}, \hat{\underset{\sim}{p}}) = \underset{\sim}{B}(\underset{\sim}{x}, \hat{\underset{\sim}{p}}) , \tag{1.6b}$$

$$\underset{\sim}{B}(\underset{\sim}{x}, \hat{\underset{\sim}{p}}) = \int d\Omega_{\hat{\underset{\sim}{p}}'} \underset{\sim}{M}(\hat{\underset{\sim}{p}}, \hat{\underset{\sim}{p}}') \underset{\sim}{I}(\underset{\sim}{x}, \hat{\underset{\sim}{p}}') . \tag{1.7b}$$

Equations (1.6) and (1.7) are applicable when the frequency change with scattering may be neglected.

The case specifically considered in I was scattering by a plasma containing N electrons with coordinates $\underset{\sim}{z}_1, \underset{\sim}{z}_2, \cdots \underset{\sim}{z}_N$. The probability distribution for these coordinates was written as $P_N(\underset{\sim}{z}_1, \cdots \underset{\sim}{z}_N)$, normalized so that

$$\int P_N \, d^3z_1 \cdots d^3z_N = 1.$$

A set of distributions $P_1(\underset{\sim}{z}_1)$, $P_2(\underset{\sim}{z}_1, \underset{\sim}{z}_2)$, $\cdots P_{N-1}(\underset{\sim}{z}_1, \cdots \underset{\sim}{z}_{N-1})$ may evidently be obtained from P_N by integration, and P_2, $P_3 \cdots$ were developed in terms of two-particle, three-particle, $\cdots$ correlation functions. In particular, P_2 was written as

$$P_2(\underset{\sim}{z}_1, \underset{\sim}{z}_2) = P_1(\underset{\sim}{z}_1)\, P_1(\underset{\sim}{z}_2)[1 + g(\underset{\sim}{z}_1, \underset{\sim}{z}_2)] \,. \tag{1.8}$$

We shall here assume, as was done in I, that the pair correlation function g has the approximate form

$$g(\underset{\sim}{z}_1, \underset{\sim}{z}_2) \cong g\left(\underset{\sim}{z}_1;\ |\underset{\sim}{z}_1 - \underset{\sim}{z}_2|\right) \,. \tag{1.9}$$

This assumption is not necessary for the derivation of the transport equation, but does simplify the equations by leading to a scalar, isotropic refractive index.

The quantity $\underset{\sim}{M}$ was then obtained in I in the form

$$\underset{\sim}{M}(\hat{\underset{\sim}{p}}, \hat{\underset{\sim}{p}}') = \sigma_g(\hat{\underset{\sim}{p}}\cdot\hat{\underset{\sim}{p}}')\underset{\sim}{m} \,, \tag{1.10}$$

where

$$(ij \mid m \mid st) = [\hat{\underset{\sim}{e}}_{\hat{\underset{\sim}{p}}}(i) \cdot \hat{\underset{\sim}{e}}_{\hat{\underset{\sim}{p}}'}(s)][\hat{\underset{\sim}{e}}_{\hat{\underset{\sim}{p}}}(j)\cdot\hat{\underset{\sim}{e}}_{\hat{\underset{\sim}{p}}'}(t)] \tag{1.11}$$

and

$$\sigma_g(\hat{\underset{\sim}{p}}\cdot\hat{\underset{\sim}{p}}') = \rho^2(\underset{\sim}{x})\, \sigma(\hat{\underset{\sim}{p}}\cdot\hat{\underset{\sim}{p}}') \int d^3R\, g(\underset{\sim}{x};\, R) \exp\left\{ i\, n_r(\underset{\sim}{x}) k(\hat{\underset{\sim}{p}}' - \hat{\underset{\sim}{p}})\cdot\underset{\sim}{R} \right\} . \tag{1.12}$$

Here

$$\rho(\underset{\sim}{x}) = N\, P_1(\underset{\sim}{x}) \tag{1.13}$$

is the electron density and σ is the appropriate cross section for Thomson scattering:

$$\sigma(\hat{\underset{\sim}{p}}\cdot\hat{\underset{\sim}{p}}') = r_0^2 \Big/ (1 + \nu_c^2/\omega^2) \,, \tag{1.14}$$

where r_0 is the classical electron radius and ν_c is the electron

collision frequency. [Evidently the theory may be adapted to other elementary scatterers.] Finally, $n_r(\underset{\sim}{x})$ is the real part of the refractive index given in first approximation by [see Eq. (4.13)]

$$n_r^2 \cong 1 - \omega_p^2/(\omega^2 + \nu_c^2) , \tag{1.15}$$

with ω_p the electron plasma frequency.

The absorption coefficient has the form

$$\frac{1}{\ell} = \frac{1}{\ell_c} + \frac{1}{\ell_t} = 2k\, n_i \tag{1.16}$$

where n_i is the imaginary part of the refractive index,

$$\sum_{s=1}^{2} \int d\Omega_{\hat{\underset{\sim}{p}}'} \, (ij \; |M(\hat{\underset{\sim}{p}},\hat{\underset{\sim}{p}}')| \; ss) = \delta_{ij} \left[\frac{1}{2} \int d\Omega_{\hat{\underset{\sim}{p}}'} \, \sigma_g(\hat{\underset{\sim}{p}}\cdot\hat{\underset{\sim}{p}}')(1 + (\hat{\underset{\sim}{p}}\cdot\hat{\underset{\sim}{p}}')^2 \right]$$

$$\equiv \delta_{ij} \frac{1}{\ell_t} . \tag{1.17}$$

and

$$\frac{1}{\ell_c} = \left[\omega_p^2/(\omega^2 + \nu_c^2) \right] (\nu_c/c) \tag{1.18}$$

(c is the speed of light).

As an alternative to Eq. (1.12) we may suppose the medium to be characterized by a dielectric constant, depending on certain random variables, and at time t and point $\underset{\sim}{r}$ to have the value $\epsilon(\underset{\sim}{r}, t)$. This may be written in terms of its fluctuations $\delta\epsilon$ as

$$\epsilon = \delta\epsilon + \langle \epsilon \rangle , \tag{1.19}$$

where "$\langle \cdots \rangle$" represents an average over the random variables.

In this case Eq. (1.12) may be re-written in the form

$$\sigma_g = \left(\frac{k^2}{4\pi}\right)^2 \int d^3R \, \langle \, \delta \, \epsilon(\underset{\sim}{x}, 0) \, \delta \, \epsilon(\underset{\sim}{x} + \underset{\sim}{R}, 0) \, \rangle \times \exp \left\{ i \, n_r(\underset{\sim}{x}) \, k(\hat{\underset{\sim}{p}}' - \hat{\underset{\sim}{p}}) \cdot R \right\} . \tag{1.20}$$

The transport equation for the case that frequency shift must be considered was obtained in III. This has the form

$$\frac{d}{ds} \underset{\sim}{I}(\underset{\sim}{x}, \hat{\underset{\sim}{p}}, \omega) + \frac{1}{\ell} \underset{\sim}{I}(\underset{\sim}{x}, \hat{\underset{\sim}{p}}, \omega) = \underset{\sim}{B}(\underset{\sim}{x}, \hat{\underset{\sim}{p}}, \omega) \tag{1.21}$$

where now

$$\underset{\sim}{B}(\underset{\sim}{x}, \hat{\underset{\sim}{p}}, \omega) = \int_0^\infty d\omega' \int d\Omega_{\hat{\underset{\sim}{p}}'} \, \underset{\sim}{M}(\hat{\underset{\sim}{p}}, \hat{\underset{\sim}{p}}'; \omega - \omega') \, \underset{\sim}{I}(\underset{\sim}{x}, \hat{\underset{\sim}{p}}', \omega'), \tag{1.22}$$

and

$$\underset{\sim}{M}(\hat{\underset{\sim}{p}}, \hat{\underset{\sim}{p}}'; \Omega) = \sigma_g(\hat{\underset{\sim}{p}} \cdot \hat{\underset{\sim}{p}}', \Omega) \underset{\sim}{m} . \tag{1.23}$$

For the case corresponding to Eq. (1.12) we have

$$\sigma_g(\hat{\underset{\sim}{p}} \cdot \hat{\underset{\sim}{p}}', \Omega) = \rho^2(\underset{\sim}{x}) \, \sigma(\hat{\underset{\sim}{p}} \cdot \hat{\underset{\sim}{p}}') \int_{-\infty}^{\infty} d\tau \int d^3R \, g(\underset{\sim}{z}; R, \tau) \, e^{i\Omega\tau} \times \exp \left\{ i \, n_r(\underset{\sim}{x}) k(\hat{\underset{\sim}{p}}' - \hat{\underset{\sim}{p}}) \cdot \underset{\sim}{R} \right\} . \tag{1.24}$$

Here $g(\underset{\sim}{z}; R, \tau)$ is the time dependent pair correlation function for a stationary random process. Alternatively for the case corresponding to Eq. (1.20) we have

$$\sigma_g(\hat{\underset{\sim}{p}}\cdot\hat{\underset{\sim}{p}}', \Omega) = \left(\frac{k^2}{4\pi}\right)^2 \int_{-\infty}^{\infty} d\tau \int d^3R \, \langle \delta\, \epsilon(\underset{\sim}{x}, t)\, \delta\, \epsilon(\underset{\sim}{x} + \underset{\sim}{R}, t + \tau)\rangle$$

$$\times\, e^{i\Omega\tau} \exp\left\{ i\, n_r(\underset{\sim}{x})\, k(\hat{\underset{\sim}{p}}' - \hat{\underset{\sim}{p}})\cdot\underset{\sim}{R} \right\}, \qquad (1.25)$$

where $k = \omega/c$.

The approximations required to derive the transport equation from the wave equation are given in detail in references I and III. We briefly review these here:

1. $\ell_t \gg R_c$, where R_c is the correlation range, or the characteristic distance over which correlations contribute to σ_g. As noted in II, for plasmas this condition can often be replaced by the condition

$$\zeta(\omega_p/\omega)^4 \ll 1,$$

where ζ is the relative mean square fluctuation in electron density.

2. $n_i \ll n_r$. This assumption is interpreted as permitting us to neglect n_i in Eqs. (1.12), (1.20), (1.23), and (1.24), where only n_r was kept in the exponentials. That is, we assumed that $(R_c/\ell) \ll 1$ in these equations (this is convenient, but not of course essential to our discussion). Assumption (2) will be used also in the next section [see Eq. (2.17), for example]. In addition to the above condition, we shall require that $|\nabla n_i|$ be small, a restriction made more precise in Eq. (2.29).

3. The eikonal approximation may be used to describe coherent wave propagation. We express the condition for validity of this approximation

in the form

$$|\nabla n_r| \ll k\, n_r^2 .$$

4. The bending of eikonal ray paths may be neglected and these may be considered to be straight lines.

5. $k\ell \gg 1$, which we interpret as meaning that successive scatterings occur in the wave zone. This assumption is necessary if we are to use the eikonal approximation to describe wave propagation between scatterings.

The purpose of this paper is to obtain the transport equation without making assumption (4).

II. THE EIKONAL APPROXIMATION

In this Section we review several aspects of the eikonal approximation which will be needed for obtaining the transport equation. We shall follow the treatment of Born and Wolf,[7] generalizing this to obtain the Green's function and to include a complex refractive index.

For an electromagnetic wave of frequency $\omega = kc$ propagating in a medium of finite extent and having a refractive index $n(\underset{\sim}{r})$, the Maxwell equation for the electric field $\underset{\sim}{E}(\underset{\sim}{r})$ is

$$[k^2 n^2(\underset{\sim}{r}) - \nabla \times \nabla \times] \underset{\sim}{E}(\underset{\sim}{r}) = 0 . \tag{2.1}$$

Let us suppose that in the absence of the refractive medium the field is $\underset{\sim}{E}_0(\underset{\sim}{r})$, where

$$[k^2 - \nabla \times \nabla \times] \underset{\sim}{E}_0(\underset{\sim}{r}) = 0 . \tag{2.2}$$

As illustrated in Fig. 1, we suppose the source of $\underset{\sim}{E}_0$ to lie far outside the refractive medium.

The field $\underset{\sim}{E}$ may be expressed in terms of $\underset{\sim}{E}_0$ and the dyadic Green's function $\underset{\sim}{G}^0(\underset{\sim}{r}, \underset{\sim}{x})$ for infinite space:[8]

$$\underset{\sim}{E}(\underset{\sim}{r}) = \underset{\sim}{E}_0(\underset{\sim}{r}) + \int d^3x \; \underset{\sim}{G}^0(\underset{\sim}{r},\underset{\sim}{x}) \cdot \underset{\sim}{E}(\underset{\sim}{x}) \times \left\{ (k^2/4\pi)[n^2(\underset{\sim}{x}) - 1] \right\} . \tag{2.3}$$

Here $\underset{\sim}{G}^0$ satisfies the equation

$$(k^2 - \nabla_r \times \nabla_r \times)\underset{\sim}{G}^0(\underset{\sim}{r}, \underset{\sim}{x}) = -4\pi \mathcal{I} \, \delta(\underset{\sim}{r} - \underset{\sim}{x}), \tag{2.4}$$

where $\mathcal{I}$ is the unit dyadic. That is,

$$\underset{\sim}{G}^0(\underset{\sim}{r}, \underset{\sim}{x}) = [\mathcal{I} + k^{-2} \nabla_x \nabla_x] (e^{ikR}/R) - \mathcal{I} [4\pi/(3k^2)] \delta(\underset{\sim}{R}) \tag{2.5a}$$

with $\underset{\sim}{R} \equiv \underset{\sim}{r} - \underset{\sim}{x}$. For $kR \gg 1$ we have

$$\underset{\sim}{G}^0(\underset{\sim}{r}, \underset{\sim}{x}) \cong [\mathcal{I} - \hat{\underset{\sim}{R}} \hat{\underset{\sim}{R}}] G^0(\underset{\sim}{r}, \underset{\sim}{x}) , \tag{2.5b}$$

where

$$G^0(\underset{\sim}{r}, \underset{\sim}{x}) = \frac{e^{ikR}}{R} . \tag{2.6}$$

Finally, we desire the dyadic Green's function $\underset{\sim}{G}(\underset{\sim}{r}, \underset{\sim}{x})$ for the refracting medium. This satisfies the equation

$$[k^2 n^2(\underset{\sim}{r}) - \nabla_{\underset{\sim}{r}} x \ \nabla_{\underset{\sim}{r}} x] \underset{\sim}{G}(\underset{\sim}{r}, \underset{\sim}{x}) = -4\pi \mathcal{I} \delta(\underset{\sim}{r} - \underset{\sim}{x}). \tag{2.7}$$

Let us now write the electric and magnetic field vectors in the respective forms

$$\underset{\sim}{E}(\underset{\sim}{r}) = \underset{\sim}{e}(\underset{\sim}{r}) e^{ikS(\underset{\sim}{r})} ,$$

$$\underset{\sim}{H}(\underset{\sim}{r}) = \underset{\sim}{h}(\underset{\sim}{r}) e^{ikS(\underset{\sim}{r})} , \tag{2.8}$$

where the eikonal S is expressed in terms of its real and imaginary parts as[9]

$$S = S_r + i S_i . \tag{2.9}$$

Since $\underset{\sim}{G}$ and $\underset{\sim}{E}$ satisfy the same differential equation, except for boundary conditions, we may write

$$\underset{\sim}{G}(\underset{\sim}{r}, \underset{\sim}{x}) = \underset{\sim}{e}(\underset{\sim}{r}, \underset{\sim}{x}) \exp \left\{ ik\, S(\underset{\sim}{r}, \underset{\sim}{x}) \right\} . \tag{2.10}$$

Now we consider $\underset{\sim}{e}$ to be a dyadic satisfying the same differential equation [Eq. (2.12)] as the vector $\underset{\sim}{e}$.

Substitution of (2.8) into Maxwell's equations give the two differential equations of the eikonal approximation for S and $\underset{\sim}{e}$ [we are using the notation of Born and Wolf]

$$(\nabla S)^2 = n^2 , \tag{2.11}$$

$$[(\nabla^2 S) + (\nabla S)(\nabla \ln n^2) + 2(\nabla S)\cdot\nabla] \underset{\sim}{e} = 0. \tag{2.12}$$

In addition, we have

$$\underset{\sim}{h} = (\nabla S) \times \underset{\sim}{e} ,$$

$$(\nabla S) \cdot \underset{\sim}{e} = 0$$

$$(\nabla S) \cdot \underset{\sim}{h} = 0 \tag{2.13}$$

in the first order.

The time-averaged Poynting vector is [following Ref. 7, we are using unrationalized Gaussian units]

$$\underset{\sim}{J} = \frac{c}{8\pi} \{ \mathrm{Re}[\underset{\sim}{e} \times \underset{\sim}{h}^*] \} e^{-2kS_i}$$

$$= \frac{c}{8\pi} (\nabla S_r) \underset{\sim}{e}\cdot\underset{\sim}{e}^* e^{-2ikS_i} . \tag{2.14}$$

The real and imaginary parts of Eq. (2.11) lead to the two real equations

$$(\nabla S_r)^2 - (\nabla S_i)^2 = n_r^2 - n_i^2 , \tag{2.15}$$

$$(\nabla S_r) \cdot (\nabla S_i) = n_r n_i . \tag{2.16}$$

Because of our assumptions that $n_i^2 \ll n_r^2$ we look for a solution to these of the form

$$(\nabla S_r)^2 \cong n_r^2 \tag{2.17}$$

with $(\nabla S_i)^2 \ll (\nabla S_r)^2$.

Following the discussion of Ref. 7, we write the equation of a given ray path as $\underset{\sim}{r} = \underset{\sim}{r}(s)$, where s is the path length measured from some reference point on the ray. We let

$$\hat{\underset{\sim}{p}}(\underset{\sim}{r}) \equiv \frac{d\underset{\sim}{r}}{ds} \tag{2.18}$$

be the tangent vector to the given ray path at the point $\underset{\sim}{r}$ and write

$$\nabla S_r = \hat{\underset{\sim}{p}}\, n_r \,. \tag{2.19}$$

Equations (2.18) and (2.19) permit us to "solve" (2.17) in the form[7]

$$\frac{d\hat{\underset{\sim}{p}}}{ds} = \nabla_\perp \ln n_r \,, \tag{2.20}$$

$$S_r(\underset{\sim}{r}) = \int_p n_r \, ds \,, \tag{2.21}$$

$$\nabla_\perp \equiv \nabla - \hat{\underset{\sim}{p}}\, \hat{\underset{\sim}{p}}\cdot\nabla \,. \tag{2.22}$$

Equation (2.20) determines a given ray path and (2.21) permits us to construct S_r , the integral being taken along the ray path passing through $\underset{\sim}{r}$.

Equation (2.16) may now be re-written in the form

$$\hat{\underset{\sim}{p}}\cdot\nabla S_i = n_i \,. \tag{2.23}$$

This may be integrated to give

$$S_i = \int_p n_i \, ds + \delta S_i ,$$

where δS_i is subject to the condition

$$\hat{\underset{\sim}{p}} \cdot \nabla \; \delta S_i = 0.$$

Thus, δS_i is constant along each ray path. The boundary condition $\underset{\sim}{E}$ becomes equal to $\underset{\sim}{E}_0$ if we follow a ray path backwards outside the medium tells us then that $\delta S_i = 0$. For the Green's function the boundary condition at $\underset{\sim}{r} \cong \underset{\sim}{x}$, implied by Eq. (2.7), also then specifies that $\delta S_i = 0$. We therefore have

$$S_i = \int_p n_i \, ds . \tag{2.24}$$

We now turn to Eq. (2.12) for $\underset{\sim}{e}$. Let us write $\underset{\sim}{e}$ in the form

$$\underset{\sim}{e} = A \, e^{i\alpha} \, \hat{\underset{\sim}{e}} , \tag{2.25}$$

where A is the magnitude of $\hat{\underset{\sim}{e}}$, $\hat{\underset{\sim}{e}}$ is a unit vector, and α is a phase.

The discussion given in Born and Wolf[7] permits us to obtain directly the equation

$$\frac{dA}{ds} = -\left(\frac{\nabla^2 S_r}{2n_r}\right) A . \tag{2.26}$$

For α and $\hat{\underset{\sim}{e}}$ we obtain the equations

$$\frac{d\alpha}{ds} = -\frac{\nabla^2 S_i}{2n_r} \tag{2.27}$$

and

$$\frac{d\hat{\underset{\sim}{e}}}{ds} = -\hat{\underset{\sim}{p}}\,(\nabla\,\ell n\,n)\cdot\hat{\underset{\sim}{e}}\ . \tag{2.28}$$

To check the consistency with Eqs. (2.13), we use Eqs. (2.20) and (2.28) to obtain

$$\frac{d}{ds}\,(\hat{\underset{\sim}{e}}\cdot\hat{\underset{\sim}{p}}) = \hat{\underset{\sim}{e}}\cdot\nabla_{\perp}\,\ell n\,n_r - \hat{\underset{\sim}{e}}\cdot\nabla\,\ell n\,n\ .$$

This vanishes if we can consider the contribution from n_i in Eq. (2.28) to be negligible. This will be the case if

$$\left|\int_p \frac{|\nabla\,n_i|}{n_r}\,ds\right| \ll 1 \tag{2.29}$$

everywhere along a given path. We shall assume that the condition (2.29) is satisfied and that we can write (2.29) in the approximate form

$$\frac{d\hat{\underset{\sim}{e}}}{ds} = -\hat{\underset{\sim}{p}}\,(\nabla_{\perp}\,\ell n\,n_r)\cdot\hat{\underset{\sim}{e}}\ . \tag{2.30}$$

The same approximation lets us set $\alpha = 0$ in Eq. (2.25).

If $\underset{\sim}{r}$ and $\underset{\sim}{x}$ are two points along a ray path, we may integrate (2.30) to give

$$\hat{\underset{\sim}{e}}(\underset{\sim}{r}) = \underset{\sim}{P}(\underset{\sim}{r},\,\underset{\sim}{x})\cdot\hat{\underset{\sim}{e}}(\underset{\sim}{x})\ ,$$

$$\underset{\sim}{P}(\underset{\sim}{r},\,\underset{\sim}{x}) = \exp\left\{-\int_{\underset{\sim}{x}}^{\underset{\sim}{r}}\hat{\underset{\sim}{p}}(\nabla_{\perp}\,\ell n\,n_r)ds\right\}\ , \tag{2.31}$$

using a somewhat terse notation.

We areparticularly interested in the eikonal representation for the Green's function $\underset{\sim}{G}(\underset{\sim}{r}, \underset{\sim}{x})$. To simplify the appearance of some of our equations we shall use interchangeably two notations for $\underset{\sim}{G}$, A, etc. For a set of coordinates $\underset{\sim}{z}_1, \underset{\sim}{z}_2, \cdots \underset{\sim}{z}_N$, we write

$$\underset{\sim}{G}_{\alpha\beta} \equiv \underset{\sim}{G}(\underset{\sim}{z}_\alpha, \underset{\sim}{z}_\beta) , \tag{2.32}$$

with $\alpha \neq \beta = 1, 2, \cdots N$. The unit tangent vectors $\hat{\underset{\sim}{p}} = \hat{\underset{\sim}{p}}(\underset{\sim}{x})$ in Eqs. (1.3) satisfy Eq. (2.20). These will be written as $\hat{\underset{\sim}{p}}_\alpha \equiv \hat{\underset{\sim}{p}}(\underset{\sim}{z}_\alpha)$, etc. The eikonal is written as

$$S_{\alpha\beta} = \int_{\underset{\sim}{z}_\alpha}^{\underset{\sim}{z}_\beta} n \, ds .$$

In matrix notation we use the representation (1.3) to write, for $i, j = 1,2$,

$$\begin{aligned} (j \mid G_{\alpha\beta} \mid i) &\equiv \hat{\underset{\sim}{e}}_{\hat{\underset{\sim}{p}}_\alpha}(j) \cdot \underset{\sim}{G}_{\alpha\beta} \cdot \hat{\underset{\sim}{e}}_{\hat{\underset{\sim}{p}}_\beta}(i) \\ &= A_{\alpha\beta} \exp(ikS_{\alpha\beta}) \, \hat{\underset{\sim}{e}}_{\hat{\underset{\sim}{p}}_\alpha}(j) \cdot \underset{\sim}{P}_{\alpha\beta} \cdot \hat{\underset{\sim}{e}}_{\hat{\underset{\sim}{p}}_\beta}(i) , \end{aligned} \tag{2.33}$$

using Eqs. (2.25) and (2.31). We emphasize that $S_{\alpha\beta}$ and $A_{\alpha\beta}$ here are determined by Eqs. (2.21), (2.24), and (2.26). We may also write

$$(j \mid G_{\alpha\beta} \mid i) \equiv G_{\alpha\beta} \, \hat{\underset{\sim}{e}}_{\hat{\underset{\sim}{p}}_\alpha}(j) \cdot \underset{\sim}{P}_{\alpha\beta} \cdot \hat{\underset{\sim}{e}}_{\hat{\underset{\sim}{p}}_\beta}(i), \tag{2.34}$$

where

$$G_{\alpha\beta} \equiv G(\underset{\sim}{z}_\alpha, \underset{\sim}{z}_\beta) = A_{\alpha\beta} \exp(ikS_{\alpha\beta}) . \tag{2.35}$$

The expression (2.35) represents the eikonal approximation to the scalar equation

$$[\nabla_z^2 + k^2 n^2(\underset{\sim}{z})]\, G(\underset{\sim}{z}, \underset{\sim}{x}) = -4\pi\, \delta(\underset{\sim}{z} - \underset{\sim}{x}). \qquad (2.36)$$

The observation that Eq. (2.26) does not depend on n_i lets us introduce the Green's function, expressed in the eikonal approximation,

$$\hat{G}(\underset{\sim}{r}, \underset{\sim}{x}) = A(\underset{\sim}{r}, \underset{\sim}{x}) \exp[ik\, S_r(\underset{\sim}{r}, \underset{\sim}{x})] \,. \qquad (2.37)$$

This is the eikonal approximation to the solution of

$$[\nabla_r^2 + k^2 n_r^2]\, \hat{G}(\underset{\sim}{r}, \underset{\sim}{x}) = -4\pi\, \delta(\underset{\sim}{r} - \underset{\sim}{x}). \qquad (2.38)$$

It is known[10] that

$$\hat{G}(\underset{\sim}{r}, \underset{\sim}{x}) = \hat{G}(\underset{\sim}{x}, \underset{\sim}{r}) \,, \qquad (2.39)$$

from which we obtain the important symmetry relation

$$A_{\alpha\beta} = A_{\beta\alpha} \,. \qquad (2.40)$$

Let us now choose three points $\underset{\sim}{z}_0$, $\underset{\sim}{z}_1$, and $\underset{\sim}{z}_2$ all on a given ray path. We construct a flux tube of ray paths all passing through $\underset{\sim}{z}_0$ and enclosing the original path on which $\underset{\sim}{z}_1$ and $\underset{\sim}{z}_2$ lie. We suppose the solid angle $\delta\,\Omega_0$ formed by the tube at $\underset{\sim}{z}_0$ to be very small. The respective cross sectional areas of the flux tube at $\underset{\sim}{z}_1$ and $\underset{\sim}{z}_2$ are $\delta\,\Sigma_1$ and $\delta\,\Sigma_2$. This geometry is illustrated in Fig. 2.

We assume for the moment that $n_i = 0$. Then we can represent the Green's function G by $\hat{G}$, Eqs. (2.37) and (2.38). The condition

that energy is conserved [i.e., $n_i = 0$] in the flux tube [see Eq. (2.14)] leads us to the relation

$$\delta \Sigma_2 A_{20}^{\ 2} n_r(2) = \delta \Sigma_1 A_{10}^{\ 2} n_r(1) . \tag{2.41}$$

Here we have written $n_r(1) \equiv n_r(\underset{\sim}{z}_1)$, etc. If we choose $\underset{\sim}{z}_1$ to lie sufficiently close to $\underset{\sim}{z}_0$, we have

$$A_{10} = R_{10}^{-1}$$

and

$$\delta \Sigma_{10} = R_{10}^{\ 2} \delta \Omega_0 ,$$

where

$$\underset{\sim}{R}_{\alpha\beta} \equiv \underset{\sim}{z}_\alpha - \underset{\sim}{z}_\beta . \tag{2.42}$$

Thus, we obtain from (2.41) the result

$$A_{20} = \left[\left(\frac{n_r(0)}{n_r(2)} \right) \left(\frac{\partial \Sigma_2}{\partial \Omega_0} \right)^{-1} \right]^{1/2} ;$$

for two arbitrary points $\underset{\sim}{z}_\alpha$ and $\underset{\sim}{z}_\beta$ we have

$$A_{\alpha\beta} = \left[\left(\frac{n_r(\beta)}{n_r(\alpha)} \right) \left(\frac{\partial \Sigma_\alpha}{\partial \Omega_\beta} \right)^{-1} \right]^{1/2} . \tag{2.43}$$

Since Eq. (2.26) does not involve n_i, we can consider Eq. (2.43) to be valid even when $n_i \neq 0$ [but of course subject to the conditions imposed on it].

Equations (2.34), (2.35), (2.43), etc. provide the eikonal representation of the Green's function which will be required for our applications.

III. HEURISTIC DERIVATION OF THE TRANSPORT EQUATION

In this Section we shall give a phenomenological derivation of the transport equation for the case that curvature of the ray paths must be taken into account. Consistent with our assumption [see Eq. (1.9)] that mean properties of the medium do not change significantly over a correlation distance R_c, we shall assume that the scattering strength $\underset{\sim}{B}$ in Eq. (1.6) and the scattering function $\underset{\sim}{M}$ are not modified.

Let us assume that the dielectric properties of the scatterer are described by the dielectric constant ϵ [see Eq. (1.19)] which depends on certain random parameters. The electric field $\underset{\sim}{E}$, which depends parametrically on these same parameters, satisfies the equation

$$[k^2 \epsilon - \nabla x \nabla x]\underset{\sim}{E} = 0 . \tag{3.1}$$

Remembering that "$\langle \cdots \rangle$" is considered to represent an average over these random variables, we introduce the <u>coherent</u> field at $\underset{\sim}{x}$ as

$$\underset{\sim}{E}_c(\underset{\sim}{x}) \equiv \langle \underset{\sim}{E}(\underset{\sim}{x}) \rangle . \tag{3.2}$$

The coherent field satisfies the wave equation (2.1); that is, as was shown in I,[11]

$$[k^2 n^2 - \nabla x \nabla x]\underset{\sim}{E}_c = 0 . \tag{3.3}$$

The field $\underset{\sim}{E}$ may be written as

$$\underset{\sim}{E} = \sum_{\lambda} \underset{\sim}{E}_{\lambda} , \tag{3.4}$$

where the sum runs over scattering from different fluctuations in ϵ and includes $\underset{\sim}{E}_c$. We suppose that these terms are so chosen that the different $\underset{\sim}{E}_{\lambda}$ are mutually incoherent. Thus,

$$\langle \underset{\sim}{E}_\lambda \underset{\sim}{E}_{\lambda'} \rangle = 0 \quad \text{for} \quad \lambda \neq \lambda' . \tag{3.5}$$

An expression is required for the radiation intensity $I_{ij}(\underset{\sim}{z}, \hat{\underset{\sim}{p}})$ [Eq. (1.5); we do not write explicitly the frequency dependence here]. Referring to Fig. 3, we consider a flux tube of ray paths all passing through a given point $\underset{\sim}{z}_0$ within a small solid angle $\delta \Omega_0$. The electric field at $\underset{\sim}{z}_0$ associated with these ray paths will be of the form (3.4). The index λ is assumed to refer to a scatterer at the point $\underset{\sim}{z}_\lambda$ contained within the volume of the flux tube. At the point $\underset{\sim}{z}_\lambda$ the area of the flux tube is $\delta \Sigma_\lambda$.

If we write each $\underset{\sim}{E}_\lambda$ in (3.4) in the eikonal form (2.8) and use Eqs. (2.14) and (3.5), we see that

$$I_{ij}(\underset{\sim}{z}_0, \hat{\underset{\sim}{p}}_0) = \frac{c}{8\pi} (n_r(0)/\delta \Omega_0) \left\langle \sum_\lambda \exp[-2k\, S_i(\underset{\sim}{z}_0, \underset{\sim}{z}_\lambda)] \hat{\underset{\sim}{e}}_{\hat{\underset{\sim}{p}}_0}(i) \cdot \underset{\sim}{e}_\lambda^* \times \underset{\sim}{e}_\lambda \cdot \hat{\underset{\sim}{e}}_{\hat{\underset{\sim}{p}}_0}(j) \right\rangle . \tag{3.6}$$

The unit vectors here are defined by Eqs. (1.3) with $\hat{\underset{\sim}{p}}_0 \equiv \hat{\underset{\sim}{p}}(\underset{\sim}{z}_0)$.

Using Eq. (2.43), we have

$$\begin{aligned} \delta \Omega_0 &= \left(\frac{d\Sigma_\lambda}{d\Omega_0} \right)^{-1} \delta \Sigma_\lambda \\ &= A_{\lambda 0}^{\;2} \frac{n_r(\lambda)}{n_r(0)} \delta \Sigma_\lambda \\ &= A_{0\lambda}^{\;2} \frac{n_r(\lambda)}{n_r(0)} \delta \Sigma_\lambda . \end{aligned} \tag{3.7}$$

To obtain the final form here, we have used Eq. (2.40). Equation (2.25), with $\alpha = 0$, lets us write

$$\underset{\sim}{e}_\lambda = A_{0\lambda}\, \hat{\underset{\sim}{e}}_\lambda(\underset{\sim}{z}_0)\, s_\lambda^{1/2} , \tag{3.8}$$

where s_λ is the source intensity at $\underset{\sim}{z}_\lambda$. Finally, (3.7) and (3.8) let us rewrite (3.6), replacing $\underset{\sim}{z}_0$ by a general point $\underset{\sim}{z}$, as

$$I_{ij}(\underset{\sim}{z}, \hat{\underset{\sim}{p}}) = n_r^2(\underset{\sim}{z}) \exp[\, -2k\, S_i(\underset{\sim}{z})\,] \left\langle \sum_\lambda C_\lambda\, \hat{\underset{\sim}{e}}_{\hat{\underset{\sim}{p}}}(i)\cdot\hat{\underset{\sim}{e}}_\lambda\, \hat{\underset{\sim}{e}}_{\hat{\underset{\sim}{p}}}(j)\cdot\hat{\underset{\sim}{e}}_\lambda \right\rangle , \tag{3.9}$$

where the C_λ are independent of $\underset{\sim}{z}$.

To obtain the transport equation we consider the change in $\underset{\sim}{I}$ if we displace the point $\underset{\sim}{z}$ by a small distance δs along the ray path. First, let us assume that in the interval δs then is no energy added by scattering from other ray paths. [This will later be accounted for by including the scattering strength $\underset{\sim}{B}$.] Use of Eqs. (1.16) and (2.24) gives

$$\frac{d}{ds}(2k\, S_i) = 1/\ell .$$

From Eqs. (2.20), (2.38), and (1.3) we obtain

$$\frac{d}{ds}[\hat{\underset{\sim}{e}}_{\hat{\underset{\sim}{p}}}(i)\cdot\hat{\underset{\sim}{e}}] = (-1)^i\, \phi\, \hat{\underset{\sim}{e}}_{\hat{\underset{\sim}{p}}}(i+1)\cdot\hat{\underset{\sim}{e}} , \tag{3.10}$$

where

$$\phi = (\hat{\underset{\sim}{p}}\cdot\hat{\underset{\sim}{k}}/|\hat{\underset{\sim}{p}} \times \hat{\underset{\sim}{k}}|)\, \hat{\underset{\sim}{e}}_{\hat{\underset{\sim}{p}}}(2)\cdot\nabla\, \ell n\, n_r . \tag{3.11}$$

In Eq. (3.10) we have adopted the following special notation. When $i = 1$,

$$\hat{\underset{\sim}{e}}_{\hat{\underset{\sim}{p}}}(i+1) = \hat{\underset{\sim}{e}}_{\hat{\underset{\sim}{p}}}(2) ;$$

when $i = 2$,

$$\hat{\underset{\sim}{e}}_{\underset{\sim}{p}}(i+1) = \hat{\underset{\sim}{e}}_{\underset{\sim}{p}}(1) .$$

With these expressions, we obtain from (3.9) the expression

$$\frac{d}{ds} I_{ij}(\underset{\sim}{z}, \hat{\underset{\sim}{p}}) = -\frac{1}{L} I_{ij}(\underset{\sim}{z}, \hat{\underset{\sim}{p}}) + (-1)^j \phi\, I_{ij+1}(\underset{\sim}{z}, \hat{\underset{\sim}{p}}) + (-1)^i \phi\, I_{i+1j}(\underset{\sim}{z}, \hat{\underset{\sim}{p}})$$

$$\equiv -\frac{1}{L} I_{ij} - \sum_{s,t=1}^{2} (ij\ |R|\ st)\, I_{st} . \qquad (3.12)$$

In the first term we have introduced the notation

$$\frac{1}{L} \equiv \frac{1}{\ell} - \frac{d}{ds} \ell n\, n_r^{\,2} . \qquad (3.13)$$

The second term describes the rotation of the polarization vectors as the radiation moves along a ray path.

Scattering within the interval δs was omitted in obtaining Eq. (3.12). If we include this, we must add $\underset{\sim}{B}$ to the right-hand side of (3.12). This gives the generalized form of the transport equation (1.6):

$$\frac{d}{ds} \underset{\sim}{I}(\underset{\sim}{x}, \hat{\underset{\sim}{p}}) + \frac{1}{L} \underset{\sim}{I}(\underset{\sim}{x}, \hat{\underset{\sim}{p}}) + \underset{\sim}{R}\, \underset{\sim}{I}(\underset{\sim}{x}, \hat{\underset{\sim}{p}}) = \underset{\sim}{B}(\underset{\sim}{x}, \hat{\underset{\sim}{p}}) , \qquad (3.14)$$

where $\underset{\sim}{B}$ is given by Eq. (1.7). If frequency shift due to scattering must be taken into account, we obtain the generalized form of Eq. (1.21):

$$\frac{d}{ds} \underset{\sim}{I}(\underset{\sim}{x}, \hat{\underset{\sim}{p}}, \omega) + \frac{1}{L} \underset{\sim}{I}(\underset{\sim}{x}, \hat{\underset{\sim}{p}}, \omega) + \underset{\sim}{R}\, \underset{\sim}{I}(\underset{\sim}{x}, \hat{\underset{\sim}{p}}, \omega) = \underset{\sim}{B}(\underset{\sim}{x}, \hat{\underset{\sim}{p}}, \omega) . \qquad (3.15)$$

The scattering strength $\underset{\sim}{B}$ is now given by Eq. (1.22).

Let us write out in detail Eqs. (3.14) and (3.15). For either case, the form is

$$\frac{d}{ds} I_{11} = -\frac{1}{L} I_{11} - \phi [I_{12} + I_{21}] + B_{11}$$

$$\frac{d}{ds} I_{12} = -\frac{1}{L} I_{12} + \phi [I_{11} - I_{22}] + B_{12}$$

$$\frac{d}{ds} I_{21} = -\frac{1}{L} I_{21} + \phi [I_{11} - I_{22}] + B_{21}$$

$$\frac{d}{ds} I_{22} = -\frac{1}{L} I_{22} + \phi [I_{12} + I_{21}] + B_{22} . \tag{3.16}$$

For many applications the Stokes parameter representation is more convenient than that of Eqs. (1.3). In this representation we write

$$I_1 = I_{11}$$

$$I_2 = I_{22}$$

$$I_{12} = \frac{1}{2} (I_3 - i\, I_4) = I_{21}^{*} . \tag{3.17}$$

[The detailed form of $\underset{\sim}{B}$ in this representation was given in II.] Equations (3.16), when expressed in terms of the Stokes parameters, become

$$\frac{dI_1}{ds} = -\frac{1}{L} I_1 - \phi I_3 + B_1$$

$$\frac{dI_2}{ds} = -\frac{1}{L} I_2 + \phi I_3 + B_2$$

$$\frac{dI_3}{ds} = -\frac{1}{L} I_3 + 2\phi[I_1 - I_2] + B_3$$

$$\frac{dI_4}{ds} = -\frac{1}{L} I_4 + B_4 \, . \tag{3.18}$$

A somewhat more "natural" choice than (1.3) for the unit polarization vectors is illustrated in Fig. 4. At each point on a ray path $\hat{e}_2$ lies in the osculating plane along the principal radius of curvature. Thus,

$$\frac{d\hat{\underset{\sim}{p}}}{ds} = \frac{\hat{\underset{\sim}{e}}_2}{R_p} \, , \tag{3.19}$$

where R_p is the principal radius of curvature. Then

$$\hat{\underset{\sim}{e}}_1 = \hat{\underset{\sim}{e}}_2 \times \hat{\underset{\sim}{p}} \, , \tag{3.20}$$

and we obtain

$$\frac{d\hat{\underset{\sim}{e}}_1}{ds} = -\frac{\hat{\underset{\sim}{e}}_2}{R_t} \, ,$$

$$\frac{d\hat{\underset{\sim}{e}}_2}{ds} = -\frac{\hat{\underset{\sim}{p}}}{R_p} + \frac{\hat{\underset{\sim}{e}}_1}{R_t} \, . \tag{3.21}$$

Here R_t is the radius of torsion of the ray path.

If we define $\underset{\sim}{m}$, Eq. (1.11), and $\underset{\sim}{I}$, Eq. (3.6), with $\hat{\underset{\sim}{e}}_1$ and $\hat{\underset{\sim}{e}}_2$ replacing the polarization vectors (1.3), the only change in the transport equation [Eqs. (3.14) or (3.15)] occurs in the rotation matrix $\underset{\sim}{R}$. After a simple calculation using Eqs. (3.21), we obtain

$$\frac{dI_{ij}}{ds} = -\frac{1}{L} I_{ij} + \frac{1}{R_t}[(-1)^i I_{i+1j} + (-1)^j I_{ij+1}] + B_{ij}. \tag{3.22}$$

These have the same form as do Eqs. (3.16), but with $\emptyset$ replaced by R_t^{-1}.

The form (3.22) will not be used in this paper.

IV. DERIVATION OF THE TRANSPORT EQUATION

We turn now to the derivation of Eq. (3.14) from Maxwell's equations[12] (the corresponding derivation of Eq. (3.15) following the method used in III, is straightforward]. For this purpose we can use the development given in I with only superficial changes--mostly of notation.[13] To avoid repetition of the complete development in I we shall just indicate here these changes.

As in I, we shall suppose the scattering system (plasma) to consist of N nonrelativistic electrons confined to a finite volume and having the probability distribution $P_N(\underset{\sim}{z}_1, \cdots \underset{\sim}{z}_N)$ described in Sec. I. This is illustrated in Fig. 1, where the source and detector are shown as being a large distance from the plasma. [The extension to other scattering systems is evidently straightforward. It is only necessary to use the appropriate scattering amplitudes f_{ij} in Eqs. (4.3), etc.] The multiple scattering representation for the electric field was given by Eqs. (I3.1) and (I3.2)[14]

$$\underset{\sim}{E}(\underset{\sim}{z}_\alpha) = \underset{\sim}{E}_I(\underset{\sim}{z}_\alpha) + \sum_{\beta(\neq\alpha)=1}^{N} \sum_{j=1}^{2} \hat{\underset{\sim}{e}}_{\alpha\beta}(j)\, F_{\alpha\beta}(j) \,. \qquad (4.1)$$

Here $\hat{\underset{\sim}{e}}_{\alpha\beta}(j)$, $j = 1, 2,$ defined by Eqs. (1.3) with $\hat{\underset{\sim}{p}}$ parallel to $\underset{\sim}{z}_\alpha - \underset{\sim}{z}_\beta$ and

$$\underset{\sim}{E}_I(\underset{\sim}{x}) = \hat{e}(1)\, E_I(\underset{\sim}{x}) \qquad (4.2)$$

represents the incident radiation, taken to be a plane wave at the scatterer. The $F_{\alpha\beta}$ are defined by the equations

$$F_{\alpha\beta}(i) = G_{\alpha\beta}{}^{0} f_{i1}(\alpha\beta, \beta 0) E_I(\underset{\sim}{z}_\beta) + \sum_{\sigma(\neq\beta)=1}^{N} \sum_{j=1}^{2} G_{\alpha\beta}{}^{0} f_{ij}(\alpha\beta, \beta\sigma) F_{\beta\sigma}(j). \tag{4.3}$$

The notation of I has been used here, with

$$G_{\alpha\beta}{}^{0} \equiv G^{0}(\underset{\sim}{z}_\alpha, \underset{\sim}{z}_\beta) , \tag{4.4}$$

as defined by Eq. (2.6), and

$$f_{ij}(\alpha\beta, \beta\sigma) = f_0 \; \hat{\underset{\sim}{e}}_{\alpha\beta}(i) \cdot \hat{\underset{\sim}{e}}_{\beta\sigma}(j) ,$$

$$f_{i1}(\alpha\beta, \beta 0) = f_0 \; \hat{\underset{\sim}{e}}_{\alpha\beta}(i) \cdot \hat{\underset{\sim}{e}}(1) . \tag{4.5}$$

The quantity $(-f_0)$ is the classical electron radius divided by $(1 - i\nu_c/\omega)$.

If we expand Eqs. (4.1) and (4.3) in a sequence of scatterings and express the result in vector form, we have

$$\underset{\sim}{E}(\underset{\sim}{z}_\alpha) = \underset{\sim}{E}_I(\underset{\sim}{z}_\alpha) + \sum_{\sigma} \Bigg\{ f_0 \, \underset{\sim}{G}_{\alpha\sigma}{}^{0} + f_0^{2} \sum_{\beta} \underset{\sim}{G}_{\alpha\beta}{}^{0} \cdot \underset{\sim}{G}_{\beta\sigma}{}^{0} + f_0^{3} \sum_{\beta_1,\beta_2} \underset{\sim}{G}^{0}_{\alpha\beta_1} \cdot \underset{\sim}{G}^{0}_{\beta_1\beta_2} \cdot \underset{\sim}{G}^{0}_{\beta_2\sigma} + \cdots \Bigg\} \cdot \underset{\sim}{E}_I(\underset{\sim}{z}_\sigma) , \tag{4.6}$$

where $\underset{\sim}{G}_{\alpha\beta}{}^{0} \equiv \underset{\sim}{G}^{0}(\underset{\sim}{z}_\alpha, \underset{\sim}{z}_\beta)$ is the expression (2.5b) and no two adjacent subscripts $\alpha, \beta_1, \cdots$ above are equal.

For a coherent sequence of scatterings we obtain (I3.22) in the form

$$\underset{\sim}{G}_{\alpha\sigma} = \underset{\sim}{G}_{\alpha\sigma}^{0} + f_0 \sum_{\beta} \int \underset{\sim}{G}_{\alpha\beta}^{0} \cdot \underset{\sim}{G}_{\beta\sigma} \, P_1(\underset{\sim}{z}_\beta) \, d^3 z_\beta$$

$$+ f_0^2 \sum_{\beta_1, \beta_2 (\neq \beta_1)} \int \underset{\sim}{G}_{\alpha\beta_1}^{0} \cdot \underset{\sim}{G}_{\beta_1\beta_2} \cdot \underset{\sim}{G}_{\beta_2\sigma}$$

$$\times \, P_1(\underset{\sim}{z}_{\beta_1}) \, P_1(\underset{\sim}{z}_{\beta_2}) \, g(\underset{\sim}{z}_{\beta_1}, \underset{\sim}{z}_{\beta_2}) \, d^3 z_{\beta_1} \, d^3 z_{\beta_2} \, , \tag{4.7}$$

where g is the pair correlation function of Eq. (1.8), etc. We emphasize that Eq. (4.7) is just Eq. (I3.22) rewritten in vector notation.

For $|\underset{\sim}{z}_{\beta_1} - \underset{\sim}{z}_{\beta_2}| \lesssim R_c$, we can rewrite Eqs. (I3.23) and (I3.24) as

$$\underset{\sim}{G}_{\beta_2\sigma} \cong \left\{ \exp\left[i \, n_r(\underset{\sim}{z}_{\beta_1}) \, \underset{\sim}{k}_{\beta_1\sigma} \cdot \underset{\sim}{R}_{\beta_2\beta_1} \right] \right\} \underset{\sim}{G}_{\beta_1\sigma} \, , \tag{4.8}$$

$$\underset{\sim}{G}_{\beta_1\beta_2} \cong \left[\mathfrak{I} - \hat{\underset{\sim}{R}}_{\beta_1\beta_2} \, \hat{\underset{\sim}{R}}_{\beta_1\beta_2} \right] \left[\exp\left(i \, n_r(\underset{\sim}{z}_{\beta_1}) k \, R_{\beta_1\beta_2} \right) \right] \Big/ R_{\beta_1\beta_2} \, . \tag{4.9}$$

Here $\underset{\sim}{k}_{\beta_1\sigma}$ is the wave number vector at $\underset{\sim}{z}_{\beta_1}$ for a ray originating at $\underset{\sim}{z}_\sigma$. This represents a generalization approximate to curved ray paths of the notation used in I. We observe that [see Eqs. (2.13)]

$$\underset{\sim}{k}_{\beta_1\sigma} \cdot \underset{\sim}{G}_{\beta_1\sigma} = 0 \, . \tag{4.10}$$

Use of Eqs. (4.8), (4.9) and (4.10) lets us rewrite (4.7) as the vector generalization of Eq. (I3.25):

$$\underset{\sim}{G}_{\alpha\sigma} = \underset{\sim}{G}_{\alpha\sigma}{}^{0} + \int \underset{\sim}{G}_{\alpha\beta}{}^{0} \cdot \underset{\sim}{G}_{\beta\sigma}\, \gamma(\underset{\sim}{z}_{\beta}) d^{3}z_{\beta} \,, \tag{4.11}$$

where

$$\gamma(\underset{\sim}{z}) = \rho(\underset{\sim}{z}) f_{0} + \rho^{2}(\underset{\sim}{z})\, f_{0}^{2} \int d^{3}R \left\{ \tfrac{1}{2}[1 + (\hat{\underset{\sim}{q}} \cdot \hat{\underset{\sim}{R}})^{2}\,] \right\}$$

$$\times\, g(\underset{\sim}{z};\, R) \left\{ \exp[\, i\, n_{r}(\underset{\sim}{z})\, k(R - \hat{\underset{\sim}{q}} \cdot \underset{\sim}{R})\,] \right\} \Big/ R \,. \tag{4.12}$$

is just the expression (I3.26b).

It follows from Eq. (2.4) that $\underset{\sim}{G}_{\alpha\sigma}$ [defined by Eq. (4.11)] satisfies Eq. (2.7) with

$$n^{2}(\underset{\sim}{z}) = 1 + (4\pi/k^{2})\, \gamma(\underset{\sim}{z}) \,. \tag{4.13}$$

We may therefore use the eikonal representation (2.33) for $\underset{\sim}{G}_{\alpha\sigma}$.

Equations (I3.31) and (I3.32) may now be rewritten as

$$\underset{\sim}{E}(\underset{\sim}{z}_{\alpha}) = \underset{\sim}{E}_{c}(\underset{\sim}{z}_{\alpha}) + \sum_{\beta(\neq\alpha)=1}^{N} \hat{\underset{\sim}{e}}_{\hat{\underset{\sim}{p}}_{\alpha}}(j)\, E_{\alpha\beta}(j) \,, \tag{4.14}$$

$$E_{\alpha\beta}(i) = \sum_{\ell,\, j=1}^{2} \left\{ (i \mid G_{\alpha\beta} \mid \ell)\, f_{\ell j}(\alpha\beta,\, \beta 0)\, E_{c}(\underset{\sim}{z}_{\beta},\, j) \right.$$

$$\left. + \sum_{\sigma(\neq\beta)=1}^{N} (i \mid G_{\alpha\beta} \mid \ell)\, f_{\ell j}(\alpha\beta,\, \beta\sigma)\, E_{\beta\sigma}(j) \right\} . \tag{4.15}$$

In Eq. (4.14) we have written $\hat{\underset{\sim}{e}}_{\hat{\underset{\sim}{p}}_\alpha}(j) \equiv \hat{\underset{\sim}{e}}_{\hat{\underset{\sim}{p}}_\alpha}(j;\, \alpha,\beta)$, $j = 1, 2$, to indicate the unit vectors (1.3) on the ray path joining $\underset{\sim}{z}_\beta$ and $\underset{\sim}{z}_\alpha$. [When no confusion will result we shall use the abbreviated notation of Eqs. (4.14) and (2.33).] In Eq. (4.15) we now take

$$f_{\ell j}(\alpha\beta,\, \beta\sigma) = f_0\, \hat{\underset{\sim}{e}}_{\hat{\underset{\sim}{p}}_\beta}(\ell;\, \alpha\beta) \cdot \hat{\underset{\sim}{e}}_{\hat{\underset{\sim}{p}}_\beta}(j;\, \beta\sigma)\,, \tag{4.16}$$

etc. The <u>coherent field</u> $\underset{\sim}{E}_c$ is just the quantity (3.2). It satisfies Eq. (3.3) with $\underset{\sim}{E}_I$ as the incident field. At a point $\underset{\sim}{z}$ the tangent vector to the ray path of $\underset{\sim}{E}_c$ will be written as

$$\hat{\underset{\sim}{p}}_c \equiv \hat{\underset{\sim}{p}}_c(\underset{\sim}{z})\,. \tag{4.17}$$

Then,

$$E_c(\underset{\sim}{z},\, j) \equiv \hat{\underset{\sim}{e}}_{\hat{\underset{\sim}{p}}_c}(j) \cdot \underset{\sim}{E}_c(\underset{\sim}{z})\,. \tag{4.18}$$

Finally the Green's function $(i \mid G_{\alpha\beta} \mid \ell)$ in Eqs. (4.15) is defined by Eq. (2.33).

The transport equation is now derived just as in I. To describe the coherent intensity we use the δ-function [as in I] $\delta[\hat{\underset{\sim}{p}},\, \hat{\underset{\sim}{p}}_c(\underset{\sim}{z})]$, having the property

$$\int f(\hat{\underset{\sim}{p}})\, \delta[\hat{\underset{\sim}{p}},\, \hat{\underset{\sim}{p}}_c(\underset{\sim}{z})]\, d\Omega_{\hat{\underset{\sim}{p}}} = f(\hat{\underset{\sim}{p}}_c)\,, \tag{4.19}$$

where $f(\hat{\underset{\sim}{p}})$ is nonsingular at $\hat{\underset{\sim}{p}} = \hat{\underset{\sim}{p}}_c$. Then, Eq. (I5.8) is written in the form

$$\underset{\sim}{I}(\underset{\sim}{z}_\alpha,\, \hat{\underset{\sim}{p}}) = \underset{\sim}{J}_c(\underset{\sim}{z}_\alpha)\, \delta[\hat{\underset{\sim}{p}},\, \hat{\underset{\sim}{p}}_c(\underset{\sim}{z}_\alpha)] + c\, n_r(\underset{\sim}{z}_\alpha) \int \left(\frac{\partial \Sigma_\beta}{\partial \Omega_\alpha}\right) ds(\underset{\sim}{z}_\beta)\, \underset{\sim}{U}(\alpha,\beta). \tag{4.20}$$

The first term here represents the intensity of the coherent field at $\underset{\sim}{z}_\alpha$. From Eq. (2.14) we see that

$$(J_c(\underset{\sim}{z}))_{ij} = (c/8\pi)\, n_r(\underset{\sim}{z})\; E_c^*(\underset{\sim}{z},\, i)\, E_c(\underset{\sim}{z},\, j).$$

The integral in the second term in Eq. (4.20) is taken in the reverse direction along the ray path which passes through $\underset{\sim}{z}_\alpha$ with the local tangent $\hat{\underset{\sim}{p}}$. The notation of Eq. (2.43) has been used, with $\delta\Sigma_\beta$ being the area at $\underset{\sim}{z}_\beta$ of a flux tube of ray paths which pass through $\underset{\sim}{z}_\alpha$ with tangent $\hat{\underset{\sim}{p}}$ and lie within the small solid angle $\delta\,\Omega_\alpha$.

The quantity $\underset{\sim}{U}(\alpha,\, \beta)$ is defined by [Eq. (I5.14)]

$$\underset{\sim}{U}(\alpha,\, \beta) = \underset{\sim}{G}_{\alpha\beta}^{\,*} \otimes \underset{\sim}{G}_{\alpha\beta} \left\{ [c\; n_r(\underset{\sim}{z}_\beta)]^{-1}\; \underset{\sim}{M}(\alpha\beta,\, \beta 0)\, \underset{\sim}{J}_c(\underset{\sim}{z}_\beta) + \int d^3 z_\sigma\; \underset{\sim}{M}(\alpha\beta,\, \beta\sigma)\, \underset{\sim}{U}(\beta,\, \sigma) \right\} . \tag{4.21}$$

Here $\underset{\sim}{M}$ is defined by Eqs. (1.10), (1.11), and (1.12), and we have used the index pairs $(\alpha\beta)$ and $(\beta\sigma)$ to indicate the direction of the unit vectors $\hat{\underset{\sim}{p}}$ and $\hat{\underset{\sim}{p}}'$ at $\underset{\sim}{z}_\beta$ in Eq. (1.10). The notation $\underset{\sim}{G}_{\alpha\beta}^{\,*} \otimes \underset{\sim}{G}_{\alpha\beta}$ is used to denote the open product $(i\; |G_{\alpha\beta}|\; j)^*(\ell\, |G_{\alpha\beta}|\, s)$.

From Eqs. (4.20) and (4.21) we obtain the transport equation

$$\underset{\sim}{I}(\underset{\sim}{z},\, \hat{\underset{\sim}{p}}) = \underset{\sim}{J}_c(\underset{\sim}{z})\; \delta\,[\,\hat{\underset{\sim}{p}},\, \hat{\underset{\sim}{p}}_c(\underset{\sim}{z})] + \int \left(\frac{\partial \Sigma_x}{\partial \Omega_z} \right) ds(\underset{\sim}{x})\; \underset{\sim}{G}^*(\underset{\sim}{z},\, \underset{\sim}{x}) \otimes \underset{\sim}{G}(\underset{\sim}{z},\, \underset{\sim}{x}) \times \left[\frac{n_r(\underset{\sim}{z})}{n_r(\underset{\sim}{x})} \right] \int d\Omega_{\hat{\underset{\sim}{p}}'}\; \underset{\sim}{M}(\hat{\underset{\sim}{p}}(\underset{\sim}{x}),\, \hat{\underset{\sim}{p}}')\; \underset{\sim}{I}(\underset{\sim}{x},\, \hat{\underset{\sim}{p}}') . \tag{4.22}$$

Here $\hat{\underset{\sim}{p}}(\underset{\sim}{x})$ is the tangent at $\underset{\sim}{x}$ of the ray path passing through $\underset{\sim}{z}$ with local tangent $\hat{\underset{\sim}{p}}$.

Equation (4.22) can be simplified using Eqs. (2.33), (2.39), and (2.41). Since

$$\left(\frac{\partial\Sigma_x}{\partial\Omega_z}\right) A^2(\underset{\sim}{z}, \underset{\sim}{x}) = \frac{n_r(\underset{\sim}{z})}{n_r(\underset{\sim}{x})},$$

we obtain

$$\underset{\sim}{I}(\underset{\sim}{z}, \hat{\underset{\sim}{p}}) = \hat{\underset{\sim}{I}}_c(\underset{\sim}{z})\ \delta[\hat{\underset{\sim}{p}}, \hat{\underset{\sim}{p}}_c(\underset{\sim}{z})] + \int^{\underset{\sim}{z}} ds(\underset{\sim}{x}) \left[\exp\left(-\int_{\underset{\sim}{x}}^{\underset{\sim}{z}} ds/\ell\right)\right] [n_r^2(\underset{\sim}{z})/n_r^2(\underset{\sim}{x})]$$

$$\times\ \underset{\sim}{P}(\underset{\sim}{z}, \underset{\sim}{x}) \otimes \underset{\sim}{P}(\underset{\sim}{z}, \underset{\sim}{x}) \int d\Omega_{\hat{\underset{\sim}{p}}'}\ \underset{\sim}{M}(\hat{\underset{\sim}{p}}(\underset{\sim}{x}), \hat{\underset{\sim}{p}}')\ \underset{\sim}{I}(\underset{\sim}{x}, \hat{\underset{\sim}{p}}') .$$

Both path integrals here are taken over the ray passing through $\underset{\sim}{z}$ with local tangent $\hat{\underset{\sim}{p}}$. The quantities $\underset{\sim}{P}$ here are matrices with elements

$$(i \mid P(\underset{\sim}{z}_\alpha, \underset{\sim}{z}_\beta) \mid s) = \hat{\underset{\sim}{e}}_{\hat{\underset{\sim}{p}}_\alpha}(i;\ \alpha,\beta) \cdot \underset{\sim}{P}_{\alpha\beta} \cdot \hat{\underset{\sim}{e}}_{\hat{\underset{\sim}{p}}_\beta}(s;\ \alpha,\beta), \tag{4.24}$$

where $\underset{\sim}{P}_{\alpha\beta}$ is the quantity (2.31).

To further simplify Eq. (4.23) let us define the matrix

$$\underset{\sim}{V}(\underset{\sim}{z}, \underset{\sim}{x}) \equiv \underset{\sim}{P}(\underset{\sim}{z}, \underset{\sim}{x}) \otimes \underset{\sim}{P}(\underset{\sim}{z}, \underset{\sim}{x}) . \tag{4.25}$$

This has matrix elements, for $i, j, s, t = 1, 2$,

$$(ij \mid V(\underset{\sim}{z}_\alpha, \underset{\sim}{z}_\beta) \mid st) = [\ \hat{\underset{\sim}{e}}_{\hat{\underset{\sim}{p}}_\alpha}(i) \cdot \underset{\sim}{P}_{\alpha\beta} \cdot \hat{\underset{\sim}{e}}_{\hat{\underset{\sim}{p}}_\beta}(s)\ ]$$

$$\times\ [\ \hat{\underset{\sim}{e}}_{\hat{\underset{\sim}{p}}_\alpha}(j) \cdot \underset{\sim}{P}_{\alpha\beta} \cdot \hat{\underset{\sim}{e}}_{\hat{\underset{\sim}{p}}_\beta}(s)\ ] . \tag{4.26}$$

Use of Eqs. (3.10) and (3.11) lets us obtain

$$\frac{d}{ds}\,(ij\ |V(\underset{\sim}{z},\ \underset{\sim}{x})|\ st) \;=\; (-1)^{i}\ \phi(i+1j\ \ |V|\ st) + (-1)^{j}\ \phi(ij+1\ \ |V|\ st). \tag{4.27a}$$

This is equivalent to

$$\frac{d}{ds}\ \underset{\sim}{V}(\underset{\sim}{z},\ \underset{\sim}{x}) \;=\; -\ \underset{\sim}{R}(\underset{\sim}{z})\ \underset{\sim}{V}(\underset{\sim}{z},\ \underset{\sim}{x})\ , \tag{4.27b}$$

where $\underset{\sim}{R}$ is the matrix introduced in Eq. (3.12).

Finally, then, we rewrite (4.23) as

$$\underset{\sim}{I}(\underset{\sim}{z},\ \hat{\underset{\sim}{p}}) \;=\; \underset{\sim}{I}_c(\underset{\sim}{z},\ \hat{\underset{\sim}{p}}) + \int^{\underset{\sim}{z}} ds(\underset{\sim}{x}) \left[\exp\left(-\int_{\underset{\sim}{x}}^{\underset{\sim}{z}} ds/\ell\right)\right] [n_r^{\,2}(\underset{\sim}{z})/n_r^{\,2}(\underset{\sim}{x})]$$

$$\times\ \underset{\sim}{V}(\underset{\sim}{z},\ \underset{\sim}{x}) \int d\Omega_{\hat{\underset{\sim}{p}}'}\ \underset{\sim}{M}\left(\hat{\underset{\sim}{p}}(\underset{\sim}{x}),\ \hat{\underset{\sim}{p}}'\right)\ \underset{\sim}{I}(\underset{\sim}{x},\ \hat{\underset{\sim}{p}}')\ . \tag{4.28}$$

The coherent intensity in Eq. (4.23) has here been abbreviated by writing

$$\underset{\sim}{I}_c(\underset{\sim}{z},\ \hat{\underset{\sim}{p}}) \;=\; \underset{\sim}{J}_c(\underset{\sim}{z})\ \delta[\hat{\underset{\sim}{p}},\ \hat{\underset{\sim}{p}}_c(\underset{\sim}{z})\ ]\ . \tag{4.29}$$

More generally, when there are several sources or <u>incoherent</u> sources, we may interpret $\underset{\sim}{I}_c$ in Eq. (4.28) as the intensity of unscattered radiation.

Equation (4.23) represents the integral form of Eq. (3.14). Indeed, on differentiating (4.28) along the ray path with tangent $\hat{\underset{\sim}{p}}$, we obtain the differential form (3.14). [Comparison of Eqs. (3.9) and (4.28) makes this obvious.]

V. DISCUSSION OF THE COHERENT INTENSITY

Let us first suppose that scattering can be neglected so that $\underset{\sim}{B} = 0$ in Eq. (3.14). In this case we have only the coherent intensity $\underset{\sim}{I}_c$ of Eq. (4.28). This satisfies the differential equation

$$\frac{d}{ds}\, \underset{\sim}{I}_c(\underset{\sim}{z}, \hat{\underset{\sim}{p}}) + \left(\frac{1}{L} + \underset{\sim}{R} \right) \underset{\sim}{I}_c(\underset{\sim}{z}, \hat{\underset{\sim}{p}}) = 0 . \tag{5.1}$$

We consider the case of a single coherent source, so through any point $\underset{\sim}{z}$ there passes only one ray path [with the exception of possible singular points].

In terms of the Stokes parameters [see Eq. (3.17)]

$$\begin{aligned} I &\equiv I_1 + I_2 \\ Q &\equiv I_1 - I_2 \\ U &= I_3 \\ V &= I_4 , \end{aligned} \tag{5.2}$$

Eq. (5.1) becomes [in the remainder of this section we drop the subscript "c" from $\underset{\sim}{I}_c$]

$$\frac{dI}{ds} + \frac{1}{L}\, I = 0 , \tag{5.3}$$

$$\frac{dV}{ds} + \frac{1}{L}\, V = 0 , \tag{5.4}$$

$$\frac{d\underset{\sim}{\psi}}{ds} + \frac{1}{L}\, \underset{\sim}{\psi} + \underset{\sim}{r}\, \underset{\sim}{\psi} = 0 . \tag{5.5}$$

Here we have written

$$\underset{\sim}{\psi} \equiv \begin{pmatrix} Q \\ U \end{pmatrix}, \tag{5.6}$$

$$\underset{\sim}{r} \equiv 2i\,\phi\,\sigma_y \equiv .2\,\phi \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \tag{5.7}$$

If $\underset{\sim}{z}_2$ and $\underset{\sim}{z}_1$ are two points on a given ray path, we may integrate Eq. (5.3):

$$I(\underset{\sim}{z}_2, \hat{\underset{\sim}{p}}_2) = \left\{ \exp\left[- \int_{\underset{\sim}{z}_1}^{\underset{\sim}{z}_2} ds/\ell \right] \right\} \left(n_r^{\,2}(2)/n_r^{\,2}(1) \right) I(\underset{\sim}{z}_1, \hat{\underset{\sim}{p}}_1)$$

$$\equiv T(2, 1)\, I(\underset{\sim}{z}_1, \hat{\underset{\sim}{p}}_1) . \tag{5.8}$$

In the second writing here we have introduced $T(2, 1)$ for the function which translates I at $\underset{\sim}{z}_1$ to the point $\underset{\sim}{z}_2$.

From Eq. (5.4) we evidently obtain

$$V(\underset{\sim}{z}_2, \hat{\underset{\sim}{p}}_2) = T(2, 1)\, V(\underset{\sim}{z}_1, \hat{\underset{\sim}{p}}_1) . \tag{5.9}$$

To integrate Eq. (5.5) we first define

$$\Phi(2, 1) \equiv 2 \int_{\underset{\sim}{z}_1}^{\underset{\sim}{z}_2} \phi\, ds . \tag{5.10}$$

Then the integrated form of (5.5) is

$$Q(\underset{\sim}{z}_2, \hat{\underset{\sim}{p}}_2) = T(2, 1)\, [\cos \Phi(2,1)\, Q(\underset{\sim}{z}_1, \hat{\underset{\sim}{p}}_1) + \sin \Phi(2,1)\, U(\underset{\sim}{z}_1, \hat{\underset{\sim}{p}}_1)] ,$$

$$U(\underset{\sim}{z}_2, \hat{\underset{\sim}{p}}_2) = T(2, 1)\, [\cos \Phi(2,1)\, U(\underset{\sim}{z}_1, \hat{\underset{\sim}{p}}_1) - \sin \Phi(2,1)\, Q(\underset{\sim}{z}_1, \hat{\underset{\sim}{p}}_1)] . \tag{5.11}$$

The above results let us write the solution to Eq. (5.1) in the form

$$\underset{\sim}{I}_c(\underset{\sim}{z}_2, \hat{\underset{\sim}{p}}_2) = T(\underset{\sim}{z}_2, \underset{\sim}{z}_1)\, \underset{\sim}{V}(\underset{\sim}{z}_2, \underset{\sim}{z}_1)\, \underset{\sim}{I}_c(\underset{\sim}{z}_1, \hat{\underset{\sim}{p}}_1) \,. \tag{5.12}$$

Here $T(\underset{\sim}{z}_2, \underset{\sim}{z}_1) \equiv T(2, 1)$ is defined by Eq. (5.8) and $\underset{\sim}{V}$ by Eq. (4.27). From Eq. (5.11) we can extract the matrix elements of $\underset{\sim}{V}$. These are givein in Table I for the I, Q, U, V representation and in Table II for the I_1, I_2, I_3, I_4 representation.

The energy flux vector at a point $\underset{\sim}{z}_1$ is

$$\underset{\sim}{J}(1) = \int \underset{\sim}{I}\,(\underset{\sim}{z}_1, \hat{\underset{\sim}{p}})\, d\Omega_{\hat{\underset{\sim}{p}}} \,. \tag{5.13}$$

Referring to Fig. 5, we construct a tube of ray paths passing through $\underset{\sim}{z}_1$ with solid angle $\delta\,\Omega_1$. The tube has a cross section of area $\delta\,\Sigma_0$ at some fixed reference point $\underset{\sim}{z}_0$. Differentiation of $\underset{\sim}{J}/n_r$ along the ray path then leads to the equation

$$\begin{aligned} \frac{d}{ds}\left[\frac{\underset{\sim}{J}(1)}{n_r(1)}\right] &= \frac{d}{ds}\int (\underset{\sim}{I}/n_r)\, d\Omega_1 \\ &= \frac{d}{ds}\int (\underset{\sim}{I}/n_r)\left(\frac{\partial\Sigma_0}{\partial\Omega_1}\right)^{-1} d\Sigma_0 \,. \end{aligned} \tag{5.14}$$

Now, according to Eq. (2.43),

$$\begin{aligned} \left(\frac{\partial\Sigma_0}{\partial\Omega_1}\right) &= A_{01}^{\;2}\,\frac{n_r(0)}{n_r(1)} \\ &= A_{10}^{\;2}\,\frac{n_r(0)}{n_r(1)} \,. \end{aligned} \tag{5.15}$$

Substitution of this into (5.14) and use of Eqs. (5.3) and (2.26) leads to the differential equation

$$\frac{d}{ds}\left(\frac{\underset{\sim}{J}(\underset{\sim}{z})}{n_r(\underset{\sim}{z})}\right) = -\left(\frac{1}{\ell} + \frac{\nabla^2 S_r}{n_r}\right)\left(\frac{\underset{\sim}{J}}{n_r}\right) - \underset{\sim}{R}\left(\frac{\underset{\sim}{J}}{n_r}\right). \tag{5.16}$$

The scalar flux

$$J(\underset{\sim}{z}) = \int I(\underset{\sim}{z}, \hat{\underset{\sim}{p}})\, d\Omega_{\hat{\underset{\sim}{p}}} \tag{5.17}$$

is seen from (5.16) to satisfy the equation

$$\frac{d}{ds}\left(\frac{J}{n_r}\right) = -\left[\frac{1}{\ell} + \frac{\nabla^2 S_r}{n_r}\right]\left(\frac{J}{n_r}\right). \tag{5.18}$$

This agrees with Eq. (37), p. 116, of Born and Wolf.[7,15]

A slightly different version of the above discussion can be given as follows. We first integrate both sides of Eq. (5.8) over the solid angles of $\hat{\underset{\sim}{p}}_1$. There results

$$\int I(\underset{\sim}{z}_2, \hat{\underset{\sim}{p}}_2)\, d\Omega_1 = T(2, 1)\, J(1). \tag{5.19}$$

Now,

$$d\Omega_1 = d\Omega_2 \left(\frac{\partial\Sigma_0}{\partial\Omega_2}\right)\left(\frac{\partial\Sigma_0}{\partial\Omega_1}\right)^{-1}$$

$$= d\Omega_2\ (A_{01}{}^2/A_{02}{}^2)\ [n_r(2)/n_r(1)].$$

From Eq. (2.26) we obtain

$$A_{01}{}^2/A_{02}{}^2 = \exp\left[\int_{\underset{\sim}{z}_1}^{\underset{\sim}{z}_2} \frac{\nabla^2 S_r}{n_r}\, ds\right]. \tag{5.20}$$

Substitution into Eq. (5.19) leads to

$$J(2) = [n_r(2)/n_r(1)] \left\{ \exp\left[- \int_{\underset{\sim}{z}_1}^{\underset{\sim}{z}_2} \left(\frac{1}{\ell} + \frac{\nabla^2 S_r}{n_r} \right) ds \right] \right\} J(1), \tag{5.21}$$

in agreement with the similar result of Ref. 7 and an obvious integral of Eq. (5.18).

The result (5.21) permits us to integrate the set of Eqs. (5.16) in the form

$$\underset{\sim}{J}(\underset{\sim}{z}) = J(\underset{\sim}{z})\, \underset{\sim}{V}(\underset{\sim}{z}, \underset{\sim}{z}_1)\, \hat{\underset{\sim}{J}} \,. \tag{5.22}$$

Here $J(\underset{\sim}{z})$ is specified by Eq. (5.21), $\underset{\sim}{V}$ by Eq. (4.27), and $\hat{\underset{\sim}{J}}$ is a constant column matrix specifying the polarization at $\underset{\sim}{z}_1$:

$$\hat{\underset{\sim}{J}} = \begin{pmatrix} \hat{J}_1 \\ \hat{J}_2 \\ \hat{J}_3 \\ \hat{J}_4 \end{pmatrix} . \tag{5.23}$$

Since $J(\underset{\sim}{z})$ is the total flux, we must have

$$\hat{J}_1 + \hat{J}_2 = 1 \tag{5.24}$$

in Eq. (5.23).

VI. RADAR BACKSCATTER IN THE DWBA

In this Section we discuss radar backscatter in the distorted wave Born approximation (DWBA). This is illustrated in Fig. 6 where the transmitting and receiving antenna is located a point $\underset{\sim}{y}$ very far from a plasma scatterer. In the DWBA only a single scattering is assumed to occur. This means that the flux of energy may be obtained from Eq. (4.22) as [assuming no coherent backscatter]

$$\underset{\sim}{J}(\underset{\sim}{y}) = \int \underset{\sim}{I}(\underset{\sim}{y}, \hat{\underset{\sim}{p}})\, d\Omega_{\hat{\underset{\sim}{p}}}$$

$$= \int d^3x\, \underset{\sim}{G}^*(\underset{\sim}{y}, \underset{\sim}{x}) \otimes \underset{\sim}{G}(\underset{\sim}{y}, \underset{\sim}{x}) \frac{n_r(\underset{\sim}{y})}{n_r(\underset{\sim}{x})} \int d\Omega_{\hat{\underset{\sim}{p}}'}\, \underset{\sim}{M}\left(\hat{\underset{\sim}{p}}(\underset{\sim}{x}), \hat{\underset{\sim}{p}}'\right) \underset{\sim}{I}_c(\underset{\sim}{x}, \hat{\underset{\sim}{p}}'). \tag{6.1}$$

Since

$$\underset{\sim}{I}_c(\underset{\sim}{x}, \hat{\underset{\sim}{p}}') = \underset{\sim}{J}_c(\underset{\sim}{x})\, \delta\left(\underset{\sim}{p}', \hat{\underset{\sim}{p}}_c(x)\right),$$

this may be re-written in the form

$$\underset{\sim}{J}(\underset{\sim}{y}) = \int d^3x\, \exp\left[-\int_{\underset{\sim}{x}}^{\underset{\sim}{y}} ds/\ell\right] \underset{\sim}{V}(\underset{\sim}{y}, \underset{\sim}{x})\, \underset{\sim}{M}\left(-\hat{\underset{\sim}{p}}_c(\underset{\sim}{x}), \hat{\underset{\sim}{p}}_c(\underset{\sim}{x})\right)$$

$$\times \frac{n_r(\underset{\sim}{y})}{n_r(\underset{\sim}{x})} A^2(\underset{\sim}{y}, \underset{\sim}{x})\, \underset{\sim}{J}_c(\underset{\sim}{x}). \tag{6.2}$$

Nos, from Eqs. (1.10) and (1.11) we see that

$$\underset{\sim}{M}\left(-\hat{\underset{\sim}{p}}_c(\underset{\sim}{x}), \hat{\underset{\sim}{p}}_c(\underset{\sim}{x})\right) = \sigma_g(-1)\, \underset{\sim}{\mathcal{P}}, \tag{6.3}$$

where

$$(ij \mid \mathcal{P} \mid st) = (-1)^{i+j}\, \delta_{is}\, \delta_{jt} \tag{6.4}$$

in the representation of Eq. (1.11).

We shall assume the incident radiation to be a plane electromagnetic wave. The quantity $\underset{\sim}{J}_c(\underset{\sim}{x})$ has, according to Eq. (5.22), the form

$$\underset{\sim}{J}_c(\underset{\sim}{x}) = J_c(\underset{\sim}{x})\, \underset{\sim}{V}(\underset{\sim}{x}, \underset{\sim}{y})\, \hat{\underset{\sim}{J}}\,, \tag{6.5}$$

where $\hat{\underset{\sim}{J}}$ has the form (5.23) and specifies the polarization of the incident radiation. As the point $\underset{\sim}{x}$ moves outside the plasma along an incident ray path, $J_c(\underset{\sim}{x})$ assumes the <u>constant</u> value $J_c(\underset{\sim}{y})$.

Since $\underset{\sim}{y}$ lies outside the plasma, $n_r(\underset{\sim}{y}) = 1$. When $\underset{\sim}{x}$ and $\underset{\sim}{y}$ are both outside the plasma

$$A(\underset{\sim}{y}, \underset{\sim}{x}) = |\underset{\sim}{x} - \underset{\sim}{y}|^{-1}.$$

Thus,

$$A^2(\underset{\sim}{x}, \underset{\sim}{y})y^2 \to 1, \qquad x \ll y,$$

as $\underset{\sim}{x}$ moves outside the plasma along an incident ray path. Therefore, we may write

$$J_c(\underset{\sim}{y})\, A^2(\underset{\sim}{x}, \underset{\sim}{y})y^2 \exp\left[- \int_{\underset{\sim}{y}}^{\underset{\sim}{x}} ds/\ell \right] = \frac{J_c(\underset{\sim}{x})}{n_r(\underset{\sim}{x})}\,. \tag{6.6}$$

That is, the quantity on the left satisfies the differential equation (5.18) and the appropriate boundary condition on an incident ray path.

The above results let us write Eq. (6.2) in the form

$$\underset{\sim}{J}(\underset{\sim}{y}) = (y^{-2}) \int d^3x\, \sigma_g(-1)\, J_c^{\,2}(\underset{\sim}{x}) \left[n_r^{\,2}(\underset{\sim}{x})\, J_c(\underset{\sim}{y}) \right]^{-1} \left[\underset{\sim}{V}(\underset{\sim}{y}, \underset{\sim}{x})\, \underset{\sim}{\rho}\, \underset{\sim}{V}(\underset{\sim}{x}, \underset{\sim}{y})\hat{\underset{\sim}{J}} \right]. \tag{6.7}$$

In the usual Born approximation $\underset{\sim}{V} = \underset{\sim}{1}$, the unit matrix, $n_r(\underset{\sim}{x}) = 1$, and $J_c(\underset{\sim}{x}) = J_c(\underset{\sim}{y})$. Thus, in the Born approximation

$$\underset{\sim}{J}(\underset{\sim}{y}) = \left[J_c(\underset{\sim}{y}) \underset{\sim}{\mathcal{P}} \hat{\underset{\sim}{J}}/y^2 \right] \int d^3x \, \sigma_g(-1) \, . \qquad (6.8)$$

To further simplify Eq. (6.7) let us use the I, Q, U, V representation of Eq. (5.2). In this representation

$$\underset{\sim}{\mathcal{P}} \begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = \begin{pmatrix} I \\ Q \\ -U \\ -V \end{pmatrix} . \qquad (6.9)$$

With the grouping of terms implied by Eqs. (5.3), (5.4) and (5.5), we may write

$$\underset{\sim}{I} \equiv \begin{pmatrix} I \\ V \end{pmatrix} \otimes \begin{pmatrix} Q \\ U \end{pmatrix}$$

$$\underset{\sim}{\mathcal{P}} = \underset{\sim}{\sigma}_z \otimes \underset{\sim}{\sigma}_z$$

$$\underset{\sim}{R} = 2i \, \phi \; \underset{\sim}{1} \otimes \underset{\sim}{\sigma}_y \, , \qquad (6.10)$$

where

$$\underset{\sim}{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

Also,

$$\underset{\sim}{V}(2,1) = \exp\left[-i \, \underset{\sim}{\phi}(2,1) \; \underset{\sim}{1} \otimes \underset{\sim}{\sigma}_y \right] , \qquad (6.11)$$

where $\underset{\sim}{\phi}$ is defined by Eq. (5.10). We see that $\underset{\sim}{\mathcal{P}}^2 = \underset{\sim}{1} \otimes \underset{\sim}{1}$ and

$$\underset{\sim}{\mathcal{P}} \, \underset{\sim}{R} \, \underset{\sim}{\mathcal{P}} = -\underset{\sim}{R} \, .$$

Thus,

$$\mathcal{P}_{\sim} \, \underset{\sim}{V}(\underset{\sim}{y}, \underset{\sim}{x}) \, \mathcal{P}_{\sim} = [\underset{\sim}{V}(\underset{\sim}{x}, \underset{\sim}{y})]^{-1} , \tag{6.12}$$

and we may re-write Eq. (6.7) as

$$\underset{\sim}{J}(\underset{\sim}{y}) = \left(\mathcal{P}_{\sim} \hat{\underset{\sim}{J}}/y^2\right) \int d^3x \, \sigma_g(-1) \, J_c^{\,2}(\underset{\sim}{x}) [n_r^{\,2}(\underset{\sim}{x}) \, J_c(\underset{\sim}{y})]^{-1} . \tag{6.13}$$

The lack of cross-polarization on backscatter characteristic of the Born approximation is therefore also found in the DWBA. We emphasize, of course, that we have shown this only in the eikonal form of the DWBA and also when the approximation (1.9) is valid for the pair correlation function.

VII. THE DIFFUSION APPROXIMATION

When the cross section $\sigma_g(\hat{\underset{\sim}{p}} \cdot \hat{\underset{\sim}{p}}')$ is sufficiently strongly peaked in the forward direction the scattering function $\underset{\sim}{B}$ can be simplified by making the diffusion approximation. The appropriate form was derived in II. We quote this here.

Using the representation (3.17) and polar coordinates for $\hat{\underset{\sim}{p}}$ [that is, writing $\underset{\sim}{I} = \underset{\sim}{I}(\underset{\sim}{x}, \theta, \phi)$] with $\hat{\underset{\sim}{k}}$ as polar axis, we have

$$\underset{\sim}{B}(\underset{\sim}{x}, \theta, \phi) = 2\,\sigma_t\,\underset{\sim}{P}\left[(\sin\theta)^{-1}\frac{\partial \underset{\sim}{I}}{\partial\phi}\right] + \sigma_t\frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial \underset{\sim}{I}}{\partial\mu}\right] + \sigma_t(1-\mu^2)^{-1}\frac{\partial^2 \underset{\sim}{I}}{\partial\phi^2}\,. \tag{7.1}$$

Here

$$\sigma_t = \frac{1}{3}\int d\Omega_{\hat{\underset{\sim}{p}}'}\,(1 - \hat{\underset{\sim}{p}} \cdot \hat{\underset{\sim}{p}}')\,\sigma_g(\hat{\underset{\sim}{p}} \cdot \hat{\underset{\sim}{p}}')\,, \tag{7.2}$$

and $\underset{\sim}{P}$ is the matrix with elements

$$P_{23} = -P_{13} = \cos\theta\,,$$

$$P_{31} = -P_{32} = 2\cos\theta\,, \tag{7.3}$$

all others vanishing.

TABLE I

Matrix elements of $\underset{\sim}{V}(\underset{\sim}{z}_2, \underset{\sim}{z}_1)$ in I, Q, U, V representation. Here $\bar{\phi} \equiv \bar{\phi}(1,2)$ in the notation of Eq. (5.10).

	I(1)	Q(1)	U(1)	V(1)
I(2)	1	0	0	0
Q(2)	0	$\cos \bar{\phi}$	$\sin \bar{\phi}$	0
U(2)	0	$-\sin \bar{\phi}$	$\cos \bar{\phi}$	0
V(2)	0	0	0	1

TABLE II

Matrix elements of $\underset{\sim}{V}(\underset{\sim}{z}_2, \underset{\sim}{z}_1)$ in I_1, I_2, I_3, I_4 representation. Here $\phi \equiv \phi(1,2)$ in the notation of Eq. (5.10).

	$I_1(1)$	$I_2(1)$	$I_3(1)$	$I_4(1)$
$I_1(2)$	$\frac{1}{2}(1 + \cos\phi)$	$\frac{1}{2}(1 - \cos\phi)$	$\frac{1}{2}\sin\phi$	0
$I_2(2)$	$\frac{1}{2}(1 - \cos\phi)$	$\frac{1}{2}(1 + \cos\phi)$	$-\frac{1}{2}\sin\phi$	0
$I_3(2)$	$-\sin\phi$	$\sin\phi$	$\cos\phi$	0
$I_4(2)$	0	0	0	1

FOOTNOTES AND REFERENCES

* This work was supported in part by the U. S. Atomic Energy Commission and in part by a Grant from the Air Force Office of Scientific Research.

1. K. M. Watson, J. Math. Phys. 10, 688 (1969). This paper will henceforth be referred to as I.

2. K. M. Watson, Electromagnetic Wave Scattering Within a Plasma in the Transport Approximation, submitted to Phys. Fluids. This paper will henceforth be referred to as II.

3. J. L. Peacher and K. M. Watson, Doppler Shift in Frequency in the Transport of Electromagnetic Waves Through an Underdense Plasma, UCRL-19212, to appear in J. Math. Phys. This paper will henceforth be referred to as III.

4. The derivations given in I and III are an adaptation to Maxwell's equations of a general derivation for quantum systems given by K. M. Watson, Phys. Rev. 118, 886 (1960).

5. To our knowledge the first published paper deriving a transport equation from a wave equation is that of L. L. Foldy, Phys. Rev. 67, 107 (1945). Foldy assumed uncorrelated point scatterers. A transport theory for scalar waves was published by Yu. N. Barabanenkov and V. N. Finkel'berg, Sov. Phys. JETP 26, 587 (1968). These authors began with a Bethe-Salpeter equation for the product of two wave amplitudes. A different derivation was given by P. E. Stott, J. Phys. A, 1, 675 (1968). Stott used the Wigner transformation to transform the field autocorrelation function (density matrix) into a transport equation.

6. See, for example, S. Chandrasekhar, Radiative Transfer (Oxford University Press, London, 1950), or V. V. Sobolev, A Treatise on Radiative Transfer (D. Van Nostrand Co., Inc., Princeton, N. J., 1963).

7. Max Born and Emil Wolf, Principles of Optics (Pergamon Press, New York, 1959), Chap. 3.

8. J. Van Bladel, IRE Trans. A and P , Nov. 1961, p. 563. To use (2.5a) in Eq. (2.3), a spherical cavity of very small radius is excluded at $\underset{\sim}{x} = \underset{\sim}{r}$ for the $\nabla_x \nabla_x$ term.

9. Born and Wolf, Ref. 8, considered only the case of a real refractive index; that is, they took $n_i = S_i = 0$.

10. This is proven, for example, in P. M. Morse and H. Feshbach, Methods of Theoretical Physics (McGraw-Hill Book Co., New York, 1953), p. 873.

11. A derivation, showing that no approximation need be made in obtaining Eq. (3.3), was given by N. Francis and K. M. Watson, Phys. Rev. 92, 291 (1953), [see in particular the appendix to this paper].

12. The reader who is not interested in following this derivation may omit Section IV. The principal result is the integral form of Eq. (3.14) given by Eq. (4.23) or (4.28).

13. The corresponding quantum mechanical derivation was given in Ref. 4.

14. Equations of I will be referred to as Eq. (I3.1), etc.

15. Since the refractive index was assumed to be real in Ref. 7, the ℓ^{-1} term of Eq. (5.5) did not occur there.

FIGURE CAPTIONS

Fig. 1. Illustration of the conditions for scattering considered in this paper.

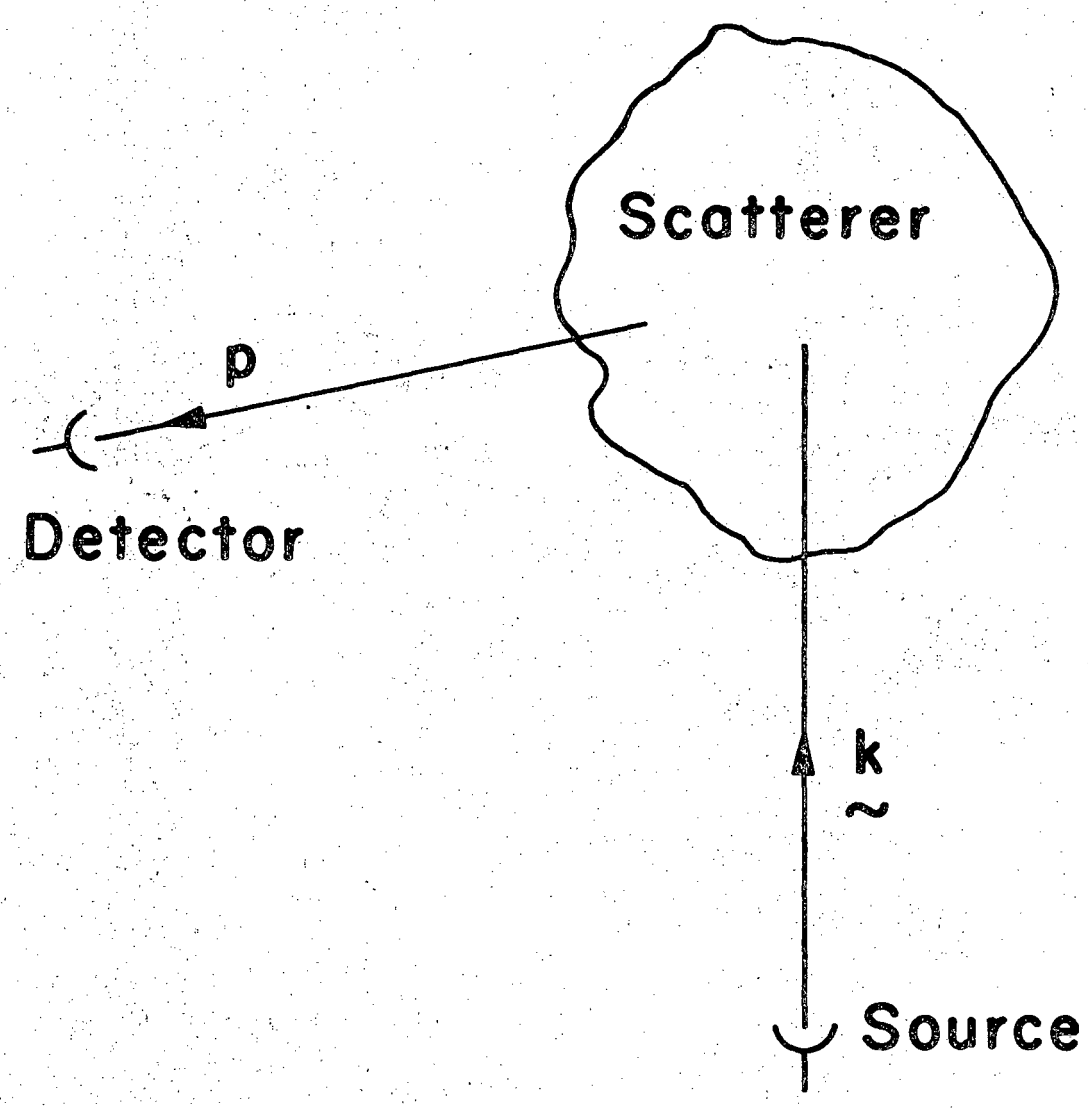
Fig. 2. A narrow tube of ray paths all passing through the point $\underset{\sim}{z}_0$.

Fig. 3. The intensity at $\underset{\sim}{z}_0$ results from sources at points $\underset{\sim}{z}_\lambda$ in the flux tube.

Fig. 4. Illustration of the unit vectors introduced in Eqs. (3.19) and (3.20).

Fig. 5. Illustration of the flux tube on which Eq. (5.14) describes the intensity.

Fig. 6. Backscatter, using the same antenna for transmitting and receiving.



XBL701-2139

Fig. 1

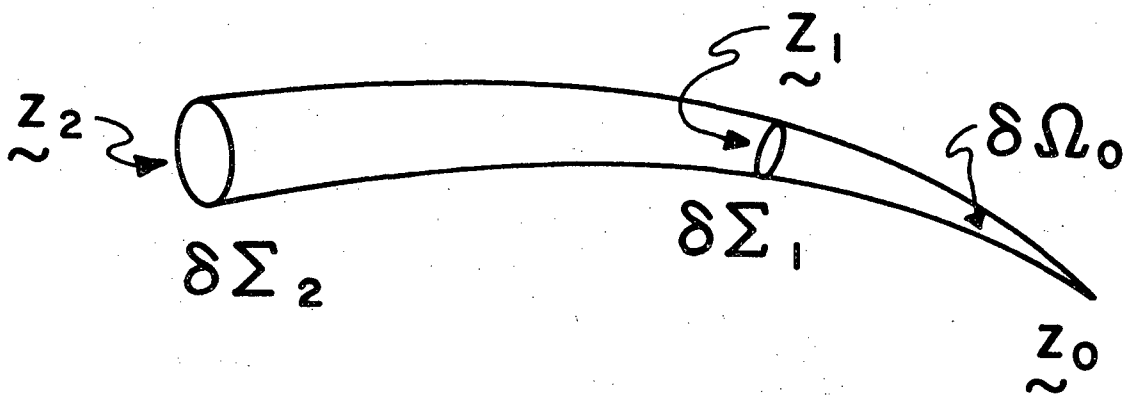


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Fig. 2

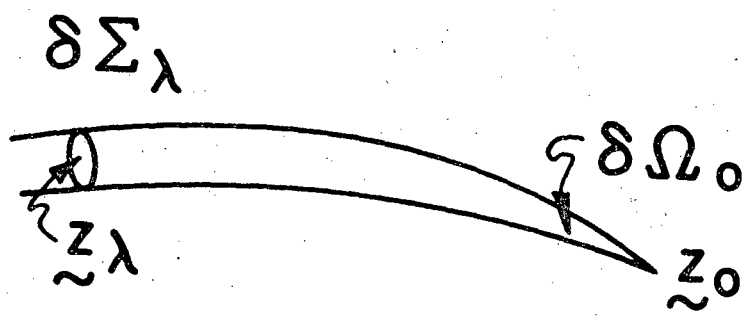

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Fig. 3

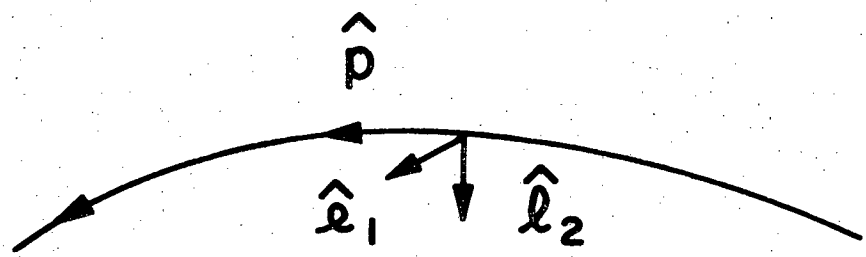

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Fig. 4

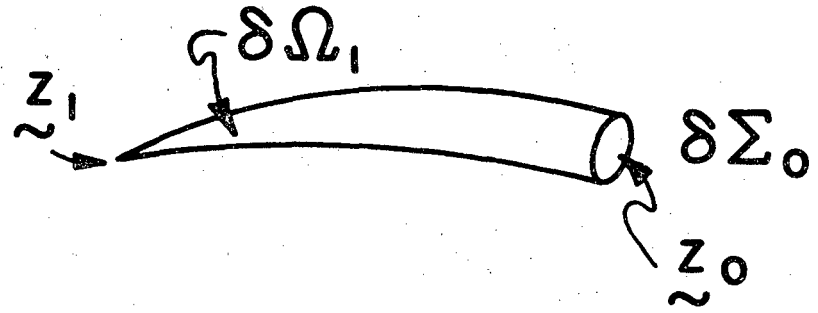

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Fig. 5

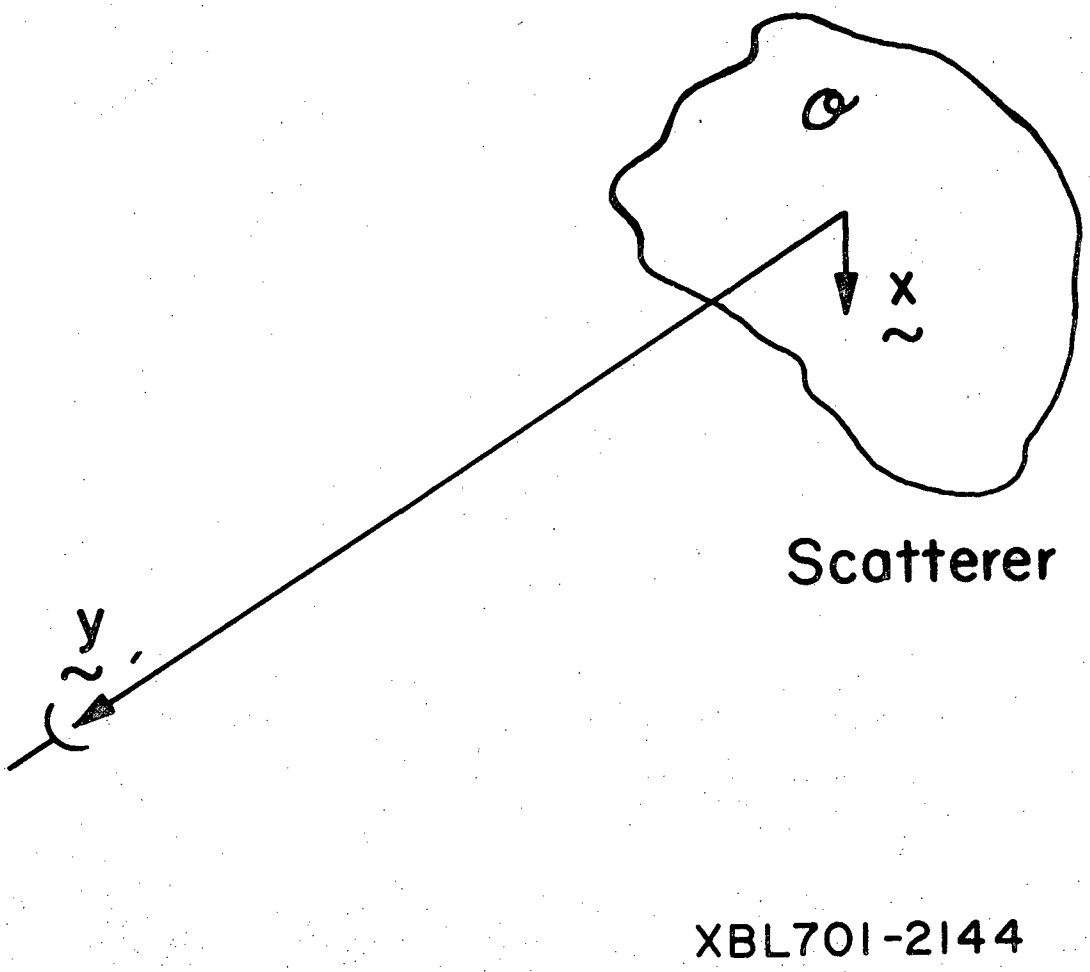

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Fig. 6

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